M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique

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M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 26, nº 6 (1992), p. 673-708

<http://www.numdam.org/item?id=M2AN_1992__26_6_673_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 26, n° 6, 1992, p. 673 à 708)

A PARALLEL SPLITTING-UP METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS TO NAVIER-STOKES EQUATIONS (*)

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Communicated by R TEMAM

Abstract — *The traditional splitting-up method or fractional step method is stuitable for sequential computing This means that the computing of the present fractional step needs the value of the previous fractional steps In this paper we propose a new splitting-up scheme for which the computing of the fractional steps is independent of each other and therefore can be computed by parallel processors We have proved the convergence estimates of this scheme both for steady state and nonsteady state linear and nonlinear problems To use finite element method to solve Navier-Stokes problems it is always difficult to handle the zero-divergent finite element spaces Here, by using the splitting-up method we can use the usual finite element spaces to solve it Moreover, the proposed method can solve the steady and nonsteady state Navier-Stokes problem by only solving some one dimensional linear systems All these one dimensional systems are independent of each other, so they can be computed by parallel processors*

Résumé — *La méthode des pas fractionnaires est souhaitable pour le calcul séquentiel Le pas, à une étape donnée, se calcule en fonction du pas à l'étape précédente On propose ici, un nouveau schéma de calcul, qui permet de déterminer les pas fractionnaires indépendamment les uns des autres On montre la convergence de la methode, et on donne une estimation du rang de convergence Enfin on applique la méthode proposée pour résoudre des problèmes de Navier-Stokes en utilisant la méthode des éléments finis usuelle On est amené alors à résoudre des systèmes linéaires, à une dimension, indépendants, et qui peuvent être traités par des processeurs en parallèle*

(*) Received for publication March 13, 1990, accepted October 3, 1991
AMS (MOS) *subject classifications* 65N30, 65N50
Supported by the Academy of Finland and TEKES

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1 INTRODUCTION

In the 1950's, the alternating-direction method was proposed by Douglas, Peaceman and Rachford [3, 13] Later, Soviet mathematicians such as Marchuk and Yanenko proposed the so-called local-one-dimensional method, [12, 19] Both these methods are generally called fractional step methods or splitting-up methods Because all these methods reduce multi-dimensional problems into a series of 1-D problems, it offers a greater efficiency and simplicity in solving the multidimensional problems These classical splitting-up methods cannot be used for parallel processors as the computing of the present fractional step always needs the value of the previous fractional step

In this paper we propose some new splitting-up schemes for which the computation of the fractional steps are independent of each other and therefore their computations can be carried out by parallel processors As proved in a paper by Tai and Neittaanmaki [15], each of the fractional steps can again be computed by parallel processors for a class of equations The number of processors depending on how many lines are used to solve the equation on each direction If the computation is reduced to S steps and in each step we solve the equation on L lines, then we need $S \times L$ processors to solve the problem We have proved the convergence estimates of this scheme both for steady state and nonsteady state linear and nonlinear problems

In this paper we first study the convergence properties of the proposed parallel splitting-up method in the case of time independent problems This first part is organized as follows

1 Introduction
2 Linear elliptic problems
3 Nonlinear elliptic problems
4 Applications of the parallel splitting-up method to steady state Navier-Stokes problems

In the second part of this paper we discuss the convergence property for nonsteady state problems The study is organized as follows

5 Parallel splitting-up methods for linear evolution equations
6 The quasilinear evolution equation and its parallel splitting-up method
7 Parallel splitting-up methods for evolution Navier-Stokes equations

In Section 2 we first estimate the condition number of the iteration matrix in the case where the split matrices mutually commute and then we can see the convergence rate of the parallel splitting iterative method Next, the Chebyshev acceleration technique is discussed In Section 3 the convergence properties of the parallel splitting-up methods are studied in the case of nonlinear strongly monotone operators

When using finite element methods to solve Navier-Stokes problems it is always difficult to handle the divergent free finite element spaces. By using the splitting-up method we can use the usual one dimensional finite element spaces to solve the Navier-Stokes equations. Moreover, the proposed method can solve the steady and nonsteady state nonlinear Navier-Stokes problems by only solving a series of linearized one dimensional problems and all these one dimensional problems are independent of each other, so they can be computed by parallel processors. In Section 4 we prove the convergence of the parallel splitting-up method for steady state Navier-Stokes problems.

The idea of splitting the divergence free condition from the Navier-Stokes equations was originally introduced by Chorin and Temam [1, 16]. This method is now refered to as the splitting-up method or projection method for the Navier-Stokes equations [14, 18]. In Temam [18] it is proved that the splitting-up method is convergent in two dimensional problems, but no convergence order is given. For three dimensional problems only a subsequence was proved to convergent to the true solution. Moreover the splitting method in [18] is not a parallel one.

In Section 5 the method is applied to linear parabolic problems. First order and second order schemes are proposed. Again the multidimensional problems are reduced to one dimensional problems. In Section 6 we give a short description of quasilinear evolution problems. In Section 7 we consider evolution Navier-Stokes problems. For an outline of the schemes of this paper, we refer to [11].

2. LINEAR ELLIPTIC PROBLEMS

In this section we consider the linear equation

$$Ax = f \tag{2.1}$$

which is obtained by discretizing elliptic boundary value problems. We assume that A is a symmetric positive definite $N \times N$ matrix and that $N = N(h)$, where h is the discretization parameter. Furthermore, we assume that A can be split into $A = A_1 + \cdots + A_m$. For the splitting of the matrix A we refer to [5, 12, 19]. On the use of the splitting-up method to solve (2.1), see for example references in [12]. In this paper we shall propose some new methods for which the computation of the fractional steps can be performed by parallel processors in solving (2.1). We first propose the following algorithm :

ALGORITHM 2.1 : *(Parallel splitting-up method with finite τ) :*

Step 1. *Choose an initial approximation $x^0 \in \mathbb{R}^N$ and a parameter $\tau > 0$ large enough.*

Step 2. *If x^j is known, compute the fractional step value $x^{j+\frac{i}{2m}}$ $(i = 1, \ldots, m)$ in the following parallel way :*

$$(I + \tau A_i)\, x^{j+\frac{i}{2m}} = \left(I - \tau \sum_{k=1,\, k \neq i}^{m} A_k\right) x^j + \tau f\,, \quad i = 1, \ldots, m\,. \tag{2.2}$$

Step 3. *Choose a parameter ω_j and set*

$$x^{j+1} = \frac{\omega_j}{m} \sum_{i=1}^{m} x^{j+\frac{i}{2m}} + (1 - \omega_j)\, x^j\,. \tag{2.3}$$

For many problems, see [15] for example, (2.2) is a series of independent one dimensional problems, so they can be computed by parallel processors again. Next, we discuss the convergence of Algorithm 2.1 under the following assumptions :

(A1) The matrices A_i, $(i = 1, \ldots, m)$ are symmetric positive definite and there exists $\beta > 0$ and $\alpha > 0$, such that the eigenvalues λ_{in} of A_i, $n = 1, \ldots, N$, $i = 1, \ldots, m$ satisfy :

$$0 < \alpha \leq \lambda_{in} \leq \beta\,. \tag{2.4}$$

(A2) The A_i commute with each other, i.e. $A_i A_j = A_j A_i$. Therefore the matrices $A_i (i = 1, \ldots, m)$ have the same eigenfunctions $\{\psi_n\}$:

$$A_i\, \psi_n = \lambda_{in}\, \psi_n\,, \quad i = 1, \ldots, m\,, \quad n = 1, \ldots, N\,. \tag{2.5}$$

(A3) There exists a positive number $c_1 < \infty$ such that

$$\max_{1 \leq n \leq N} \frac{\lambda_{jn}}{\lambda_{in}} \leq c_1 \quad \forall i, j = 1, \ldots, m\,, \tag{2.6}$$

where c_1 may depend on h.

Let $e^{j+\frac{i}{2m}} = x^{j+\frac{i}{2m}} - x$ and $e^j = x^j - x$ denote the errors. From (2.2) we get

$$e^{j+\frac{i}{2m}} = (I + \tau A_i)^{-1} \left(I - \tau \sum_{k=1,\, k \neq i}^{m} A_k\right) e^j\,.$$

By (2.3) and (2.2),

$$\begin{aligned} e^{j+1} &= (1 - \omega_j)\, e^j + \frac{\omega_j}{m} \sum_{i=1}^{m} (I + \tau A_i)^{-1} \left(I - \tau \sum_{k=1,\, k \neq i}^{m} A_k\right) e^j \\ &= (1 - \omega_j)\, e^j + \frac{\omega_j}{m} \sum_{i=1}^{m} (I + \tau A_i)^{-1} (I + \tau A_i - \tau A)\, e^j \\ &= (1 - \omega_j)\, e^j + \frac{\omega_j}{m} \sum_{i=1}^{m} (I - \tau (I + \tau A_i)^{-1} A)\, e^j \\ &= (I - \omega_j B_\tau)\, e^j\,. \end{aligned} \tag{2.7}$$

Here

$$B_\tau = \frac{\tau}{m} \sum_{i=1}^{m} (I + \tau A_i)^{-1} A . \tag{2.8}$$

Thus the convergence of Algorithm 2.1 depends on the parameter ω_j and the condition number of B_r. For the condition number of B_r, we have the following theorem :

THEOREM 2.1 : *Under the assumptions* (A1), (A2) *and* (A3), *we have*

$$\begin{aligned}\overline{\lim_{r \to \infty}}\, p(\tau) &\leq \frac{1}{m^2} \max_{1 \leq n \leq N} \left(\sum_{i=1}^{m} \lambda_{in}^{-1}\right)\left(\sum_{i=1}^{m} \lambda_{in}\right) \\ &\leq c_1 . \end{aligned}\tag{2.9}$$

Here $p(\tau)$ *is the condition number of* B_τ *and* $\overline{\lim_{r \to \infty}}\, p(\tau) \equiv \lim_{r \to \infty} \sup_{r > 0} p(\tau)$.

Proof : Under the condition (A2) we know that A_i, $i = 1, \ldots, m$ has the same eigenfunction system $\{\psi_n\}$ with A. For simplicity, we write λ_i $(i = 1, \ldots, m)$ to denote λ_{in} which is the eigenvalue of A_i corresponding to the eigenfunction ψ_n. Thus, the corresponding eigenvalue of B_τ is :

$$\begin{aligned}\mu &= \frac{\tau}{m} \sum_{i=1}^{m} \frac{1}{1 + \tau\lambda_i} \sum_{k=1}^{m} \lambda_k \\ &= \frac{\tau}{m} \sum_{i=1}^{m} \sum_{k=1}^{m} \frac{\lambda_k}{1 + \tau\lambda_i} \\ &= \frac{1}{m} \sum_{i=1}^{m} \sum_{k=1}^{m} \left(\frac{\tau\lambda_i}{1 + \tau\lambda_i} + \frac{\tau\lambda_k - \tau\lambda_i}{1 + \tau\lambda_i}\right) \\ &= \sum_{i=1}^{m} \frac{\tau\lambda_i}{1 + \tau\lambda_i} + \frac{1}{2m} \sum_{i=1}^{m} \sum_{k=1}^{m} \left(\frac{\tau(\lambda_k - \lambda_i)}{1 + \tau\lambda_i} + \frac{\tau(\lambda_i - \lambda_k)}{1 + \tau\lambda_k}\right) \\ &= \sum_{i=1}^{m} \frac{\tau\lambda_i}{1 + \tau\lambda_i} + \frac{1}{2m} \sum_{i=1}^{m} \sum_{k=1}^{m} \frac{\tau^2(\lambda_k - \lambda_i)^2}{(1 + \tau\lambda_i)(1 + \tau\lambda_k)} \\ &= \sum_{i=1}^{m} \frac{\tau\lambda_i}{1 + \tau\lambda_i} + \frac{1}{2m} \sum_{i=1}^{m} \sum_{k=1}^{m} \frac{\tau\lambda_i}{1 + \tau\lambda_i} \frac{\tau\lambda_k}{1 + \tau\lambda_k} \left(\frac{\lambda_k - \lambda_i}{\sqrt{\lambda_k \lambda_i}}\right)^2 . \end{aligned}\tag{2.10}$$

