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SPECTRAL STUDY OF A COUPLED COMPACT-NONCOMPACT PROBLEM (*)

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Abstract — *We consider the coupled problem of acoustic vibration of air in a porous medium Ω_p, made of infinitely close thin sheets, parallel to the plane (x_1, x_3), in contact with free air in some region Ω_f We assume that there is no interaction between the sheets unless by the region Ω_f*

The case of a porous medium made of thin channels parallel to the x_1-axis was considered in [1, 2, 3] In this paper, we consider a somewhat more complicated problem because completely explicit solutions are not available in general

Let us denote by A the operator associated with the coupled eigenvalue problem $(-Au = \omega^2 u)$ and by $A_p(x_2)$ the operator associated in the sheet $x_2 =$ Const in Ω_p In order to study the spectrum of A we consider two cases according to the values of ω^2. In the first case (when ω^2 is not an eigenvalue of the problem in Ω_p), the problem reduces to an implicit eigenvalue problem in Ω_f, in the second case (when ω^2 is an eigenvalue of $A_p(a_2)$ for some value a_2 of x_2), we show that ω^2 belongs to the essential spectrum of A

Résumé. — *Nous étudions la structure du spectre d'un opérateur associé à un problème couplé de vibrations acoustiques Plus précisément, nous considérons un milieu poreux Ω_p, constitué par un grand nombre de lamelles planes uniformément distribuées, en contact avec une cavité remplie d'air, que nous désignerons par Ω_f Nous supposons qu'il n'y a pas d'interaction entre les lamelles, sauf par la région Ω_f*

Le cas d'un milieu poreux constitué de canaux parallèles a été considéré en [1, 2, 3], le problème présenté ici est plus compliqué du fait de l'absence, en général, de solutions complètement explicites

Si nous désignons par A l'opérateur associé au problème couplé $(-Au = \omega^2 u)$ et par $A_p(x_2)$ l'opérateur associé au problème dans une lamelle de Ω_p, nous considérons deux cas suivant les valeurs de ω^2 Dans le premier cas (où ω^2 n'est pas valeur propre de A_p), nous montrons que le problème aux valeurs propres pour A se ramène à un problème aux valeurs propres implicites dans Ω_f Dans le second cas (lorsque ω^2 est valeur propre de $A_p(a_2)$, pour une certaine valeur a_2 de x_2) nous montrons que ω^2 est un point du spectre essentiel de A

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1. INTRODUCTION

The equations describing the acoustic vibration in a porous medium, made by channels in a solid body, were obtained by using homogenization techniques [1, 3, 8, 9]. The spectral properties of the associated operator are classical in the case of a porous medium made by channels in all directions, i.e. when the fluid region is connected. In the case of channels in one direction, the properties of the homogenized equations are very different [1] because the waves propagate only in the direction of the channels. As a consequence the compactness properties are lost. The same occurs in the case of parallel plane sheets which we consider here. Certain proofs are technically cumbersome, we only give an outline which is sufficient for the logic understanding of them. Complete proofs are given in [7].

In the first section, we set the problem and give its variational formulation ; in the following sections we study the structure of the spectrum. So, we shall show that :

1. $\omega^2 = 0$ is a simple eigenvalue of the operator A associated with the coupled eigenvalue problem.

2. When $\Omega_f = \varnothing$, the set of the points ω^2 which are eigenvalues of the Neumann-Dirichlet problem in any sheet located in the plane $x_2 =$ Const. constitutes the essential spectrum of A_p (associated with the problem in Ω_p).

3. When $\Omega_f \neq \varnothing$, for particular geometries (see Sect. 5), we show that the set defined by

$$E = \{\omega^2 ; \omega^2 \text{ is an eigenvalue of the problem in a sheet } \}$$

belongs to the essential spectrum of A.

4. For a particular geometry (see Sect. 5), we prove that the points ω^2 which belong to the resolvent set $\rho(A_p(x_2))$, for any $x_2 \in [0, 1]$, are either eigenvalues of finite multiplicity of A, or points of the resolvent set $\rho(A)$.

