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FINITE ELEMENT COMPUTATION
OF HYPERELASTIC RODS IN LARGE DISPLACEMENTS (*)

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Communicated by R TEMAM

Abstract. — *The present paper deals with the Numerical Analysis of the equilibrium problem of an elastic rod subjected to very large displacements, involving traction, flexion and torsion The model which is used describes the motion and the deformation of the rod by orthogonal directors, which leads to an hyperelastic problem written in an objective and intrinsic framework This problem is then approximated by Mixed Finite Elements and the corresponding approximation is proved to be well posed and to produce solutions which strongly converge towards any isolated stable equilibrium position of the rod Finally, the discrete problem is numerically solved by a Newton's type procedure which respects the orthogonality of the directors and the inextensibility of the rod*

Résumé. — *Le sujet de cet article est l'Analyse Numérique du problème d'équilibre d'une poutre soumise à de très grands déplacements, et à l'action combinée d'efforts de traction, de flexion et de torsion Le modèle utilisé décrit le mouvement et les déformations de la poutre à l'aide de vecteurs directeurs orthogonaux, ce qui conduit à un problème hyperélastique écrit sous forme variationnelle, objective et intrinsèque Le problème est ensuite approché par une méthode d'Éléments Finis mixtes, et on démontre que le problème approché est bien posé et qu'il possède des solutions discrètes qui convergent fortement vers toute configuration d'équilibre stable de la poutre Enfin, on propose une méthode de résolution numérique du problème discret à l'aide d'une méthode de Newton qui respecte l'orthogonalité des vecteurs directeurs et l'inextensibilité de la poutre*

1. INTRODUCTION

This paper is concerned with the Numerical Analysis of the equilibrium problem of a hyperelastic rod subjected to large displacements, flexion and torsion. The relevant mechanical model, introduced in [1], [2] or [3], uses

directors to describe the motion of the rod and to measure torsion and flexion in an objective and intrinsic framework.

In a previous paper [4], we have written such a problem under a variational form and proved it to be well-posed.

Our goal is to derive here a consistent Finite Element approximation of this problem (Sect. 3), to prove its convergence (Sect. 3), its consistency (Sect. 4) and to describe a Newton's type solution procedure (Sect. 5) which respects its geometric constraints (orthonormality of the directors and, if relevant, inextensibility of the rod).

2. THE MECHANICAL PROBLEM
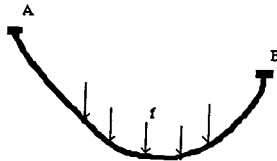
2.1. Reference configuration

Figure 1. — The Physical Problem.

We consider herein the deformation of a rod which, in its reference configuration is straight, of length L and of cross sections $\Omega(S)$. By choosing an adequate cartesian basis in the physical space E, this reference configuration can be identified to the domain $\bar{\Omega}$ of $\mathbb{R}^3$ given by

$$\bar{\Omega} = \left\{ X \in R^3, X = (X_1, X_2, S), 0 \leq S \leq L, (X_1, X_2) \in \bar{\Omega}(S) \right\} . \quad (2.1)$$

2.2. Kinematics of the rod

Following [4], we will suppose that the position $x(X)$ of any particle X after deformation is given by

$$x(X_1, X_2, S) = r(S) + X_1 d_1(S) + X_2 d_2(S) , \quad (2.2)$$

$$d_\alpha(S) . d_\beta(S) = \delta_{\alpha\beta} , \quad \forall 1 \leq \alpha, \beta \leq 2 , \quad \forall S \in [0, L] . \quad (2.3)$$

Above, $r(S)$ is the final position of the rod center line and $\{d_\alpha(S)\}$ can be viewed as the images after deformation of two material vectors engraved on the section $X_3 = S$, initially parallel to the coordinate axes and of norm unity. The above equality implies that the cross sections do not change

shape but does not require them to stay normal to the center line after deformation. In other words, shear is permitted.

In addition to (2.2)-(2.3), the deformation $x(X)$ must satisfy the kinematic boundary conditions imposed to the rod under study. Introducing the director $d_3 = d_1 \times d_2$, we suppose that these conditions consist in

$$r(0) = r_0 \quad \text{(rod fixed in 0)}, \tag{2.4}$$

and in one or more of the following

$$r(L) = r_L \quad \text{(rod fixed at both ends)}, \tag{2.5}$$

$$d_k(0) = d_k^0 \text{ for } k \text{ given in } \{1, 2, 3\}$$

$$\text{(rod rotating around } d_k^0)$$

or $\quad d_k(0) = d_k^0, \ \forall k = 1, 2, 3 \quad \text{(rod clamped)}, \tag{2.6}$

$$d_k(L) = d_k^L \text{ for } k \text{ given in } \{1, 2, 3\},$$

or $\quad d_k(L) = d_k^L, \ \forall k = 1, 2, 3. \tag{2.7}$

Finally, the set of kinematically admissible deformations is obtained by combining the kinematic assumption (2.2)-(2.3) with the above boundary conditions, and is therefore given by

$$K = \{ \{r, d_1, d_2, d_3\} = \{r, d_i\} \in W^{1, p}(0, L; E^4),$$
$$r(0) = r_0, d_i(s) \cdot d_j(s) = \delta_{ij}, \ \forall i, j = 1, 3, \ \forall s \in [0, L],$$
$$\{r, d_i\} \text{ satisfies one or more of the}$$
$$\text{boundary condition (2.5)-(2.7) } \}. \tag{2.8}$$

Herein, $W^{1, p}(0, L)$ denotes the usual Banach space

$$W^{1, p}(0, L) = \left\{ f \in L^p(0, L), f' = \frac{df}{dS} \in L^p(0, L) \right\},$$

with norms
$$\|f\|_{0, p} = \left(\int_0^L |f|^p \, dS \right)^{1/p},$$

$$\|f\|_{1, p} = (\|f\|_{0, p}^p + \|f'\|_{0, p}^p)^{1/p}.$$

2.3. Measuring strains

Following [4], to the deformation $x(X)$ given by (2.2), we associate the strains $\{u_j\}$ and $\{w_k\}$ defined by

$$u_j(d_i) = \frac{1}{2} e_{jkl} d_k' \cdot d_l, \tag{2.9}$$

$$v_k(d_i) = r' \cdot d_k. \tag{2.10}$$

When $\{r, d_i\} \in K$, the above definitions reduce to

$$d_i' = (u_j d_j) \times d_i,\tag{2.11}$$

$$r' = v_k d_k.\tag{2.12}$$

As indicated in [4], $\{u_j\}$ and $\{v_k\}$ are an objective measure of strains. They are defined independently of any observer and their knowledge characterizes $x(X)$ within a rigid body motion. From a geometric point of view, u_α measure flexion, u_3 measures torsion, v_α measure shear and v_3 measures volume changes.

From the compact injection from $W^{1,p}(0, L)$ into $L^\infty(0, L)$, it is also a simple matter to prove that u_j and v_k are weakly continuous maps from $W^{1,p}(0, L; E^4)$ into $L^p(0, L)$.

2.4. Variational formulation

In the above kinematic framework, the equilibrium positions of the rod under study are characterized by the equilibrium equations

$$m' + r \times n = 0,\tag{2.13}$$

$$n' + f = 0,\tag{2.14}$$

and by the hyperelastic constitutive law

$$m = \frac{\partial w}{\partial u_j} d_j,\tag{2.15}$$

$$n = \frac{\partial w}{\partial v_k} d_k.\tag{2.16}$$

Above, m and n denote the stress couple and stress resultant inside the rod, f is the lineic density of applied forces, and w is the elastic stored energy potential of the rod. A typical choice for w is :

$$w(S, u_j, v_k) = \frac{1}{2} \left\{ GA_\alpha(S) v_\alpha^2 + EA(S)(v_3 - 1)^2 \right.$$
$$\left. + EI_\alpha(S) u_\alpha^2 + GI(S) u_3^2 \right\}.\tag{2.17}$$

Such an energy potential involves shear $(GA_\alpha v_\alpha^2)$, tension $(EA(v_3 - 1)^2)$, flexion $(EI_\alpha u_\alpha^2)$ and torsion (GIu_3^2).

If f is independent of r, the above equilibrium problem takes the variational form [4]

$$\begin{cases} \text{Find} \quad \{r, d_i\} \in \bar{K} \quad \text{such that} \\ J(r, d_i) \leqslant J(p, g_i), \quad \forall \{p, g_i\} \in \bar{K}, \end{cases}\tag{2.18}$$

under the notation $\bar{K} = K$ and

$$J(r, d_i) = \int_0^L w(S, u_j(d_i), v_k(r, d_i))\, dS - \int_0^L f \cdot r\, dS\,. \qquad (2.19)$$

In our analysis, we will suppose that w is so chosen that the energy J satisfies :

(H1) J is continuous on $W^{1,P}(0, L\,; E^4)$;

(H2) $J(r, d_i) = C_0\left(\sum_{i=1}^3 \|d_i'\|_{0,p}^p + \|r' - d_3\|_{0,p}^p \right) + J_0(r, d_i)$,

with J_0 positive and weakly lower semicontinuous

on $W^{1,P}(0, L\,; E^4)$.

