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SPECTRAL-FINITE ELEMENT METHOD FOR COMPRESSIBLE FLUID FLOWS (*)

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Abstract. — *In this paper, a combined Fourier spectral-finite element method is proposed for solving n-dimensional* ($n = 2, 3$), *semi-periodic compressible fluid flow problems. The strict error estimation, as well as the convergence rate, is presented.*

Résumé. — *Dans cet article, on propose une méthode combinant les approches spectrales (de type Fourier) et les éléments finis pour résoudre des problèmes semi-périodiques de fluides compressibles en dimensions 2 et 3. On obtient des majorations d'erreur, ainsi que la vitesse de convergence.*

I. INTRODUCTION

n-dimensional ($n = 2, 3$) compressible fluid flow problems satisfy the following equations [1, 2] :

$$\left\{\begin{array}{l} \partial_t U^{(\ell)} + (U\,.\,\nabla)\,U^{(\ell)} - \dfrac{1}{\rho}\,\partial_\ell(\kappa\,\nabla\,.\,U) - \dfrac{1}{\rho}\displaystyle\sum_{j=1}^{n}\partial_j[\nu(\partial_j U^{(\ell)} + \partial_\ell U^{(j)})] + \\ + \dfrac{1}{\rho}\,\partial_\ell p = f^{(\ell)}, \quad \ell = 1, \ldots, n\,, \\ \partial_t T + (U\,.\,\nabla)\,T - \dfrac{1}{\rho T S_T}(\nabla\,.\,\mu\nabla)\,T - \dfrac{\nu}{2\,\rho T S_T}\displaystyle\sum_{\ell, j=1}^{n}(\partial_\ell U^{(j)} + \partial_j U^{(\ell)})^2 \\ - \dfrac{\kappa}{\rho T S_T}(\nabla\,.\,U)^2 - \dfrac{\rho S_\rho}{S_T}(\nabla\,.\,U) = 0\,, \\ \partial_t \rho + (U\,.\,\nabla)\,\rho + \rho(\nabla\,.\,U) = 0\,, \end{array}\right. \tag{1.1}$$

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where $\partial_t = \partial/\partial t$, $\partial_\ell = \partial/\partial x_\ell$, $U = (U_1, \ldots, U_n)$ is the velocity. T is the absolute temperature. $\nu(T, \rho)$ is the viscous coefficient. $\kappa(T, \rho) = \nu'(T, \rho) - \frac{2}{3}\nu(T, \rho)$, with $\nu'(T, \rho)$ being the second viscous coefficient. $\mu(T, \rho)$ is the coefficient of heat conduction. $S(T, \rho)$ is the entropy, $S_T = \partial S/\partial T$, $S_\rho = \partial S/\partial \rho$.

Under certain conditions, Tani [2] proved the local existence of the classical solution to (1.1). Much work has been done to the numerical solutions of this problem. The most classical one is the finite difference method [1, 3]. By using this method, Guo Ben-yu proposed a class of schemes to solve the periodic problem and the first kind boundary condition problem of (1.1), and proved the error estimations strictly [4, 5]. Finite element method is often used to solve (1.1) [6], too. This method is particularly suitable for problems with irregular domains. Recently, the Fourier spectral method has also been applied to solve the periodic solution of (1.1) [7]. However, for many practical problems, the boundary conditions are neither fully periodic nor fully nonperiodic. A appropriate strategy to tackle such problems is to combine the Fourier spectral method with finite difference method or finite element method. Now, there are a lot of papers devoted to the analysis of such combined methods [8-10]. The present authors once used the Fourier spectral-finite element method for solving the semi-periodic problems of incompressible fluid flow. The numerical results were quite encouraging [11-13].

