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**FINITE ELEMENT APPROXIMATION OF A NON-LIPSCHITZ
 NONLINEAR EIGENVALUE PROBLEM (*)**

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Abstract — Given $p \in (0, 1)$, we consider the following problem find $u \neq 0$, such that

$$-\Delta u = [u]_+^p = u^p \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a $C^{2,1}$ domain. We prove a near optimal L^∞ error bound for the standard continuous piecewise linear Galerkin finite element approximation with an acute triangulation. In addition we analyse a more practical approximation using numerical integration on the nonlinear term, proving a near optimal interior L^∞ error bound.

Résumé — Étant donné $p \in (0, 1)$, considérons le problème suivant trouver $u \neq 0$, tel que

$$-\Delta u = [u]_+^p = u^p \quad \text{dans } \Omega \quad u = 0 \quad \text{sur } \partial\Omega,$$

où $\Omega \subset \mathbb{R}^2$ est un domaine $C^{2,1}$. Nous montrons l'existence d'une borne d'erreur L^∞ quasi-optimale pour la méthode standard d'approximation de Galerkin par éléments finis continus et linéaires par morceaux avec une triangularisation aiguë. De plus nous étudions une méthode d'approximation plus appliquée utilisant une intégration numérique sur le terme non linéaire, qui montre une borne d'erreur L^∞ et quasi optimale à l'intérieur du domaine.

1. INTRODUCTION

The finite element approximation of the semilinear elliptic equation : given λ and $g \in \mathbb{R}$, find u such that

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega \subset \mathbb{R}^2 \tag{1.1a}$$

$$u = g \quad \text{on } \partial\Omega; \tag{1.1b}$$

where $f \in C^1(\mathbb{R})$ is relatively well understood. The error analysis is based upon the implicit function theorem in general and hence the need for

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f to be C^1 . For a less smooth nonlinearity the theory is not so well developed. We illustrate the situation on some model problems.

If the nonlinearity λf is monotonically decreasing then there exists a unique solution to (1.1). In addition the implicit function theorem can be avoided and the need for f to be C^1 can be relaxed to being locally Lipschitz. Under these circumstances it is relatively straightforward to prove optimal H^1 , L^2 and L^∞ error bounds for the standard continuous piecewise linear Galerkin approximation and for a more practical scheme involving numerical integration, see Crouzeix and Rappaz (1990). For a non-Lipschitz nonlinearity it is not so straightforward. For $p \in \mathbb{R}^+$, let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(t) \equiv [t]_+^p \equiv \begin{cases} t^p & 0 \leq t \\ 0 & t \leq 0. \end{cases} \quad (1.2)$$

If $\lambda \in \mathbb{R}^-$ and $g \in \mathbb{R}^+$, (1.1) and (1.2) can be viewed as a model reaction-diffusion problem. For $p \geq 1$ f is locally Lipschitz and the above theory yields optimal error bounds. For $p \in (0, 1)$ f is not locally Lipschitz and the above theory yields pessimistic error bounds for the standard linear Galerkin approximation; $h^{\min(1, 2p)}$ in H^1 and $h^{2p} \ln \frac{1}{h}$ in L^∞ . Barrett and Shanahan (1991) have recently proved optimal order H^1 and L^∞ error bounds for the standard linear Galerkin approximation in this case. Their L^∞ error analysis is based on a $L^1 - L^\infty$ duality argument used by Nochetto (1988) for a regularized version of the nonlinearity f . In the presence of numerical integration however, Barrett and Shanahan (1991) managed to prove only an optimal order H^1 error bound.

For λf not monotonically decreasing the relationship between u and λ for fixed g is far more complicated. There can be lack of existence and/or bifurcation can take place. However, if f is C^1 one can employ the implicit function theorem to derive optimal order H^1 , L^2 and L^∞ error bounds along regular branches and at simple turning and bifurcation points for the standard linear Galerkin approximation and for a more practical scheme involving numerical integration, see Crouzeix and Rappaz (1990).

For λf not monotonically decreasing and $f \notin C^1$ there appears to be no error bounds at present in the literature, except for the well-studied « plasma problem »: given $\lambda \in \mathbb{R}^+$, $\gamma \in \mathbb{R}^+$, find $(g, u) \in \mathbb{R} \times H_0^1(\Omega)$ such that

$$-\Delta u = \lambda f(u - g) \quad \text{in } \Omega, \quad \lambda \int_{\Omega} f(u - g) = \gamma; \quad (1.3)$$

where f is given by (1.2) with $p = 1$ and hence $f \in C^{0,1}(\mathbb{R})$, but

$f \notin C^1(\mathbb{R})$. Let $\lambda_2 > \lambda_1 > 0$ be the first two eigenvalues of the associated linear eigenvalue problem : find $(\lambda_i, \phi_i) \in \mathbb{R} \times H_0^1(\Omega)$ such that

$$-\Delta \phi_i = \lambda_i \phi_i \quad \text{in } \Omega . \tag{1.4}$$

Then one can show that there exists a unique solution to (1.3) for $\lambda \in (0, \lambda_2)$ and using the generalised implicit function theorem, see Girault and Raviart (1982), one can derive optimal order H^1, L^2 and L^∞ error bounds to (1.3) with $\lambda \in (0, \lambda_2)$ for the standard linear Galerkin approximation and for a more practical scheme involving numerical integration, see Barrett and Elliott (1989) and Caloz (1991). For a variational approach, avoiding the generalised implicit function theorem, see Barrett and Elliott (1991). One can view (1.3) as a free boundary problem since $\Omega_+ \equiv \{x \in \Omega : u(x) > g\}$ and $\Gamma \equiv \partial\Omega_+$ are unknown.

It would be of interest to analyse the finite element approximation of (1.3) when f is given by (1.2) with $p \in [0, 1]$; this being a model vortex problem, see Eydeland and Turkington (1988) for example. In this paper we analyse the finite element approximation of a simpler problem, the nonlinear eigenvalue problem $(P(\lambda))$: Find $(\lambda, u(\lambda)) \in \mathbb{R} \times H_0^1(\Omega)$, $u \neq 0$, such that (1.1a) holds, where f is given by (1.2) with $p \in (0, 1)$. This nonlinear eigenvalue problem has a non-Lipschitz nonlinearity with λf monotonically increasing for $\lambda > 0$; $\lambda \leq 0$ only yields the trivial solution $u \equiv 0$. Therefore it has the same important character as the above vortex problem. It is simpler though in that the « free boundary » where $u = g \equiv 0$ occurs on $\partial\Omega$ and that one can prove for all $\lambda \in \mathbb{R}^+$ there exists a unique non-trivial solution. However, we believe it to be a useful model to analyse in order to see what can be achieved and what is required to analyse the vortex problem.

A simple calculation yields that the generalised implicit function is not applicable to $(P(\lambda))$ for $p \in (0, 1)$. The monotonicity and maximum principle approach that we adopt in this paper to analyse the finite element approximation of $(P(\lambda))$ has been motivated by some of the techniques used by Conrad and Cortey-Dumont (1987a, b) to analyse the continuous piecewise linear finite element approximation of the nonlinear variational inequality : given $\lambda, g \in \mathbb{R}^+$ and a nondecreasing C^2 function $f(\cdot)$, $f(0) > 0$; find $u \in K \equiv \{v \in H_0^1(\Omega) : v \leq g \text{ a.e. in } \Omega\}$ such that

$$\int_{\Omega} \nabla u \cdot \nabla (v - u) \geq \lambda \int_{\Omega} f(u) (v - u) \quad \forall v \in K . \tag{1.5}$$

We stress that our analysis exploits the fact that λf is monotonically increasing and is not applicable to the problem studied in Barrett and Shanahan (1991), where λf is monotonically decreasing. Similarly the analysis given there is not applicable to $(P(\lambda))$. The extension of the

present approach to the model vortex problem will be the subject of a forthcoming paper.

The layout of this paper is as follows. In the next section we study the continuous problem $(P(\lambda))$. In section 3 we prove a near optimal L^∞ error bound for the standard continuous piecewise linear Galerkin finite element approximation, with an acute triangulation, of $(P(\lambda))$. Finally in section 4, we study a more practical approximation using numerical integration on the nonlinear term, proving a near optimal interior L^∞ error bound. This result exploits the fact that the lack of Lipschitz continuity of $f(u)$ occurs in the vicinity of $\partial\Omega$.

Throughout this paper we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,p,\Omega}$ and semi-norm $|\cdot|_{m,p,\Omega}$. For $p = 2$, we adopt the convention $H^m(\Omega) \equiv W^{m,2}(\Omega)$, $\|\cdot\|_{m,\Omega} \equiv \|\cdot\|_{m,2,\Omega}$, and $|\cdot|_{m,\Omega} \equiv |\cdot|_{m,2,\Omega}$. We set $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$. Finally C denotes a generic positive constant independent of h , but possibly dependent on p .

2. ANALYSIS OF THE CONTINUOUS PROBLEM

Let Ω be a bounded domain in \mathbb{R}^2 with a boundary $\partial\Omega$ of class $C^{2,1}$. The problem we wish to study is: $(P(\lambda))$ given $p \in (0, 1)$ and $\lambda \in \mathbb{R}$ find $u(\lambda) \neq 0 \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = \lambda (f(u), v)_\Omega \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where f is given by (1.2) and $(v_1, v_2)_D \equiv \int_D v_1 v_2$. This problem has been studied by Aronson and Peletier (1981) in connection with the long time behaviour of solutions of the porous medium equation. Below we recall and extend some of their results.

Firstly we recall versions of the strong and Hopf maximum principles suitable for our needs, see for example Gilbarg and Trudinger (1983): (S.M.P.) if $v \in H^2(\Omega)$ with

$$-\Delta v \geq 0 (\leq 0) \quad \text{a.e. in } \Omega \tag{2.2a}$$

$$v \geq 0 (\leq 0) \quad \text{on } \partial\Omega \tag{2.2b}$$

then $v \geq 0 (\leq 0)$ in $\bar{\Omega}$. (2.2c)

In addition if $v(x) = 0$ for some $x \in \Omega$ then $v \equiv 0$ in $\bar{\Omega}$.

(H.M.P.) if $v \in C^2(\bar{\Omega})$ satisfies (2.2a) with $v = 0$ on $\partial\Omega$ and $v \neq 0$ then

$$\frac{\partial v}{\partial \nu} < 0 \quad (> 0) \quad \text{on } \partial\Omega, \tag{2.3}$$

where ν is the outward unit normal to $\partial\Omega$.