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2. SETTING OF THE PROBLEM. VARIATIONAL FORMULATION

We consider a porous medium, made of very many thin sheets disposed as in figure 2.1, which occupies the domain Ω_p of $\mathbb{R}^3$ defined by

$$\Omega_p = \{(x_1, x_2, x_3), x_1 \in]-\ell(x_2, x_3), 0[, \quad x_2 \in]0, 1[, x_3 \in]0, 1[\}$$

where $\ell(x_2, x_3)$ is a smooth strictly positive function.

That porous medium is in contact with free air contained in some region Ω_f of $\mathbb{R}^3$. The interface Γ is disposed as in figure 2.1.

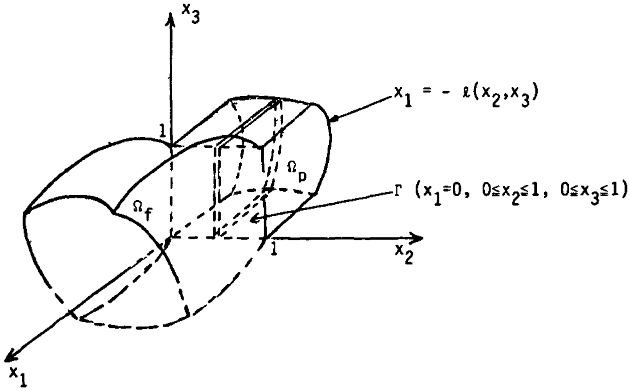

Figure 2.1.

In the sequel, we shall denote by ν the outer unit normal to the curve $x_1 = - \ell(x_2, x_3)$ in its plane, and by $\mathbf{n}$ the outer normal to the boundary $\partial\Omega_f$ of Ω_f.

The equations and boundary conditions of the homogenized problem are immediately deduced from [1], they are :

$$- \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_3^2} - \omega^2 u = 0 \quad \text{in} \quad \Omega_p \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad x_1 = - \ell(x_2, x_3) \tag{2.2}$$

$$\frac{\partial u}{\partial x_3} = 0 \quad \text{on} \quad x_3 = 0 \quad \text{and} \quad x_3 = 1 \tag{2.3}$$

$$- \Delta u - \omega^2 u = 0 \quad \text{in} \quad \Omega_f \tag{2.4}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_f \backslash \Gamma . \tag{2.5}$$

As for the transmission conditions on Γ, they are :

$$[\![u]\!] = 0 , \qquad \left[\!\!\left[\frac{\partial u}{\partial x_1} \right]\!\!\right] = 0 \quad \text{on} \quad \Gamma \tag{2.6}$$

where $[\![\, . \,]\!]$ denotes the jump across the interface Γ. The unknown u denotes the velocity potential.

We note that (2.1)-(2.6) was written in terms of a classical eigenvalue problem, i.e. for an eigenfunction u and an eigenvalue ω^2. We shall refer to this system in the sequel even in the case when the points ω^2 belong to the essential spectrum of the corresponding operator A (defined later) for which evident modifications must be considered.

Let us define

$$\Omega = \Omega_p \cup \Omega_f \cup \Gamma$$

and

$$H \equiv L^2(\Omega)$$

$$V = \left\{ v \in L^2(\Omega)\ ;\ \frac{\partial v}{\partial x_\alpha} \in L^2(\Omega),\ \alpha = 1, 3\ ;\ \frac{\partial}{\partial x_2}(v|_{\Omega_f}) \in L^2(\Omega_f) \right\} \tag{2.7}$$

It is easily proved that the problem (2.1)-(2 6) is equivalent to the following one :

$$\left\{ \begin{array}{l} \text{Find } u \in V \text{ and } \omega \in \mathbb{R} \text{ such that :} \\ \displaystyle\int_{\Omega_f} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx + \int_{\Omega_p} \frac{\partial u}{\partial x_\alpha}\frac{\partial v}{\partial x_\alpha}\,dx = \omega^2 \int_\Omega uv\,dx \quad \forall v \in V \end{array} \right. \tag{2.8}$$

with $i = 1, 2, 3$ and $\alpha = 1, 3$.