These two assumptions are satisfied by the choice (2.17) if we take $p = 2$ and $C_0 = \min (GA_\alpha, EA, EI_\alpha, GI)$.

2.5. The case with no extension and no shear

The case with no extension and no shear is an important limit case in practice. It is simply obtained by adding the constraint $r' = d_3$ in the definition of kinematically admissible deformation fields. In other words, we introduce

$$K^{\text{inc}} = \{\{r, d_i\} \in K, r' = d_3\} \qquad (2.20)$$

and define our equilibrium problem by (2.18) with $\bar{K}$ now given by

$$\bar{K} = K^{\text{inc}}\,.$$

In this case, we have $v_1 = v_2 = v_3 - 1 = 0$ and thus w is only a function of $\{u_j\}$.

3. THE FINITE ELEMENT PROBLEM

3.1. Finite Element formulation

The classical Finite Element approximation of the equilibrium problem (2.18) approximates the position r of the rod center line by C^1 piecewise cubic Finite Elements and the director fields d_i by C^0 piecewise quadratic Finite Elements.
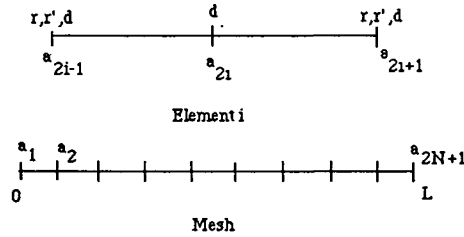
Figure 2. — The Finite Element Mesh.

With this choice, the Finite Element formulation of our problem is simply

$$\begin{cases} \text{Find} \quad \{r, d_i\} \in \bar{K}_h \quad \text{such that} \\ J(r, d_i) \leqslant J(p, g_i), \quad \forall \{p, g_i\} \in \bar{K}_h, \end{cases} \tag{3.21}$$

with

$$\begin{aligned} \bar{K}_h = K_h = \{ \{r, d_i\} \in V_h \times W_h^3, d_i(M_p) . d_j(M_p) = \delta_{ij}, \\ \forall i, j = 1, 3, \quad \forall M_p \in M_h, \{r, d_i\} \text{ satisfies the} \\ \text{boundary conditions imposed for } K \}, \end{aligned} \tag{3.22}$$

in the general case and

$$\bar{K}_h = K_h^{\text{inc}} = \{ \{r, d_i\} \in K_h, r' = d_3 \} \tag{3.23}$$

in the case with no shear and no extension. Above, the Finite Element spaces V_h and $\bar{W}_h$ are respectively given by

$$V_h = \{ v \in C^1(0, L; E), v|_{[a_{2i-1}, a_{2i+1}]} \in P_3, \forall i = 1, N_h \}, \tag{3.24}$$

$$W_h = \{ v \in C^0(0, L; E), v|_{[a_{2i-1}, a_{2i+1}]} \in P_2, \forall i = 1, N_h \}, \tag{3.25}$$

with $([a_{2i-1}, a_{2i+1}])_{i=1, N_h}$ a given subdivision of $[0, L]$, P_n the space of polynomials of degree less than or equal to n, and M_h the set of vertices and centers of the subdivision.

3.2. Lemmas

LEMMA 1 : *Let* $(\{r_h, d_{ih}\})_h$ *be a sequence of elements with* $\{r_h, d_{ih}\} \in K_h, \forall h$ *and* $(\{r_h, d_{ih}\})_h$ *weakly converging towards* $\{r, d_i\}$ *in* $W^{1, p}(0, L; E^4)$. *Then*

$$d_j(S) . d_k(S) = \delta_{jk}, \quad \forall S \in [0, L], \quad \forall j, k = 1, 3. \tag{3.26}$$

Proof : Let $h = \max |a_{2i+1} - a_{2i-1}|$, q be such that $1/p + 1/q = 1$ and M_p be the point of M_h closest to S. We then have

$$d_{jh}(S) . d_{kh}(S) - d_{jh}(M_p) . d_{kh}(M_p) =$$

$$= d_{jh}(M_p) . \int_{M_p}^S d'_{kh}(\sigma) \, d\sigma + d_{kh}(M_p) . \int_{M_p}^S d'_{jh}(\sigma) \, d\sigma$$

$$+ \left(\int_{M_p}^S d'_{jh}(\sigma) \, d\sigma \right) . \left(\int_{M_p}^S d'_{kh}(\sigma) \, d\sigma \right) .$$

From the Hölder inequality and since $\{r_h, d_{ih}\} \in K_h$, this implies

$$|d_{jh}(S) . d_{kh}(S) - \delta_{jk}| \leqslant h^{1/q} \|d'_{kh}\|_{0,p} + h^{1/q} \|d'_{jh}\|_{0,p}$$
$$+ h^{2/q} \|d'_{kh}\|_{0,p} \|d'_{jh}\|_{0,p} .$$

Since $(\{d_{ih}\})_h$ is weakly converging in $W^{1,p}(0, L ; E^3)$, $\|d'_{kh}\|_{0,p}$ is bounded and $d_{jh}(S) . d_{kh}(S)$ converges pointwise towards $d_j(S) . d_k(S)$. By going to the limit as h goes to zero in the above inequality, we then obtain the desired result. □

LEMMA 2 : *For any $\{r, d_i\}$ in $\bar{K}$, one can construct an element $\pi_h(\tau, d_i) \in K_h$ such that*

$$\lim_{h \to 0} \| \pi_h(r, d_i) - \{r, d_i\} \|_{1,p} = 0 .$$

The proof of this lemma is more technical and will be the purpose of the following section.

3.3. Convergence result

We now are able to prove the following convergence result.

THEOREM 1 : *Any isolated minimum of the continuous equilibrium problem (2.18) is the strong limit in $W^{1,p}(0, L ; E^4)$ of a sequence of local minima of the Finite Element problem (3.21).*

Proof : The above theorem is valid for both the general case and the case with no shear and no extension. Its proof is based on standard compactness arguments and requires five steps.

Step 1. *Interpolation of the minimum.*

Let $\{\bar{r}, \bar{d}_i\}$ be an isolated minimum of (2.18). In other words, we suppose

that there exists a closed ball $\bar{B}_\delta$ in $W^{1, p}(0, L ; E^4)$ with center $\{\bar{r}, \bar{d}_\iota\}$ and radius δ such that

$$J(\bar{r}, \bar{d}_\iota) < J(p, g_\iota) , \quad \forall \{p, g_\iota\} \neq \{\bar{r}, \bar{d}_\iota\} \in \bar{B}_\delta \cap \bar{K} .$$

From Lemma 2, and the continuity of J on $W^{1, p}(0, L ; E^4)$, we have, for h sufficiently small,

$$\pi_h(\bar{r}, \bar{d}_\iota) \in \bar{B}_\delta \cap \bar{K}_h ,$$

$$\lim_{h \to 0} J(\pi_h(\bar{r}, \bar{d}_\iota)) = J(\bar{r}, \bar{d}_\iota) .$$

Step 2. *Finite Element minimizing sequence.*

Since J is continuous and since $\bar{K}_h \cap \bar{B}_\delta$ is closed and bounded in $V_h \times W_h^3$, there exists a minimizer $\{r_h, d_{\iota h}\}$ of J on $\bar{K}_h \cap \bar{B}_\delta$. By construction, the sequence $(\{r_h, d_{\iota h}\})_h$ is bounded in $W^{1, p}(0, L ; E^4)$. It can therefore be decomposed into subsequences, still denoted by $(\{r_h, d_{\iota h}\})_h$ which weakly converge towards elements $\{r^*, d_\iota^*\}$ of $W^{1, p}(0, L ; E^4)$.

Since $\bar{B}_\delta$ is weakly closed, any such limit $\{r^*, d_\iota^*\}$ is in $\bar{B}_\delta$. Moreover, since $\bar{K}_h$ is included in K_h by definition, Lemma 1 implies that we also have

$$d_\iota^*(S) . d_j^*(S) - \delta_{\iota j} = 0 , \quad \forall S \in [0, L] , \quad \forall i, j = 1, 3 .$$

By definition of $\bar{K}_h$ and of $\bar{K}$, we therefore have

$$\{r^*, d_\iota^*\} \in \bar{K} \cap \bar{B}_\delta .$$

Step 3. *Characterization of $\{r^*, d_\iota^*\}$.*

By construction of $\{r_h, d_{\iota h}\}$ and of $\pi_h(\bar{r}, \bar{d}_\iota)$, we have

$$J(r_h, d_{\iota h}) \leqslant J(\pi_h(\bar{r}, \bar{d}_\iota)), \forall h . \tag{3.27}$$

Going to the limit as h goes to zero, and taking into account the weak lower semicontinuity of J_0 and hence of J, we get

$$J(r^*, d_\iota^*) \leqslant \lim \inf J(r_h, d_{\iota h}) \leqslant \lim \sup J(r_h, d_{\iota h})$$

$$\leqslant \lim J(\pi_h(\bar{r}, \bar{d}_\iota)) = J(\bar{r}, \bar{d}_\iota) . \tag{3.28}$$

But $\{r^*, d_i^*\} \in \bar{K} \cap \bar{B}_\delta$ and $\{\bar{r}, \bar{d}_i\}$ is the unique minimizer of J over $\bar{K} \cap \bar{B}_\delta$. Therefore, we must have

$$\{r^*, d_i^*\} = \{\bar{r}, \bar{d}_i\} .$$

In other words, all the subsequences of $(\{r_h, d_{ih}\})_h$ converge weakly towards $\{\bar{r}, \bar{d}_i\}$ in $W^{1, p}(0, L ; E^4)$ and then so does the whole sequence.