From (S.M.P.) and (H.M.P.) one can prove a non-degeneracy condition : (N.D.C.) if $v \in C^2(\bar{\Omega})$ satisfies

$$-\Delta v \geq 0 \text{ in } \Omega \quad v = 0 \text{ on } \partial\Omega \tag{2.4a}$$

and $v \neq 0$ then there exist constants $\sigma_* > 0$ and $C_1 > 0$ such that for $\sigma \in (0, \sigma_*)$

$$v(x) \geq C_1 \text{dist}(x, \partial\Omega) \quad \forall x \in \Omega_\sigma \tag{2.4b}$$

$$v(x) \geq C_2 \sigma \quad \forall x \in \bar{\Omega} \setminus \Omega_\sigma, \tag{2.4c}$$

where

$$\Omega_\sigma \equiv \{x \in \bar{\Omega} : \text{dist}(x, \partial\Omega) < \sigma\}. \tag{2.4d}$$

We note from elliptic regularity, as $f \in C^{0,p}(\mathbb{R})$, that any solution of (2.1) is such that $u \in C^{2,p}(\bar{\Omega})$ and from (S.M.P.) that $\lambda \leq 0$ implies that $u \equiv 0$. Moreover, it is easily seen that

$$u(\lambda) = \lambda^{\frac{1}{1-p}} u(1) \quad \forall \lambda \in \mathbb{R}^+ \tag{2.5}$$

and hence without loss of generality we study $(P) \equiv (P(1))$ only.

For later use we consider a slightly more general problem than (P) : (Q) given $p \in (0, 1)$ and $\xi \in \mathbb{R} \setminus \mathbb{R}^-$ find $w(\xi) \in H_0^1(\Omega)$, $w(0) \neq 0$, such that

$$(\nabla w, \nabla v)_\Omega = (f(w + \xi), v)_\Omega \quad \forall v \in H_0^1(\Omega), \tag{2.6}$$

where f is given by (1.2).

In Theorem 2.1 below we prove existence and uniqueness of a solution to (Q) for all $\xi \in \mathbb{R} \setminus \mathbb{R}^-$, and hence to (P) . The proof is an extension of that given by Aronson and Peletier (1981) for (P) . Firstly we gather together some results concerning the first eigenpair (λ_1, ϕ_1) of (1.4), which will be useful later.

LEMMA 2.1 : *The first eigenpair (λ_1, ϕ_1) of (1.4) are such that*

(i) $\lambda_1 > 0$ is simple and ϕ_1 is of one sign (2.7a)

(ii) $\phi_1 \in C^{2,\tau}(\bar{\Omega})$, $0 < \tau < 1$ (2.7b)

(iii) *normalising so that*

$$\max \{ \phi_1(x) : x \in \Omega \} = 1 \tag{2.7c}$$

it follows that ϕ_1 satisfies the non-degeneracy conditions (N D C) for some constants σ_* and C_1 and

$$(iv) \quad \int_{\Omega} \phi_1^{\alpha-1} < \infty \quad \text{for all } \alpha > 0. \tag{2.7d}$$

Proof From classical eigenfunction theory we have that (i) holds Elliptic regularity yields (ii) (iii) follows from (i), (ii) and (2.7c). For σ sufficiently small it follows that for all $x \in \Omega_\sigma$ there exists a unique $y(x) \in \partial\Omega$ such that $\text{dist}(x, \partial\Omega) = |x - y(x)|$, see Gilbarg and Trudinger (1983), p. 355. Hence it follows from (N.D.C.) that for all $\alpha > 0$

$$\begin{aligned} \int_{\Omega} \phi_1^{\alpha-1} &\equiv \int_{\Omega_\sigma} \phi_1^{\alpha-1} + \int_{\Omega \setminus \Omega_\sigma} \phi_1^{\alpha-1} \leq \\ &\leq C \int_{\partial\Omega} \left(\int_0^\sigma \eta^{\alpha-1} d\eta \right) ds + C \sigma^{\alpha-1} \leq C. \end{aligned}$$

Therefore (iv) holds. ■

Throughout this paper let $\tilde{\Omega}$ be a bounded convex domain in \mathbb{R}^2 with a boundary $\partial\tilde{\Omega}$ of class $C^{2,1}$ such that $\bar{\Omega} \subset \tilde{\Omega}$. Let $\{\tilde{\lambda}_1, \tilde{\phi}_1\}$ be the first eigenpair to the eigenvalue problem (1.4) with Ω replaced by $\tilde{\Omega}$. Clearly they satisfy (2.7) with Ω replaced by $\tilde{\Omega}$. In addition we set

$$\beta \equiv \inf \{ \tilde{\phi}_1(x) \cdot x \in \Omega \} > 0. \tag{2.8}$$

THEOREM 2.1 : *There exists a unique solution w to (Q), $w \in C^{2,p}(\bar{\Omega})$ and*

$$C_2 \tilde{\phi}_1 \geq w \geq C_1 \phi_1 \quad \text{in } \bar{\Omega}, \tag{2.9}$$

where the positive constants C_1 and C_2 depend only on p , Ω and $\tilde{\Omega}$ for ξ sufficiently small

Proof Let $\bar{w} \equiv k \tilde{\phi}_1$, where $k \in \mathbb{R}^+$ is such that

$$k \geq \max \left\{ \xi, \left(\frac{2^p}{\beta \tilde{\lambda}_1} \right)^{\frac{1}{1-p}} \right\}, \tag{2.10a}$$

then it follows that

$$\begin{aligned}
 -\Delta \bar{w} - f(\bar{w} + \xi) &= k \tilde{\lambda}_1 \tilde{\phi}_1 - [k \tilde{\phi}_1 + \xi]^p \geq k \tilde{\lambda}_1 \tilde{\phi}_1 - (2k)^p \\
 &\geq \tilde{\lambda}_1 \beta k^p \left\{ k^{1-p} - \frac{2^p}{\beta \tilde{\lambda}_1} \right\} \geq 0.
 \end{aligned}
 \tag{2.10b}$$

Let $w \equiv c\phi_1$, where $c \in \mathbb{R}^+$ is such that

$$c \leq \min \left\{ \left(\frac{1}{\lambda_1} \right)^{\frac{1}{1-p}}, k\beta \right\},
 \tag{2.11a}$$

then it follows that

$$\begin{aligned}
 -\Delta w - f(w + \xi) &= c\lambda_1 \phi_1 - [c\phi_1 + \xi]^p \\
 &\leq \lambda_1 (c\phi_1)^p \left\{ (c\phi_1)^{1-p} - \frac{1}{\lambda_1} \right\} \leq 0
 \end{aligned}
 \tag{2.11b}$$

and

$$\bar{w} \geq w > 0 \quad \text{in } \Omega.
 \tag{2.11c}$$

Setting $\bar{w}_0 \equiv \bar{w}$ and $w_0 \equiv w$, we define for $i \geq 0$

$$-\Delta \bar{w}_{i+1} = f(\bar{w}_i + \xi) \quad \text{in } \Omega \quad \bar{w}_{i+1} = 0 \quad \text{on } \partial\Omega
 \tag{2.12a}$$

and

$$-\Delta w_{i+1} = f(w_i + \xi) \quad \text{in } \Omega \quad w_{i+1} = 0 \quad \text{on } \partial\Omega.
 \tag{2.12b}$$

Elliptic regularity yields that $\bar{w}_i, w_i \in C^{2,p}(\bar{\Omega})$ for all $i \geq 1$. Since f is monotonically increasing it follows from (S.M.P.) that

$$\begin{aligned}
 \bar{w} \equiv w_0 \geq \bar{w}_1 \geq \dots \bar{w}_i \geq \bar{w}_{i+1} \geq \dots w_{i+1} \geq \\
 w_i \geq \dots w_1 \geq w_0 \equiv w \quad \text{in } \bar{\Omega}.
 \end{aligned}
 \tag{2.13}$$

Therefore $\{\bar{w}_j\}_{j \in \mathbb{N}}$ is a pointwise decreasing sequence bounded below by w and thus it converges pointwise to w :

$$\bar{w} \geq w \geq w > 0 \quad \text{in } \Omega.
 \tag{2.14}$$

From the continuity of f and elliptic regularity it follows that $w \in C^{2,p}(\bar{\Omega})$ solves (Q) .

We now prove uniqueness. Assume the contrary there exist two solutions

w_i $i = 1$ and 2 to (Q) for a given ξ . Elliptic regularity yields that $w_i \in C^{2,p}(\bar{\Omega})$ and hence (N.D.C.) holds. From (2.6) we have that

$$(f(w_1 + \xi), w_2)_\Omega = (\nabla w_1, \nabla w_2)_\Omega = (f(w_2 + \xi), w_1)_\Omega \quad (2.15a)$$

and hence that

$$\int_\Omega w_1(w_2 + \xi)^p \left\{ \left(\frac{1 + \frac{\xi}{w_1}}{1 + \frac{\xi}{w_2}} \right)^p \left(\frac{w_2}{w_1} \right)^{1-p} - 1 \right\} = 0. \quad (2.15b)$$

Therefore we have, for $w_2 \not\equiv w_1$, that there exist points $x, y \in \Omega$ such that

$$w_2(x) > w_1(x) \quad \text{and} \quad w_1(y) > w_2(y) \quad (2.16a)$$

and hence by continuity and (N.D.C.)

$$\gamma \equiv \sup \{ \tau > 0 : \tau w_2 \geq w_1 \text{ in } \bar{\Omega} \} \in (1, \infty). \quad (2.16b)$$

Let $z \equiv \gamma w_2 - w_1 \in C^{2,p}(\bar{\Omega})$ and hence $z \geq 0$ in $\bar{\Omega}$. In addition we have

$$\begin{aligned} -\Delta z &= [\gamma(w_2 + \xi)^p - (w_1 + \xi)^p] \\ &= [(\gamma - \gamma^p)(w_2 + \xi)^p + (\gamma(w_2 + \xi))^p - (w_1 + \xi)^p] \\ &\geq 0 \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (2.17)$$

As $z \not\equiv 0$ by (2.16), (S.M.P.) and (H.M.P.) yield that

$$z > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial z}{\partial \nu} < 0 \quad \text{on } \partial\Omega. \quad (2.18)$$

However by the construction of z , (2.16), it follows that either

$$z(x) = 0 \quad \text{for some } x \in \Omega \quad (2.19a)$$

or

$$\frac{\partial z}{\partial \nu}(x) = 0 \quad \text{for some } x \in \partial\Omega. \quad (2.19b)$$

This contradicts (2.18). Hence $w_2 \equiv w_1$ and we have uniqueness. For full details, see Aronson and Peletier (1981), where uniqueness to (P) is proved under the assumption $\partial\Omega \in C^3$; but this can be relaxed to $C^{2,1}$ if one uses the results on the distance function given by Gilbarg and Trudinger (1983), p. 355, to deduce (2.19b). ■

We end this section by proving two results that will be useful later for the finite element error analysis.