In this paper, we continue the work of [11-13] by applying the Fourier spectral-finite element method to solve the semi-periodic problem of (1.1). Assume $Q \subset \mathbb{R}^{n-1}$ is a polygonal domain (it is an open interval, when $n = 2$). $I = (0, 2\pi)$,

$$\Omega = Q \times I = \left\{x = (x_1, \ldots, x_{n-1}, x_n)/(x_1, \ldots, x_{n-1}) \in Q,\ x_n \in I\right\} .$$

We consider the solution of (1.1) in the domain $(x, t) \in \Omega \times [0, t_0]$. We suppose that all the functions in (1.1) have the periodicity 2π in x_n direction, and that U and T satisfy the homogeneous first kind boundary conditions. These mean that

$$\begin{cases} \eta\,|_{x_n = 0} = \eta\,|_{x_n = 2\pi}, & \forall (x_1, \ldots, x_{n-1}, t) \in Q \times [0, t_0], \quad \eta = U, T, \rho, \\ \eta\,|_{(x_1, \ldots, x_{n-1}) \in \partial Q} = 0, & \forall (x_n, t) \in I \times [0, t_0], \quad \eta = U, T. \end{cases} \tag{1.2}$$

Besides, we assume that the initial values of (1.1) are the following,

$$\eta\,|_{t=0} = \eta_0, \qquad \eta = U, T, \rho . \tag{1.3}$$

To avoid « negative density » (i.e. $\rho < 0$), which is likely caused by the round off errors during the computations, and which generates a nonphysical solution and instabilize the computations, we adopt the idea of Guo Ben-yu [4, 5], i.e., we seek $\varphi = \ln \rho$ by (1.1) instead of calculating ρ directly. Besides, we assume the fluid satisfies the following equation of state,

$$p = R\rho T ,$$

where R is a positive constant. Consequently (1.1) can be rewritten into the following form,

$$\left\{ \begin{aligned} & \partial_t U^{(\ell)} + (U \cdot \nabla) U^{(\ell)} - e^{-\varphi} \partial_\ell (\kappa \nabla \cdot U) - \\ & - e^{-\varphi} \sum_{j=1}^{n} \partial_j [\nu (\partial_j U^{(\ell)} + \partial_\ell U^{(j)})] + R \partial_\ell T + RT \partial_\ell \varphi = \\ & = f^{(\ell)}, \quad \ell = 1, \ldots, n , \\ & \partial_t T + (U \cdot \nabla) T - e^{-\varphi} T^{-1} S_T^{-1} (\nabla \cdot \mu \nabla) T - \\ & - \frac{1}{2} \nu e^{-\varphi} T^{-1} S_T^{-1} \sum_{\ell, j = 1}^{n} (\partial_\ell U^{(j)} + \partial_j U^{(\ell)})^2 \\ & - \kappa e^{-\varphi} T^{-1} S_T^{-1} (\nabla \cdot U)^2 - S_\varphi S_T^{-1} (\nabla \cdot U) = 0 , \\ & \partial_t \varphi + (U \cdot \nabla) \varphi + \nabla \cdot U = 0 . \end{aligned} \right. \tag{1.4}$$

We suppose ν, μ, κ and s are sufficiently smooth to each of their variables, and there exist positive constants B_0, B_1, B_2, ν_0, ν_1, μ_0, μ_1, κ_1, A_0, A_1, S_0, S_1, S_2, Φ_0 and Φ_1, such that for $B_0 < T < B_1$ and $|\varphi| \leq B_2$,

$$\left\{ \begin{aligned} & \nu_0 < \nu < \nu_1, \ \mu_0 < \mu < \mu_1, \ |\kappa| < \kappa_1, \ \min (nk + (n+1)\nu, \nu) > A_0 \\ & S_0 < S_T < S_1, \ |S_\varphi| < S_2, \ \Phi_0 < e^{-\varphi} < \Phi_1 , \\ & \left| \frac{\partial \eta}{\partial z} \right| \leq A_1 , \quad \text{where} \quad \eta = \nu, \kappa, \mu, S_T, S_\varphi , \quad z = T, \varphi . \end{aligned} \right. \tag{1.5}$$