Moreover, since $\{r^*, d_i^*\} = \{\bar{r}, \bar{d}_i\}$, (3.28) also implies that we have

$$\lim_{h \to 0} J(r_h, d_{ih}) = J(\bar{r}, \bar{d}_i) . \tag{3.29}$$

Step 4. *Strong convergence.*

From assumption (H2), J is the sum of the weakly lower semicontinuous functions $C_0 \|d_i'\|_{0, p}^p$, $C_0 \|r' - d_3\|_{0, p}^p$ and J_0. Since $J(r_h, d_{ih})$ converges towards $J(\bar{r}, \bar{d}_i)$, each of its component must also converge, that is we must have

$$\lim_{h \to 0} C_0 \|d_{ih}'\|_{0, p}^p = C_0 \|\bar{d}_i'\|_{0, p}^p ,$$

$$\lim_{h \to 0} C_0 \|r_h' - d_{3 h}\|_{0, p}^p = C_0 \|\bar{r}' - \bar{d}_3\|_{0, p}^p ,$$

$$\lim_{h \to 0} J_0(r_h, d_{ih}) = J_0(\bar{r}, \bar{d}_i) . \tag{3.30}$$

If it were not the case, then the weak convergence of $(\{r_h, d_{ih}\})_h$ and the weak lowersemicontinuity of the different components of J would forbid (3.29) to take place.

From step 3 and from above, we first observe that $(d_{ih}')_h$ converges weakly and in norm towards $\bar{d}_i'$ in L^p. We have therefore strong convergence of $(d_{ih}')_h$ towards $\bar{d}_i'$ in L^p which, combined with the weak convergence of $(d_{ih})_h$ towards $\bar{d}_i$ in $W^{1, p}$ implies the strong convergence of $(d_{ih})_h$ towards $\bar{d}_i$ in $W^{1, p}$.

Similarly, $(r_h' - d_{3 h})_h$ converges weakly and in norm towards $\bar{r}' - \bar{d}_3$ in L^p, therefore converges strongly towards $\bar{r}' - \bar{d}_3$ in L^p. This, combined with the strong convergence of $(d_{3 h})_h$ towards $\bar{d}_3$ in L^p and the weak convergence of $(r_h)_h$ towards $\bar{r}$ in $W^{1, p}$, guarantees the strong convergence of $(r_h)_h$ towards $\bar{r}$ in $W^{1, p}$.

Step 5. $\{r_h, d_{ih}\}$ *is a local minimizer of* (3.1).

From the strong convergence of $(\{r_h, d_{ih}\})_h$ towards $\{\bar{r}, \bar{d}_i\}$ proved in step 4, we can deduce that for h sufficiently small, $\{r_h, d_{ih}\}$ will be in the interior of $\bar{B}_\delta$. Being by construction a minimum of J on $\bar{K}_h \cap \bar{B}_\delta$, and being in the interior of $\bar{B}_\delta$, $\{r_h, d_{ih}\}$ is then a local minimum of J on $\bar{K}_h$.

Therefore, finally, $\{\bar{r}, \bar{d}_i\}$ appears as the strong limit of the local minima $\{r_h, d_{ih}\}$ of J on $\bar{K}_h$.

4. APPROXIMATION RESULTS

4.1. Synopsis

The convergence result of the preceding section was based on Lemma 2, which states that for any kinematically admissible deformation $\{r, d_i\}$ of $\bar{K}$, there exists a Finite Element approximation $\pi_h(r, d_i) \in \bar{K}_h$ such that

$$\lim_{h \to 0} \| \pi_h(r, d_i) - \{r, d_i\} \|_{1, p} = 0 . \tag{4.31}$$

In the general case $(\bar{K}_h = K_h)$, which allows shear and extension, such a result is easy to prove. Indeed, there are interpolants $\pi_h r$ of r and $\pi_h d_i$ of d_i which satisfy ([5]) :

$$\pi_h r \in V_h , \quad \forall h ,$$

$$\pi_h r(a_{2i-1}) = r(a_{2i-1}) , \quad \forall i = 1, N_h + 1 ,$$

$$\lim_{h \to 0} \| \pi_h r - r \|_{1, p} = 0 ,$$

$$\pi_h d_i \in W_h , \quad \forall h ,$$

$$\pi_h d_i(M_p) = d_i(M_p) , \quad \forall M_p \in M_h ,$$

$$\lim_{h \to 0} \| \pi_h d_i - d_i \|_{1, p} = 0 .$$

By construction, $\pi_h(r, d_i) = (\pi_h r, \pi_h d_i)$ belongs to K_h and satisfies the desired approximation property (4.31).

The situation is more complicate in the limit case when shear or extension are no longer allowed $(\bar{K}_h = K_h^{\text{inc}})$. There, the element $\{\pi_h r, \pi_h d_i\}$ does not satisfy the inextensibility constraint $(\pi_h r)' = \pi_h d_3$ and therefore does

not belong to K_h^{inc}. Imposing $d_{3\,h} = (\pi_h\,r)'$ does not solve the problem because the element $\{\pi_h\,r,\ \pi_h\,d_\alpha,\ (\pi_h\,r)'\}$ does not satisfy the orthonormality constraint

$$d_{ih}(M_p) \cdot d_{jh}(M_p) = \delta_{ij}\,, \quad \forall M_p \in M_h\,.$$

On the other hand, the field r_h defined by

$$r_h(S) = r_0 + \int_0^S \pi_h\,d_3(\sigma)\,d\sigma$$

does not satisfy the boundary condition $r(L) = r_L$.

To overcome this problem, we propose below a construction of $\pi_h(r, d_i)$ which first interpolates d_i on W_h and then iteratively and locally rotates this interpolate so that we can satisfy the boundary condition

$$r_h(L) = r_0 + \int_0^L d_{3\,h}(\sigma)\,d\sigma = r_L\,.$$

4.2. Notation

Let $\{r, d_i\}$ be the element of K^{inc} that we want to approximate. By definition of K^{inc}, $\{d_i(S)\}$ is a direct orthonormal basis of E (it can thus be identified to an element of the rotation group SO_3) and we have

$$r' = d_3\,. \tag{4.32}$$

Moreover, if we want the set K^{inc} of kinematically admissible deformations to be nonempty and to contain more than a single element, we must have

$$|r(L) - r(0)| = |r_L - r_0| = \left| \int_0^L d_3(\sigma)\,d\sigma \right| < L\,. \tag{4.33}$$

This implies in particular that d_3 does not stay parallel to itself, that is there exists two points S_1 and S_2 in $(0, L)$ and two unit vectors e_1 and e_2 in E such that

$$d_3(S_\alpha) = e_\alpha\,, \quad \forall \alpha = 1, 2\,, \tag{4.34}$$

$$|e_3| = |e_1 \times e_2| = C_1 > 0\,. \tag{4.35}$$

Let now $\varepsilon > 0$ be given. By continuity of d_3, there exists two disjoint intervals I_1 and I_2 in $(0, L)$ such that

$$|d_3(S) - e_\alpha| < \varepsilon\,, \quad \forall S \in I_\alpha\,, \quad \forall \alpha = 1, 2\,. \tag{4.36}$$

To each interval I_α, we now associate a subinterval J_α with same center but of half length and a function $q_\alpha \in W_0^{1,P}(I_\alpha)$ such that

$$q_\alpha|_{J_\alpha} = 1/\text{meas } (J_\alpha), \qquad (4.37)$$

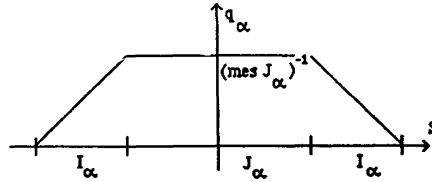$$0 \leq q_\alpha \leq 1/\text{meas } (J_\alpha). \qquad (4.38)$$

Figure 3. — The function q_α.