LEMMA 2.2 : *Let*

$$\mathcal{R}(v) \equiv \frac{|v|_{1,\Omega}^2}{(f'(u)v, v)_\Omega}, \tag{2.20a}$$

then it follows that

$$\frac{1}{p} = \mathcal{R}(u) \leq \mathcal{R}(v) \quad \forall v \in H_0^1(\Omega) \tag{2.20b}$$

and hence

$$|v|_{1,\Omega}^2 - (f'(u)v, v)_\Omega \geq (1-p) |v|_{1,\Omega}^2 \quad \forall v \in H_0^1(\Omega). \tag{2.20c}$$

Proof : It follows from (2.9) that

$$C \leq C \tilde{\phi}_1^{p-1} \leq f'(u) = pu^{p-1} \leq C \phi_1^{p-1}. \tag{2.21}$$

From Sobolev's embedding theorem and (2.7d) we have that for $\varepsilon \in \left(0, \frac{p}{1-p}\right)$

$$(\phi_1^{p-1}v, v)_\Omega \leq |\phi_1^{p-1}|_{0,1+\varepsilon,\Omega} |v^2|_{0,1+\frac{1}{\varepsilon},\Omega} \leq C |v|_{1,\Omega}^2 \quad \forall v \in H_0^1(\Omega). \tag{2.22}$$

Therefore from (2.21) and (2.22) it follows that

$$C |v|_{0,\Omega}^2 \leq (f'(u)v, v)_\Omega \leq C |v|_{1,\Omega}^2 \quad \forall v \in H_0^1(\Omega) \tag{2.23}$$

and hence $\mathcal{R}(v)$ is well-defined on $H_0^1(\Omega)$.

From classical eigenfunction theory we have that the first eigenpair (μ_1, ψ_1) of $\mathcal{R}(\cdot)$; that is,

$$\mu_1 = \mathcal{R}(\psi_1) \leq \mathcal{R}(v) \quad \forall v \in H_0^1(\Omega) \tag{2.24a}$$

is such that

$$\mu_1 > 0 \quad \text{is simple and } \psi_1 \text{ is of one sign.} \tag{2.24b}$$

In addition any eigenpair $(\mu_i, \psi_i) \in \mathbb{R}^+ \times H_0^1(\Omega)$ satisfy

$$(\nabla \psi_i, \nabla v)_\Omega = \mu_i (f'(u) \psi_i, v)_\Omega \quad \forall v \in H_0^1(\Omega) \tag{2.25a}$$

and

$$(\psi_i, \psi_j)_\Omega = 0 \quad i \neq j. \tag{2.25b}$$

From (2.1) we see that $\left(\frac{1}{p}, u\right)$ is an eigenpair of (2.25) and as u is of one sign it follows that it is the first eigenpair. Therefore $\mu_1 = \frac{1}{p}$ and the results (2.20b and c) follow. ■

LEMMA 2.3 : Given $p \in (0, 1)$ and $\xi \in \mathbb{R}^+$ the solution $w \equiv w(\xi)$ to (Q) and the solution $u \equiv w(0)$ to (P) are such that

$$(i) w \geq u > 0 \text{ in } \Omega \text{ and } (ii) |w - u|_{1, \Omega} \leq C \xi, \tag{2.26}$$

where C depends only on p and Ω .

Proof: From (2.14), (2.10) and (2.11) the solution $w(\xi)$ to (Q) constructed in Theorem 2.1 above satisfies

$$-\Delta w = f(w + \xi) \geq f(w) \text{ in } \Omega \quad w = 0 \text{ on } \partial\Omega, \tag{2.27a}$$

$$w(\xi) \geq \underline{w}(\xi) \geq \underline{w}(0) \equiv \underline{u} > 0 \text{ in } \Omega \tag{2.27b}$$

and

$$-\Delta \underline{u} \leq f(\underline{u}) \text{ in } \Omega \quad \underline{u} = 0 \text{ on } \partial\Omega. \tag{2.27c}$$

Therefore the desired result (2.26) (i) follows by the construction given in the proof of Theorem 2.1.

From (2.1), (2.6) and the fact that $f'(\cdot)$ is strictly decreasing on \mathbb{R}^+ we obtain that

$$\begin{aligned} |w - u|_{1, \Omega}^2 &= (f(w + \xi) - f(u), w - u)_\Omega \leq \\ &\leq (f'(u)(w + \xi - u), w - u)_\Omega. \end{aligned} \tag{2.28}$$

Combining (2.28) with (2.20c) yields

$$(1 - p)|w - u|_{1, \Omega}^2 \leq \xi (f'(u), w - u). \tag{2.29}$$

The desired result (2.26) (ii) follows from (2.29), (2.21), Sobolev's embedding theorem and (2.7d). ■

In the next section we consider the continuous piecewise linear finite element approximation of (P).

3. FINITE ELEMENT APPROXIMATION

Let Ω^h be a polygonal approximation to Ω defined by $\bar{\Omega}^h \equiv \bigcup_{\tau \in T^h} \bar{\tau}$, where

T^h is a quasi-uniform triangulation consisting of acute-angled triangles τ with a maximum diameter not exceeding h . Let I be the set of nodes and

$\{x_i\}_{i \in I}$ the coordinates of the vertices of this triangulation. If $x_i \in \partial\Omega^h$ then $x_i \in \partial\Omega$ so that $\text{dist}(\partial\Omega, \partial\Omega^h) \leq Ch^2$. We assume for all h that no triangle has all three vertices on $\partial\Omega^h$. For ease of exposition we assume that Ω is convex so that $\Omega^h \subseteq \Omega$. We introduce

$$S^h \equiv \{ \chi \in C(\bar{\Omega}^h) : \chi|_{\tau} \text{ is linear } \forall \tau \in T^h \} \tag{3.1}$$

and

$$S_0^h \equiv \{ \chi \in C(\bar{\Omega}) : \chi|_{\bar{\Omega}^h} \in S^h \text{ and } \chi|_{\bar{\Omega} \setminus \bar{\Omega}^h} = 0 \} . \tag{3.2}$$

Let $\{\chi_j\}_{j \in I}$ be the continuous piecewise linear basis functions for S^h satisfying $\chi_j(x_i) = \delta_{ij} \forall i, j \in I$ and hence $\chi_j \geq 0$ in $\bar{\Omega}$ for all $j \in I$.

The approximation to (P) that we wish to consider is :

(P^h) Given $p \in (0, 1)$, find $u^h \neq 0 \in S_0^h$ such that

$$(\nabla u^h, \nabla \chi)_{\Omega^h} = (f(u^h), \chi)_{\Omega^h} \quad \forall \chi \in S_0^h, \tag{3.3}$$

where f is given by (1.2).

As the triangulation T^h consists solely of acute-angled triangles we have that

$$(\nabla \chi_j, \nabla \chi_i)_{\Omega^h} \leq 0 \quad i \neq j \tag{3.4}$$

$\forall i, j \in I,$

$$0 < (\nabla \chi_i, \nabla \chi_i)_{\Omega^h} = - \sum_{j \neq i} (\nabla \chi_j, \nabla \chi_i)_{\Omega^h}$$

see Ciarlet and Raviart (1973), and this yields a discrete maximum principle, a discrete analogue of (S.M.P.) :

(D.M.P.) If $v^h \in S^h$ with

$$(\nabla v^h, \nabla \chi_i)_{\Omega^h} \geq 0 (\leq 0) \quad \forall \chi_i \in S_0^h \tag{3.5a}$$

$$v^h \geq 0 (\leq 0) \quad \text{on } \partial\Omega^h \tag{3.5b}$$

then

$$v^h \geq 0 (\leq 0) \quad \text{in } \bar{\Omega}^h. \tag{3.5c}$$

In addition if $v^h(x_j) = 0$ for some $x_j \in \Omega^h$ then $v^h \equiv 0$ in $\bar{\Omega}^h$.

It follows immediately from (D.M.P.) that if a solution u^h exists to (P^h) then

$$u^h > 0 \quad \text{in } \Omega^h. \tag{3.6}$$

For later use we consider the discretization of (Q) :

(Q^h) Given $p \in (0, 1)$ and $\xi \in \mathbb{R} \setminus \mathbb{R}^-$ find $w^h(\xi) \in S_0^h$, $w^h(0) \neq 0$, such that

$$(\nabla w^h, \nabla \chi)_{\Omega^h} = (f(w^h + \xi), \chi)_{\Omega^h} \quad \forall \chi \in S_0^h, \tag{3.7}$$

where f is given by (1.2).

In Theorem 3.1 below we prove existence and uniqueness of a solution to (Q^h) for all $\xi \in \mathbb{R} \setminus \mathbb{R}^-$, and hence to (P^h) . The proof is a discrete analogue of that of Theorem 2.1.

Firstly we introduce the corresponding finite element approximation of (1.4) : find $(\lambda_1^h, \phi_1^h) \in \mathbb{R} \times S_0^h$ such that

$$(\nabla \phi_1^h, \nabla \chi)_{\Omega^h} = \lambda_1^h (\phi_1^h, \chi)_{\Omega^h} \quad \forall \chi \in S_0^h. \tag{3.8}$$

In the lemma below we gather together some results concerning the first eigenpair (λ_1^h, ϕ_1^h) which will be useful later.