As the function $\dfrac{x}{1+x}$ is increasing on $(0, \infty)$, we get by (2.4) and (2.10)

$$\mu \leq m \frac{\tau\beta}{1 + \tau\beta} + \frac{1}{2m} \left(\frac{\tau\beta}{1 + \tau\beta}\right)^2 \sum_{i=1}^{m} \sum_{k=1}^{m} \left(\sqrt{\frac{\lambda_k}{\lambda_i}} - \sqrt{\frac{\lambda_i}{\lambda_k}}\right)^2 . \tag{2.11}$$

But

$$\begin{aligned}\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\sqrt{\frac{\lambda_k}{\lambda_i}}-\sqrt{\frac{\lambda_i}{\lambda_k}}\right)^2 &= \sum_{i=1}^{m}\sum_{k=1}^{m}\left(\frac{\lambda_k}{\lambda_i}+\frac{\lambda_i}{\lambda_k}-2\right)\\ &= 2\sum_{i=1}^{m}\sum_{k=1}^{m}\frac{\lambda_k}{\lambda_i}-2m^2\\ &= 2\left(\sum_{i=1}^{m}\lambda_i^{-1}\right)\left(\sum_{i=1}^{m}\lambda_i\right)-2m^2 \qquad (2.12)\\ &\leqslant 2s-2m^2 .\end{aligned}$$

Here

$$s=\max_{1\leqslant n\leqslant N}\left(\sum_{i=1}^{m}\lambda_{in}^{-1}\right)\left(\sum_{i=1}^{m}\lambda_{in}\right). \qquad (2.13)$$

By using the facts that $s \geqslant m^2$ and $\frac{\tau\beta}{1+\tau\beta}<1$, we get from (2.11) and (2.12)

$$\begin{aligned}\mu &\leqslant \left(\frac{\tau\beta}{1+\tau\beta}\right)\left(m+\frac{\tau\beta}{1+\tau\beta}\left(\frac{s}{m}-m\right)\right)\\ &\leqslant \frac{\tau\beta}{1+\tau\beta}\,\frac{s}{m}. \qquad (2.14)\end{aligned}$$

On the other hand, by (2.10)

$$\begin{aligned}\mu &\geqslant m\,\frac{\tau\alpha}{1+\tau\alpha}+\frac{1}{2m}\left(\frac{\tau\alpha}{1+\tau\alpha}\right)^2\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\sqrt{\frac{\lambda_k}{\lambda_i}}-\sqrt{\frac{\lambda_i}{\lambda_k}}\right)^2\\ &\geqslant m\,\frac{\tau\alpha}{1+\tau\alpha}. \qquad (2.15)\end{aligned}$$

Consequently, the condition number $p(\tau)$ of B_τ satisfies

$$1\leqslant p(\tau)\leqslant\frac{\mu_{\max}}{\mu_{\min}}\leqslant\frac{\left(\frac{\tau\beta}{1+\tau\beta}\right)\frac{s}{m}}{m\,\frac{\tau\alpha}{1+\tau\alpha}}=\frac{\beta}{\alpha}\,\frac{(1+\tau\alpha)}{(1+\tau\beta)}\,\frac{s}{m^2}. \qquad (2.16)$$

Thus

$$\overline{\lim_{\tau\to\infty}}\, p(\tau)\leqslant\frac{1}{m^2}s .$$

Noting that

$$s=\max_{1\leqslant n\leqslant N}\left(\sum_{i=1}^{m}\lambda_{in}^{-1}\right)\left(\sum_{i=1}^{m}\lambda_{in}\right)=\max_{1\leqslant n\leqslant N}\sum_{i=1}^{m}\sum_{k=1}^{m}\frac{\lambda_{kn}}{\lambda_{in}}\leqslant m^2c_1 ,$$

we obtain

$$\overline{\lim_{\tau \to \infty}}\, p(\tau) \leqslant \frac{1}{m^2} s \leqslant c_1 . \tag{2.17}$$

□

Remark 2.1 : Usually authors let $\tau \to 0$ in the iterations to make Algorithm 2.1 convergent. From the above consideration we see that the condition number corresponding to $\tau \to \infty$ is better than the condition number corresponding to $\tau \to 0$. In (2.16), let $\tau \to 0$, then $1 \leqslant \overline{\lim_{\tau \to 0}}\, p(\tau) \leqslant \frac{\beta}{\alpha} \frac{s}{m^2}$. Usually $\frac{\beta}{\alpha}$ is a very big factor.

As we can see, when $\tau \to \infty$,

$$\lim_{\tau \to \infty} B_\tau = \lim_{\tau \to \infty} \frac{1}{m} \sum_{i=1}^{m} \left(\frac{I}{\tau} + A_i \right)^{-1} A = \frac{1}{m} \sum_{i=1}^{m} A_i^{-1} A \equiv B . \tag{2.18}$$

Consequently, we can propose the following algorithm, which can be regarded as a limiting case of Algorithm 2.1 as $\tau \to \infty$.

ALGORITHM 2.2 : *(Improved parallel splitting-up method) :*

Step 1. *Choose an initial approximation* $x^0 \in \mathbb{R}^N$.

Step 2. *If* x^j *is known, compute* $x^{j + \frac{i}{2m}}$ $(i = 1, \ldots, m)$ *by parallel processors as follows :*

$$A_i x^{j + \frac{i}{2m}} = f - \sum_{k=1,\, k \neq i}^{m} A_k x^j , \quad i = 1, \ldots, m . \tag{2.19}$$

Step 3. *Choose a parameter* ω_j *and set*

$$x^{j+1} = \frac{\omega_j}{m} \sum_{i=1}^{m} x^{j + \frac{i}{2m}} + (1 - \omega_j) x^j . \tag{2.20}$$

For different i the computations of (2.19) are independent and can be computed by parallel processors. Moreover, for each i, (2.19) is a series of independent one dimensional problems. This means the computations of (2.19) for different i can be computed by parallel processors and for each i they can again be computed by parallel processors. Obviously, the error of Algorithm 2.2 behaves in the following way :

$$e^{j+1} = (I - \omega_j B) e^j . \tag{2.21}$$

THEOREM 2.2 : *The condition number* p *of the matrix* B *in* (2.21) *is bounded by the constant* c_1 *as defined in* (2.6) :

$$p \leq c_1 . \qquad \square$$

When we know the condition number, the convergence of Algorithm 2.1 and 2.2 still depends on the choice of the parameter ω_j. When we choose ω_j to have the same value for all j, the iteration is called stationary ; otherwise we call it unstationary. For the stationary iteration the following result is known from [12, p. 92].

THEOREM 2.3 : *Let the lower and upper bounds of the eigenvalue* μ_n *of* B *(or* B_τ*) be* a *and* b*, respectively :*

$$0 < a \leq \mu_n \leq b . \tag{2.22}$$

Set $\omega_j = \omega = \dfrac{2}{a+b}$. *Then the asymptotic convergence rate of Algorithm* 2.1 *(resp. Algorithm* 2.2*) is* $2/p(\tau)$ *(resp.* $2/p$*).* $\square$

For the nonstationary iteration the Chebyshev acceleration method is always used. Under the assumption (2.22) we define

$$P_s(\lambda) = \frac{T_s\left(\dfrac{b+a-2\lambda}{b-a}\right)}{T_s\left(\dfrac{b+a}{b-a}\right)}, \quad \lambda \in (a, b), \tag{2.23}$$

where

$$T_s = \frac{(y+\sqrt{y^2-1})^s + (y-\sqrt{y^2-1})^s}{2}, \quad |y| \leq 1$$

is the Chebyshev polynomial of degree s. Consequently $P_s(\lambda)$ is a polynomial of degree s. Then we choose ω_j to be the inverse of the zero point of $P_s(\lambda)$, i.e.

$$\omega_j = \frac{2}{b+a-(b-a)\cos\dfrac{(2j-1)\pi}{2s}}, \quad j = 1, \ldots, s . \tag{2.24}$$

Moreover, let $\omega_{ks+j} = \omega_j$, $1 \leq j \leq s$, $k = 1, 2, \ldots$ Then the convergence of this kind of iteration is also known [12, p. 92] :

THEOREM 2.4 : *Let* s *be large enough and the condition number* $p \gg 1$ *and let the parameter* ω_j *be selected as in* (2.24). *Then the asymptotic* *convergence rate of Algorithm* 2.2 (*resp. Algorithm* 2.1) *is* $\frac{2}{\sqrt{p}}$ $\left(resp. \frac{2}{\sqrt{p(\tau)}}\right)$.

3. NONLINEAR ELLIPTIC PROBLEMS

Let V and H be two Hilbert spaces. Morever, let $((.,.))$ and $(.,.)$ be the inner product in V and H, respectively, and $\|.\|$, $|.|$ the corresponding norms. We assume that V is dense in H and the imbedding mapping from V to H is continuous, i.e. $V \hookrightarrow H \hookrightarrow V^*$. Here V^* is the dual space of V. Let $A : V \to V^*$ be a nonlinear strongly monotone Lipschitz continuous mapping [7]. Then, from the theory of monotone operators we know that the equation

$$Au = f \tag{3.1}$$

has a unique solution for any $f \in H$ [7, Theorem 18.5].

As in [17], we suppose A can be split into $A = A_1 + \cdots + A_m$ and

(B1) A_i $(i = 1, \ldots, m)$ are Lipschtiz continuous mappings from the Hilbert space V_i to V_i^*. Here $V \subset V_i \subset H \subset V_i^* \subset V^*$ and $V = \bigcap_{i=1}^{m} V_i$. The solution u of (3.1) satisfies the regularity hypothesis : $A_i u \in H$, $i = 1, \ldots, m$.

(B2) There exists $r_i > 0$ such that

$$(A_i w - A_i v, w - v) \geq r_i |w - v|^2 ,$$

$$\text{for} \quad w, v \in V_i , \quad A_i w , \quad A_i v \in H . \tag{3.2}$$

(B3) Let $f \in H$ and $f = \sum_{i=1}^{m} f_i$, here $f_i \in H$. In particular we can have $f_1 = f$ and $f_i = 0$ $(i = 2, \ldots, m)$.

A parallel splitting-up algorithm can be constructed in the following way :

ALGORITHM 3.1 : *(Parallel splitting-up algorithm for the nonlinear problem) :*

Step 1. *Choose an initial approximation* $u^0 \in H$ *and a parameter* $\tau > 0$ *small enough.*

Step 2. *If u^j is known, compute $u^{j+\frac{i}{2m}} \in V_i$ $(i = 1, \dots, m)$ in the following parallel way :*

$$u^{j+\frac{i}{2m}} - u^j + \tau A_i u^{j+\frac{i}{2m}} = \tau f_i \,, \quad i = 1, \dots, m \,. \tag{3.3}$$

Step 3. *Set*

$$u^{j+1} = \frac{1}{m} \sum_{i=1}^{m} u^{j+\frac{i}{2m}} \,. \tag{3.4}$$

We remark that a similar idea to this algorithm was used in paper [10, p. 206] in connection with variational inequalities.