Finally, for a given subdivision $([a_{2i-1}, a_{2i+1}])_{i=1, N_h}$, we denote by $(\varphi_p)_{p=1, 2N_h+1}$ the Finite Element nodal basis of W_h, defined by

$$\varphi_p(M_q) = \delta_{pq}, \qquad \forall M_q \in M_h,$$

and by p_p the positive number

$$p_p = \int_0^L \varphi_p(\sigma)\, d\sigma.$$

We then have

$$\pi_h(d_i) = \sum_p d_i(M_p)\, \varphi_p,$$

$$\| \pi_h(\dot{v}) \|_{1,p} \leq C_2 \| v \|_{1,p}, \qquad \forall v \in W^{1,P}(0, L). \qquad (4.39)$$

4.3. Iterative construction of $\pi_h(r, d_i)$

The proposed construction of the Finite Element approximation of $\{r, d_i\}$ proceeds as follows :

• set $d_i^0 = \pi_h d_i$,
• then, for $n \geq 0$ and $d_i^n(M_p)$ known,

1. compute the residual

$$R^n = r_L - r_0 - \int_0^L d_3^n(\sigma)\, d\sigma$$

$$= r_L - r_0 - \sum_p p_p\, d_3^n(M_p)$$

$$= R_1^n e_3 \times e_1 + R_2^n e_3 \times e_2 + R_3^n e_3; \qquad (4.40)$$

2. compute the rotation vector

$$\omega_p^n = \lambda_2^n q_1(M_p) e_3 - \lambda_3^n q_1(M_p) e_2 + \lambda_1^n q_2(M_p) e_3 \qquad (4.41)$$

which cancels the residual (Newton update)

$$\sum_p p_p \, \omega_p^n \times d_3^n(M_p) \approx \sum_\alpha \sum_{M_p} p_p \, \omega_p^n \times e_\alpha = R^n \, .$$

From (4.40) and (4.41), this amounts to set

$$\lambda_1^n = R_1^{n/} \left(\sum_p p_p \, q_2(M_p) \right) \, ,$$

$$\lambda_{\alpha+1}^n = R_{\alpha+1}^n / \left(\sum_p p_p \, q_1(M_p) \right) \, ; \qquad (4.42)$$

3. update $d_i(M_p)$ by rotation :

$$d_i^{n+1}(M_p) = \exp(\Omega_p^n) \, d_i^n(M_p) \, , \qquad (4.43)$$

the skew symmetric operator Ω_p^n being defined by

$$\Omega_p^n X = \omega_p^n \times X \, , \quad \forall X \in E \, ;$$

- end loop on n,
- set :

$$d_{ih} = \lim_{n \to 0} \sum_p d_i^n(M_p) \, \varphi_p \, ,$$

$$r_h(S) = r_0 + \int_0^S d_{3h}(\sigma) \, d\sigma \, ,$$

$$\pi_h(r, d_i) = \{ r_h, d_{ih} \} \, . \qquad (4.44)$$

The above construction can be proved to satisfy Lemma 2. More precisely, we have the following technical theorem, whose proof can be found in annex.

THEOREM 2 : *There exists two constants C_3 and C_4 such that, $\forall \varepsilon < 1/8 \, C_4$, there exists h_0 such that, $\forall h < h_0$, we have :*

$$\pi_h(r, d_i) \quad \text{is well defined and belongs to } K_h^{inc} \, ,$$

$$\| r - r_h \|_{2,p} + \sum_i \| d_i - d_{ih} \|_{1,p} \leq \frac{\varepsilon}{C_3} (3 + (1 + L)^{1/p}) \, . \qquad (4.45)$$

5. NUMERICAL ALGORITHM

5.1. The problem to solve

Within an arclength continuation framework, we have to solve the following problem at each load increment n [6] :

$$
\begin{cases}
J(\mu^{n+1}, r^{n+1}, d_i^{n+1}) \leqslant J(\mu^{n+1}, p, g_i), \quad \forall (p, g_i) \in K_h^{\text{inc}}, \\
(r^{n+1}, d_i^{n+1}) \in K_h^{\text{inc}}, \\
\langle d_i^{n+1} - d_i^n, d_i^n - d_i^{n-1} \rangle + \mu_u^2(\mu^{n+1} - \mu^n)(\mu^n - \mu^{n-1}) = ds^2,
\end{cases}
$$

$$(5.46)$$

under the notation (3.23) and

$$
J(\mu, r, d_i) = \int_0^L w(S, u_j(d_i)) \, dS - \mu \int_0^L f \cdot r \, dS . \qquad (5.47)
$$

Here, we have replaced the problem of finding one solution by the problem of building a solution curve associated to a variable load factor. On this solution curve, we build points μ^{n+1}, r^{n+1}, d_i^{n+1} which are at a distance ds one from another.

More precisely, μ is the incremental load factor, ds the stepsize along the solution curve, $\langle \cdot, \cdot \rangle$ an adequate scalar product on W_h, and μ_u is a scaling factor.
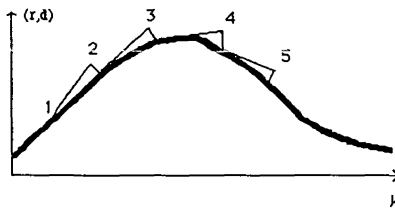
Figure 4. — Arclength Continuation.

We have specialized above in the case with no shear and no extension which we think is the most interesting. It is better conditioned than the slightly extensible case and since extensibility has very little effect on the physical solution, using an inextensible approximation leads overall to a faster and more accurate solution.

In the inextensible case, the constraint on r implies that we have

$$
r(S) = r_0 + \int_0^S d_3(\sigma) \, d\sigma
$$

and thus r can be eliminated from our problem. Under the notation

$$F(S) = - \int_S^L f(t) \, dt \, , \qquad (5.48)$$

the energy J reduces to

$$
\begin{aligned}
J(\mu, d_i) &= \int_0^L w(S, u_j(d_i)) \, dS - \mu \int_0^L F'(S) \cdot r \, dS \\
&= \int_0^L w(S, u_j(d_i)) \, dS \\
&\quad + \mu \int_0^L F(S) \cdot d_3(S) \, dS - \mu F(0) \cdot r_0 . \qquad (5.49)
\end{aligned}
$$

Similarly, the elimination of r from the definition of K_h^{inc} leads to the new definition

$$
\begin{aligned}
K_h^{\text{inc}} = \big\{ \, \{d_i\} \in X_h, \, d_i(M_p) \cdot d_j(M_p) = \delta_{ij} \, , \quad \forall i, j = 1, 3 \, , \\
\forall M_p \in M_h, \, r_L = r_0 + \int_0^L d_3(\sigma) \, d\sigma \big\} \, , \qquad (5.50)
\end{aligned}
$$

under the notation

$$
\begin{aligned}
X_h = \big\{ \, \{d_1, d_2, d_3\} \in W_h^3, \, \{d_i\} \text{ satisfies the boundary conditions} \\
(2.6)\text{-}(2.7) \text{ used in the definition of } K \, \big\} \, , \qquad (5.51)
\end{aligned}
$$

$$
\begin{aligned}
M_h &= \text{set of all vertices and centers of the subdivision} \\
&\quad ([a_{2i-1}, a_{2i+1}]) \, , \\
&= \big\{ a_1, a_2, a_3, ..., a_{2N_h}, a_{2N_h+1} \big\} \, . \qquad (5.52)
\end{aligned}
$$

5.2. Euler-Lagrange Equations

By construction of K_h^{inc}, the Euler-Lagrange equations associated to (5.46) are simply

$$
\frac{\partial J}{\partial d_j} (\mu^{n+1}, d_i^{n+1}) \cdot \hat{g}_j + \sum_p \sum_{i,j=1}^{3} \lambda_{ijp}^{n+1} (d_i^{n+1} \cdot \hat{g}_j + \hat{g}_i \cdot d_j^{n+1})(M_p) +
$$

$$
+ N^{n+1} \cdot \int_0^L \hat{g}_3(\sigma) \, d\sigma = 0 \, , \quad \forall (\hat{g}_j) \in dX_h, \quad (5.53)
$$

$$
(d_i^{n+1} \cdot d_j^{n+1})(M_p) = \delta_{ij} \, , \quad \forall i, j = 1, 3 \, , \quad \forall M_p \in M_h \, , \quad (5.54)
$$

$$
\int_0^L d_3^{n+1}(\sigma) \, d\sigma = r_L - r_0 \, , \qquad (5.55)
$$

$$\left\langle (d_i^{n+1} - d_i^n), (d_i^n - d_i^{n-1}) \right\rangle + \mu_u^2 (\mu^{n+1} - \mu^n)(\mu^n - \mu^{n-1}) = ds^2 , \quad (5.56)$$

with λ_{ijp} and N the Lagrange multipliers associated to the constraints (5.54) and (5.55). Observe that N is actually the reaction force exerted on the rod by its support in L. After linearization around $(d_i, \mu, \lambda_{ijp}, N)$, and if d_i satisfies (5.54), these equations reduce to the system

$$\left[\frac{\partial^2 J}{\partial d_j \, \partial d_k} (\mu, d_i) \cdot g_k \right] \cdot \hat{g}_j + \sum_p \sum_{ij} \lambda_{ijp} (g_i \cdot \hat{g}_j + \hat{g}_i \cdot g_j)(M_p) +$$

$$+ \, dN \cdot \int_0^L \hat{g}_3(\sigma) \, d\sigma + d\mu \int_0^L F(\sigma) \cdot \hat{g}_3(\sigma) \, d\sigma$$

$$+ \sum_p \sum_{ij} d\lambda_{ijp}(d_i \cdot \hat{g}_j + \hat{g}_i \cdot d_j)(M_p) =$$

$$= - \frac{\partial J}{\partial d_j} (\mu, d_i) \cdot \hat{g}_j - \sum_p \sum_{ij} \lambda_{ijp}(d_i \cdot \hat{g}_j + d_j \cdot \hat{g}_i)(M_p) -$$

$$- N \cdot \int_0^L \hat{g}_3(\sigma) \, d\sigma , \quad \forall \{\hat{g}_j\} \in dX_h, \ \{g_k\} \in dX_h , \quad (5.57)$$

$$(d_i \cdot g_j + d_j \cdot g_i)(M_p) = 0 , \quad \forall i, j = 1, 3 , \quad \forall M_p \in M_h , \quad (5.58)$$

$$\int_0^L g_3(\sigma) \, d\sigma = - \int_0^L d_3(\sigma) \, d\sigma + r_L - r_0 , \quad (5.59)$$

$$\left\langle g_i, d_i^n - d_i^{n-1} \right\rangle + \mu_u^2 \, d\mu \, (\mu^n - \mu^{n-1}) = ds^2 -$$

$$- \left\langle (d_i - d_i^n), (d_i^n - d_i^{n-1}) \right\rangle - \mu_u^2 (\mu - \mu^n)(\mu^n - \mu^{n-1}) , \quad (5.60)$$

with unknowns $\{g_i\}$, $d\mu$, $d\lambda_{ijp}$ and dN.