LEMMA 3.1 : *The first eigenpair (λ_1^h, ϕ_1^h) of (3.8) are such that*

(i) $\lambda_1^h > \lambda_1 > 0$ is simple and ϕ_1^h is of one sign, (3.9a)

(ii) normalising so that

$$\max \{ \phi_1^h(x_j) : j \in I \} = 1, \tag{3.9b}$$

it follows that for $h \leq h_0$

$$| \phi_1 - \phi_1^h |_{0, \infty, \Omega} \leq Ch^2 \ln \frac{1}{h} \tag{3.9c}$$

and

$$\phi_1^h(x_j) \geq C \phi_1(x_j) \quad \forall j \in I; \tag{3.9d}$$

and

(iii) for $h \leq h_0$

$$\int_{\Omega^h} (\phi_1^h)^\alpha < \infty \quad \text{for all } \alpha > 0. \tag{3.9e}$$

Proof : From (3.4) and the Perron-Frobenius theorem we have that (i) holds, see e.g. Barrett and Elliott (1989) for details. An optimal L^2 error bound for the first eigenfunction is given in Strang and Fix (1973). From this it is a simple matter, using standard L^∞ error estimates, to

generate the L^∞ bound (3.9c). Since the triangulation is quasi-uniform and $\Omega^h \subseteq \Omega$ we have that

$$\text{dist}(x_j, \partial\Omega) \geq \text{dist}(x_j, \partial\Omega^h) \geq Ch \quad \forall x_j \in \Omega. \tag{3.10}$$

Therefore (3.9d) is a direct consequence of (3.9c), (N.D.C.) for ϕ_1 and (3.10). The result (3.9e) is clearly true for $\alpha \geq 1$, we know prove it for $\alpha \in (0, 1)$. Let

$$B^h \equiv \{\tau \in T^h : \bar{\tau} \cap \partial\Omega^h \neq \emptyset\} \quad \text{and} \quad \bar{\Omega}_B^h \equiv \bigcup_{\tau \in B^h} \bar{\tau}. \tag{3.11}$$

For the same reasons as for (3.10) it follows that

$$\text{dist}(\tau, \partial\Omega) \geq \text{dist}(\tau, \partial\Omega^h) \geq Ch \quad \forall \tau \in T^h \setminus B^h, \tag{3.12}$$

and hence

$$\phi_1^h(x) \geq C \phi_1(x) \quad \forall x \in \Omega^h \setminus \Omega_B^h. \tag{3.13}$$

Therefore combining (2.7d) and (3.13) we have for all $\alpha \in (0, 1)$ that

$$\int_{\Omega^h \setminus \Omega_B^h} (\phi_1^h)^{\alpha-1} \leq C \int_{\Omega^h \setminus \Omega_B^h} \phi_1^{\alpha-1} < \infty. \tag{3.14}$$

A simple calculation ; using the quasi-uniformity of T^h , (3.9d), (N.D.C.) for ϕ_1 and (3.10) ; yields for all $\alpha \in (0, 1)$

$$\int_{\Omega_B^h} (\phi_1^h)^{\alpha-1} = \sum_{\tau \in B^h} \int_{\tau} (\phi_1^h)^{\alpha-1} \leq C \sum_{\tau \in B^h} [m(\tau) h^{\alpha-1}] \leq Ch^\alpha. \tag{3.15}$$

Combining (3.14) and (3.15) yields the desired result (3.9e). ■

If one imagines extending the triangulation T^h of Ω^h to \tilde{T}^h of $\tilde{\Omega}^h$ where $\tilde{\Omega}^h$ is a polygonal approximation of $\tilde{\Omega}$, such that \tilde{T}^h of $\tilde{\Omega}^h$ satisfies the properties given at the start of this section with \tilde{T}^h , $\tilde{\Omega}^h$ and \tilde{I} replacing T^h , Ω^h and I . Then $\{\tilde{\lambda}_1^h, \tilde{\phi}_1^h\}$, the corresponding finite element approximation of $\{\tilde{\lambda}_1, \tilde{\phi}_1\}$, satisfies the corresponding versions of (3.9) and in addition we have the discrete analogue of (2.8)

$$\beta^h \equiv \inf \{ \tilde{\phi}_1^h(x_j) : x_j \in \tilde{I} \} \geq C > 0. \tag{3.16}$$

THEOREM 3.1: For $h \leq h_0$ there exists a unique solution w^h to (Q^h) and

$$C_3 \geq C_2 \tilde{\phi}_1^h \geq w^h \geq C_1 \phi_1^h \quad \text{in } \bar{\Omega}, \tag{3.17}$$

where the positive constants C_1, C_2 and C_3 depend only on p, Ω and $\bar{\Omega}$ for ξ sufficiently small.

Proof: The proof of existence is a discrete analogue of the proof of Theorem 2.1 using (D.M.P.) instead of (S.M.P.). Similarly for uniqueness, we have from (D.M.P.) that $w_i^h > 0$ in Ω^h . Following the argument (2.15) we have, for $w_2^h \neq w_1^h$, that there exist $k, \ell \in I$ such that $w_2^h(x_k) > w_1^h(x_k)$ and $w_1^h(x_\ell) > w_2^h(x_\ell)$ and hence

$$\gamma \equiv \max_{x_j \in \Omega^h} \frac{w_1^h(x_j)}{w_2^h(x_j)} > 1. \tag{3.18}$$

Let $z^h \equiv \gamma w_2^h - w_1^h \in S_0^h$, then it follows from (D.M.P.) that $z^h \equiv 0$ in $\bar{\Omega}^h$ and hence we have uniqueness. ■

Remark 3.1: It is not necessary for h to be sufficiently small to guarantee that there exists a unique solution w^h to (Q^h) . We imposed it in order to simplify the proof and to establish at the same time (3.17) for later use. ■

We now prove discrete analogues of Lemmas 2.2 and 2.3.

LEMMA 3.2 : *Let*

$$\mathcal{R}^h(\chi) \equiv \frac{|\chi|_{1, \Omega^h}^2}{(f'(u^h)\chi, \chi)_{\Omega^h}}, \tag{3.19a}$$

then it follows that for $h \leq h_0$

$$\frac{1}{p} = \mathcal{R}^h(u^h) \leq \mathcal{R}^h(\chi) \quad \forall \chi \in S_0^h \tag{3.19b}$$

and hence

$$|\chi|_{1, \Omega^h}^2 - (f'(u^h)\chi, \chi)_{\Omega^h} \geq (1-p)|\chi|_{1, \Omega^h}^2 \quad \forall \chi \in S_0^h. \tag{3.19c}$$

Proof: The proof is a discrete analogue of the proof of Lemma 2.2. It follows from (3.17) and (3.9e) that $\mathcal{R}^h(\chi)$ is well-defined on S_0^h . From (3.4) and the Perron-Frobenius theorem we have that the first eigenpair (μ_1^h, ψ_1^h) of $\mathcal{R}^h(\cdot)$ is such that $\mu_1^h > 0$ is simple and ψ_1^h is of one sign. It follows that $(\frac{1}{p}, u^h)$ is the first eigenpair of $\mathcal{R}^h(\cdot)$ and hence the desired results (3.19b and c). ■

LEMMA 3.3 : *Given $p \in (0, 1)$ and $\xi \in \mathbb{R}^+$ then for $h \leq h_0$ the solution $w^h \equiv w^h(\xi)$ to (Q^h) and the solution $u^h \equiv w^h(0)$ to (P^h) are such that*

$$(i) \ w^h \geq u^h > 0 \text{ in } \Omega^h \text{ and } (ii) \ |w^h - u^h|_{1, \Omega^h} \leq C \xi, \tag{3.20}$$

where C depends only on p and Ω .

Proof : The proof of (3.20) (i) is a discrete analogue of the proof of (2.26) (i). Similarly, from (3.3), (3.7), (3.17) and (3.19c) it follows that

$$(1 - p) |w^h - u^h|_{1, \Omega^h}^2 \leq \lambda \xi (f'(u^h), w^h - u^h)_{\Omega^h} \leq C \xi ((\phi_1^h)^{p-1}, w^h - u^h)_{\Omega^h}.$$

The desired result (3.20) (ii) follows from Sobolev's embedding theorem and (3.9e). ■

We are now in a position to analyse the error in the approximation of (P) by (P^h) . Firstly, we introduce some more notation.

Let $\pi_h : C(\bar{\Omega}) \rightarrow S^h$ denote the interpolation operator such that for any $v \in C(\bar{\Omega})$, $\pi_h v \in S^h$ satisfies

$$\pi_h v(x_i) = v(x_i) \quad \forall i \in I. \tag{3.21}$$

We recall the standard approximation result : for $m = 0$ or 1 , $q \in [1, \infty]$ and $r \in (1, \infty)$ provided $W^{2,r}(\tau) \hookrightarrow W^{m,q}(\tau)$

$$|v - \pi_h v|_{m,q,\tau} \leq Ch^2 \left(\frac{1}{q} - \frac{1}{r} + 1\right)^{-m} |v|_{2,r,\tau} \quad \forall v \in W^{2,r}(\tau), \quad \forall \tau \in T^h. \tag{3.22}$$

Let $\mathcal{G} \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega))$ be the « inverse Laplacian » defined by

$$(\nabla \mathcal{G} \eta, \nabla v)_{\Omega} = (\eta, v)_{\Omega} \quad \forall v \in H_0^1(\Omega). \tag{3.23}$$

Let $\mathcal{G}^h \in \mathcal{L}(L^2(\Omega^h), S_0^h)$ be the « discrete inverse Laplacian » defined by

$$(\nabla \mathcal{G}^h \eta, \nabla \chi)_{\Omega^h} = (\eta, \chi)_{\Omega^h} \quad \forall \chi \in S_0^h. \tag{3.24}$$

We recall the following standard finite element error bounds for \mathcal{G}^h :

$$|(\mathcal{G} - \mathcal{G}^h) \eta|_{0,\Omega} + h |(\mathcal{G} - \mathcal{G}^h) \eta|_{1,\Omega} \leq Ch^2 |\eta|_{0,\Omega}, \tag{3.25a}$$

$$\|(\mathcal{G} - \mathcal{G}^h) \eta\|_{0,\infty,\Omega} \leq Ch |\eta|_{0,\Omega} \tag{3.25b}$$

and for $h \leq h_0$

$$\|(\mathcal{G} - \mathcal{G}^h) \eta\|_{0,\infty,\Omega} \leq C \ln \frac{1}{h} \|(I - \pi_h) \mathcal{G} \eta\|_{0,\infty,\Omega}. \tag{3.25c}$$

In addition we require the well-known discrete Sobolev embedding result

$$\|\chi\|_{0,\infty,\Omega} \leq C \left(\ln \frac{1}{h}\right)^{1/2} |\chi|_{1,\Omega} \tag{3.26}$$

and the Gagliardo-Nirenberg inequality, see Friedman (1969), for all $v \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$

$$\|v\|_{0,\infty,\Omega} \leq C |v|_{1,\infty,\Omega}^\varepsilon |v|_{1,\Omega}^{1-\varepsilon} \quad \text{for any } \varepsilon \in (0, 1]. \quad (3.27)$$

LEMMA 3.4 : Given $p \in (0, 1)$ then the unique solutions u and u^h of (P) and (P^h) , respectively, satisfy for $h \leq h_0$ and any $\varepsilon \in (0, 1]$

$$\|u - u^h\|_{0,\infty,\Omega} \leq C \left\{ \left(\ln \frac{1}{h} \right)^{1/2} \|(\mathcal{G} - \mathcal{G}^h) f(u)\|_{0,\infty,\Omega} + \|(\mathcal{G} - \mathcal{G}^h) f(u^h)\|_{0,\infty,\Omega}^{1-\varepsilon} \right\} \quad (3.28)$$

where C depends only on p and Ω .