Then, classically ([3] Chap. IV for instance) we have :

PROPOSITION 2.1 : *The space V, defined by* (2.7), *equipped with the scalar product*

$$(u, v)_V = a(u, v) + (u, v)_{L^2(\Omega)}$$

where $a(u, v)$ is the bilinear form defined by the left hand side of (2.8), *is a Hilbert space and the imbedding of V in H is dense, continuous but not compact*

The associated selfadjoint operator A is defined in the domain

$$D(A) = \left\{ v \in L^2(\Omega)\ ;\ \Delta(v|_{\Omega_f}) \in L^2(\Omega_f),\ \frac{\partial}{\partial x_\alpha^2}(v|_{\Omega_p}) \in L^2(\Omega_p),\ \alpha = 1, 3, \right.$$
$$\left. v \text{ satisfying the conditions } (2.2), (2.3), (2.5), (2.6) \right\}$$

and Av is defined by

$$Av = \begin{cases} -\Delta v & in \quad \Omega_f \\ -\dfrac{\partial^2 v}{\partial x_1^2} - \dfrac{\partial^2 v}{\partial x_3^2} & in \quad \Omega_p . \end{cases}$$

It is well known that the spectrum $\sigma(A)$ may have a somewhat complicated structure, essential spectrum ([3], Sects. III.7 and IV.3), containing eigenvalues of infinite multiplicity, accumulation points of eigenvalues or a continuous spectrum. We now study the structure of that spectrum.

From the definition of the operator A, it is easily seen that $\omega^2 = 0$ *is a simple eigenvalue,* the corresponding eigenfunctions being $u =$ Const.

We now search for eigenvalues $\lambda = \omega^2 \neq 0$. We first consider *the problem in* Ω_p, *with* $\Omega_f = \varnothing$, and denote by $A_p(x_2)$ the associated operator (in a sheet situated in the plane $x_2 =$ Const.) with the boundary condition

$$u|_\Gamma = 0 \,. \tag{2.9}$$

Then, for the spectral study of the system (2.1)-(2.6), we have to consider the two following cases :

1) ω^2 is a point of the resolvent set $\rho(A_p(x_2))$ for any $x_2 \in [0, 1]$,

2) ω^2 is such that : $\exists a_2 \in [0, 1]$ for which ω^2 is an eigenvalue of the operator $A_p(a_2)$.

3. SPECTRAL STUDY OF THE COUPLED SYSTEM WHEN ω^2 SATISFIES 1)

Our purpose is, as in [2], to show that the points $\lambda = \omega^2$ are isolated eigenvalues with finite multiplicity or points of the resolvent set $\rho(A)$. To this end, we first prove that the spectral problem (2.1)-(2.6) reduces to an implicit eigenvalue problem in Ω_f.

Since ω^2 belongs to the resolvent set of the operator associated with the problem in each sheet, by using classical results (see [6] and for details [7]) we have :

PROPOSITION 3.1 : *Let be* φ *a given function*

$$\varphi \in L^2_{x_2}((0, 1) ; H^{1/2}(0, 1)) \,,$$

then, the problem

$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_3^2} - \omega^2 u = 0 \quad in \quad \Omega_p \tag{3.1}$$

$$\frac{\partial u}{\partial x_3} = 0 \quad on \quad x_3 = 0 \quad and \quad x_3 = 1 \tag{3.2}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad on \quad x_1 = -\ell(x_2, x_3) \tag{3.3}$$

$$u = \varphi \quad on \quad \Gamma \tag{3.4}$$

has a unique solution u_ω^φ for any ω^2 satisfying 1) *and*

$$u_\omega^\varphi \in \mathscr{H}(\Omega_p) \equiv \int_0^1 H_{x_2}^1(\Omega_p)\, dx_2$$

where the classical notation (Cf [5], Chap IV, Sect 5) *was used for the space integral*