THEOREM 3.1 : *Under the conditions* (B1), (B2) *and* (B3) *we have the following error estimates :*

$$|u^n - u|^2 \leq \varepsilon_n \quad n = 1, 2, \dots \,.$$

Here

$$\varepsilon_n = e^0(1 + \tau r)^{-n} + \tau^2 \delta \sum_{k=0}^{n-1} (1 + 2\,\tau r)^{k-n} \,, \quad n = 1, 2, \dots \,,$$

$$e^0 = |u^0 - u|^2 \,,$$

and

$$r = \min_{1 \leq i \leq m} r_i \,, \quad \delta = \frac{1}{m} \sum_{i=1}^{m} |f_i - A_i u| \,.$$

Proof : Let $e^{j+\frac{i}{2m}} = u^{j+\frac{i}{2m}} - u$, $e^j = u^j - u$. By (3.3)

$$e^{j+\frac{i}{2m}} - e^j + \tau A_i u^{j+\frac{i}{2m}} - \tau A_i u = \tau (f_i - A_i u) \,. \tag{3.5}$$

Taking the inner product with $2\, e^{j+\frac{i}{2m}}$ for both sides of (3.5) and using the equality :

$$2\left(e^{j+\frac{i}{2m}} - e^j, e^{j+\frac{i}{2m}}\right) = \left|e^{j+\frac{i}{2m}}\right|^2 - |e^j|^2 + \left|e^{j+\frac{i}{2m}} - e^j\right|^2 , \tag{3.6}$$

we can get

$$\left|e^{J+\frac{i}{2m}}\right|^2 - |e^J|^2 + \left|e^{J+\frac{i}{2m}} - e^J\right|^2 + 2\tau\left(A_i u^{J+\frac{i}{2m}} - A_i u, u^{J+\frac{i}{2m}} - u\right)$$

$$= 2\tau\left(f_i - A_i u, e^{J+\frac{i}{2m}}\right)$$

$$= 2\tau(f_i - A_i u, e^J) + 2\tau\left(f_i - A_i u, e^{J+\frac{i}{2m}} - e^J\right). \qquad (3.7)$$

Summing up (3.7) for $i = 1, \ldots, m$ and using (B2) and the inequality

$$|e^{J+1}| = \left|\frac{1}{m}\sum_{i=1}^{m} e^{J+\frac{i}{2m}}\right| \leqslant \frac{1}{m}\sum_{i=1}^{m}\left|e^{J+\frac{i}{2m}}\right|$$

$$\leqslant \frac{1}{m}\sqrt{m}\left(\sum_{i=1}^{m}\left|e^{J+\frac{i}{2m}}\right|^2\right)^{1/2},$$

we obtain

$$m|e^{J+1}|^2 - m|e^J|^2 + \sum_{i=1}^{m}\left|e^{J+\frac{i}{2m}} - e^J\right|^2 + 2m\tau r|e^{J+1}|^2 \leqslant$$

$$\leqslant 2\tau\sum_{i=1}^{m}\left(f_i - A_i u, e^{J+\frac{i}{2m}} - e^J\right)$$

$$\leqslant 2\tau\sum_{i=1}^{m}|f_i - A_i u|\left|e^{J+\frac{i}{2m}} - e^J\right|$$

$$\leqslant \tau^2\sum_{i=1}^{m}|f_i - A_i u|^2 + \sum_{i=1}^{m}\left|e^{J+\frac{i}{2m}} - e^J\right|^2. \qquad (3.8)$$

Therefore,

$$(m + 2m\tau r)|e^{J+1}|^2 \leqslant m|e^J|^2 + \tau^2\sum_{i=1}^{m}|f_i - A_i u|^2$$

or

$$|e^{J+1}|^2 \leqslant \frac{1}{1+2\tau r}|e^J|^2 + \frac{\tau^2}{1+2\tau r}\delta.$$

Finally, by induction we get

$$|e^n|^2 \leqslant (1+2\tau r)^{-n}|e^0| + \tau^2\delta\sum_{k=0}^{n-1}(1+2\tau r)^{k-n} = \varepsilon_n. \qquad (3.9)$$

□

THEOREM 3.2 : *For any error tolerance* $\varepsilon \geqslant 0$, *there exists a* $\tau > 0$ *and an iteration number* $N \in \mathbb{N}$ *(the set of positive integer numbers) such that*

$$|u^N - u| \leqslant \varepsilon . \tag{3.10}$$

Proof : The result can be obtained from Theorem 3.1. By taking $\tau > 0$ small enough, for any $\varepsilon > 0$, we can ensure that the second term in (3.9) is less than $\varepsilon/2$. Letting $n \to \infty$ we can also ensure that the first term is less than $\varepsilon/2$. □

Remark 3.1 : We can introduce ω in (3.4), i.e.

$$u^{j+1} = \frac{\omega}{m} \sum_{i=1}^{m} u^{j+\frac{i}{2m}} + (1-\omega) u^j . \tag{3.11}$$

For a suitably chosen ω we can improve the convergence rate.

4. APPLICATIONS OF THE PARALLEL SPLITTING-UP METHODS TO STEADY STATE NAVIER-STOKES PROBLEMS

It is well known (see [8, 18], for example) that the m-dimensional steady state incompressible fluid flow can be described by the following Navier-Stokes equations :

$$\begin{cases} -\nu\, \Delta u + \sum_{i=1}^{m} u_i D_i u + \operatorname{grad} p = f & \text{in} \quad \Omega \\ \operatorname{div} u = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega . \end{cases} \tag{4.1}$$

Here the domain $\Omega \subset \mathbb{R}^m$ is convex, bounded and open, with a Lipschtiz boundary $\partial\Omega$; $\nu > 0$ is a constant ; $u = (u_1, u_2, \ldots, u_m)$ is the velocity of the fluid, $f = (f_1, f_2, \ldots, f_m)$ is the prescribed external force and both are m-dimensional vector functions ; p is the pressure which is a scalar function and $D_i = \dfrac{\partial}{\partial x_i}$.

Let $\mathbf{L}^2(\Omega)$ be a vector Hilbert space with inner product and norm :

$$\begin{cases} (u, v) = \sum_{i=1}^{m} \int_\Omega u_i v_i \, dx \\ |u|^2 = (u, u) . \end{cases} \tag{4.2}$$

Let $\mathbf{H}_0^1(\Omega)$ be the closure of $\mathbb{C}_0^\infty(\Omega) = C_0^\infty(\Omega) \times \cdots \times C_0^\infty(\Omega)$ under the inner product

$$((u, v)) = \sum_{i=1}^{m} (D_i u, D_i v) . \tag{4.3}$$

The corresponding norm is $\|u\|^2 = ((u, u))$. In proofs, we will use spaces $\mathbf{L}^\infty(\Omega)$, $\mathbf{H}^1(\Omega)$, $\mathbf{H}^{-1}(\Omega)$, $\mathbf{H}^2(\Omega)$, $\mathbf{W}^{k,q}(\Omega)$, etc. Their definitions and associated norms should be understood according to the standard definitions. We will use $\vec{n}$ to denote the outer normal vector of $\partial\Omega$.

Set $\tilde{\mathbf{V}} = \{u \in \mathbb{C}_0^\infty(\Omega) | \operatorname{div} u = 0\}$. We use $\mathbf{H}$ to denote the closure of $\tilde{\mathbf{V}}$ in the $\mathbf{L}^2(\Omega)$ norm and use $\mathbf{V}$ to denote the closure of $\tilde{\mathbf{V}}$ in the $\mathbf{H}_0^1(\Omega)$ norm. The following lemma is well known (see, for example [18]).

LEMMA 4.1 : *The space* $\mathbf{H}$ *has the following expression*

$$\mathbf{H} = \{u \in \mathbf{L}^2, \operatorname{div} u = 0, u \cdot \vec{n}|_{\partial\Omega} = 0\},$$

and the following decomposition is valid

$$\mathbf{L}^2(\Omega) = \mathbf{H} \oplus \mathbf{H}^\perp, \tag{4.4}$$

where

$$\mathbf{H}^\perp = \{u \in \mathbf{L}^2(\Omega) : \exists p \in H^1(\Omega), u = \operatorname{grad} p\}. \tag{4.5}$$

The Navier-Stokes equations (4.1) have the following variational formulation : Find $u \in \mathbf{V}$ such that

$$\nu((u, v)) + b(u, u, v) = (f, v) \quad \forall v \in \mathbf{V}. \tag{4.6}$$

Here

$$b(u, w, v) = \sum_{i=1}^{m} \int_\Omega u_i D_i wv \, dx. \tag{4.7}$$

Later in the algorithms, we will use a symmetric form of $b(\cdot, \cdot, \cdot)$, which is defined as

$$\hat{b}(u, v, w) = 1/2(b(u, v, w) - b(u, w, v)) \quad \forall u, v, w \in \mathbf{H}^1(\Omega).$$

It is obvious that

$$\begin{aligned} b(u, v, w) &= \hat{b}(u, v, w) & \forall u \in \mathbf{H} \cap \mathbf{H}^1(\Omega) \quad & \forall v, w \in \mathbf{H}_0^1(\Omega), \\ b(u, v, v) &= 0 & \forall u \in \mathbf{H} \cap \mathbf{H}^1(\Omega) \quad & \forall v \in \mathbf{H}_0^1(\Omega), \\ \hat{b}(u, v, v) &= 0 & \forall u \in \mathbf{H} \cap \mathbf{H}^1(\Omega) \quad & \forall v \in \mathbf{H}_0^1(\Omega). \end{aligned}$$

LEMMA 4.2 : *Suppose that* $m \leq 4$. *Then* $b(u, w, v)$ *is a trilinear functional in* $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ *and there exists a constant* $c(m) > 0$, *which is only related to* m, *such that*

$$|b(u, w, v)| \leq c(m) \|u\| \|w\| \|v\|. \tag{4.8}$$

For the proof we refer to [8, 18] It is also known that the solution of (4 6) is unique under the following condition

LEMMA 4 3 *If* $m \leq 4$ *and if* ν *large enough or* f *« small » enough such that*

$$\nu^2 > c(m) \|f\|_{\mathbf{H}^{-1}(\Omega)}, \tag{4 9}$$

then the solution of (4 6) *is unique*

It is difficult to obtain the numerical solution of (4 1) There already exist a lot of research results on numerical solutions of the Navier-Stokes equations In using the finite element method for the Navier-Stokes problems, it is difficult to handle a finite element space for **V** In the literature nonconforming finite element methods or mixed finite element methods have been used [2, 4, 6]

Here we will use the splitting-up method and the finite element method to solve (4 1) in such a way that it is not necessary to construct a finite element space for **V** Moreover, in the proposed method the computation can also be done by parallel processors

In [18] the splitting-up method was applied to solve nonstationary discretized Navier-Stokes problems In this section we will propose the splitting-up method for the steady state Navier-Stokes problem (4 1) The Algorithm 4 1 given below is a two-step iteration scheme in the first step it solves a nonlinear elliptic system and in the second step it solves a Poisson equation We can prove the convergence of the algorithm requiring only the condition of Lemma 4 3 In Algorithm 4 2 we solve a linear elliptic system in the first step instead of solving a nonlinear elliptic system Under appropriate regularity assumptions about u, we also prove the convergence of Algorithm 4 2 In Algorithm 4 3 we split the problem of Algorithm 4 2 into a serie of independent one dimensional problems We will prove that, under suitable assumptions about the regularity of u, Algorithm 4 3 is also convergent

ALGORITHM 4 1 *(Nonlinear splitting-up method for Navier-Stokes equations)*

Step 1 *Choose an initial function* $u^{1/2} \in \mathbf{H}_0^1(\Omega)$, *a parameter* $\tau > 0$ *and an error tolerance* $\varepsilon_0 > 0$

Step 2 *If* $u^{j-1/2} (j \geq 1)$ *is known, solve the Poisson equation*

$$\begin{cases} \Delta p^j = \operatorname{div} u^{j-1/2} & in \quad \Omega, \\ \dfrac{\partial p^j}{\partial n} = 0 & on \quad \partial\Omega \end{cases} \tag{4 10}$$

Here $\dfrac{\partial p^j}{\partial n}$ *denotes the outer normal derivative to* $\partial\Omega$

Step 3. *Let* $u^j = u^{j-1/2} - \operatorname{grad} p^j$.
If $|u^j - u^{j+1}| \leqslant \varepsilon_0$ *then stop, otherwise go to Step* 4.

Step 4. *Solve the following nonlinear elliptic system : find* $u^{j+1/2} \in \mathbf{H}_0^1(\Omega)$ *such that*

$$(u^{j+1/2} - u^j, v) + \tau\nu((u^{j+1/2}, v)) + \tau\hat{b}(u^{j+1/2}, u^{j+1/2}, v) = = \tau(f, v) \quad \forall v \in \mathbf{H}_0^1(\Omega) \quad (4.11)$$

and go to Step 2.