5.3. Solving the linearized Euler Lagrange equations

When (5.54) is satisfied, (5.58) simply imposes that $\{g_i\} (M_p)$ be in the tangent space to SO_3 at $\{d_i\} (M_p)$, and hence it implies that $\{g_i\}$ is of the form

$$g_i (M_p) = U_p \times d_i(M_p) , \quad (5.61)$$

with U_p an arbitrary rotation vector. If we now restrict (5.57) to test functions $\hat{g}_j(\hat{U})$ which are also of the form

$$\hat{g}_j (M_p) = \hat{U}_p \times d_j(M_p) ,$$

with $\hat{U}_p$ arbitrary, then the linearized system (5.57)-(5.58) simply reduces to the new system

$$a(U, \hat{U}) + dN^l \int_0^L \hat{g}_3^l(\hat{U}) \, d\sigma + d\mu \int_0^L F \cdot \hat{g}_3(\hat{U}) \, d\sigma = L_1(\hat{g}_j(\hat{U})) \quad (5.62)$$

$$\int_0^L g_3(U) \, d\sigma = L_2 , \qquad\qquad (5.63)$$

$$\langle g_i(U), \Delta d_i \rangle + d\mu \, \Delta\mu = L_3 , \qquad\qquad (5.64)$$

with new unknowns U, $(dN^l)_{l=1,3}$ and $d\mu$. Above, $L_1(\hat{g}_j)$, L_2 and L_3 respectively denote the right-hand sides of (5.57), (5.59) and (5.60), $(dN^l)_l$ and $(\hat{g}_3^l)_l$ denote the cartesian components of the vectors dN and $\hat{g}_3$, and we have used the notation

$$a(U, \hat{U}) = \left(\frac{\partial^2 J}{\partial d_j \, \partial d_k} \cdot g_k(U) \right) \cdot \hat{g}_j(\hat{U}) +$$

$$+ \sum_p \sum_{ij} \lambda_{ijp}(U_p \times d_i \cdot \hat{U}_p \times d_j + U_p \times d_j \cdot \hat{U}_p \times d_i) , \quad (5.65)$$

$$\Delta d_i = (d_i^n - d_i^{n-1}) , \qquad\qquad (5.66)$$

$$\Delta\mu = \mu_u^2(\mu^n - \mu^{n-1}) . \qquad\qquad (5.67)$$

Compared to the usual linearized system encountered in rod computations, this new system :

• also uses as basic unknown the rotation vector U_p, defined in each node $M_p \in M_h$ of the Finite Element mesh (nine unknowns per Finite Element),

• does not involve any unknown associated to the position of the rod center line, these being eliminated by the inextensibility condition,

• adds four scalar unknowns which are the variations of the reaction force N in L and of the load factor μ.

These last unknowns play the role of the scaling factor usually encountered in continuation techniques. They can therefore be treated by the bordering algorithm proposed in ([6]). Then, the solution of the linearized system (5.62)-(5.63) can be obtained by the following block Gaussian elimination :

• Solve $\qquad\qquad a(U^1, \hat{U}) = L_1(\hat{g}_j(\hat{U})) , \quad \forall \hat{U} ;$ $\qquad\qquad$ (5.68)

• For $l = 1, 3$, solve $\quad a(U^{2,l}, \hat{U}) = - \int_0^L \hat{g}_3^l(\hat{U}) \, d\sigma , \quad \forall \hat{U} ;$ $\qquad$ (5.69)

• Solve $\qquad a(U^3, \hat{U}) = - \int_0^L F \cdot \hat{g}_3(\hat{U}) \, d\sigma , \quad \forall \hat{U} ;$ $\qquad$ (5.70)

- Solve the 4×4 linear system (of unknown $dN^{\,l}$ and $d\mu$)   (5.71)

$$dN^{\,l} \int_0^L g_3(U^{2,\,l})\, d\sigma + d\mu \int_0^L g_3(U^3)\, d\sigma = L^2 - \int_0^L g_3(U^1)\, d\sigma,$$

$$dN^{\,l} \langle g_{\imath}(U^{2,\,l}), \Delta d_{\imath} \rangle + d\mu \,(\langle g_{\imath}(U^3), \Delta d_{\imath} \rangle + \Delta \mu) =$$
$$= L_3 - \langle g_{\imath}(U^1), \Delta d_{\imath} \rangle;$$

- Set $U = U^1 + dN^{\,l} U^{2,\,l} + d\mu\, U^3$;   (5.72)

- Set $g_{\imath}(M_p) = U_p \times d_{\imath}(M_p), \quad \forall M_p \in M_h$.   (5.73)

5.4. Computation of λ

The above solution procedure does not give the value of $d\lambda$. To obtain it, we rewrite the Euler Lagrange equation (5.53) at point $\{\pi(d_k + g_k), \mu + d\mu, N + dN\}$ with $\{\pi(d_k + g_k)\}(M_p)$ a projection of $\{d_k + g_k\}(M_p)$ onto SO_3. At first order, we get by construction of $\{g_k\}$, $d\mu$, $d\lambda_{\imath jp}$ and dN

$$\frac{\partial J}{\partial d_j}(\mu + d\mu, \pi(d_k + g_k)) \cdot \hat{g}_j + (N + dN) \cdot \int \hat{g}_3(\sigma)\, d\sigma +$$

$$\sum_p \sum_{\imath j} (\lambda + d\lambda)_{\imath jp} (\pi(d_k + g_k)_{\imath} \cdot \hat{g}_j + \pi(d_k + g_k)_j \cdot \hat{g}_{\imath})(M_p) =$$

$$= 0(\|g_{\imath}\|^2 + \|dN\|^2 + d\mu^2 + |d\lambda|^2), \quad \forall \{\hat{g}_{\imath}\} \in dX_h.$$

A direct identification then yields:

$$2(\lambda + d\lambda)_{\imath jp} \approx -\frac{\partial J}{\partial d_l}(\mu + d\mu, \pi(d_k + g_k)) \cdot \hat{g}_l^{\imath jp}$$

$$- (N + dN) \cdot \int_0^L \hat{g}_3^{\imath jp}(\sigma)\, d\sigma \qquad (5.74)$$

under the notation

$$\hat{g}_l^{\imath jp}(M_q) = \frac{1}{2}(\pi(d_k + g_k)_j \, \delta_{\imath l} + \pi(d_k + g_k)_{\imath}\, \delta_{jl})(M_q)\, \delta_{pq}.$$

5.5. Final Solution Procedure

With the above results, the Newton and arclength continuation method takes the following form when applied to the solution of our Finite Element equilibrium problem (3.21):

For $n = 1$ until $\mu^n \geqslant 1$ do :

• For $m = 1$ until (5.53)-(5.56) is satisfied, do (Newton loop) :

1. Compute the rotation vector U, the variation of reaction force dN and of load factor $d\mu$ by (5.68)-(5.72) ;

2. Obtain the linear variation g_t of d_t by (5.73) ;

3. Set $\{d_t\}$, N, $\mu = \{\pi(d_t + g_t)\}$, $N + dN$, $\mu + d\mu$;

4. Compute $(\lambda + d\lambda)_{ijp}$ by (5.74) ;

• End of Newton loop ;

• Update Δd, $\Delta \mu$ and ds ;

End of continuation loop.

Remark 1 : Formula (5.74) is first order accurate as is the linear expansion (5.57)-(5.60) of the nonlinear system (5.53)-(5.56). Therefore, as confirmed by our numerical experiments, the above Newton loop has a quadratic speed of convergence. The use of (5.74) for computing $d\lambda$ is much cheaper in CPU time than would be the solution of (5.57) : it is explicit and only involves the first gradient of J. Moreover, the same gradient is used in solving (5.68) at the next step, which results in further savings.