Now, we define the family of operators $T(\omega)$ by

$$T(\omega)\,\varphi = \left.\frac{\partial u_\omega^\varphi}{\partial n}\right|_\Gamma \tag{3 5}$$

where u_ω^φ is the unique solution of (3 2)-(3 5), and we denote by $E_1(x_2)$ and by $E_1'(x_2)$ (dual of $E_1(x_2)$ the two spaces, defined for fixed x_2 in $[0, 1]$ by

$$E_1(x_2) \equiv L_{x_2}^2([0, 1], H^{1/2}(0, 1))$$
$$E_1'(x_2) \equiv L_{x_2}^2([0, 1], (H^{1/2})'\,(0, 1))$$

Then we classically ([6], [8]) have

PROPOSITION 3 2 *The operator T, defined by* (3 5), *enjoys the properties*

a) $T \in \mathscr{L}(E_1, E_1')$

b) *T is holomorphic with respect to ω*

And, solving in Ω_p, we have

PROPOSITION 3 3 *Let be ω^2 satisfying* 1), *then the spectral problem* (2 1)-(2 6) *is equivalent to the implicit eigenvalue problem in Ω_f*

$$\left\{\begin{array}{l} \textit{Find } u \in H^1(\Omega_f),\ u \neq 0 \quad \textit{and} \quad \omega^2 \in \mathbb{R}^+ \ \textit{such that} \\ \displaystyle\int_{\Omega_f} \nabla u \,.\, \nabla v\, dx + \langle T(\omega)\, u, v\rangle_{E_1' E_1} = \\ \qquad\qquad = \omega^2 \displaystyle\int_{\Omega_f} u \,.\, v\, dx \quad \forall v \in H^1(\Omega_f) \end{array}\right. \tag{3 6}$$

Now, we have to prove that the points ω^2 which verify 1) are either eigenvalues of finite multiplicity or points of the resolvent set of the operator $A(\Omega_f)$, associated with the form $a_f(\omega, u, v)$ defined by the left hand side of (3 7) This follows from Proposition V, 7 5 in [3] provided that the coerciveness of $a_f(\omega, v, v)$ holds at some point The property of coerciveness was proved in the case of a porous medium made of channels

[2]. In the case of sheets, explicit computations were performed for particular geometries. It is the case for the problem associated with the figure 5.1 where the sheets are circular rings defined, in cylindrical coordinates, by Const. $= r_0 < r < \ell(z)$. By writing the problem in cylindrical coordinates r, θ, z, using asymptotic expansion of Bessel functions as the index tends to infinity and Fourier expansions in $L^2(\Gamma)$ it is possible to prove, thanks to [5], the coerciveness of a_f. In short, whenever it is possible to give explicit solutions, the coerciveness is proved. Consequently, we can reasonably think that the form a_f is also coercive in any case, but no technically easy to prove.

4. SPECTRAL STUDY OF THE PROBLEM IN Ω_p WHEN ω^2 SATISFIES 2)

In this section, we consider the eigenvalue problem in Ω_p with $\Omega_f = \varnothing$.

For fixed x_2, $x_2 = a_2$, let us denote by $A_p(a_2)$ the operator associated with the problem in the corresponding sheet :

$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_3^2} - \omega^2 u = 0 \quad \text{in the sheet} \quad x_2 = a_2 \tag{4.1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad x_1 = -\ell(a_2, x_3) \tag{4.2}$$

$$\frac{\partial u}{\partial x_3} = 0 \quad \text{on} \quad x_3 = 0 \quad \text{and} \quad x_3 = 1 \tag{4.3}$$

$$u = 0 \quad \text{on} \quad \Gamma \,(\text{i.e. } x_1 = 0,\, x_2 = a_2)\,. \tag{4.4}$$

The operator $A_p(a_2)$ has a compact inverse and, consequently, possesses a countable infinity of positive eigenvalues such that

$$0 < \omega_0^2(a_2) \leqslant \omega_1^2(a_2) \leqslant \cdots \to \infty\,.$$

We shall denote by $u_{a_2}(x_1, x_3)$ an associated eigenfunction.

Our purpose is to show that ω^2, satisfying 2), belongs to the essential spectrum $\sigma_{\text{ess}}(A_p)$ of A_p (operator associated with the problem in Ω_p with $\Omega_f = \varnothing$). To this end we have to construct a Weyl sequence (Proposition IV.3.2 in [3]).