THEOREM 4.1 : *If* $m \leqslant 4$ *and the assumptions of Lemma* 4.3 *are valid, then for any error tolerance* $\varepsilon_0 > 0$, *we can choose* τ *very small and the iteration number* k *very large such that* $|u^k - u| \leqslant \varepsilon_0$.

Proof : Let P_H be the orthogonal projection operator from $\mathbf{L}^2(\Omega)$ to its subspace $\mathbf{H}$. We can show that functions $u^{j-1/2}$ and u^j in Step 2 and Step 3 satisfy

$$u^j = P_H u^{j-1/2}. \quad (4.12)$$

In fact, from Lemma 4.3 we see that there exists $p^j \in H^1(\Omega)$ such that

$$u^{j-1/2} - P_H u^{j-1/2} = \operatorname{grad} p^j, \quad (4.13)$$

As $\operatorname{div} P_H u^{j-1/2} = 0$, $\left.\frac{\partial p^j}{\partial n}\right|_{\partial\Omega} = u^{j-1/2} \cdot \vec{n}|_{\partial\Omega} - P_H u^{j-1/2} \cdot \vec{n}|_{\partial\Omega} = 0$, we know that p^j is just the solution of (4.10). This proves (4.12).

Let, in the following, $e^{j+1/2} = u^{j+1/2} - u$, $e^j = u^j - u$. If we set $v = 2\, e^{j+1/2}$ in (4.11) we will get

$$\begin{aligned}
&2(e^{j+1/2} - e^j, e^{j+1/2}) + 2\,\tau\nu((e^{j+1/2}, e^{j+1/2})) + \\
&\qquad + 2\,\tau\hat{b}(u^{j+1/2}, u^{j+1/2}, e^{j+1/2}) - 2\,\tau\hat{b}(u, u, e^{j+1/2}) = \\
&= 2\,\tau(f, e^{j+1/2}) - 2\,\tau\nu((u, e^{j+1/2})) - 2\,\tau b(u, u, e^{j+1/2}) \\
&= 2\,\tau(\operatorname{grad} p, e^{j+1/2}) \\
&= 2\,\tau(\operatorname{grad} p, e^j) + 2\,\tau(\operatorname{grad} p, e^{j+1/2} - e^j) \\
&= 2\,\tau(\operatorname{grad} p, e^{j+1/2} - e^j)\,. \qquad (4.14)
\end{aligned}$$

Here we use the property that $\hat{b}(u, v, w) = b(u, v, w)$ for $u \in \mathbf{V}$, v, $w \in \mathbf{H}_0^1$ and $e^j \in \mathbf{H}$ which implies $(\operatorname{grad} p, e^j) = 0$. By using the property $\hat{b}(u^{j+1/2}, e^{j+1/2}, e^{j+1/2}) = 0$ and the well known estimate (see [18]) :

$$\|u\| \leqslant \frac{\|f\|_{-1,2}}{\nu}, \quad \text{with} \quad \|f\|_{-1,2} := \|f\|_{\mathbf{H}^{-1}(\Omega)}\,. \quad (4.15)$$

We can get

$$
\begin{aligned}
&\hat{b}(u^{j+1/2}, u^{j+1/2}, e^{j+1/2}) - \hat{b}(u, u, e^{j+1/2}) = \\
&\quad = \hat{b}(u^{j+1/2}, u^{j+1/2}, e^{j+1/2}) - \hat{b}(u^{j+1/2}, u, e^{j+1/2}) + \\
&\qquad + \hat{b}(u^{j+1/2}, u, e^{j+1/2}) - \hat{b}(u, u, e^{j+1/2}) \\
&\quad = \hat{b}(u^{j+1/2}, e^{j+1/2}, e^{j+1/2}) + \hat{b}(e^{j+1/2}, u, e^{j+1/2}) \\
&\quad = \hat{b}(e^{j+1/2}, u, e^{j+1/2}) \\
&\quad \leqslant c(m)\|u\|\,\|e^{j+1/2}\|^2 \\
&\quad \leqslant c(m)\frac{\|f\|_{-1,2}}{\nu}\|e^{j+1/2}\|^2 . \qquad (4.16)
\end{aligned}
$$

But (4.9) means that there exists β, $0 < \beta < 1$, such that

$$\beta\nu^2 = c(m)\|f\|_{-1,2} . \qquad (4.17)$$

Substituting (4.17) and (4.16) into (4.14) and also using (3.6) we get

$$
\begin{aligned}
&|e^{j+1/2}|^2 - |e^j|^2 + |e^{j+1/2} - e^j|^2 + 2\tau\nu\|e^{j+1/2}\|^2 - \\
&\qquad - 2\tau\beta\nu\|e^{j+1/2}\|^2 \leqslant \tau^2|\operatorname{grad} p|^2 + |e^{j+1/2} - e^j|^2 . \qquad (4.18)
\end{aligned}
$$

Let $\gamma > 0$ be the Sobolev constant such that

$$\|w\|^2 \geqslant \gamma|w|^2 \quad \forall w \in H_0^1(\Omega) , \qquad (4.19)$$

then we can get from (4.18)

$$(1 + 2\tau\nu\gamma(1-\beta))|e^{j+1/2}|^2 \leqslant |e^j|^2 + \tau^2|\operatorname{grad} p|^2 . \qquad (4.20)$$

But $|e^{j+1}|^2 = |P_H e^{j+1/2}|^2 \leqslant |e^{j+1/2}|^2$. Thus

$$
\begin{aligned}
|e^{j+1}|^2 &\leqslant (1 + 2\tau\nu\gamma(1-\beta))^{-1}|e^j|^2 + \\
&\qquad + \tau^2(1 + 2\tau\nu\gamma(1-\beta))^{-1}|\operatorname{grad} p|^2
\end{aligned}
$$

or

$$
\begin{aligned}
|e^k|^2 &\leqslant (1 + 2\tau\nu\gamma(1-\beta))^{-k}|e^0|^2 + \\
&\qquad + \tau^2|\operatorname{grad} p|^2 \sum_{j=0}^{k} (1 + 2\tau\nu\gamma(1-\beta))^{-j} . \qquad (4.21)
\end{aligned}
$$

As $1 - \beta > 0$, obviously we can always choose τ small enough such that

$$\tau^2|\operatorname{grad} p|^2 \sum_{j=0}^{k} (1 + 2\tau\nu\gamma(1-\beta))^{-j} \leqslant \tau^2|\operatorname{grad} p|\frac{1}{2\tau\nu\gamma(1-\beta)} \leqslant \frac{\varepsilon_0}{2} .$$

For such a $\tau > 0$, we then let k be large enough such that

$$(1 + 2\,\tau \nu \gamma (1 - \beta))^{-k}\,|e^0| \leqslant \frac{\varepsilon_0}{2}.$$

Then $|e^k| \leqslant \varepsilon_0$. This proves the theorem. □

Algorithm 4.1 involves the solving of a nonlinear elliptic system. The following algorithm reduces everything to linear elliptic systems.

ALGORITHM 4.2 : *(Linearized splitting-up method for Navier-Stokes equations) :*

Step 1, *Step* 2 *and Step* 3 *are the same as for Algorithm* 4.1.

Step 4. *Find* $u^{j+1/2} \in \mathbf{H}_0^1(\Omega)$ *such that*

$$(u^{j+1/2} - u^j, v) + \tau \nu ((u^{j+1/2}, v)) + \tau b(u^j, u^{j+1/2}, v) = \\ = \tau (f, v) \quad \forall v \in H_0^1(\Omega)\,. \quad (4.22)$$

As u^j is known, (4.22) is really a linear equation system. In order to prove the convergence of Algorithm 4.2 we need to assume that u satisfies the following condition :

(C1) The solution u of (4.1) is in $\mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ and there exists a constant $\alpha > 1$ such that

$$\nu^2 \gamma^2 = \alpha \|u\|_{1,\infty}^2\,. \quad (4.23)$$

Here γ is the constant as in (4.19), $\|u\|_{1,\infty} = \max\limits_{1 \leqslant i \leqslant m} \|D_i u\|_{0,\infty} + \|u\|_{0,\infty}$ and $\|\cdot\|_{0,\infty} = \|\cdot\|_{L^\infty(\Omega)}$.

THEOREM 4.2 : *Under condition* (C1) *Algorithm* 4.2 *is convergent in the same sense as in Theorem* 4.1.

Proof : Similar to the proof of Theorem 4.1 we can get for Algorithm 4.2

$$|e^{j+1/2}|^2 - |e^j|^2 + 2\,\tau \nu \|e^{j+1/2}\|^2 - \\ - 2\,\tau |b(u^j, u^{j+1/2}, e^{j+1/2}) - b(u, u, e^{j+1/2})| \leqslant \tau^2 |\operatorname{grad} p|\,. \quad (4.24)$$

Furthermore, because $b(u^j, e^{j+1/2}, e^{j+1/2}) = 0$, we get

$$\begin{aligned} |b(u^j, u^{j+1/2}, e^{j+1/2}) - b(u, u, e^{j+1/2})| &= \\ &= |b(e^j, u, e^{j+1/2})| \\ &\leqslant \|u\|_{1,\infty}\,|e^j|\,|e^{j+1/2}| \\ &\leqslant \varepsilon \|u\|_{1,\infty}^2\,|e^{j+1/2}|^2 + \frac{1}{4\,\varepsilon}\,|e^j|^2\,. \quad (4.25) \end{aligned}$$

Here $\varepsilon > 0$ is a constant to be chosen. If we substitute (4.25) into (4.24) we find that

$$[1 + 2\,\tau(\nu\gamma - \varepsilon\,\|u\|_{1,\infty}^2)]\,|e^{j+1/2}|^2 \leq$$
$$\leq \left(1 + \frac{\tau}{2\,\varepsilon}\right)|e^j|^2 + \tau^2|\operatorname{grad} p\,|^2 . \quad (4.26)$$

Now choosing ε such that

$$\nu\gamma - \varepsilon\,\|u\|_{1,\infty}^2 = \frac{\alpha}{4\,\varepsilon}, \quad (4.27)$$

we get from (4.23)

$$\varepsilon = \frac{\nu\gamma}{2\|u\|_{1,\infty}^2} > 0 .$$

Substituting this into (4.26) we obtain

$$\left(1 + \frac{\tau\alpha}{2\,\varepsilon}\right)|e^{j+1/2}|^2 \leq \left(1 + \frac{\tau}{2\,\varepsilon}\right)|e^j|^2 + \tau^2|\operatorname{grad} p\,|^2 . \quad (4.28)$$

As we can see under condition (**C**1) we have $\alpha > 1$. So we obtain the result of the theorem. □

In order to simplify and improve the efficiency of Algorithm 4.2, we further split all the multidimensional problems into a series of one dimensional problems. This also improves the parallel degree of the computation.

First we introduce the following split trilinear forms $b_i : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}$

$$\begin{aligned} b_i(u, w, v) &= \frac{1}{2}\sum_{k=1}^{m}\int_\Omega u_i \frac{\partial w_k}{\partial x_i} v_k\,dx - \frac{1}{2}\sum_{k=1}^{m}\int_\Omega u_i\, w_k \frac{\partial v_k}{\partial x_i}\,dx \\ &= \frac{1}{2}\int_\Omega u_i\, D_i\, wv\,dx - \frac{1}{2}\int_\Omega u_i\, wD_i\, v\,dx . \end{aligned} \quad (4.29)$$

It is obvious that

$$b_i(u, w, w) = 0 \quad \forall u, w \in \mathbf{H}_0^1(\Omega) . \quad (4.30)$$

and

$$\sum_{i=1}^{m} b_i(u, w, v) = b(u, w, v) \quad \forall u \in \mathbf{H} \cap \mathbf{H}^1,\ w, v \in \mathbf{H}_0^1(\Omega) . \quad (4.31)$$

Next we split f into $f = f_1 + \cdots + f_m$. In the following we will use the notation

$$((v, w))_i = (D_i v, D_i w) . \tag{4.32}$$

ALGORITHM 4.3 : *(A parallel linear splitting-up method for Navier-Stokes equations)*

Step 1, *Step* 2 *and Step* 3 *is the same as in Algorithm* 4.1.