Proof : Since f is monotonically increasing it follows from (3.23) and (3.24) that

$$\begin{aligned} (\nabla \mathcal{G}^h f(u), \nabla \chi_i)_{\Omega^h} &= (f(u), \chi_i)_{\Omega^h} \leq \\ &\leq (f(\lambda \mathcal{G}^h f(u) + \xi_1^h), \chi_i)_{\Omega^h} \quad \forall \chi_i \in S_0^h, \end{aligned} \quad (3.29a)$$

where

$$\xi_1^h \equiv \|u - \mathcal{G}^h f(u)\|_{0,\infty,\Omega} \equiv \|(\mathcal{G} - \mathcal{G}^h) f(u)\|_{0,\infty,\Omega}. \quad (3.29b)$$

It follows from (3.20) and (3.26) that

$$0 < u^h \leq w^h(\xi_1^h) \leq u^h + C \left(\ln \frac{1}{h} \right)^{1/2} \xi_1^h \quad \text{in } \Omega^h. \quad (3.30)$$

From (D.M.P.) and by choosing $\bar{w}^h \equiv k^h \phi_1^h$, with k^h sufficiently large in the construction used in the proof of Theorem 3.1, it follows that

$$w^h(\xi_1^h) \geq \mathcal{G}^h f(u) > 0 \quad \text{in } \Omega^h. \quad (3.31)$$

Therefore combining (3.29) \rightarrow (3.31) yields

$$\begin{aligned} u - u^h &= [u - \mathcal{G}^h f(u)] + [\mathcal{G}^h f(u) - u^h] \\ &\leq (\mathcal{G} - \mathcal{G}^h) f(u) + (w^h(\xi_1^h) - u^h) \leq C \left(\ln \frac{1}{h} \right)^{1/2} \xi_1^h. \end{aligned} \quad (3.32)$$

Similarly we have from (3.23) and (3.24) that

$$\begin{aligned} (\nabla \mathcal{G} f(u^h), \nabla v)_\Omega &= (f(u^h), v)_\Omega \leq \\ &\leq (f(\mathcal{G} f(u^h) + \xi_2^h), v)_\Omega \quad \forall v \in H_0^1(\Omega), \end{aligned} \quad (3.33a)$$

where

$$\xi_2^h \equiv \|u^h - \mathcal{G}f(u^h)\|_{0, \infty, \Omega} \equiv \|(\mathcal{G} - \mathcal{G}^h)f(u^h)\|_{0, \infty, \Omega}. \quad (3.33b)$$

From Sobolev's embedding theorem, elliptic regularity, (2.9), (2.10a), (3.33b), (3.25b) and (3.17) we have for $r \in (2, \infty)$ that

$$\begin{aligned} |w(\xi_2^h)|_{1, \infty, \Omega} &\leq C \|w(\xi_2^h)\|_{2, r, \Omega} \leq C \|f(w(\xi_2^h) + \xi_2^h)\|_{0, r, \Omega} \\ &\leq C \|w(\xi_2^h) + \xi_2^h\|_{0, \infty, \Omega} \leq C. \end{aligned} \quad (3.34)$$

From (2.26), (3.27) and (3.34) we have for any $\varepsilon \in (0, 1]$ that

$$0 < u \leq w(\xi_2^h) \leq u + C(\xi_2^h)^{1-\varepsilon}. \quad (3.35)$$

From (S.M.P.) and by choosing k sufficiently large in the construction used in the proof of Theorem 2.1, it follows that

$$w(\xi_2^h) \geq \mathcal{G}f(u^h) > 0 \quad \text{in } \Omega. \quad (3.36)$$

Therefore combining (3.33), (3.35), (3.25b), (3.29a) and (3.36) yields

$$\begin{aligned} u^h - u &= [u^h - \mathcal{G}f(u^h)] + [\mathcal{G}f(u^h) - u] \\ &\leq (\mathcal{G}^h - \mathcal{G})f(u^h) + w(\xi_2^h) - u \leq C(\xi_2^h)^{1-\varepsilon}. \end{aligned} \quad (3.37)$$

Combining (3.29), (3.32), (3.33) and (3.37) yields the desired result (3.40). ■

Finally we have the main result of this section.

THEOREM 3.2: *Given $p \in (0, 1)$ then the unique solutions u and u^h of (P) and (P^h), respectively, satisfy for $h \leq h_0$ and any $\varepsilon > 0$*

$$\|u - u^h\|_{0, \infty, \Omega} + h\|u - u^h\|_{1, \infty, \Omega} \leq Ch^{2-\varepsilon}, \quad (3.38)$$

where C depends only on p and Ω .

Proof: The L^∞ result follows directly from (3.28), (3.25c), (3.22) and (3.17) as $\|\mathcal{G}f(u^h)\|_{2, r, \Omega} \leq C$ for all $r \in (1, \infty)$. The $W^{1, \infty}$ result then follows from the L^∞ result, (3.22) and the inverse inequality

$$\|\chi\|_{1, \infty, \Omega} \leq Ch^{-1} \|\chi\|_{0, \infty, \Omega} \quad \forall \chi \in S^h. \quad (3.39) \quad \blacksquare$$

4. A MORE PRACTICAL APPROXIMATION

The standard Galerkin approximation analysed in the previous section requires the term $(f(u^h), \chi)_{\Omega^h}$ to be integrated exactly. This is obviously difficult in practice and it is computationally more convenient to consider a scheme where numerical integration is applied to this term. Below we introduce and analyse such a scheme

For $v_1, v_2 \in C(\bar{\Omega}^h)$ we approximate $(v_1, v_2)_{\Omega^h}$ by

$$(v_1, v_2)^h \equiv \int_{\Omega^h} \pi_h(v_1 v_2) = \sum_{j \in I} \omega_j v_1(x_j) v_2(x_j), \tag{4.1}$$

where $\omega_j \equiv \int_{\Omega^h} \chi_j \forall j \in I$. Introducing the interior nodes $I_0 \equiv \{j \in I : x_j \in \Omega^h\}$, we set

$$(v_1, v_2)_I^h \equiv \sum_{j \in I_0} \omega_j v_1(x_j) v_2(x_j). \tag{4.2}$$

It is easy to show using (3.22) and inverse inequalities that for all non-negative $\chi \in S^h$ and $r \in [1, \infty)$

$$C_1 \|\chi\|_{0,r,\Omega^h}^r \leq (\chi^r, 1)^h \leq C_2 \|\chi\|_{0,r,\Omega^h}^r. \tag{4.3}$$

We now define a more practical approximation of (P) than (P^h) :
 (\hat{P}^h) Given $p \in (0, 1)$, find $\hat{u}^h \neq 0 \in S_0^h$ such that

$$(\nabla \hat{u}^h, \nabla \chi)_{\Omega^h} = (f(\hat{u}^h), \chi)^h \quad \forall \chi \in S_0^h, \tag{4.4}$$

where f is given by (1.2).

We introduce the corresponding discretizations of (Q) and (1.4):

(\hat{Q}^h) Given $p \in (0, 1)$ and $\xi \in \mathbb{R} \setminus \mathbb{R}^-$ find $\hat{w}^h(\xi) \in S_0^h$, $\hat{w}^h(0) \neq 0$, such that

$$(\nabla \hat{w}^h, \nabla \chi)_{\Omega^h} = (f(\hat{w}^h + \xi), \chi)^h \quad \forall \chi \in S_0^h, \tag{4.5}$$

where f is given by (1.2).

Find $(\hat{\lambda}_i^h, \hat{\phi}_i^h) \in \mathbb{R} \times S_0^h$ such that

$$(\nabla \hat{\phi}_i^h, \nabla \chi)_{\Omega^h} = \hat{\lambda}_i^h (\hat{\phi}_i^h, \chi)^h \quad \forall \chi \in S_0^h. \tag{4.6}$$

It is convenient to introduce $\mathcal{G} \in \mathcal{L}(C(\bar{\Omega}^h), S_0^h)$, the « discrete inverse Laplacian in the presence of numerical integration » defined by

$$(\nabla \mathcal{G}^h \eta, \nabla \chi)_{\Omega^h} = (\eta, \chi)^h \quad \forall \chi \in S_0^h. \tag{4.7}$$

We note that $\mathcal{G}^h \eta \equiv \hat{\mathcal{G}}^h(\pi_h \eta)$ and recall the well-known error bound for $\hat{\mathcal{G}}^h$, which follows immediately from (3.24), (4.7), (4.1) and (3.22) : for any $\eta^h \in S^h$

$$|(\mathcal{G}^h - \hat{\mathcal{G}}^h) \eta^h|_{1, \Omega} \leq Ch^{1+m} |\eta^h|_{m, \Omega^h} \quad m = 0 \quad \text{or} \quad 1. \tag{4.8}$$

We now have the discrete analogues of Lemmas 3.1 → 3.4 and Theorems 3.1 → 3.2. In the majority of cases we do not give proofs as they are a straightforward modification of their counterparts in the previous section.