In order to simplify the computations, we suppose that the function ℓ does not depend on x_3 so, we define ℓ_1 by

$$\ell_1(x_2) \equiv \ell(x_2, x_3)\,. \tag{4.5}$$

Let us remark that, in that case, the eigenvalues and eigenvectors are explicitely known :

$$\omega^2 = k^2 \pi^2 + \frac{(2m+1)^2 \pi^2}{4 \ell_1^2(a_2)} ; \quad u_{a_2} = f(x_2) \sin \frac{(2m+1)\pi}{2 \ell_1(a_2)} x_1 \cos(2kx_3)$$

but these expressions will not be used in the sequel.

It is clear that, if $u_{a_2}(x_1, x_3)$ is solution of (4.1)-(4.4) then, the function $w(x_1, x_2, w_3)$ defined in Ω_p by

$$w(x_1, x_2, x_3) = f(x_2)\, u_{a_2}\left(\frac{\ell_1(a_2)}{\ell_1(x_2)} x_1, x_3 \right) \tag{4.6}$$

satisfies the boundary conditions (4.2)-(4.4).

We then easily see that the distribution defined in Ω_p by

$$\mathcal{T} \equiv C\ \delta(x_2 - a_2)\, u_{a_2}\left(\frac{\ell_1(a_2)}{\ell_1(x_2)} x_1, x_3 \right) \tag{4.7}$$

where C is an arbitrary constant, is a solution of the problem (4.1)-(4.4) in the sense of distributions.

But, as $\mathcal{T}$ does not belong to

$$D(A_p) = \Big\{ v \in L^2(\Omega_p) ; \quad \frac{\partial^2 v}{\partial x_1^2} - \frac{\partial^2 v}{\partial x_3^2} \in L^2(\Omega_p) ,$$
$$v \text{ satisfying the boundary conditions (4.2)- (4.4)} \Big\}$$

$\mathcal{T}$ is not an eigenfunction. We shall replace δ by a sequence of smooth functions tending to δ in order to prove that the corresponding $\mu = \omega^2$ is a point of $\sigma_{\text{ess}}(A_p)$.

4.1. Construction of a Weyl sequence

Let $\psi \in \mathcal{D}(\mathbb{R})$ and c be respectively such that

$$\int_{\mathbb{R}} \psi(\xi)\, d\xi = 1 , \quad c = \int_{\mathbb{R}} \psi^2(\xi)\, d\xi$$

and let us define the sequence

$$\psi_k(\xi) \equiv \psi(k\xi) , \quad k = 1, 2, \ldots \tag{4.8}$$

which enjoys the properties

$$\begin{cases} k\psi_k \to \delta \quad \text{as} \quad k \to \infty \quad \text{in} \quad \mathcal{D}'(\mathbb{R}) \\ \displaystyle\int_{\mathbb{R}} \psi_k^2(\xi)\, d\xi = \frac{c}{k} \\ \text{Supp. } \psi_k \subset [-1/k,\ 1/k]\,. \end{cases} \tag{4.9}$$

Then, in Ω_p, we define the sequence $w_k(x_1, x_2, x_3)$ by

$$w_k(x) = \frac{k^{1/2}\, \psi_k(x_2 - a_2)}{\sqrt{c}\, \| u_{a_2} \|_{L^2([-\ell_1(a_2),\, 0] \times [0,\, 1])}}\, u_{a_2}\left(\frac{\ell_1(a_2)\, x_1}{\ell_1(x_2)},\, x_3 \right). \tag{4.10}$$

Now, we have still to prove that the sequence defined by (4.10) satisfies the hypotheses of the Weyl's theorem of characterization of the essential spectrum, namely :

$$\| w_k \|_{L^2(\Omega_p)} \to 1 \quad \text{as} \quad k \to +\infty \tag{4.11}$$

$$w_k \to 0 \quad \text{in} \quad L^2(\Omega_p) \text{ weakly} \tag{4.12}$$

$$\| (A_p - \omega^2 I)\, w_k \|_{L^2(\Omega_p)} \to 0 \quad \text{as} \quad k \to +\infty\,. \tag{4.13}$$

This is easily checked from (4.10).