Step 4. *Solve the following linear systems in parallel : Find* $u^{j+i/q} \in \mathbf{H}_0^1(\Omega)$, $q = 2(m+1)$, $i = 1, \dots, m$, *such that*

$$(u^{j+i/q} - u^j, v) + \tau\nu((u^{j+i/q}, v))_i + \tau b_i(u^j, u^{j+i/q}, v) = \\ = \tau(f_i, v) \quad \forall v \in \mathbf{H}_0^1(\Omega), \quad i = 1, \dots, m . \tag{4.33}$$

Step 5. *Set* $u^{j+1/2} = \frac{1}{m} \sum_{i=1}^{m} u^{j+i/q}$ *and go to Step* 2.

Here $b_i(\cdot, \cdot, \cdot)$ is the split trilinear form. For every i, (4.33) can again be solved by parallel processors by a one dimensional method as in [15]. In order to get the convergence of Algorithm 4.3 we need the following assumption :

(C2) The solution u of (4.1) is in $\mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ and there exists a number $\alpha > 1$ such that

$$4\nu^2\gamma_0^2 = \alpha \|u\|_{1,\infty}^2 . \tag{4.34}$$

Here $\gamma_0 = \min_{1 \le i \le m} (1, \gamma_i)$, and $\gamma_i > 0$ is the Sobolev constant such that

$$|D_i v| \geq \gamma_i |v|^2 \quad \forall v \in \mathbf{H}_0^1(\Omega) . \tag{4.35}$$

Obviously $\gamma_0 > 0$.

THEOREM 4.3 : *Under the condition* (C2) *Algorithm* 4.3 *is convergent in the same sense as in Theorem* 4.1.

Proof : Let $e^{j+i/q} = u^{j+i/q} - u$, $e^j = u^j - u$ and take $v = 2e^{j+i/q}$ in (4.33). By using (3.6), we get

$$\begin{aligned} &|e^{j+i/q}|^2 - |e^j|^2 + |e^{j+i/q} - e^j|^2 + 2\tau\nu|D_i e^{j+i/q}|^2 + \\ &\quad + 2\tau[b_i(u^j, u^{j+i/q}, e^{j+i/q}) - b_i(u, u, e^{j+i/q})] \\ &= 2\tau(f_i, e^{j+i/q}) - 2\tau\nu(D_i u, D_i e^{j+i/q}) - 2\tau b_i(u, u, e^{j+i/q}) \\ &\le 2\tau(f_i, e^j) - 2\tau\nu(D_i u, D_i e^j) - 2\tau b_i(u, u, e^j) + \\ &\quad + \tau^2 C(u, f) + |e^{j+i/q} - e^j|^2 . \end{aligned} \tag{4.36}$$

Here $C(u, f)$ is a constant which is related to u and f.

By (4.29) and (4.30) we have

$$\begin{aligned} &\left|b_i(u^j, u^{j+i/q}, e^{j+i/q}) - b_i(u, u, e^{j+i/q})\right| = \\ &= \left|b_i(e^j, u, e^{j+i/q})\right| \\ &\leq \frac{1}{2}\|u\|_{1,\infty} |e^j| |e^{j+i/q}| + \frac{1}{2}\|u\|_{0,\infty} |e^j| \|e^{j+i/q}\| \qquad (4.37) \\ &\leq \frac{\varepsilon}{2}\|D_i u\|_{0,\infty}^2 |e^{j+i/q}|^2 + \frac{1}{8\varepsilon}|e^j|^2 + \frac{\varepsilon}{2}\|u\|_{0,\infty}^2 \|e^{j+i/q}\|^2 + \frac{1}{8\varepsilon}|e^j|^2 . \end{aligned}$$

Substituting (4.37) into (4.36) and also using (4.35), we get

$$\begin{aligned} &(1 + 2\tau\gamma_0\nu - \varepsilon\tau\gamma_0\|u\|_{1,\infty}^2)|e^{j+i/q}|^2 - \left(1 + \frac{\tau}{4\varepsilon}\right)|e^j|^2 \leq \\ &\quad \leq 2\tau(f_i, e^j) - 2\tau\nu(D_i u, D_i e^j) - 2\tau b_i(u, u, e^j) + \tau^2 C(u, f) . \qquad (4.38) \end{aligned}$$

We choose ε such that

$$2\nu\gamma_0 - \varepsilon\gamma_0\|u\|_{1,\infty}^2 = \frac{\alpha}{4\varepsilon} . \qquad (4.39)$$

From condition (C2) we get $\varepsilon > 0$ and

$$\varepsilon = \frac{\nu}{\|u\|_{1,\infty}^2} . \qquad (4.40)$$

Therefore, by (4.38) we have

$$\begin{aligned} &\left(1 + \frac{\alpha\tau}{4\varepsilon}\right)|e^{j+i/q}|^2 - \left(1 + \frac{\tau}{4\varepsilon}\right)|e^j|^2 \leq \\ &\quad \leq 2\tau(f_i, e^j) - 2\tau\nu(D_i u, D_i e^j) - 2\tau b_i(u, u, e^j) + \tau^2 C(u, f) . \qquad (4.41) \end{aligned}$$

Summing up both sides of (4.41) for $i = 1, \ldots, m$ and using estimates $\sum_{i=1}^{m} |e^{j+i/q}|^2 \geq m|e^{j+1/2}|^2$ and (4.1), we obain

$$\begin{aligned} m\left(1 + \frac{\alpha\tau}{4\varepsilon}\right)|e^{j+1/2}|^2 - m\left(1 + \frac{\tau}{4\varepsilon}\right)|e^j|^2 &\leq 2\tau(\operatorname{grad} p, e^j) + \\ &+ m\tau^2 C(u, f) . \qquad (4.42) \end{aligned}$$

But $|e^{j+1}| = |P_H e^{j+1/2}| \leq |e^{j+1/2}|$ and $(\operatorname{grad} p, e^j) = 0$, so we get

$$|e^{j+1}| \leq \frac{1 + \frac{\tau}{4\varepsilon}}{1 + \frac{\alpha\tau}{4\varepsilon}}|e^j| + \tau^2\frac{C(u, f)}{1 + \frac{\alpha\tau}{4\varepsilon}} . \qquad (4.43)$$

As $\alpha > 1$, we obtain the result of the theorem. □

Remark 4.1 : In condition (C1) we require the existence of $\alpha > 1$ such that $\nu^2 \gamma^2 = \alpha \|u\|_{1,\infty}^2$. In the condition (C2) we need the existence of $\alpha > 1$ such that $4 \nu^2 \gamma_0^2 = \alpha \|u\|_{1,\infty}$. As u also is related to ν, we should show that these are not « empty conditions ». Next we will show that if Ω is smooth, f regular and small enough or ν big enough, then conditions (C1) and (C2) can be satisfied.

In fact from the theory of the steady state Navier-Stokes equations [8, 18], we know that if Ω is smooth, $f \in W^{k,q}(\Omega)$, $k \geq -1$, $1 < q < \infty$, then there exists a constant $C_0 > 0$ which is independent of ν and f such that

$$\|u\|_{k+2,q} \leq C_0 \left(\frac{\|f\|_{k,q}}{\nu} + \frac{1}{\nu} \sum_{j=1}^{m} \left\| u_j \frac{\partial u}{\partial x_j} \right\|_{k,q} + \|u\|_{0,q} \right) . \tag{4.44}$$

Here $\|\cdot\|_{k,q}$ is the norm in $\mathbf{W}^{k,q}(\Omega)$. Now take $q = 3/2$, $k = 0$. For three dimensional problems (two dimensional problems are much simpler)

$$\begin{aligned} \|u\|_{2,q} &\leq C_0 \left(\frac{\|f\|_{0,q}}{\nu} + \frac{1}{\nu} \sum_{j=1}^{3} \left\| u_j \frac{\partial u}{\partial x_j} \right\|_{0,q} + \|u\|_{0,q} \right) \\ &\leq C_0 \left(\frac{\|f\|_{0,q}}{\nu} + \frac{1}{\nu} \|u\|_{1,2} \|u\|_{0,6} + \|u\|_{0,q} \right) . \end{aligned} \tag{4.45}$$

In (4.15), we have

$$\|u\|_{1,2} \leq C \|u\| \leq C \frac{\|f\|_{-1,2}}{\nu}$$

and the Sobolev imbedding theorem implies that there exists a constant $C_1 > 0$, which is also independent of ν and f, such that

$$\|u\|_{2,q} \leq C_1 \left(\frac{\|f\|_{0,q}}{\nu} + \frac{\|f\|_{-1,2}^2}{\nu^3} \right) . \tag{4.46}$$

Taking $k = 1$ in (4.44), we can get an estimate for $\|u\|_{3,q}$ from (4.46). Similarly we can get an estimate for $\|u\|_{4,q}$. All these estimates show that if ν is very large or f is very small, then $\|u\|_{3,q}$, $\|u\|_{4,q}$, etc. will be very small. So, from the Sobolev imbedding theorem we can see that (C1) and (C2) are true for ν big enough or f small enough.

Remark 4.2 : The conditions (C1) and (C2) guarantee that the Navier-Stokes equations are of an elliptic type. From our computing experiences, we also find that when the equations have a hyperbolic property, they produce difficulties in the computations. However, as in Temam [18], if the solution of the Navier-Stokes equations is unique, then we can prove that the splitting-up solution converges to the true solution. The convergence is very slow in the hyperbolic case. In order to get a convergent solution, we also need to use a good initial solution. So in real computations, we first take a large viscosity number and get an approximate solution and then gradually reduce the viscosity number to its actual value. For each new viscosity number, we use the solution from the previous one as the initial solution. In this way we may get a convergent solution even for equations with a small viscosity number.

5. PARALLEL SPLITTING-UP METHODS FOR THE EVOLUTION EQUATIONS

In this section we will consider the following linear evolution equation :

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + A\phi = f \quad \text{in} \quad Q_T = \Omega \times [0, T] \\ \phi(0) = \phi_0 . \end{cases} \tag{5.1}$$

Here A can be either a linear differential operator or a matrix. We assume that A is time independent and A and f can be split into :

$$\begin{aligned} A &= A_1 + A_2 + \cdots + A_m \\ f &= f_1 + f_2 + \cdots + f_m . \end{aligned} \tag{5.2}$$

In this section, if A is a matrix in $\mathbb{R}^n$, then the norm used will be the Euclidean $\mathbb{R}^n$-norm. If A is assumed to be a differential operator, then in order to use the semigroup theory, the norm will indicate the L^2-norm. When A is a differential operator, the arguments used in this section are informal. The alternating-direction method or local one-dimensional method for (5.1) is already well-known, see [12, 19]. Here the splitting-up method we propose is a new one. The computations of the fractional steps are again independent of each other and so they can be done by parallel processors. In the algorithms we use f_i^j to indicate $f_i\left(\left(j+\frac{1}{2}\right)\tau\right)$ and f^j to indicate $f\left(\left(j+\frac{1}{2}\right)\tau\right)$.

ALGORITHM 5.1 : *(Parallel splitting-up method for evolution equations) :*

Step 1. *Choose a step size $\tau > 0$. If ϕ^j is already computed, compute $\phi^{j+\frac{i}{2m}}$ for $i = 1, \dots, m$ in a parallel way as*

$$\frac{\phi^{j+\frac{i}{2m}} - \phi^j}{m\tau} + A_i \phi^{j+\frac{i}{2m}} = f_i^j, \quad i = 1, \dots, m . \tag{5.3}$$

Step 2. *Set*

$$\phi^{j+1} = \frac{1}{m} \sum_{i=1}^{m} \phi^{j+\frac{i}{2m}}. \tag{5.4}$$

Step 3. *If* $T = (j+1)\tau$ *then stop, otherwise go to Step 1.*

The classical splitting-up method can always be regarded as a perturbation of some classical implicit scheme, for example, the Crank-Nicholson scheme, but it seems that Algorithm 5.1 cannot be regarded as a perturbation of the classical scheme.