Remark 2 : The solution procedure (5.68)-(5.73) only involves 3 unknowns by node M_p (the values of U_p) instead of 9 (the values of $g_t(M_p)$) for (5.57)-(5.60). But it only works if $\{d_t\} (M_p)$ is in SO_3 which explains why we have to project $\{d_t + g_t\} (M_p)$ back to SO_3 at each step. The projection procedure that we have used is from [7] and is given by

$$\{\pi(d_t + g_t)\} (M_p) = \pi(d_t(M_p) + U_p \times d_t(M_p))$$
$$= Q_p \cdot \{d_t\} (M_p)$$

with Q_p the rotation matrix defined by

$$Q_p = Id + (U_p + U_p^2/2)/(1 + \|U_p\|^2/4) \, ,$$

$$U_p = \begin{pmatrix} 0 & -U_p^3 & U_p^2 \\ U_p^3 & 0 & -U_p^1 \\ -U_p^2 & U_p^1 & 0 \end{pmatrix} .$$

Remark 3 : In the case with shear and extension, the unknown field r cannot be eliminated. The application of the above Newton procedure to this case leads to the algorithm described in [8].

Remark 4 : By construction of J and u, we have

$$\frac{\partial J}{\partial d_j} \cdot \hat{g}_i = \int_0^L \frac{\partial w}{\partial u_r} \cdot (d'_{r+1} \cdot \hat{g}_{r-1} + \hat{g}'_{r+1} \cdot d_{r-1})\, dS + \mu \int_0^L F \cdot \hat{g}_3\, dS,$$

$$\left(\frac{\partial^2 J}{\partial d_j\, \partial d_k} \cdot g_k(U) \right) \cdot g_j(\hat{U}) = \int_0^L \left\{ \frac{\partial^2 w}{\partial u_r\, \partial u_s} \left(\frac{\partial u_r}{\partial d_k} \cdot g_k(U) \right) \left(\frac{\partial u_s}{\partial d_j} \cdot g_j(\hat{U}) \right) + \right.$$

$$\left. + \frac{\partial w}{\partial u_r} \left(\frac{\partial^2 \tilde{u}_r}{\partial d_j\, \partial d_k} \cdot g_k(U) \right) \cdot \hat{g}_j(\hat{U}) \right\} dS.$$

In the case of nonconservative external loads such as drag forces, the force resultant F depends on d_3. This case can still be handled by the above solution procedure, the only change being the addition of the nonsymmetric term

$$\int_0^L \left(\frac{\partial F}{\partial d_3} \cdot g_3(U) \right) \cdot \hat{g}_3(\hat{U})\, dS$$

to the second derivative of J.

5.6. Initialization

We initialize our continuation loop by setting $\{\Delta d_i\} = 0$ and by picking a reasonable value for the first load increment. The choice of $\{d_i^0\}$ is more delicate since it must satisfy the boundary conditions of X_h, the orthonormality condition (2.3) and be reasonably close to K_h^{inc}. Our current strategy :

1. takes a circular initial configuration r^0, coplanar with the load $F(0)$, of length L, and satisfying the boundary conditions imposed on r,
   2. sets $d_3^0 = (r^0)'$,
   3. computes d_α^0 by linearly interpolating the angle defining d_α in the cross section between the values imposed at the extremities of the rod,
   4. and, if needed, changes locally the values of the Euler angles of $\{d_i^0\}$ in order to satisfy the boundary conditions imposed on $\{d_i\}$, in a way very similar to the construction of $\pi_h(r, d_i)$ in Section 4.
   If needed, as is the case for example for the steep wave problem, this strategy can be supplemented by an incremental variation of the kinematic boundary condition on r_L.

5.7. Numerical Result

All tests were run on a Macintosh SE/30 personal computer.
The first numerical test deals with the Cantilever beam in large displacements whose analytical solution has been presented in [9]. For a

beam of length $L = 10$ m, of flexural stiffness $EI = 100$ daN $\times$ m^2 and subjected to a terminal load of $P = -10$ daN, the exact solution corresponding to a vertical displacement of 8.10 m was obtained without continuation after five iterations of the Newton loop, and this for a mesh of fifty Finite Elements.

The second test is more challenging since it deals with a very flexible rod such as those encountered in flexible riser systems. The data are taken from [4]. We consider a very flexible rod, of length $L = 32.6$ m, of flexural
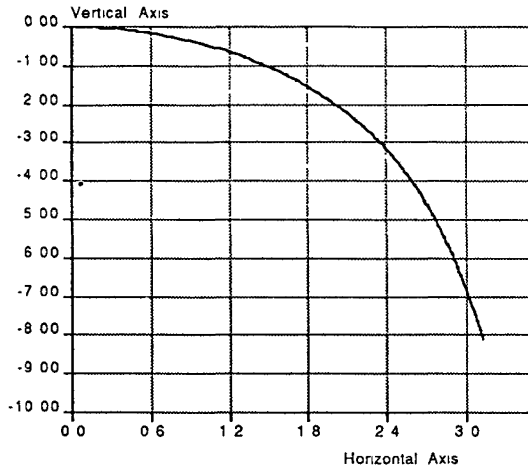

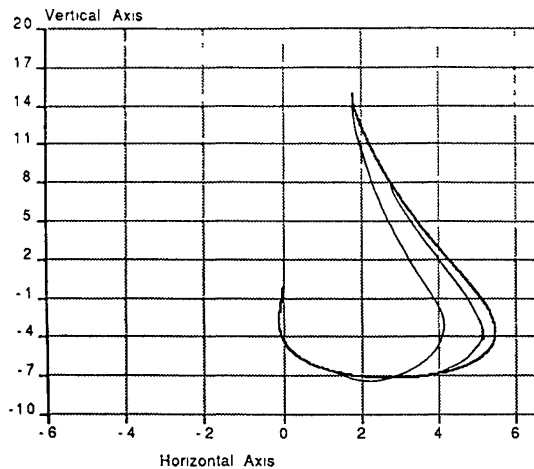Figure 5. — The Cantilever beam under Terminal load.

Figure 6. — In Plane Configuration of the Flexible Rod.

stiffness $EI = 435$ daN $\times$ m^2, of torsional stiffness $GI = 400$ daN $\times$ m^2 or
4 000 daN $\times$ m^2, and subjected to a uniform loading $f = -7.5$ daN/m. The
rod hangs between point $A = (0, 0, 0)$ and $B = (1.8, 0, 15)$. Both extre-
mities are vertically clamped and a rotation of 90 degrees is imposed to the
rod section in B. The Finite Element mesh consists again of fifty Finite
Elements. Here, we have first computed the solution at zero imposed
torsion, solution which was obtained after 8 continuation steps of 3 to
4 Newton iterations each. Starting from this solution, we have then directly
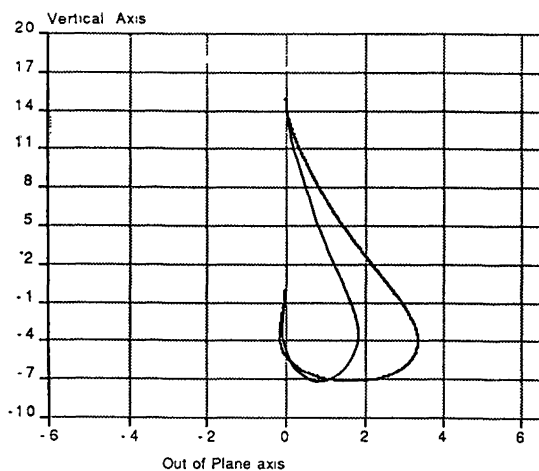

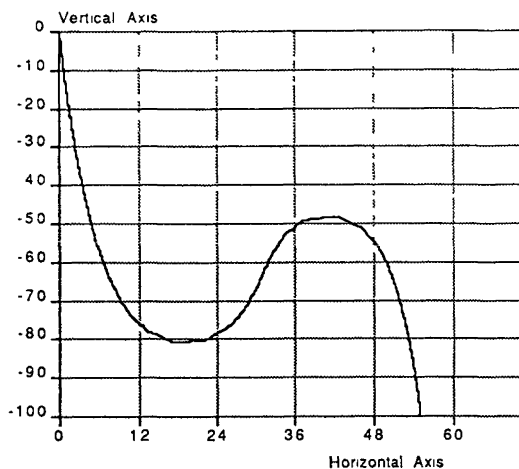Figure 7. — Out of Plane displacement of the Flexible Rod.

Figure 8. — Final Configuration of the steep wave problem.

imposed the 90 degrees rotation, which was obtained in 5 Newton iterations for each value of the torsional stiffness. As expected, with imposed torsion, the rod configuration is no longer coplanar, the out of plane displacement being larger when GI is larger. The first figure displays the in plane configuration for the case without torsion and for the two cases with imposed torsion. The second one displays the out of plane displacements observed in torsion for the two values of GI.