Moreover, we have :

PROPOSITION 4.1 : *Let us denote by* $\mathcal{E}$ *the set defined by*

$\mathcal{E} \equiv \{\omega^2 \in \mathbb{R}^+$; ω^2 *is an eigenvalue of the problem* (4.1)-(4.4) *in a sheet*$\}$,

and by $\bar{\mathcal{E}}$ *its closure then, we have*

$$\bar{\mathcal{E}} = \sigma_{\text{ess}}(A_p)\,. \tag{4.14}$$

Proof : From the previous results, if $\omega^2 \in \mathcal{E}$ then ω^2 is a point of $\sigma_{\text{ess}}(A_p)$, consequently

$$\mathcal{E} \subset \sigma_{\text{ess}}(A_p) \Rightarrow \bar{\mathcal{E}} \subset \sigma_{\text{ess}}(A_p)\,.$$

Conversely, we have

$$\bar{\mathcal{E}} \supset \sigma_{\text{ess}}(A_p)\,.$$

Indeed, it is easily proved, by integrating in x_2 that, if $\omega^2 \notin \bar{\mathcal{E}}$, then ω^2 belongs to the resolvent set $\rho(A_p)$. ■

Remark 4.2 : Hypothesis (4.5) is not essential, we obtain Proposition 4.1 in the general case, where $x_1 = -\ell(x_2, x_3)$, by using the theory of perturbation of the boundary (see [3], Sect. V.5).

5. SPECTRAL STUDY OF THE COUPLED PROBLEM WHEN ω^2 SATISFIES 2)

We consider now the porous medium Ω_p in contact with the air contained in a bounded domain Ω_f of $\mathbb{R}^3$. We show that if ω^2 is an eigenvalue of the problem in a sheet, then ω^2 belongs to the $\sigma_{\mathrm{ess}}(A)$, where A denotes the operator associated with the coupled problem in Ω. To this end, as in the preceding section, we construct a Weyl sequence.

5.1. Construction of a Weyl sequence v_k

The sequence v_k is obtained by means of its restrictions to Ω_p and Ω_f.

Construction of v_k *in* Ω_p : We search for $v_{k\,|\,\Omega_p}$ of the form

$$v_{k\,|\,\Omega_p} = w_k(x_1, x_2, x_3) + \hat{w}_k(x_2, x_3) \tag{5.1}$$

where w_k is the sequence defined in (4.10) and $\hat{w}_k$ a function to be defined later, such that $[\![v_k]\!] = 0$ on Γ.

Construction of v_k *in* Ω_f : We take, as restriction to Ω_f, $v_{k\,|\,\Omega_f}$ solution of the Neumann problem in Ω_f :

$$(-\Delta - \omega^2 I)\, v_k = 0 \quad \text{in} \quad \Omega_f \tag{5.2}$$

$$\frac{\partial v_k}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_f \backslash \Gamma \tag{5.3}$$

$$\frac{\partial v_k}{\partial x_1} = \frac{\partial w_k}{\partial x_1}(0, x_2, x_3) \quad \text{on} \quad \Gamma \tag{5.4}$$

which has a unique solution when ω^2 *is not an eigenvalue* of (5.2)-(5.4), that we shall suppose in the sequel. Then the trace of $v_{k\,|\,\Omega_f}$ is well defined and we take

$$\hat{w}_k(x_2, x_3) = v_{k\,|\,\Omega_f}(0, x_2, x_3)\,. \tag{5.5}$$

Consequently the sequence v_k is well determined and we immediately verify that $v_k \in D(A)$.