THEOREM 5.1 : *If* A *is the infinitesimal generator of a* C_0*-semigroup and* A_i, $i = 1, \dots, m$ *are symmetric positive definite, then for any* $\tau > 0$ *Algorithm* 5.1 *is stable and the error is*

$$e^j = \phi(j\tau) - \phi^j = o(\tau). \tag{5.5}$$

Proof : From (5.3) and (5.4) we know

$$\begin{aligned} \phi^{j+1} &= \frac{1}{m} \sum_{i=1}^{m} \phi^{j+\frac{i}{2m}} \\ &= \frac{1}{m} \left[\sum_{i=1}^{m} (I + m\tau A_i)^{-1} \phi^j \right] + \tau \sum_{i=1}^{m} (I + m\tau A_i)^{-1} f_i^j. \end{aligned} \tag{5.6}$$

As A_i is symmetric positive definite, we have $\|(I + m\tau A_i)^{-1}\| \leq 1$. Thus

$$\|\phi^{j+1}\| \leq \|\phi^j\| + \tau \sum_{i=1}^{m} \|f_i^j\|. \tag{5.7}$$

This shows that the algorithm is stable. Next we analyse the local error of the scheme. Let us define

$$\hat{\phi}^{j+1} = \frac{1}{m} \sum_{i=1}^{m} \hat{\phi}^{j+\frac{i}{2m}}, \tag{5.8}$$

where
$$\frac{\hat{\phi}^{j+\frac{i}{2m}} - \phi(j\tau)}{m\tau} + A_i \hat{\phi}^{j+\frac{i}{2m}} = f_i^j, \quad i = 1, \dots, m. \tag{5.9}$$

i.e., $\hat{\phi}^{j+\frac{i}{2m}}$ and $\hat{\phi}^{j+\frac{i}{2m}}$ are the solutions of (5.3) and (5.4) with ϕ^j being replaced by $\phi(j\tau)$. Similar as (5.6), we have

$$\begin{aligned} \hat{\phi}^{j+1} &= \frac{1}{m} \sum_{i=1}^{m} \hat{\phi}^{j+\frac{i}{2m}} \\ &= \frac{1}{m} \left[\sum_{i=1}^{m} (I + m\tau A_i)^{-1} \phi(j\tau) \right] + \tau \sum_{i=1}^{m} (I + m\tau A_i)^{-1} f_i^j. \end{aligned} \tag{5.10}$$

By Taylor expansion we see that

$$(I + m\tau A_i)^{-1}\,\phi(j\tau) = \phi(j\tau) - m\tau A_i\,\phi(j\tau) + o(\tau^2)\left\|A_i^2\,\phi(j\tau)\right\|\,, \tag{5.11}$$

$$(I + m\tau A_i)^{-1} f_i^j = f_i^j + o(\tau)\left\|A_i f_i^j\right\|\,. \tag{5.12}$$

Substituting (5.11) and (5.12) into (5.10) and also assuming that f and ϕ are regular enough and with required bounded derivatives when A and A_i are differential operators, then we obtain

$$\begin{aligned}\hat{\phi}^{j+1} &= \frac{1}{m}\left[\sum_{i=1}^{m}(\phi(j\tau) - m\tau A_i\,\phi(j\tau))\right] + \tau\sum_{i=1}^{m} f_i^j + o(\tau^2)\\ &= \phi(j\tau) - \tau A\phi(j\tau) + \tau f^j + o(\tau^2)\,.\end{aligned} \tag{5.13}$$

From the semigroup theory, we know

$$\phi((j+1)\tau) = e^{-\tau A}\,\phi(j\tau) + \int_0^\tau e^{-(\tau - t)} f(t + j\tau)\,dt\,. \tag{5.14}$$

In order to avoid being deeply involved with the semigroup theory, we simply use $e^{-\tau A}$ to indicate the semigroup generated by $-A$. According to the semigroup theory, the following approximations are true

$$e^{-\tau A}\,\phi(j\tau) = \phi(j\tau) - \tau A\phi(j\tau) + o(\tau^2)\,, \tag{5.15}$$

$$\begin{aligned}\int_0^\tau e^{-(\tau - t)} f(t + j\tau)\,dt &= \tau\, e^{-\frac{\tau}{2}} f^j + o(\tau^3) = \\ &= \tau f^j - \frac{\tau^2}{2} A f^j + o(\tau^3) = \tau f^j + o(\tau^2)\,.\end{aligned} \tag{5.16}$$

Substituting (5.15) and (5.16) into (5.14) and comparing with (5.13), it gives

$$\left\|\hat{\phi}^{j+1} - \phi((j+1)\tau)\right\| = o(\tau^2)\,. \tag{5.17}$$

This means the local error is of second order, so the global error is of first order. □

We can improve the accuracy of the algorithm by the predictor-corrector method.

ALGORITHM 5.2 : *(Predictor-corrector method)*

Step 1. *Assuming* ϕ^j *is known, compute* $\phi^{j + \frac{i}{3m}}$ *for* $i = 1, \dots, m$ *in a parallel way as*

$$\left(I + \frac{\tau}{2} m A_i\right)\phi^{j + \frac{i}{3m}} = \phi^j + \frac{m\tau}{2} f_i^j\,, \quad i = 1, \dots, m\,. \tag{5.18}$$

Step 2. *Set*

$$\phi^{j+2/3} = \frac{1}{m} \sum_{i=1}^{m} \phi^{j + \frac{i}{3m}} . \tag{5.19}$$

Step 3. *Set*

$$\phi^{j+1} = \phi^j + \tau (f^j - A\phi^{j+2/3}) . \tag{5.20}$$

THEOREM 5.2 : *For* $\tau > 0$ *small enough Algorithm* 5.2 *is stable and*

$$\phi(j\tau) - \phi^j = o(\tau^2) . \tag{5.21}$$

Proof : From (5.18)-(5.20), it is true that

$$\begin{aligned} \phi^{j+1} &= \phi^j + \tau (f^j - A\phi^{j+2/3}) \\ &= \left(I - \frac{1}{m} \sum_{i=1}^{m} \tau A \left(I + \frac{\tau}{2} mA_i \right)^{-1} \right) \phi^j + \tau f^j - \\ &\quad - \frac{\tau^2}{2} A \sum_{i=1}^{m} \left(I + \frac{\tau}{2} mA_i \right)^{-1} f_i^j \\ &= T\phi^j + \tau f^j - \frac{\tau^2}{2} A \sum_{i=1}^{m} \left(I + \frac{\tau}{2} mA_i \right)^{-1} f_i^j . \end{aligned} \tag{5.22}$$

Here $T = I - \frac{1}{m} \sum_{i=1}^{m} \tau A \left(I + \frac{\tau}{2} mA_i \right)^{-1}$. When τ is small enough, $\|T\| \leq 1$ and the algorithm is stable. To prove the convergence, we similarly define $\hat{\phi}^{j+1}$ as the solution of (5.18)-(5.20) with ϕ^j being replaced by $\phi(j\tau)$. From this definition and also by using (5.11)-(5.12), we have

$$\begin{aligned} \hat{\phi}^{j+1} &= \phi(j\tau) + \tau (f^j - A\hat{\phi}^{j+2/3}) \\ &= T\phi(j\tau) + \tau f^j - \frac{\tau^2}{2} A \sum_{i=1}^{m} \left(I + \frac{\tau}{2} mA_i \right)^{-1} f_i^j \\ &= \left(I - \frac{1}{m} \sum_{i=1}^{m} \tau A \left(I - \frac{\tau}{2} mA_i \right) \right) \phi(j\tau) + \tau f^j - \\ &\quad - \frac{\tau^2}{2} A \sum_{i=1}^{m} \left(I - \frac{\tau}{2} mA_i \right) f_i^j + o(\tau^3) \\ &= \left(I - \tau A + \frac{\tau^2}{2} A^2 \right) \phi(j\tau) + \tau \left(f^j - \frac{\tau}{2} Af^{\prime} \right) + o(\tau^3) . \end{aligned} \tag{5.23}$$

Similar as in (5.15) the following approximation is also true

$$(I + m\tau A_i)^{-1} \phi(j\tau) = \phi(j\tau) - m\tau A_i \phi(j\tau) + \frac{\tau^2}{2} A^2 \phi(j\tau) + o(\tau^3), \tag{5.24}$$

so we can also approximate (5.14) by

$$\begin{aligned} \phi((j+1)\tau) &= e^{-\tau A} \phi(j\tau) + \tau e^{-\frac{\tau}{2}} f^j + o(\tau^3) \\ &= \phi(j\tau) - \tau A\phi(j\tau) + \frac{\tau^2}{2} A^2 \phi(j\tau) + \\ &\quad + \tau \left(f^j - \frac{\tau}{2} A f^j\right) + o(\tau^3). \end{aligned} \tag{5.25}$$

Substracting (5.25) from (5.23) produces

$$\left\| \hat{\phi}^{j+1} - \phi((j+1)\tau) \right\| = o(\tau^3). \tag{5.26}$$

This means the algorithm is locally convergent of third order and so globally convergent of second order. □

Taking a combination we can achieve second order convergence as well :

ALGORITHM 5.3 : *(Second order parallel splitting-up method)*

Step 1. *Assume* ϕ^j *is known, compute* $\phi^{j+\frac{i}{2m}}$ *in a parallel way as :*

$$\left(I + \frac{m}{2}\tau A_i\right) \phi^{j+\frac{i}{2m}} = \left(I - \sum_{k=1, k\neq i}^{m} \frac{m}{2} \tau A_k\right) \phi^j + \frac{m}{2} \tau f^j, \quad i = 1, \ldots, m. \tag{5.27}$$

Step 2. *Set*

$$\phi^{j+1} = \frac{2}{m^2}\left[\left(\frac{m^2}{2} - m\right) \phi^j + \sum_{i=1}^{m} \phi^{j+\frac{i}{2m}}\right]. \tag{5.28}$$

THEOREM 5.3 : *If* $\tau > 0$ *is small enough, then Algorithm* 5.3 *is stable and is globally convergent of second order.*

Proof : The stability can be similarly proved from the relation between ϕ^{j+1} and ϕ^j when $\phi^{j+\frac{i}{2m}}$ is eliminated from (5.27)-(5.28). Next we prove the local convergence order. Let us define $\hat{\phi}^{j+1}$ and $\hat{\phi}^{j+\frac{i}{2m}}$ to be the solution of (5.27) and (5.28) with ϕ^j being replaced by $\phi(j\tau)$. From the series expansion formula (5.11)-(5.12), we get

$$
\begin{aligned}
\hat{\phi}^{j+\frac{i}{2m}} &= \left(I + \frac{m}{2}\tau A_i\right)^{-1} \left(I - \frac{m}{2}\tau \sum_{k \neq i} A_k\right) \phi(j\tau) \\
&\quad + \frac{m}{2}\tau \left(I + \frac{m}{2}\tau A_i\right)^{-1} f^j \\
&= \left(I - \frac{m}{2}\tau A_i + \frac{m^2}{4}\tau^2 A_i^2 + o(\tau^3)\right) \left(I - \frac{m}{2}\tau \sum_{k \neq i} A_k\right) \phi(j\tau) \\
&\quad + \frac{m}{2}\tau \left(I - \frac{m}{2}\tau A_i\right) f^j + o(\tau^3) \qquad (5.29) \\
&= \left(I - \frac{m}{2}\tau A + \frac{m^2}{4}\tau^2 A_i A\right) \phi(j\tau) \\
&\quad + \frac{m}{2}\tau \left(I - \frac{m}{2}\tau A_i\right) f^j + o(\tau^3), \quad i = 1, \dots, m.
\end{aligned}
$$

Furthermore, from (5.28) we obtain

$$
\begin{aligned}
\hat{\phi}^{j+1} &= \frac{2}{m^2}\left[\left(\frac{m^2}{2} - m\right)\phi(j\tau) + \sum_{i=1}^{m} \hat{\phi}^{j+\frac{i}{2m}}\right] \\
&= \frac{2}{m^2}\left\{\left[\left(\frac{m^2}{2} - m\right) I + mI - \frac{m^2}{2}\tau A + \frac{m^2}{4}\tau^2 A^2\right]\phi(j\tau)\right. \\
&\quad \left. + \left[\frac{m^2}{2}\tau I - \frac{m^2}{4}\tau A\right] f^j\right\} + o(\tau^3) \qquad (5.30) \\
&= \left(I - \tau A + \frac{1}{2}\tau^2 A^2\right)\phi(j\tau) + \tau\left(I - \frac{1}{2}\tau A\right) f^j + o(\tau^3).
\end{aligned}
$$

This again shows that the method is locally convergent of third order and so globally convergent of second order. □

By using global extrapolation, we can also improve the convergence order for the time step. Theoretical analysis will be reported elsewhere.