LEMMA 4.1 : *The first eigenpair $(\hat{\lambda}_1^h, \hat{\phi}_1^h)$ of (4.6) are such that*

(i) $\hat{\lambda}_1^h > 0$ *is simple and $\hat{\phi}_1^h$ is of one sign ;* (4.9a)

(ii) *normalising so that*

$$\max \{ \hat{\phi}_1^h(x_j) : j \in I \} = 1, \tag{4.9b}$$

it follows that for $h \leq h_0$

$$|\lambda_1 - \hat{\lambda}_1^h| \leq Ch^2, \quad |\phi_1 - \hat{\phi}_1^h|_{0, \infty, \Omega} \leq Ch^2 \ln \frac{1}{h} \tag{4.9c}$$

and

$$\hat{\phi}_1^h(x_j) \geq C \phi_1(x_j) \quad \forall j \in I ; \tag{4.9d}$$

and

(iii) *for $h \leq h_0$*

$$((\hat{\phi}_1^h)^{\alpha-1}, 1)_I^h < \infty \quad \forall \alpha > 0. \tag{4.9e}$$

Proof : From (3.4) and the Perron-Frobenius theorem we have that (i) holds, see e.g. Barrett and Elliott (1989) for details. For (4.9c) see Strang and Fix (1973), (3.9c), (4.8) and (3.26). Similarly (4.9d) follows from (3.9d), (3.9c) and (4.9c). Clearly (4.9e) holds for $\alpha \geq 1$, we now prove it for

$\alpha \in (0, 1)$. From (4.9d), the quasi-uniformity of the mesh and (N.D.C.) we have

$$\begin{aligned}
 ((\hat{\phi}_1^h)^{\alpha-1}, 1)_I^h &\leq Ch^2 \sum_{j \in I_0} [\phi_1(x_j)]^{\alpha-1} \leq \\
 &\leq Ch^2 \sum_{\substack{j \in I_0 \\ x_j \in \Omega_\sigma}} [\text{dist}(x_j, \partial\Omega)]^{\alpha-1} + C\sigma^{\alpha-1}. \quad (4.10)
 \end{aligned}$$

Then for $h \leq h_0$ we choose $K \in \mathbb{N}$ such that $(K - 1)h < \sigma \leq Kh$ and set for $k = 1 \rightarrow K$

$$I_0^{(k)} \equiv \{j \in I : (k - 1)h < \text{dist}(x_j, \partial\Omega) \leq kh\}. \quad (4.11)$$

By quasi-uniformity of the mesh it follows that the number of nodes belonging to $I_0^{(k)}$ is bounded above by Ch^{-1} . Therefore from (4.11) we have that

$$\begin{aligned}
 h^2 \sum_{\substack{j \in I_0 \\ x_j \in \Omega}} [\text{dist}(x_j, \partial\Omega)]^{\alpha-1} &\leq Ch^\alpha + Ch \sum_{k=1}^{K-1} (kh)^{\alpha-1} \leq \\
 &\leq C \int_0^{(K-1)h} y^{\alpha-1} dy + C \leq C. \quad (4.12)
 \end{aligned}$$

Combining (4.10) and (4.12) yields the desired result (4.9e). ■

THEOREM 4.1 : *For $h \leq h_0$ there exists a unique solution \hat{w}^h to (\hat{Q}^h) and*

$$C_3 \geq C_2 \tilde{\phi}_1^h \geq \hat{w}^h \geq C_1 \hat{\phi}_1^h \text{ in } \bar{\Omega}, \quad (4.13)$$

where the positive constants C_1, C_2 and C_3 depend only on p, Ω and $\tilde{\Omega}$ for ξ sufficiently small.

LEMMA 4.2 : *Let*

$$\hat{\mathcal{R}}^h(\chi) \equiv \frac{|\chi|_{1, \Omega^h}^2}{(f'(\hat{u}^h)\chi, \chi)_I^h}, \quad (4.14a)$$

then it follows that for $h \leq h_0$

$$\frac{1}{p} = \hat{\mathcal{R}}^h(\hat{u}^h) \leq \hat{\mathcal{R}}^h(\chi) \quad \forall \chi \in S_0^h \quad (4.14b)$$

and hence

$$|\chi|_{1, \Omega^h}^2 - (f'(\hat{u}^h)\chi, \chi)_I^h \geq (1 - p) |\chi|_{1, \Omega^h}^2 \quad \forall \chi \in S_0^h. \quad (4.14c)$$

Proof: For $\varepsilon \in \left(0, \frac{p}{1-p}\right)$ we have from (4.9e), (4.3) and Sobolev's embedding theorem that

$$\begin{aligned} ((\hat{\phi}_1^h)^{p-1}, \chi^2)_I^h &\leq [((\hat{\phi}_1^h)^{(p-1)(1+\varepsilon)}, 1)_I^h]^{\frac{1}{1+\varepsilon}} \left[\left(\chi^2 \left(\frac{1+\varepsilon}{\varepsilon} \right), 1 \right)_I^h \right]^{\frac{\varepsilon}{1+\varepsilon}} \\ &\leq C \|\chi\|_{0,2\left(1+\frac{1}{\varepsilon}\right),\Omega}^2 \leq C |\chi|_{1,\Omega}^2 \quad \chi \in S_0^h. \end{aligned} \tag{4.15}$$

Therefore from (4.13) and (4.15) we have that $\hat{\mathcal{R}}^h$ is well-defined on S_0^h . The remainder of the proof follows that of Lemmas 2.2 and 3.2. ■

LEMMA 4.3 : Given $p \in (0, 1)$ and $\xi \in \mathbb{R}^+$ then for $h \leq h_0$ the solution $\hat{w}^h \equiv \hat{w}^h(\xi)$ to (\hat{Q}^h) and the corresponding solution $\hat{u}^h \equiv \hat{u}^h(0)$ to (\hat{P}^h) are such that

$$(i) \quad \hat{w}^h \geq \hat{u}^h > 0 \text{ in } \Omega^h \text{ and } (ii) \quad |\hat{w}^h - \hat{u}^h|_{1,\Omega^h} \leq C \xi, \tag{4.16}$$

where C depends only on p and Ω .

Proof: The bounds (4.16) (i) follow as for (3.20) (i). (4.16) (ii) follows as for (3.20) (ii) and (2.26) (ii) by noting that

$$|\hat{w}^h - \hat{u}^h|_{1,\Omega^h}^2 = (f(\hat{w}^h + \xi) - f(\hat{u}^h), \hat{w}^h - \hat{u}^h)_I^h. \tag{4.17} \quad \blacksquare$$

LEMMA 4.4 : Given $p \in (0, 1)$ then the unique solutions u and \hat{u}^h of (P) and (\hat{P}^h) , respectively, satisfy for $h \leq h_0$ and any $\varepsilon \in (0, 1]$

$$\begin{aligned} \|u - \hat{u}^h\|_{0,\infty,\Omega} &\leq C \left\{ \left(\ln \frac{1}{h} \right)^{1/2} \|(\mathcal{G} - \hat{\mathcal{G}}^h) f(u)\|_{0,\infty,\Omega} + \right. \\ &\quad \left. + \|(\mathcal{G} - \hat{\mathcal{G}}^h) f(\hat{u}^h)\|_{0,\infty,\Omega}^{1-\varepsilon} \right\}, \end{aligned} \tag{4.18}$$

where C depends only on p and Ω .

In order to prove an L^∞ error bound for \hat{u}^h we need to bound

$$\begin{aligned} (\mathcal{G} - \hat{\mathcal{G}}^h) f(v) &\equiv (\mathcal{G} - \mathcal{G}^h) \pi_h f(v) + \\ &\quad + \mathcal{G} (I - \pi_h) f(v) + (\mathcal{G}^h - \hat{\mathcal{G}}^h) \pi_h f(v) \end{aligned} \tag{4.19}$$

for $v \equiv u$ and \hat{u}^h . This we do in the following lemmas.

LEMMA 4.5 : For $h \leq h_0$ and for $v \equiv u$ and \hat{u}^h we have that

$$\|(\mathcal{G} - \mathcal{G}^h) \pi_h f(v)\|_{0,\infty,\Omega} \leq C \left(h \ln \frac{1}{h} \right)^2, \tag{4.20}$$

where C depends on p and Ω .

Proof: It follows from (3.25c), (3.22) and the Calderon-Zygmund inequality that

$$\begin{aligned} \|(\mathcal{G} - \mathcal{G}^h) \pi_h f(v)\|_{0, \infty, \Omega} &\leq C \left(\ln \frac{1}{h}\right) h^{2\left(1-\frac{1}{r}\right)} |\mathcal{G}(\pi_h f(v))|_{2, r, \Omega} \\ &\leq C \left(\ln \frac{1}{h}\right) h^{2\left(1-\frac{1}{r}\right)} r \|\pi_h f(v)\|_{0, \infty, \Omega}. \end{aligned} \tag{4.21}$$

The desired result follows from setting $r = \ln(1/h)$ and noting the bound (4.13). ■

For the next results we need to introduce some more notation. For σ such that ϕ_1 satisfies (N.D.C.) and $h \leq h_0$ choose K such that $(K + 1)h \leq \sigma < (K + 2)h$.

We define

$$\begin{aligned} T_j^h &\equiv \{ \tau \in T^h : \exists x \in \tau \text{ with } \text{dist}(x, \partial\Omega) < jh \} \quad j = 1 \rightarrow K, \\ T_{K+1}^h &\equiv T^h \setminus T_K^h, \end{aligned} \tag{4.22a}$$

$$R_1^h \equiv \bigcup_{\tau \in T_1^h} \bar{\tau}, \quad R_j^h \equiv \bigcup_{\tau \in T_j^h \setminus T_{j-1}^h} \bar{\tau} \quad j = 2 \rightarrow K + 1. \tag{4.22b}$$

It follows that $\bar{\Omega}^h \equiv \bigcup_{j=1}^{K+1} R_j^h$, from the quasi-uniformity of the mesh that

$$\begin{aligned} R_j^h \subset A_j^h &\equiv \{x \in \bar{\Omega}^h : (j - 1)h \leq \text{dist}(x, \partial\Omega) < (j + 1)h\} \\ & \quad j = 1 \rightarrow K, \end{aligned} \tag{4.23a}$$

and hence

$$\underline{m}(R_j^h) \leq \underline{m}(A_j^h) \leq \text{Ch}. \tag{4.23b}$$

For $y \in \Omega$ let $g_y \in W^{1,1}(\Omega)$ be the Green's function such that $g_y = 0$ on $\partial\Omega$ and

$$(\nabla g_y, \nabla v)_\Omega = v(y) \quad \forall v \in W^{1,\infty}(\Omega); \tag{4.24a}$$

and $g_y^h \in S_0^h$, its Galerkin approximation; that is,

$$(\nabla g_y^h, \nabla \chi)_{\Omega^h} = \chi(y) \quad \forall \chi \in S_0^h. \tag{4.24b}$$

LEMMA 4.6 : (i) For $y \in \Omega$ and for $h \leq h_0$

$$\|g_y\|_{0,1,A_j^h} \leq \text{Ch} \left(\ln \frac{1}{h}\right) (jh) \quad j = 1 \rightarrow K. \tag{4.25a}$$

(ii) For $y \in \Omega$

$$\|g_y^h\|_{0, \infty, \Omega^h} + \left(\ln \frac{1}{h}\right)^{1/2} |g_y^h|_{1, \Omega^h} \leq C \left(\ln \frac{1}{h}\right). \tag{4.25b}$$