The last test is adapted from a test case proposed by Coflexip Company (the steep wave problem). It considers a rod of length $L = 185$ m, fixed in points $A = (0, 0, 0)$ and $B = (55, 0, -100)$, of flexural stiffness $EI = 7$ kN $\times$ m^2. It is subjected to a vertical load of density $f(S)$ given by

$$\begin{cases} f(S) = -0.3 \text{ kN/m} & \text{if} \quad 0 < S < 110\,, \\ \quad\quad = 0.38 \text{ kN/m} & \text{if} \quad 110 < S < 170\,, \\ \quad\quad = 0.60 \text{ kN/m} & \text{if} \quad 170 < S\,. \end{cases}$$
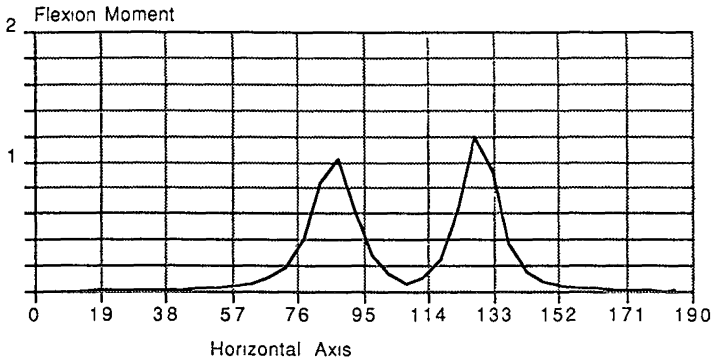

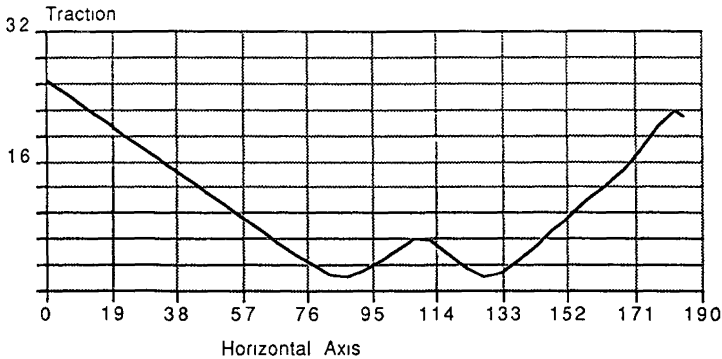Figure 9. — The flexion moment distribution in the steep wave problem.

Figure 10. — The normal tension distribution in the steep wave problem.

The computation used a Finite Element mesh of 37 elements. A first solution was obtained with B fixed in $B = (120, 0, -100)$ in 33 continuation steps of the loading factor. Then, the point B was incrementally translated to its final position in 5 increments of 10 to 15 meters. For each translation increment, the solution was obtained in 4 iterations of the standard Newton algorithm. The final configuration, together with the flexion moment and tension force are displayed on the above figures.

6. ANNEX. PROOF OF THEOREM 2

The proof uses the following lemmas.

LEMMA 3 : *For any h with $h \le$ meas $(I_\alpha)/8$ (restriction 1), we have*

$$\frac{1}{2} \le \sum_{M_p} p_p \, q_\alpha (M_p) \le \frac{5}{2} \, . \tag{6.75}$$

Proof : Let I_α^h (resp. J_α^h) be the union of all segments $[a_{2i-1}, a_{2i+1}]$ having at least one vertex in I_α (resp. included in J_α). By construction of q_α, we then have

$$\sum_p p_p \, q_\alpha (M_p) \ge \sum_{M_p \in J_\alpha^h} p_p \, q_\alpha (M_p)$$

$$= \frac{1}{\text{meas } (J_\alpha)} \sum_{M_p \in J_\alpha^h} p_p$$

$$\ge \text{meas } (J_\alpha^h)/\text{meas } (J_\alpha) \ge \frac{1}{2} \, .$$

Similarly, we have

$$\sum_{M_p} p_p \, q_\alpha (M_p) \le \sum_{M_p \in I_\alpha^h} p_p \, q_\alpha (M_p)$$

$$\le \frac{1}{\text{meas } (J_\alpha)} \sum_{M_p \in I_\alpha^h} p_p$$

$$\le \text{meas } (I_\alpha^h)/\text{meas } (J_\alpha) \le \frac{5}{2} \, .$$

LEMMA 4 : *Under restriction 1, we have*

$$\|v\|_{C^0(0, L)} \le C_3 \|v\|_{1, p}, \quad \forall v \in W^{1, p}(0, L), \tag{6.76}$$

$$\sum_{p=1}^{2N_h+1} p_p \|\omega_p^n\| \le C_4 \|R^n\|, \tag{6.77}$$

where C_3 and C_4 denote constants independent of ε and h, and $\| \cdot \|$ denotes the euclidian norm in E.

Proof : Inequality (6.76) is a direct consequence of the Sobolev embedding theorem. As for (6.77), we have by Lemma 3 and definition of ω_p^n

$$\| \omega_p^n \| \leqslant \sqrt{2}(q_1(M_p) + q_2(M_p)) \sup_\ell | \lambda_\ell^n |$$

$$\leqslant 2\sqrt{2}(q_1(M_p) + q_2(M_p)) \sup_\ell | R_\ell^n |$$

$$\leqslant \frac{2\sqrt{2}}{C_1^2} (q_1(M_p) + q_2(M_p)) \| R^n \| .$$

Using again Lemma 3, we then deduce

$$\sum_p p_p \| \omega_p^n \| \leqslant \frac{2\sqrt{2}}{C_1^2} \left(\sum_\alpha \sum_p p_p q_\alpha(M_p) \right) \| R^n \|$$

$$\leqslant \frac{10\sqrt{2}}{C_1^2} \| R^n \| .$$

LEMMA 5 : *Under restriction 1, there exists a constant $K_1(\varepsilon)$ such that*

$$\| \Omega^n \|_{1,p} = \left\| \sum_q \Omega_q^n \varphi_q \right\|_{1,p} \leqslant K_1(\varepsilon) \| R^n \| . \tag{6.78}$$

Proof : By construction of Ω_q^n and from Lemma 4, we have

$$\| \Omega^n \|_{1,p} = \| \omega^n \|_{1,p}$$

$$\leqslant \left\| \lambda_1^n e_3 \sum_q q_2(M_q) \varphi_q \right\|_{1,p}$$

$$+ \left\| \lambda_2^n e_3 \sum_q q_1(M_q) \varphi_q \right\|_{1,p}$$

$$+ \left\| \lambda_3^n e_2 \sum_q q_1(M_q) \varphi_q \right\|_{1,p}$$

$$\leqslant \sup_\ell | \lambda_\ell^n | (\| \pi_h q_2 \|_{1,p} + 2 \| \pi_h q_1 \|_{1,p})$$

$$\leqslant 2 \sup_\ell | R_\ell^n | (C_2 \| q_2 \|_{1,p} + 2 C_2 \| q_1 \|_{1,p})$$

$$\leqslant \frac{2 C_2}{C_1^2} (\| q_2 \|_{1,p} + 2 \| q_1 \|_{1,p}) \| R^n \| ,$$

which completes our proof. From Lemmas 3 and 4, we also deduce

$$\sup_{p} \left\| \Omega_p^n \right\| \leqslant \left\| \Omega^n \right\|_{C^0(0, L)} \leqslant C_3 K_1(\varepsilon) \left\| R^n \right\| . \tag{6.79}$$

LEMMA 6 :

$$\left\| R^{n+1} \right\| \leqslant \frac{1}{2} \left\| R^n \right\| , \tag{6.80}$$

$$\left\| d_i^{n+1} - d_i^n \right\|_{1, p} \leqslant K_2(\varepsilon) \left\| R^n \right\| , \tag{6.81}$$

$$\left\| d_3^{n+1} - d_3 \right\|_{C^0(I_\alpha)} \leqslant \varepsilon . \tag{6.82}$$

Proof : The proof proceeds by induction. Therefore we have to suppose that (6.82) holds for $n + 1 = 0$, which is

Restriction 2 : $\left\| \pi_h d_3 - d_3 \right\|_{C^0(I_\alpha)} \leqslant \varepsilon .$

We then proceed in 3 steps, assuming that (6.80), (6.81) and (6.82) hold for any $0 \leqslant m < n$.

Step 1. *Checking* (6.80).

By definition of R^n and by construction of Ω_p^n, we have

$$R^{n+1} = r_L - r_0 - \sum_p p_p \, d_3^{n+1}(M_p)$$

$$= r_L - r_0 - \sum_p p_p \exp(\Omega_p^n) \, d_3^n(M_p)$$

$$= R^n + \sum_p (Id - \exp(\Omega_p^n)) \, d_3^n(M_p) \, p_p$$

$$= R^n - \sum_p \omega_p^n \times d_3^n(M_p) \, p_p - \sum_p p_p \sum_{k \geqslant 2} \frac{(\Omega_p^n)^k \, d_3^n(M_p)}{k!}$$

$$= \sum_p \omega_p^n \times (e_\alpha \text{ Ind } (I_\alpha) - d_3^n(M_p)) \, p_p - \sum_p p_p \sum_{k \geqslant 2} \frac{(\Omega_p^n)^k \, d_3^n(M_p)}{k!} .$$

Above, Ind (I_α) denotes the indicator function of the interval I_α. Using (4.36) and (6.82) at step $m = n - 1$, the above identity yields

$$\left\| R^{n+1} \right\| \leqslant \sum_p 2 \, \varepsilon p_p \left\| \omega_p^n \right\| + \sum_p p_p \left\| \Omega_p^n \right\|^2 \left(\sum_{k \geqslant 0} \frac{\left\| \Omega_p^n \right\|^k}{(k+2)!} \left\| d_3^n(M_p) \right\| \right)$$

$$\leqslant \left(\sum_p p_p \left\| \omega_p^n \right\| \right) \left(2 \, \varepsilon + \sup_p \left\| \Omega_p^n \right\| \exp(\left\| \Omega_p^n \right\|) \right) .$$