Remark 5.1 : We assumed that A is time independent, which is not essential to the problem. When A is related to t, Algorithms 5.1-5.3 are still valid. Then we compute ϕ^{j+1} by taking $A((j + 1/2)\tau)$ as A.

Remark 5.2 : In the proof we have freely used operators A^2, A_i^2, $A_i A$ although for boundary value problems these operators may not be well defined because of the boundary conditions. So in order to get higher order convergence, boundary corrections may be necessary as mentioned in paper [5].

6. THE QUASILINEAR EVOLUTION EQUATION AND ITS PARALLEL SPLITTING-UP METHOD

We consider the quasilinear evolution problem

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + A(t, \phi)\,\phi = 0 & \text{in} \quad Q_T = \Omega \times [0, T]\,, \\ \phi\,|_{t=0} = \phi^0 & \text{in} \quad \Omega\,. \end{cases} \tag{6.1}$$

Here, operator A is related both to ϕ and t. We assume $A(t, \phi)$ can be split into :

$$A(t, \phi) = \sum_{i=1}^{m} A_i(t, \phi)\,. \tag{6.2}$$

Here we assume that A, $A_i(t, \phi)$ are nonnegative and smooth operators [12, p. 169]. Consequently, the solution ϕ is also smooth when ϕ_0 is smooth and Ω is smooth. We shall propose the following algorithm in order to compute the numerical solution.

ALGORITHM 6.1 : *(Parallel splitting-up method for quasilinear equation)*

Step 1. *Choose a step size $\tau > 0$. If ϕ^j is already computed, then set*

$$\bar{\phi}^{j+1} = \phi^j + \frac{\tau}{2} A(\tau j, \phi^j)\,\phi^j \tag{6.3}$$

and

$$A_i^{j+1} = A_i\left(\tau\left(j + \frac{1}{2}\right), \bar{\phi}^{j+1}\right), \quad i = 1, \ldots, m\,. \tag{6.4}$$

Step 2. *Compute $\phi^{j+\frac{i}{2m}}$ for $i = 1, \ldots, m$ in a parallel way as*

$$\frac{\phi^{j+\frac{i}{2m}} - \phi^j}{m\tau} + A_i^{j+1}\,\phi^{j+\frac{i}{2m}} = 0\,, \quad i = 1, \ldots, m\,. \tag{6.5}$$

Step 3. *Set*

$$\phi^{j+1} = \frac{1}{m}\sum_{i=1}^{m} \phi^{j+\frac{i}{2m}}\,. \tag{6.6}$$

As A_i^{j+1} is linear and time independent, Step 2 is the same as in Algorithm 5.1. It can be proved that this scheme is first order convergent and absolutely stable. A second order scheme can also be constructed by using a similar algorithm as in Algorithm 5.2 and 5.3. The proof is omitted here.

7. PARALLEL SPLITTING-UP METHODS FOR EVOLUTION NAVIER-STOKES EQUATIONS

In this section we consider the evolution Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \, \Delta u + \sum_{i=1}^{m} u_i D_i u + \operatorname{grad} p = f & \text{in} \quad Q_T = \Omega \times [0, T] \\ \operatorname{div} u = 0 & \text{in} \quad Q_T \\ u = 0 & \text{on} \quad \partial\Omega \times [0, T] \\ u(x, 0) = u_0(x) & \text{in} \quad \Omega . \end{cases} \tag{7.1}$$

We will use the same notations as in Section 4. We use $(. , .)$ and $| . |$ for the inner product and norm of $\mathbf{L}^2(\Omega)$ and use $((. , .))$ and $\| . \|$ for the inner product and norm of $\mathbf{H}_0^1(\Omega)$. They are defined in (4.2) and (4.3). Notation $((. , .))_i$ is defined in (4.32). The trilinear forms $b(. , . , .)$ and $b_i(. , . , .)$ are defined in (4.7) and (4.29). We will restrict our attention to only two and three dimensional problems. Therefore there exists $C(m) > 0$ $(m = 2, 3)$ such that

$$|b(u, v, w)| \leq C(m) \|u\| \, \|v\| \, \|w\| \quad \forall u, v, w \in \mathbf{H}_0^1(\Omega) . \tag{7.2}$$

As is well known, it is difficult to prove the uniqueness of the solution of the Navier-Stokes equations. In order to guarantee the uniqueness, the following assumption is needed as in [18, p. 304] :

$$d_4 = \frac{d_2}{\nu} + (1 + d_1^2) \left(|u_0|^2 + \frac{T d_2}{\nu} \right)^{1/2} \exp \left(\int_0^T |f'(s)| \, ds \right) < \frac{\nu^3}{C(3)^2} , \tag{7.3}$$

where

$$d_1 = |f(0)| + \nu C_0 \|u_0\|_{H^2(\Omega)} + C_1 \|u_0\|_{H^2(\Omega)}^2 , \tag{7.4}$$

$$d_2 = \|f\|_{L^\infty(0, T, V')} . \tag{7.5}$$

For the constants $C_0 > 0$, $C_1 > 0$ we refer to Temam [18].

The application of the splitting-up method to evolution Navier-Stokes problems can be found in Temam [18] and some other papers [9, 14, 20]. In [18] a two step method is considered. The first step solves a nonlinear system and the second one solves a Poisson equation with a Neumann boundary condition.

The algorithm given in Temam [18] is.

ALGORITHM 7.1 : *(Splitting-up method for evolution Navier-Stokes equations)*

Step 1. *Set* $u^0 = u_0$ *and choose a step size* $\tau > 0$.

Step 2. *If* u^j *is known, find* $u^{j+1/2} \in H_0^1(\Omega)$ *such that*

$$(u^{j+1/2} - u^j, v) + \tau\nu((u^{j+1/2}, v)) + \tau\hat{b}(u^{j+1/2}, u^{j+1/2}, v) = (\bar{f}^j, v) \quad \forall v \in \mathbf{H}_0^1(\Omega). \tag{7.6}$$

Here $\bar{f}^j = \int_{(j-1)\tau}^{j\tau} f(t)\,dt$, $\hat{b}(u, v, w) = 1/2(b(u, v, w) - b(u, w, v))$.

Step 3. *Compute* p^{j+1} *from*

$$\begin{cases} \Delta p^{j+1} = \operatorname{div} u^{j+1/2} & in \quad \Omega \\ \dfrac{\partial p^{j+1}}{\partial n} = 0 & on \quad \partial\Omega . \end{cases} \tag{7.7}$$

Step 4. *Set* $u^{j+1} = u^{j+1/2} - \operatorname{grad} p^{j+1}$. *If* $(j+1)\tau = T$ *then stop, otherwise go to Step* 2.

In Temam [18] the convergence of Algorithm 7.1 is proved only for two dimensional problems. No convergence order is given there. For three dimensional problems, it is only proved that there exists a subsequence convergent to the true solution as $\tau \to 0$. The proof in [18] is based on the fully discretized model, thus under some stability assumptions, such results are proved. For a semidiscretized model, no results were presented in [18]. In the following two algorithms, we use semidiscrete models and not only prove the convergence for two and three dimensional problems, but also find their convergence orders. The convergence order we prove here is one half. In a recent paper [14], without splitting the multidimensional problems into one dimensional problems, the author was able to prove that Scheme 7.2 below has a first order of convergence under natural assumptions.

In (7.6) we are required to solve a nonlinear system. Here we propose the following algorithm which only needs to solve linear systems.

ALGORITHM 7.2 : *(Linearized splitting-up method)*

Step 1. *Choose a step size* $\tau > 0$. *Set* $u^0 = u_0$.

Step 2. *If* u^j *is known, then find* $u^{j+1/2} \in \mathbf{H}_0^1(\Omega)$ *such that*

$$\left(\frac{u^{j+1/2} - u^j}{\tau}, v\right) + \nu((u^{j+1/2}, v)) + b(u^j, u^{j+1/2}, v) = (f^{j+1}, v) \quad \forall v \in \mathbf{H}_0^1(\Omega), \tag{7.8}$$

where $f^{j+1} = f((j+1)\tau)$.

Step 3. *Find* p^{j+1}

$$\begin{cases} \Delta p^{j+1} = \operatorname{div} u^{j+1/2} & in \quad \Omega \\ \dfrac{\partial p^{j+1}}{\partial n} = 0 & on \quad \partial\Omega . \end{cases} \tag{7.9}$$

Step 4. *Set* $u^{j+1} = u^{j+1/2} - \operatorname{grad} p^{j+1}$. *If* $(j+1)\tau = T$ *then stop, otherwise go to Step* 2.

Next we analyse the convergence of Algorithm 7.1. We need the following assumption :

(**D**1) The solution u of (7.1) is regular enough and the following relation is true

$$\nu\gamma > \alpha \|u\|_{1,\infty} .$$

Here $\gamma > 0$ is the Sobolev constant as in (4.19), $\|u\|_{1,\infty} = \max\limits_{1 \le i \le m} \|D_i u\|_{\mathbf{L}^\infty(Q_T)}$. Let

$$\begin{aligned} U^{j+1} &= u((j+1)\tau), \quad j = 0, 1, \dots \\ e^{j+1/2} &= u^{j+1/2} - U^{j+1} \\ e^j &= u^j - U^j . \end{aligned} \tag{7.10}$$

From (7.8) we see

$$\left(\frac{e^{j+1/2} - e^j}{\tau}, v\right) + \nu((e^{j+1/2}, v)) + b(u^j, u^{j+1/2}, v) - b(U^j, U^{j+1}, v) =$$
$$= (f^{j+1}, v) - \left(\frac{U^{j+1} - U^j}{\tau}, v\right) - \nu((U^{j+1}, v)) - b(U^j, U^{j+1}, v)$$
$$\forall v \in \mathbf{H}_0^1(\Omega) . \tag{7.11}$$

As

$$\frac{U^{j+1} - U^j}{\tau} = \left.\frac{du}{dt}\right|_{t=(j+1)\tau} - \left.\frac{\tau}{2}\frac{d^2u}{dt^2}\right|_{t=(j+\theta)\tau} . \tag{7.12}$$

Here $0 < \theta < 1$. We get from (7.12) that the right-hand side of (7.11) equals

$$(f^{j+1}, v) - \left(\frac{U^{j+1} - U^j}{\tau}, v\right) - \nu((U^{j+1}, v)) - b(U^{j+1}, U^{j+1}, v) -$$
$$- b(U^j - U^{j+1}, U^{j+1}, v) = (\operatorname{grad} p, v) + \frac{\tau}{2}(\lambda(u), v)$$
$$\forall v \in \mathbf{H}_0^1(\Omega) . \tag{7.13}$$

Here $|\lambda(u)|$ depends on u, u', u'' and is bounded by a constant which is independent of τ. Taking $v = 2\, e^{J+1/2}$ in (7.11), we get

$$|e^{J+1/2}|^2 - |e^J|^2 + |e^{J+1/2} - e^J|^2 + 2\,\tau\nu\,\|e^{J+1/2}\|^2 + 2\,\tau b(e^J, U^{J+1}, e^{J+1/2})$$
$$= 2\,\tau(\operatorname{grad} p, e^{J+1/2}) + \tau^2(\lambda(u), e^{J+1/2})$$
$$= 2\,\tau(\operatorname{grad} p, e^J) + 2\,\tau(\operatorname{grad} p, e^{J+1/2} - e^J) + \tau^2(\lambda(u), e^{J+1/2})$$
$$\leq \tau^2|\operatorname{grad} p|^2 + \tau^2|e^{J+1/2}|^2 + \frac{1}{4}\tau^2|\lambda(u)|^2 + |e^{J+1/2} - e^J|^2 . \tag{7.14}$$