(iii) For $y \in \Omega_j$, where $\Omega_\sigma \subset\subset \Omega \sigma_j \subset\subset \Omega$, and for $h \leq h_0$

$$\|g_y^h\|_{1, \infty, R_j^h} \leq C \quad j = 1 \rightarrow K. \tag{4.25c}$$

Proof : (i) Let $G_j^h \in H_0^1(\Omega) \cap H^2(\Omega)$ be such that

$$(\nabla G_j^h, \nabla v)_\Omega = (1, v)_{A_j^h} \quad \forall v \in H_0^1(\Omega). \tag{4.26}$$

It follows that

$$G_j^h(y) \equiv \int_{A_j^h} g_y(x) dx \equiv \|g_y\|_{0, 1, A_j^h}. \tag{4.27a}$$

and

$$|\nabla_x G_j^h(x)| \equiv \left| \int_{A_j^h} \nabla_x g_x(t) dt \right| \leq C \int_{A_j^h} |x - t|^{-1} dt, \tag{4.27b}$$

since $g_x(t) = g_t(x) = C \ln(|x - t|) + v_t(x)$ and $v_t \in C^2(\bar{\Omega})$. A simple calculation using (4.23b) yields from (4.27b) that

$$|G_j^h|_{1, \infty, \Omega} \leq Ch \ln \frac{1}{h}. \tag{4.28}$$

Therefore we have, adopting the notation (2.4d), that

$$0 \leq G_j^h(x) \leq Ch \ln \frac{1}{h} (jh) \quad \forall x \in \bar{\Omega}_{jh}. \tag{4.29}$$

As $-\Delta G_j^h = 0$ in $\Omega \setminus \bar{\Omega}_{jh}$, (S.M.P.) yields that

$$0 \leq G_j^h(y) \leq Ch \ln \frac{1}{h} (jh) \quad \forall y \in \Omega \tag{4.30}$$

and hence the desired result (4.25a) follows from (4.27a) and (4.30).

(ii) From (4.24b) we have that $|g_y^h|_{1, \Omega^h}^2 = g_y^h(y)$ and hence the desired result (4.25b) follows from this and (3.26).

(iii) From an inverse inequality and (3.22) we have that

$$\|g_y^h\|_{1, \infty, R_j^h} \leq Ch^{-1} \|g_y - g_y^h\|_{1, R_j^h} + C |g_y|_{2, R_j^h}. \tag{4.31}$$

As $y \in \Omega_I$ is far away from R_j^h one can prove that

$$\|g_y - g_y^h\|_{1, R_j^h} \leq Ch \quad (4.32)$$

using the standard techniques for local energy estimates, see Wahlbin (1990) for details. The desired result (4.25c) then follows from (4.31) and (4.32). ■

LEMMA 4.7: For $h \leq h_0$ we have

$$(i) \quad \|\mathcal{G}(I - \pi_h) f(u)\|_{0, \infty, \Omega} \leq Ch^2 \ln \frac{1}{h}, \quad (4.33a)$$

$$(ii) \quad \|\mathcal{G}(I - \pi_h) f(\hat{u}^h)\|_{0, \infty, \Omega} \leq Ch^2 \ln \frac{1}{h} [1 + |\hat{u}^h|_{1, \infty, \Omega^h}^2] \quad (4.33b)$$

and

$$(iii) \quad \|\mathcal{G}(I - \pi_h) f(\hat{u}^h)\|_{0, \infty, \Omega} \leq Ch^{p-\varepsilon} \quad \forall \varepsilon > 0; \quad (4.33c)$$

where C depends on p and Ω .

Proof: We have from (3.23), (4.24a) and (4.23) that for $v \equiv u$ and \hat{u}^h there exists $y \in \Omega$ such that

$$\begin{aligned} \|\mathcal{G}(I - \pi_h) f(v)\|_{0, \infty, \Omega} &\equiv |((I - \pi_h) f(v), g_y)_\Omega| \\ &\leq |((I - \pi_h) f(v), g_y)_{A_1^h}| \\ &\quad + \sum_{j=2}^{K+1} |((I - \pi_h) f(v), g_y)_{R_j^h}|. \end{aligned} \quad (4.34)$$

The results (4.25a) and (4.13) yield that

$$|((I - \pi_h) f(v), g_y)_{A_1^h}| \leq Ch^2 \ln \frac{1}{h}. \quad (4.35)$$

It follows from (4.25a) and (3.22) that for $j = 2 \rightarrow K$

$$\begin{aligned} |((I - \pi_h) f(v), g_y)_{R_j^h}| &\leq Ch \ln \frac{1}{h} (jh) \|(I - \pi_h) f(v)\|_{0, \infty, R_j^h} \\ &\leq Ch^3 \ln \frac{1}{h} (jh) |f(v)|_{2, \infty, R_j^h}. \end{aligned} \quad (4.36)$$

As

$$D^2 f(v) \equiv f'(v) D^2 v + f''(v) (Dv)^2, \quad (4.37)$$

it follows from (4.36), (4.22), (2.9) and (N.D.C.) for ϕ_1 that

$$\begin{aligned} \sum_{j=2}^K \left| \langle (I - \pi_h) f(u), g_y \rangle_{R_j^h} \right| &\leq Ch^3 \ln \frac{1}{h} \sum_{j=2}^K (jh) [(j-1)h]^{p-2} \\ &\leq Ch^2 \ln \frac{1}{h} \int_0^{(K-1)h} t^{p-1} dt \\ &\leq Ch^2 \ln \frac{1}{h}. \end{aligned} \tag{4.38}$$

Similarly it follows, noting (4.37), (4.13) and (4.9d), that

$$\sum_{j=2}^K \left| \langle (I - \pi_h) f(\hat{u}^h), g_y \rangle_{R_j^h} \right| \leq Ch^2 \ln \frac{1}{h} |\hat{u}^h|_{1, \infty, \Omega^h}^2. \tag{4.39}$$

In addition we have for any $\delta > 0$, noting (3.22), that

$$\begin{aligned} \left| \langle (I - \pi_h) f(v), g_y \rangle_{R_{k+1}^h} \right| &\leq C \| (I - \pi_h) f(v) \|_{0, 1+\delta, R_{k+1}^h} \\ &\leq Ch^2 |f(v)|_{2, 1+\delta, R_{k+1}^h}. \end{aligned}$$

Therefore from (4.37), (4.13), (4.9d), (2.9) and (N.D.C.) for ϕ_1 we have that

$$\left| \langle (I - \pi_h) f(u), g_y \rangle_{R_{k+1}^h} \right| \leq Ch^2 \tag{4.40a}$$

and

$$\left| \langle (I - \pi_h) f(\hat{u}^h), g_y \rangle_{R_{k+1}^h} \right| \leq Ch^2 |\hat{u}^h|_{1, \infty, \Omega^h}^2. \tag{4.40b}$$

Similarly we have for any $\delta > 0$ that

$$\begin{aligned} \left| \langle (I - \pi_h) f(\hat{u}^h), g_y \rangle_{\Omega^h \setminus R_1^h} \right| &\leq Ch^2 |\hat{u}^h|_{2, 1+\delta, \Omega^h \setminus R_1^h} \\ &\leq Ch^2 \|f''(\hat{u}^h)\|_{0, \infty, \Omega^h \setminus R_1^h} |\hat{u}^h|_{1, 2(1+\delta), \Omega^h \setminus R_1^h}^2 \\ &\leq Ch^p |\hat{u}^h|_{1, \infty, \Omega^h}^{2\delta/(1+\delta)} \leq Ch^{p-2\delta/(1+\delta)}; \end{aligned} \tag{4.41}$$

where we have employed an inverse inequality and noted that $|\hat{u}^h|_{1, \Omega^h} \leq C$, which follows immediately from (4.4), (4.3) and (4.13).

The desired results (4.33a-c) follow from (4.34) and (4.35) with (a) (4.38) and (4.40a); (b) (4.39) and (4.40b); and (c) (4.41). ■

LEMMA 4.8 : For $v \equiv u$ and \hat{u}^h and for $h \leq h_0$ we have that

$$\begin{aligned} \text{(i)} \quad &\| (\mathcal{G}^h - \hat{\mathcal{G}}^h) \pi_h f(v) \|_{0, \infty, \Omega} \leq \\ &\leq Ch^2 \left(\ln \frac{1}{h} \right)^{1/2} \left\{ h^{-1/2} \|f(v)\|_{0, \infty, A_1^h \cup A_2^h} + (1 + h^{p-1/2}) |\pi_h v|_{1, \infty, \Omega^h} \right\} \end{aligned} \tag{4.42a}$$

and

(ii) for $\Omega_I \subset\subset \Omega$

$$\|(\mathcal{G}^h - \hat{\mathcal{G}}^h) \pi_h f(v)\|_{0, \infty, \Omega_I} \leq Ch^2 \left(\ln \frac{1}{h}\right)^{1/2} \{1 + |\pi_h v|_{1, \infty, \Omega^h}\} \quad (4.42b)$$

where C depends on p and Ω .