From (6.77) and (6.79), we then have

$$\|R^{n+1}\| \leqslant C_4 \|R^n\| (2\,\varepsilon + C_3 K_1(\varepsilon) \|R^n\| \exp(C_3 K_1(\varepsilon) \|R^n\|))\,.$$

Using then (6.80) for $0 \leqslant m < n$, we finally obtain

$$\|R^{n+1}\| \leqslant C_4 (2\,\varepsilon + C_3 K_1(\varepsilon) \|R^0\| \exp(C_3 K_1(\varepsilon) \|R^0\|)) \|R^n\|\,.$$

This is (6.80), provided that we have

Restriction 3 : $\quad C_4 (2\,\varepsilon + C_3 K_1(\varepsilon) \|R^0\| \exp(C_3 K_1(\varepsilon) \|R^0\|)) \leqslant \dfrac{1}{2}\,.$

Step 2. *Checking* (6.81). By construction, we have

$$d_i^{n+1} - d_i^n = \sum_p (\exp \Omega_p^n - Id)\, d_i^n(M_p)\, \varphi_p\,,$$

$$= \pi_h((\exp \Omega^n - Id)\, d_i^n)\,,$$

which implies

$$\left\|d_i^{n+1} - d_i^n\right\|_{1,p} = \left\| \pi_h((\exp \Omega^n - Id)\, d_i^n)\right\|_{1,p}$$

$$\leqslant C_2 \left\| (\exp \Omega^n - Id)\, d_i^n\right\|_{1,p}$$

$$\leqslant C_2 \left\|\exp \Omega^n - Id\right\|_{1,p} \left\|d_i^n\right\|_{1,p}$$

$$\leqslant C_2 \left\|\sum_{k \geqslant 1} \frac{(\Omega^n)^k}{k!}\right\|_{1,p} (\left\|d_i^0\right\|_{1,p} + \left\|d_i^0 - d_i^n\right\|_{1,p})$$

$$\leqslant C_2 \left\|\Omega^n\right\|_{1,p} \exp\left\|\Omega^n\right\|_{1,p}$$

$$(\left\|\pi_h d_i\right\|_{1,p} + \left\|d_i^0 - d_i^n\right\|_{1,p})\,. \tag{6.83}$$

Moreover, writing (6.81) and (6.80) for $0 \leqslant m < n$ yields

$$\left\|d_i^0 - d_i^n\right\|_{1,p} \leqslant \sum_{m=0}^{n-1} \left\|d_i^{m+1} - d_i^m\right\|_{1,p}$$

$$\leqslant K_2(\varepsilon) \sum_{m=0}^{n-1} \|R^m\|$$

$$\leqslant K_2(\varepsilon) \|R^0\| \left(\sum_{m=0}^{n-1} 2^{-m}\right)$$

$$\leqslant 2\,K_2(\varepsilon) \|R^0\|\,. \tag{6.84}$$

Plugging (6.84) into (6.83) and using (6.78) and (6.80) finally yields

$$\left\| d_\iota^{n+1} - d_\iota^n \right\| \le C_2 K_1(\varepsilon) \left\| R^n \right\| \exp(K_1(\varepsilon) \left\| R^0 \right\|)(C_2 \left\| d_\iota \right\|_{1,p} +$$

$$+ 2 K_2(\varepsilon) \left\| R^0 \right\|),$$

which is (6.81) if we set

$$K_2(\varepsilon) = C_2 K_1(\varepsilon) \exp(1) \left(C_2 \sup_\ell \left\| d_\ell \right\|_{1,p} + 2 \right) \tag{6.85}$$

and impose the restriction

$$\text{Restriction 4}: \quad \max (K_1(\varepsilon), K_2(\varepsilon)) \left\| R^0 \right\| \le 1.$$

Step 3. *Checking* (6.82). From (6.76), (6.81) and (6.84), we get

$$\left\| d_3^{n+1} - d_3 \right\|_{C^0(I_\alpha)} = \left\| d_3^{n+1} - d_3 \right\|_{C^0(0, L)}$$

$$\le C_3 \left\| d_3^{n+1} - d_3 \right\|_{1,p}$$

$$\le C_3 (\left\| d_3^{n+1} - d_3^n \right\|_{1,p} + \left\| d_3^n - d_3^0 \right\|_{1,p} + \left\| d_3^0 - d_3 \right\|_{1,p})$$

$$\le C_3 (3 K_2(\varepsilon) \left\| R^0 \right\| + \left\| \pi_h d_3 - d_3 \right\|_{1,p}).$$

This is (6.82) if we impose the restriction

$$\text{Restriction 5}: \quad C_3 (3 K_2(\varepsilon) \left\| R^0 \right\| + \left\| \pi_h d_3 - d_3 \right\|_{1,p}) \le \varepsilon.$$

Proof of Theorem 4 : Let $\varepsilon < 1/8 \, C_4$ be given. Let

$$e_h = \sup_\ell \left\| d_\ell - \pi_h d_\ell \right\|_{1,p}.$$

We have seen that e_h goes to zero with h, and therefore, there exists h_0 such that, $\forall h < h_0$, we have

$$h \le \text{meas } (I_\alpha)/8, \tag{6.86}$$

$$C_4 C_3^2 LK_1(\varepsilon) e_h \exp(C_3^2 LK_1(\varepsilon) e_h) \le \frac{1}{4}, \tag{6.87}$$

$$C_3 L \max (K_1(\varepsilon), K_2(\varepsilon)) e_h \le 1, \tag{6.88}$$

$$(3 C_3^2 LK_2(\varepsilon) + C_3) e_h \le \varepsilon. \tag{6.89}$$

Moreover, the definition of R^n implies that we have

$$\|R^0\| = \left\| r_L - r_0 - \int_0^L d_3^0(\sigma)\, d\sigma \right\|$$

$$= \left\| \int_0^L (d_3 - d_3^0)(\sigma)\, d\sigma \right\|$$

$$\leq C_3 L \|d_3 - d_3^0\|_{1,p}$$

$$\leq C_3 L e_h. \qquad (6.90)$$

With this choice of h and ε, Restriction 1 holds because of (6.86), Restriction 2 and 5 hold because of (6.89) and (6.90), Restriction 3 follows from (6.87) and from our choice of ε and Restriction 4 is a direct consequence of (6.88) and (6.90). Therefore, all the above lemmas can be applied.

In particular, we have

$$\|d_{\iota h} - d_\iota\|_{1,p} \leq \sum_{m=0}^{\infty} \|d_\iota^{m+1} - d_\iota^m\|_{1,p} + \|d_\iota^0 - d_\iota\|_{1,p}$$

$$\leq \sum_{m=0}^{\infty} K_2(\varepsilon)\|R^m\| + \|\pi_h\, d_\iota - d_\iota\|_{1,p}$$

$$\leq 2\, K_2(\varepsilon)\|R^0\| + e_h$$

$$\leq (2\, K_2(\varepsilon)\, C_3\, L + 1)\, e_h$$

$$\leq \varepsilon/C_3,$$

$$\|r_h - r\|_{2,p} = \left(\int_0^L \left| \int_0^s (d_3 - d_{3h})(\sigma)\, d\sigma \right|^p ds + \|d_3 - d_{3h}\|_{1,p}^p \right)^{1/p}$$

$$\leq (L\, \|d_3 - d_{3h}\|_{0,p}^p + \|d_3 - d_{3h}\|_{1,p}^p)^{1/p}$$

$$\leq (L + 1)^{1/p}\, \varepsilon/C_3,$$

which proves (4.45).

To prove that $\{r_h, d_{\iota h}\} \in K_h^{\text{inc}}$, we first observe that by construction of $d_{\iota h}$, we have

$$\{d_{\iota h}(M_p)\} = \left(\prod_{m=0}^{\infty} \exp \Omega_p^m \right) \{d_\iota^0(M_p)\}$$

$$= \left(\prod_{m=0}^{\infty} \exp \Omega_p^m \right) \{d_\iota(M_p)\}.$$

Since $\{d_i(M_p)\}$ is a direct orthonormal basis of E and since $\exp \Omega_p^m$ is a rotation matrix, it then follows that $\{d_{ih}(M_p)\}$ is a direct orthonormal basis of E. Moreover, because q_α and hence ω^m and Ω^m take zero values at both ends of the rod, the above identity yields

$$\{d_{ih}(0)\} = \{d_i(0)\},$$
$$\{d_{ih}(L)\} = \{d_i(L)\},$$

and hence $\{d_{ih}\}$ satisfies the kinematic boundary conditions of K_h^{inc}.
As for r_h, by construction, we have

$$r_h(0) = 0,$$

$$r_h(L) = r_0 + \int_0^L d_{3h}(\sigma)\,d\sigma$$

$$= r_L + r_0 - r_L + \lim_{n\to\infty} \int_0^L d_3^n(\sigma)\,d\sigma$$

$$= r_L - \lim_{n\to\infty} R^n = r_L,$$

$$r_h'(S) = d_{3h}(S).$$

All conditions are satisfied for $\{r_h, d_{ih}\}$ to be in K_h^{inc} which completes our proof.

ACKNOWLEDGMENTS

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