II. NOTATIONS

Suppose $\mathscr{D} \subset \mathbb{R}^m$ ($m = 1, 2$ or 3) is a bounded open domain, with its boundary being locally Lipschitz continuous. $p \geq 1$ and $s \geq 0$ are real numbers. We denote by $W^{s,p}(\mathscr{D})$ the classical Sobolev spaces, with the norm $\| \cdot \|_{s,p,\mathscr{D}}$ and the semi-norm $| \cdot |_{s,p,\mathscr{D}}$ (see [14]). In particular, we set $H^s(\mathscr{D}) = W^{s,2}(\mathscr{D})$, $\| \cdot \|_{s,\mathscr{D}} = \| \cdot \|_{s,2,\mathscr{D}}$, $| \cdot |_{s,\mathscr{D}} = | \cdot |_{s,2,\mathscr{D}}$, and $\mathscr{L}^p(\mathscr{D}) = W^{0,p}(\mathscr{D})$, and we denote by $\| \cdot \|_{\mathscr{D}}$ and $(\cdot, \cdot)_{\mathscr{D}}$ the norm and the inner product of $\mathscr{L}^2(\mathscr{D})$. For simplicity, we shall omit the domain notations whenever $\mathscr{D} = \Omega$.

Suppose $\mathscr{B}$ is a Banach space, and $\vartheta \subset \mathbb{R}^1$ is an interval. We define the abstract measurable function spaces as follows :

$$L^2(\vartheta ; \mathscr{B}) = \Big\{\eta/\eta : \vartheta \to \mathscr{B},\ \|\eta\|_{\mathscr{L}^2(\vartheta ; \mathscr{B})} = \left(\int_\vartheta \|\eta(t')\|^2_{\mathscr{B}}\, dt'\right)^{1/2} < \infty\Big\},$$

$$H^1(\vartheta ; \mathscr{B}) = \Big\{\eta/\eta : \vartheta \to \mathscr{B},\ \|\eta\|_{H^1(\vartheta ; \mathscr{B})} = \left(\|\eta\|^2_{\mathscr{L}^2(\vartheta ; \mathscr{B})} + \left\|\frac{\partial \eta}{\partial t'}\right\|^2_{\mathscr{L}^2(\vartheta ; \mathscr{B})}\right)^{1/2} < \infty\Big\},$$

$$C(\vartheta ; \mathscr{B}) = \Big\{\eta/\eta : \vartheta \to \mathscr{B} \text{ is strongly continuous},\ \|\eta\|_{C(\vartheta ; \mathscr{B})} = \max_{t' \in \vartheta} \|\eta(t')\| < \infty\Big\}, \quad \text{etc.}.$$

For $\alpha, \beta \geqslant 0$, we define the non-isotropic Sobolev spaces as follows [15],

$$H^{\alpha, \beta}(\Omega) = L^2(I ; H^\alpha(Q)) \cap H^\beta(I ; \mathscr{L}^2(Q)),$$
$$X^{\alpha, \beta}(\Omega) = H^{\beta+1}(I ; H^\alpha(Q)) \cap H^\beta(I ; H^{\alpha+1}(Q)),$$

each equipped with the following norm, respectively,

$$\|\eta\|_{H^{\alpha,\beta}(\Omega)} = (\|\eta\|^2_{\mathscr{L}^2(I ; H^\alpha(Q))} + \|\eta\|^2_{H^\beta(I ; L^2(Q))})^{1/2},$$
$$\|\eta\|_{X^{\alpha,\beta}(\Omega)} = (\|\eta\|^2_{H^{\beta+1}(I ; H^\alpha(Q))} + \|\eta\|^2_{H^\beta(I ; H^{\alpha+1}(Q))})^{1/2}.$$

If $\alpha, \beta \geqslant 1$, we define furthermore that

$$Y^{\alpha,\beta}(\Omega) = H^{\alpha,\beta}(\Omega) \cap H^1(I ; H^{\alpha-1}(Q)) \cap H^{\beta-1}(I ; H^1(Q)),$$

equipped with the norm

$$\|\eta\|_{Y^{\alpha,\beta}(\Omega)} = (\|\eta\|^2_{H^{\alpha,\beta}(\Omega)} + \|\eta\|^2_{H^1(I ; H^{\alpha-1}(Q))} + \|\eta\|^2_{H^{\beta-1}(I ; H^1(Q))})^{1/2}.$$