Here we have used the fact that $(\operatorname{grad} p, e^J) = 0$ and that

$$b(u^J, u^{J+1/2}, e^{J+1/2}) - b(U^J, U^{J+1}, e^{J+1/2}) = b(e^J, U^{J+1}, e^{J+1/2}) . \tag{7.15}$$

But

$$|2\, b(e^J, U^{J+1}, e^{J+1/2})| \leq 2\|u\|_{1,\infty} |e^J|\,|e^{J+1/2}|^2$$
$$\leq \varepsilon\|u\|_{1,\infty} |e^{J+1/2}|^2 + \frac{1}{\varepsilon}|e^J|^2 . \tag{7.16}$$

Substituting this to (7.14), we obtain

$$(1 + 2\,\tau\nu\gamma - \varepsilon\tau\|u\|^2_{1,\infty} - \tau^2)|e^{J+1/2}|^2 \leq$$
$$\leq \left(1 + \frac{\tau}{\varepsilon}\right)|e^J|^2 + \tau^2(|\operatorname{grad} p|^2 + |\lambda(u)|^2) . \tag{7.17}$$

If the condition (**D1**) is satisfied, then for $\tau \leq \nu\gamma - \|u\|_{1,\infty}$ we can find $\varepsilon > 0$ such that

$$2\,\nu\gamma - \varepsilon\|u\|^2_{1,\infty} - \tau = \frac{1}{\varepsilon} . \tag{7.18}$$

Therefore we get from (7.17)

$$\left(1 + \frac{\tau}{\varepsilon}\right)|e^{J+1/2}|^2 \leq \left(1 + \frac{\tau}{\varepsilon}\right)|e^J|^2 + \tau^2(|\operatorname{grad} p|^2 + |\lambda(u)|^2) . \tag{7.19}$$

By using the property

$$|e^{J+1}| \leq |e^{J+1/2}| , \tag{7.20}$$

we get from (7.19)

$$|e^{J+1}|^2 - |e^J|^2 \leq \tau^2(|\operatorname{grad} p|^2 + |\lambda(u)|^2) . \tag{7.21}$$

Summing up relation (7.21), we obtain

$$|e^{J+1}|^2 \leq \tau\int_0^{(J+1)\tau}(|\operatorname{grad} p|^2 + |\lambda(u)|^2)\,dt + o(\tau^2) . \tag{7.22}$$

Here we use the fact that $e^0 = u_0 - u(0) = 0$.

Thus we have proved the following theorem :

THEOREM 7.1 : *If* $u(t)$ *is smooth enough and* (**D1**) *is valid, then Algorithm* 7.2 *is convergent of half order for* τ *and the error estimate* (7.22) *is true.*

In the following we will discuss the splitting of the Navier-Stokes problem into one dimensional problems. Here we will give the convergence estimate under the appropriate regularity assumption on $u(t)$ for the semidiscrete case.

ALGORITHM 7.3 : *(Linearized one dimensional splitting-up method) :*

Step 1. *Choose* $\tau > 0$. *Set* $u^0 = u_0$ *and split* $f = f_1 + \cdots + f_m$.

Step 2. *If* u^j *is known, find* $u^{j+i/q} \in \mathbf{H}_0^1(\Omega)$ $(q = 2(m+1))$, $i = 1, \dots, m$ *in a parallel way such that*

$$(u^{j+i/q} - u^j, v) + m\tau\nu((u^{j+i/q}, v))_i + m\tau b_i(u^j, u^{j+i/q}, v) = \\ = m\tau(f_i^{j+1}, v) \quad \forall v \in \mathbf{H}_0^1(\Omega), \quad i = 1, \dots, m. \tag{7.23}$$

Step 3. *Set* $u^{j+1/2} = \frac{1}{m}\sum_{i=1}^{m} u^{j+i/q}$.

Step 4. *Same as Step* 3 *in Algorithm* 7.1.

Step 5. *Same as Step* 4 *in Algorithm* 7.1.

Here the trilinear forms $b_i(\cdot, \cdot, \cdot)$, $i = 1, \dots, m$, are defined as in (4.29), and $f_i^{j+1} = f_i((j+1)\tau)$.

In order to obtain convergence we need an assumption similar to (**C2**) :

$$\text{(D2)} \qquad \|u\|_{0,\infty}^2 + \|u\|_{1,\infty}^2 \leq \gamma_0^2 \nu^2 .$$

Here $\|u\|_{0,\infty} = \|u\|_{\mathbf{L}^\infty(Q_T)}$, $\|u\|_{1,\infty} = \max_{1 \leq i \leq m} \|D_i u\|_{0,\infty}$,

$\gamma_0 = \min_{1 \leq i \leq m} (1, \gamma_i)$, $\gamma_i > 0$ is the Sobolev constant :

$$|D_i w|^2 \geq \gamma_i |w|^2 \quad \forall w \in \mathbf{H}_0^1(\Omega).$$

Obviously $\gamma_0 > 0$.

THEOREM 7.2 : *If* $u(t)$ *is regular and* (**D2**) *is valid, then Algorithm* 7.3 *is convergent of half order for* τ.

Proof : Take U^j, U^{j+1} as defined in (7.10), define $e^{j+i/q} = u^{j+i/q} - U^{j+1}$, $e^j = u^j - U^j$, and let $v = 2e^{j+i/q}$ in (7.23), we get

$$|e^{j+i/q}|^2 - |e^j|^2 + |e^{j+i/q} - e^j|^2 + 2m\tau\nu|D_i e^{j+i/q}|^2 +$$

$$+ 2 m\tau [b_i(u^j, u^{j+i/q}, e^{j+i/q}) - b_i(U^j, U^{j+1}, e^{j+i/q})]$$

$$= 2 m\tau (f_i^{j+1}, e^{j+i/q}) - 2 \tau \left(\frac{d}{dt} u((j+1)\tau), e^{j+i/q} \right)$$

$$- 2 m\tau\nu (D_i U^{j+1}, D_i e^{j+i/q})$$

$$- 2 m\tau b_i(U^j, U^{j+1}, e^{j+i/q}) + \tau^2(\lambda(u), e^{j+i/q})$$

$$\leqslant 2 m\tau (f_i^{j+1}, e^j) - 2 \tau \left(\frac{d}{dt} u((j+1)\tau), e^j \right)$$

$$- 2 m\tau\nu (D_i U^{j+1}, D_i e^j)$$

$$- 2 m\tau b_i(U^{j+1}, U^{j+1}, e^j) + |e^j - e^{j+i/q}|^2$$

$$+ \tau^2|\mu((j+1)\tau)|^2 + m\tau^2|e^{j+i/q}|^2 .$$

Here $\mu(t) = \mu(u(t), f(t))$ is a function related to u and f and is bounded by a constant which is independent of τ.

Notice that

$$b_i(u^j, u^{j+i/q}, e^{j+i/q}) - b_i(U^j, U^{j+1}, e^{j+i/q}) =$$

$$= b_i(e^j, U^{j+1}, e^{j+i/q})$$

$$\leqslant \frac{1}{2}\varepsilon \|u\|_{1,\infty}^2 |e^{j+i/q}|^2 + \frac{1}{8\varepsilon}|e^j|^2 + \frac{\varepsilon}{2}\|u\|_{0,\infty}^2 |D_i e^{j+i/q}|^2 + \frac{1}{8\varepsilon}|e^j|^2 . \tag{7.25}$$

Substituting this into (7.24) we get

$$(1 + 2 m\tau\nu\gamma_0 - \varepsilon m\tau\gamma_0(\|u\|_{1,\infty}^2 + \|u\|_{0,\infty}^2) - m\tau^2)|e^{j+i/q}|^2 -$$

$$- \left(1 + \frac{m\tau}{4\varepsilon}\right)|e^j|^2$$

$$\leqslant 2 m\tau (f_i^{j+1}, e^j) - 2 \tau \left(\frac{d}{dt} u((j+1)\tau), e^j \right) - 2 m\tau\nu (D_i U^{j+1}, D_i e^j)$$

$$- 2 m\tau b_i(U^{j+1}, U^{j+1}, e^j) + \tau^2|\mu((j+1)\tau)| . \tag{7.26}$$

Choose ε such that

$$2\nu\gamma_0 - \varepsilon\gamma_0(\|u\|_{0,\infty}^2 + \|u\|_{1,\infty}^2) - \tau = \frac{1}{4\varepsilon} . \tag{7.27}$$

From (**D2**), we can see that for τ, $0 < \tau < \nu\gamma_0$, we will get $\varepsilon > 0$. Therefore from (7.26) we get

$$\left(1 + \frac{m\tau}{4\varepsilon}\right)|e^{j+i/q}|^2 - \left(1 + \frac{m\tau}{4\varepsilon}\right)|e^j|^2 \leqslant$$

$$\leq 2\,m\tau\,(f_i^{j+1}, e^j) - 2\,\tau\left(\frac{d}{dt}u((j+1)\,\tau),\, e^j\right) - 2\,m\tau\nu\,(D_i\,U^{j+1}, D_i\,e^j)$$

$$- 2\,m\tau b_i(U^{j+1}, U^{j+1}, e^j) + \tau^2|\mu\,((j+1)\,\tau)|^2\,, \quad i = 1, \ldots, m\,. \quad (7.28)$$

As $\sum_{i=1}^{m} |e^{j+i/q}|^2 \geq m\,|e^{j+i/q}|^2$, a summation for $i = 1, \ldots, m$ gives

$$m\left(1 + \frac{m\tau}{4\,\varepsilon}\right)|e^{j+1/2}|^2 - m\left(1 + \frac{m\tau}{4\,\varepsilon}\right)|e^j|^2 \leq$$

$$\leq 2\,m\tau\,(\operatorname{grad} p,\, e^j) + m\tau^2|\mu\,((j+1)\,\tau)|^2$$
$$= m\tau^2|\mu\,((j+1)\,\tau)|^2\,. \quad (7.29)$$

But $|e^{j+1}|^2 \leq |e^{j+1/2}|^2$. As also $e^0 = 0$, we obtain

$$\left(1 + \frac{m\tau}{4\,\varepsilon}\right)|e^{j+1}|^2 \leq \sum_{k=0}^{j+1} \tau^2|\mu\,((j+1)\,\tau)|^2$$
$$\leq \tau \int_0^{(j+1)\,\tau} |\mu\,(t)|^2\,dt + o(\tau^2)\,.$$

In particular we have

$$|e^{j+1}|^2 \leq \tau \int_0^T |\mu\,(t)|^2\,dt + o(\tau^2)\,. \quad (7.30)$$

This gives the error estimate and proves the theorem. □

Remark 7.1 : It can be proved that if Ω is very smooth and ν is large enough or f is small enough, then (**D**1) and (**D**2) are true.

Remark 7.2 : In the proofs of Theorem 7.2 we require that the time step $\tau \leq \nu\gamma_0$ for (7.27). This is not essential. In fact it is possible to prove that $|e^{j+i/q}|$ generated in Algorithm 7.3 is bounded by a constant which is independent of τ. So

$$\tau^2(\lambda\,(u),\, e^{j+i/q}) \leq \tau^2|\lambda\,(u)|\,|e^{j+i/q}| \leq C\,\tau^2|\lambda\,(u)|\,. \quad (7.31)$$

By this inequality we can show that Theorem 7.2 is true for any $\tau > 0$. Similarly the restriction $\tau \leq \nu\gamma - \|u\|_{1,\infty}$ for (7.18) is also not essential in Theorem 7.1. Theorem 7.1 is also valid for any $\tau > 0$.

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