We have still to prove that v_k is a Weyl sequence, that is to say that v_k, defined by its restrictions $v_{k\,|\,\Omega_f}$ and $v_{k\,|\,\Omega_p}$ (respectively defined by (5.1), (5.2)-(5.4) and (5.5)) satisfies

$$\|v_k\|_{L^2(\Omega)} \to 1 \quad \text{as} \quad k \to +\infty \tag{5.6}$$

$$v_k \to 0 \quad \text{in} \quad L^2(\Omega) \text{ weakly} \tag{5.7}$$

$$\|(A - \omega^2 I)\, v_k\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad k \to +\infty \tag{5.8}$$

Since we have

$$\|v_k\|^2_{L^2(\Omega)} = \|w_k\|^2_{L^2_p(\Omega_p)} + \|v_{k\,|\,\Omega_f}\|^2_{L^2(\Omega_f)} + \|\hat{w}_k\|^2_{L^2(\Omega_p)} + \\ + 2(w_k, \hat{w}_k)_{L^2(\Omega_p)} \tag{5.9}$$

and as w_k is yet a Weyl sequence in Ω_p, (5.6) and (5.7) immediately follows from the following lemma :

LEMMA 5.1 : *Let be $v_{k\,|\,\Omega_f}$ and $\hat{w}_k$ the sequences defined respectively by* (5.2)-(5.4) *and* (5.5), *then we have*

$$\|v_{k\,|\,\Omega_f}\|_{L^2(\Omega_f)} \to 0 \quad \textit{as} \quad k \to +\infty \tag{5.10}$$

$$\|\hat{w}_k\|_{L^2(\Omega_p)} \to 0 \quad \textit{as} \quad k \to +\infty\,. \tag{5.11}$$

Proof : From classical estimates [6], we have, for $0 < \delta < 1$

$$\|v_{k\,|\,\Omega_f}\|_{L^2(\Omega_f)} \leq \|v_{k\,|\,\Omega_f}\|_{H^{\delta + 1/2}(\Omega_f)} \leq C \left\| \frac{\partial w_k}{\partial x_1} \right\|_{H^{-1+\delta}(\Gamma)} \tag{5.12}$$

consequently, the proof of (5.10) reduces to prove that

$$\frac{\partial w_k}{\partial x_1} \to 0 \quad \text{in } L^2(\Gamma) \text{ weakly } (\Rightarrow \text{in } H^{-1+\delta}(\Gamma) \text{ strongly})\,. \tag{5.13}$$

Now, from the construction of the w_k, we easily show that

$$\int_\Gamma \left(\frac{\partial w_k}{\partial x_1} \right)^2 dx_1\, dx_2$$

is bounded independently of k ; then we have still to prove that

$$\int_\Gamma \frac{\partial w_k}{\partial x_1}\, \varphi\, dx_2\, dx_3 \to 0 \quad \forall \varphi \in \mathcal{D}(\Gamma)$$

that is easily obtained from the properties of the ψ_k (*cf.* (4.10)). ■

As for (5.11), by using the property of continuity of the traces from $H^{1/2+\delta}(\Omega_f)$ into $H^{\delta}(\Gamma)$ and taking account of (5.12), we have

$$\|\hat{w}_k\|_{L^2(\Omega_p)} \leq C_1 \|\hat{w}_k\|_{L^2(\Gamma)} = C_1 \left\|v_{k\,|\,\Omega_f}(0, x_2, x_3)\right\|_{L^2(\Gamma)} \leq$$
$$\leq C_2 \left\|v_{k\,|\,\Omega_f}(0, x_2, x_3)\right\|_{H^{\delta}(\Gamma)} \leq C \left\|v_k(\Omega_f)\right\|_{H^{\delta+1/2}(\Omega_f)} \to 0 \quad \text{as} \quad k \to +\infty$$

and the Lemma is proved. ∎

Now, from the definition of the operator A, we have

$$(A - \omega^2 I)\, v_k = \begin{cases} 0 & \text{in } \Omega_f \\ -\dfrac{\partial^2 w_k}{\partial x_1^2} - \dfrac{\partial^2 w_k}{\partial x_3^2} - \omega^2 w_k - \dfrac{\partial^2 \hat{w}_k}{\partial x_3^2} - \omega^2 \hat{w}_k & \text{in } \Omega_p \end{cases}$$

and w_k is such that

$$\left\| -\frac{\partial^2 w_k}{\partial x_1^2} - \frac{\partial^2 w_k}{\partial x_3^2} - \omega^2 w_k \right\|_{L^2(\Omega_p)} \to 0 \quad \text{as} \quad k \to +\infty$$

then, taking into account (5.11), (5.8) will be proved if we show that $\hat{w}_k$, defined by (5.5), is such that

$$\left\| \frac{\partial^2 \hat{w}_k}{\partial x_3^2} \right\|_{L^2(\Gamma)} \to 0 \quad \text{as} \quad k \to \infty . \tag{5.14}$$