Proof: We have from (3.24), (4.7), (4.24b), (4.23) and (3.22) that for $v \equiv u$ and \hat{u}^h there exists $y \in \Omega^h$ such that

$$\begin{aligned} \|(\mathcal{G}^h - \hat{\mathcal{G}}^h) \pi_h f(v)\|_{0, \infty, \Omega} &\leq \sum_{j=1}^{K+1} |(I - \pi_h)(\{\pi_h f(v)\} g_y^h)|_{0, 1, R^h} \\ &\leq Ch^2 \sum_{j=1}^{K+1} \{|\pi_h f(v)|_{1, R_j^h} |g_y^h|_{1, R_j^h}\}. \end{aligned} \quad (4.43)$$

It follows from an inverse inequality and (4.23) that

$$\begin{aligned} \sum_{j=1}^2 \{|\pi_h f(v)|_{1, R_j^h} |g_y^h|_{1, R_j^h}\} &\leq \\ &\leq Ch^{-1/2} \|f(v)\|_{0, \infty, A_1^h \cup A_2^h} |g_y^h|_{1, R_1^h \cup R_2^h}. \end{aligned} \quad (4.44)$$

It follows from (4.23b), (4.13), (4.9d), (2.9) and (N.D.C.) for ϕ_1 that

$$\begin{aligned} \sum_{j=3}^K \{|\pi_h f(v)|_{1, R_j^h} |g_y^h|_{1, R_j^h}\} &\leq \\ &\leq C \sum_{j=3}^K \{ \|f'(v)\|_{0, \infty, R_j^h} |\pi_h v|_{1, R_j^h} |g_y^h|_{1, R_j^h} \} \\ &\leq C \left\{ \sum_{j=3}^K h \|f'(v)\|_{0, \infty, R_j^h}^2 \right\}^{1/2} |g_y^h|_{1, \Omega^h} |\pi_h v|_{1, \infty, \Omega^h} \\ &\leq C (1 + h^{p-1/2}) |g_y^h|_{1, \Omega^h} |\pi_h v|_{1, \infty, \Omega^h}, \end{aligned} \quad (4.45)$$

since

$$\sum_{j=3}^K h \|f'(v)\|_{0, \infty, R_j^h}^2 \leq C \int_h^{(K-1)h} t^{2(p-1)} dt.$$

Similarly we have

$$\begin{aligned} \sum_{j=3}^K \{|\pi_h f(v)|_{1, R_j^h} |g_y^h|_{1, R_j^h}\} &\leq \\ &\leq C \left\{ \sum_{j=3}^K h \|f'(v)\|_{0, \infty, R_j^h} \right\} |g_y^h|_{1, \infty, \Omega^h \setminus R_{K+1}^h} |\pi_h v|_{1, \infty, \Omega^h} \\ &\leq C \left\{ \int_h^{(K-1)h} t^{p-1} dt \right\} |g_y^h|_{1, \infty, \Omega^h \setminus R_{K+1}^h} |\pi_h v|_{1, \infty, \Omega^h} \\ &\leq C |g_y^h|_{1, \infty, \Omega^h \setminus R_{K+1}^h} |\pi_h v|_{1, \infty, \Omega^h}. \end{aligned} \quad (4.46)$$

Finally it follows from (4.13), (4.9d), (2.9) and (N.D.C.) for ϕ_1 that

$$\begin{aligned} |\pi_h f(v)|_{1, R_{k+1}^h} |g_y^h|_{1, R_{k+1}^h} &\leq C \|f'(v)\|_{0, \infty, R_{k+1}^h} |g_y^h|_{1, \Omega^h} |\pi_h v|_{1, \Omega^h} \\ &\leq C |g_y^h|_{1, \Omega^h} |\pi_h v|_{1, \Omega^h}. \end{aligned} \tag{4.47}$$

The desired result (4.42a) follows from (4.43), (4.44), (4.45), (4.47) and (4.25b). The desired result (4.42b) follows from (4.43) with $y \in \Omega_I$, (4.44) and (4.46) with (4.25c), (4.23b) and (4.13); and (4.47) with (4.25b). ■

LEMMA 4.9 : For $h \leq h_0$, $\Omega_I \subset\subset \Omega$ and for all $\varepsilon > 0$ we have

$$(i) \quad \|(\mathcal{G} - \hat{\mathcal{G}}^h) f(u)\|_{0, \infty, \Omega} \leq Ch^{2-\varepsilon}(1 + h^{p-1/2}), \tag{4.48a}$$

$$(ii) \quad \|(\mathcal{G} - \hat{\mathcal{G}}^h) f(u)\|_{0, \infty, \Omega_I} \leq Ch^{2-\varepsilon}, \tag{4.48b}$$

$$(iii) \quad \|(\mathcal{G} - \hat{\mathcal{G}}^h) f(\hat{u}^h)\|_{0, \infty, \Omega} \leq Ch^{p-\varepsilon}, \tag{4.48c}$$

$$\begin{aligned} (iv) \quad \|(\mathcal{G} - \hat{\mathcal{G}}^h) f(\hat{u}^h)\|_{0, \infty, \Omega} &\leq \\ &\leq Ch^{2-\varepsilon} \left\{ 1 + |\hat{u}^h|_{1, \infty, \Omega^h}^2 + h^{p-1/2} |\hat{u}^h|_{1, \infty, \Omega^h} + h^{-1/2} \|(\hat{u}^h)^p\|_{0, \infty, A_1^h \cup A_2^h} \right\} \end{aligned} \tag{4.48d}$$

and

$$(v) \quad \|(\mathcal{G} - \hat{\mathcal{G}}^h) f(\hat{u}^h)\|_{0, \infty, \Omega_I} \leq Ch^{2-\varepsilon}(1 + |\hat{u}^h|_{1, \infty, \Omega^h}^2), \tag{4.48e}$$

where C depends on p and Ω .

Proof: The above results follow by combining (4.19) and (4.20) with various other results :

(i) (4.33a), (4.42a) and noting that $\|u^p\|_{0, \infty, A_1^h \cup A_2^h} \leq Ch^p$.

(ii) (4.33a) and (4.42b).

(iii) (4.33c), (4.42a), (3.39) and (4.13).

(iv) (4.33b) and (4.42a).

(v) (4.33b) and (4.42b). ■

Finally we have the main result of this section.

THEOREM 4.2 : Given $p \in (0, 1)$ then the unique solutions u and \hat{u}^h of (P) and (\hat{P}^h) , respectively, satisfy for $h \leq h_0$, $\Omega_I \subset\subset \Omega$ and any $\varepsilon > 0$

$$\begin{aligned} \|u - \hat{u}^h\|_{0, \infty, \Omega} + h \|u - \hat{u}^h\|_{1, \infty, \Omega} &\leq \\ &\leq \begin{cases} Ch^{2-\varepsilon} & \text{for } p \in [1/2, 1) \\ Ch^{(3/2)+p-\varepsilon} & \text{for } p \in (0, 1/2] \end{cases} \end{aligned} \tag{4.49a}$$

and

$$\|u - \hat{u}^h\|_{0, \infty, \Omega_I} + h \|u - \hat{u}^h\|_{1, \infty, \Omega_I} \leq Ch^{2-\varepsilon}; \quad (4.49b)$$

where C depends on p and Ω .

Proof : From (4.18), (4.48a) and (4.48c) we have that

$$\|u - \hat{u}^h\|_{0, \infty, \Omega} \leq Ch^{p-\varepsilon}. \quad (4.50)$$

From (3.22) and (3.39) we have that.

$$|\hat{u}^h|_{1, \infty, \Omega^h} \leq C(1 + h^{-1} \|u - \hat{u}^h\|_{0, \infty, \Omega}) \quad (4.51a)$$

and hence

$$|\hat{u}^h|_{1, \infty, \Omega^h}^2 \leq C(1 + h^{-2} \|u - \hat{u}^h\|_{0, \infty, \Omega}^2). \quad (4.51b)$$

Combining (4.50) and (4.51b) yields

$$|\hat{u}^h|_{1, \infty, \Omega^h}^2 \leq C(1 + h^{p-2-\varepsilon} \|u - \hat{u}^h\|_{0, \infty, \Omega}). \quad (4.52)$$

Combining (4.18), (4.48a), (4.48d) and (4.51) we obtain

$$\begin{aligned} \|u - \hat{u}^h\|_{0, \infty, \Omega} &\leq Ch^{2-\varepsilon}(1 + h^{p-1/2}) + Ch^{p-\varepsilon} \|u - \hat{u}^h\|_{0, \infty, \Omega} + \\ &\quad + h^{(3/2)-\varepsilon} \|(\hat{u}^h)^p\|_{0, \infty, A_1^h \cup A_2^h}. \end{aligned} \quad (4.53)$$

Therefore (4.13) and (4.53) yield that

$$\|u - \hat{u}^h\|_{0, \infty, \Omega} \leq Ch^{(3/2)-\varepsilon} \quad (4.54a)$$

and hence

$$\|(\hat{u}^h)^p\|_{0, \infty, A_1^h \cup A_2^h} \leq Ch^p. \quad (4.54b)$$

The desired result (4.49a) then follows from (4.53), (4.54b) and (3.39). The result (4.49b) follows from (4.48b), (4.48e), (4.49a) and (3.39). ■

COROLLARY : Given $p \in (0, 1)$ then the unique solutions u and \hat{u}^h of (P) and (\hat{P}^h) in one dimension, respectively, satisfy for $h \leq h_0$

$$\|u - \hat{u}^h\|_{0, \infty, \Omega} + h \|u - \hat{u}^h\|_{1, \infty, \Omega} \leq Ch^2, \quad (4.55)$$

where C depends on p and Ω .

Proof: In one dimension the key differences are (i) for y a mesh point

$$g_y^h \equiv g_y \quad \text{and} \quad \|g_y^h\|_{1, \infty, \Omega^h} \leq C$$

and (ii) the $\left(\ln \frac{1}{h}\right)$ terms are avoided and one can set $\varepsilon = 0$ in the above. It is then a simple matter to adapt the proof of Lemma 4.8 to show that (4.42b) holds for $\Omega_I \equiv \Omega$ and hence the desired result (4.55) follows. ■

In proving the above error bounds, (4.49) and (4.55), we have exploited, and hence the lengthy argument, the fact that the lack of Lipschitz continuity of $f(u)$ and $f(\hat{u}^h)$ occurs in the vicinity of $\partial\Omega$, where the Green's functions are « small »; see (4.25a) for example. This is the reason why there is no « pollution » from the numerical integration of this rough forcing term, i.e. we have an optimal interior L^∞ error bound in two dimensions, global in one dimension. This is in marked contrast to the case where the lack of Lipschitz continuity occurs in the interior. Wahlbin (1990), § 18, studies the one dimensional linear problem :

$$-u'' = [x]_+^p \quad \text{for } x \in \Omega \equiv (-1, 1) \quad u(-1) = u(1) = 0,$$

where $p \in (0, 1)$ and shows that

$$\| \mathcal{G}[x]_+^p - \mathcal{G}^h \pi_h [x]_+^p \|_{0, \infty, \Omega} \leq Ch^{1+p}.$$

In addition he shows that the L^∞ error does not improve away from $x = 0$, where the forcing term is rough. Thus there is global pollution in this case.

Finally, we note from Lemmas 4.5, 4.7 and 4.8 that the only term that is not converging globally at the near optimal rate in L^∞ in two dimensions is the last term in (4.19). From this it would appear better to define \hat{u}^h , our fully practical approximation to u , to be

$$\hat{u}^h = \mathcal{G}^h \pi_h f(\hat{u}^h) \tag{4.56a}$$

as opposed to our present choice

$$\hat{u}^h = \hat{\mathcal{G}}^h f(\hat{u}^h) \equiv \hat{\mathcal{G}}^h \pi_h f(\hat{u}^h). \tag{4.56b}$$

We note that (4.56a) is as computationally convenient as (4.56b). However, we have not been able to generalise Lemma 4.2 to this choice.

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