As w_k is smooth in x_3, it is easily seen that $\partial^2 v_k / \partial x_3^2$ satisfies, in Ω_f, the equation (5.2) and the boundary condition obtained by differenciating (5.4) with respect to x_3 twice, but does not satisfy (5.3) except for particular geometries. Let us suppose that Ω_f satisfies the following property :

(P) : Ω_f is such that if v_k satisfies (5.2), (5.3) and (5.4), then $\partial^2 v_k / \partial x_3^2$ satisfies them too.

Then, we have

$$\frac{\partial^2 v_k}{\partial x_3^2} \to 0 \quad \text{in} \quad H^{\delta+1/2}(\Omega_f) \text{ strongly for } 0 < \delta < 1$$

which is analogous to (5.12).

Now, we have

$$\left\| \frac{\partial^2 \hat{w}_k}{\partial x_3^2} \right\|_{L^2(\Omega_p)} \leqslant C_1 \left\| \frac{\partial^2 \hat{w}_k}{\partial x_3^2} \right\|_{L^2(\Gamma)} = C_1 \left\| \frac{\partial^2 v_k}{\partial x_3^2} \right\|_{L^2(\Gamma)} \leqslant$$

$$\leqslant C_2 \left\| \frac{\partial^2 v_k}{\partial x_3^2} (0, x_2, x_3) \right\|_{H^\delta(\Gamma)} \leqslant C \left\| \frac{\partial^2 v_{k \mid \Omega_f}}{\partial x_3^2} \right\|_{H^{\delta + 1/2}(\Omega_f)}$$

and consequently (5.14).

Exemples of such a geometry are cylinders with generators parallel to x_3 and periodicity conditions with respect to x_3.

Then we have

THEOREM 5.2 : *For any domain Ω_f the geometry of which satisfies the hypothesis (P), the points $\lambda = \omega^2$ which are eigenvalues of the problem* (4.1)-(4.4) *in a sheet of the plane $x_2 =$ Const. but which are not eigenvalues of the Neumann problem in Ω_f, belong to the essential spectrum of the coupled problem in Ω.*

Remark 5.3 : Computations in cylindrical coordinates (r, θ, z) allow us to consider other geometries. In particular, domains with symmetry of revolution around of the axis z as in figure 5.1.

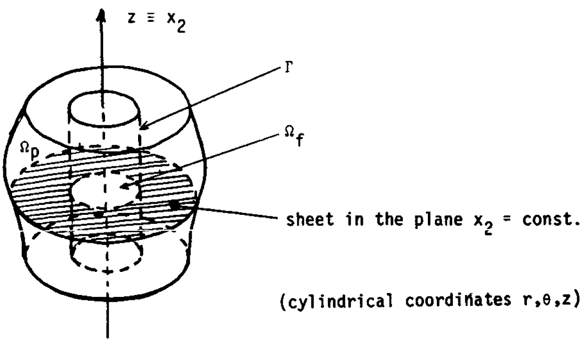

Figure 5.1.

More exactly, in the particular case of figure 5.1 we proved [7] :

THEOREM 5.4 : *When the problem is periodic with respect to θ and the function ℓ depends only on z.*

a) *If $\lambda = \omega^2$ is a point of the resolvent set $\rho(A_p(z))$ for any $z \in [0, 1]$, then $\lambda = \omega^2$ is either eigenvalue of finite multiplicity or point of the resolvent set.*

b) *If* $\lambda = \omega^2$ *is such that there exists* $a_2 \in [0, 1]$ *for which* $\lambda = \omega^2$ *is an eigenvalue of the operator* $A_p(a_2)$ *and is not an eigenvalue of the problem in* Ω_f, *then* $\lambda = \omega^2$ *belongs to the essential spectrum of the coupled problem.*

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