Define $C_p^\infty(\Omega)$ be the set of infinitely differentiable functions on $\bar{\Omega}$, which have the periodicity 2π in x_n direction. We denote by $H_p^s(\Omega)$, $H_p^{\alpha,\beta}(\Omega)$, $X_p^{\alpha,\beta}(\Omega)$ and $Y_p^{\alpha,\beta}(\Omega)$ the closure of $C_p^\infty(\Omega)$ in $H^s(\Omega)$, $H^{\alpha,\beta}(\Omega)$, $X^{\alpha,\beta}(\Omega)$ and $Y^{\alpha,\beta}(\Omega)$, respectively. Besides, we set $H^s_{0,p}(\Omega) = H^s_p(\Omega) \cap \mathscr{L}^2(I ; H^1_0(\Omega))$, $Y^{\alpha,\beta}_{0,p}(\Omega) = Y^{\alpha,\beta}_p(\Omega) \cap \mathscr{L}^2(I ; H^1_0(\Omega))$.

III. THE SCHEMES

By integeration by parts, we have that the generalized solution of (1.2)-(1.4),

$$(U, T, \varphi) \in [C(0, t; H^1_{0,p}(\Omega) \cap C(\bar{\Omega}))]^n \times$$
$$\times [C(0, t; H^1_{0,p}(\Omega) \cap C(\bar{\Omega}))] \times [C(0, t; H^1_p(\Omega) \cap C(\bar{\Omega}))],$$

satisfy the following equations :

$$\begin{cases} (\partial_t U, v) + ([U \cdot \nabla] U, v) + R(\nabla T, v) + R(T \nabla \varphi, v) + \\ \qquad + \sum_{m=1}^{3} J_m(T, \varphi, U, v) = (f, v), \quad \forall v \in (H^1_{0,p}(\Omega))^n, \\ (\partial_t T, w) + ([U \cdot \nabla] T, w) + J_4(T, \varphi, w) + J_5(T, \varphi, U, w) = 0, \\ \qquad \forall w \in H^1_{0,p}(\Omega), \\ (\partial_t \varphi, \chi) + ([U \cdot \nabla] \varphi, \chi) + (\nabla \cdot U, \chi) = 0, \quad \forall \chi \in H^1_p(\Omega), \end{cases} \tag{3.1}$$

where

$$J_1(T, \varphi, U, v) = (\kappa(T, \varphi) \nabla \cdot U, \nabla \cdot (e^{-\varphi} v)),$$

$$J_2(T, \varphi, U, v) = \sum_{\ell, j=1}^{n} (\nu(T, \varphi) \partial_j U^{(\ell)}, \partial_j (e^{-\varphi} v^{(\ell)})),$$

$$J_3(T, \varphi, U, v) = \sum_{\ell, j=1}^{n} (\nu(T, \varphi) \partial_\ell U^{(j)}, \partial_j (e^{-\varphi} v^{(\ell)})),$$

$$J_4(T, \varphi, w) = (\mu(T, \varphi) \nabla T, \nabla (e^{-\varphi} T^{-1} S_T^{-1} w)),$$

$$J_5(T, \varphi, U, w) = -\Bigg(\frac{1}{2} \nu e^{-\varphi} T^{-1} S_T^{-1} \sum_{\ell, j=1}^{n} (\partial_j U^{(\ell)} + \partial_\ell U^{(j)})^2 +$$
$$+ \kappa e^{-\varphi} T^{-1} S_T^{-1} (\nabla \cdot U)^2 + S_\varphi S_T^{-1} (\nabla \cdot U), w\Bigg).$$

Obviously, any classical solution of (1.2)-(1.4) satisfies (3.1). Now we construct the Fourier spectral-finite element approximation of (3.1). Let $\{C_h\}_h$ be a regular family of finite triangulations of $\bar{Q}$, such that

$\bar{Q} = \bigcup_{m=1}^{M_h} K_m$ (K_m are small intervals when $n = 2$, and small triangles when $n = 3$). Define

$$h_m = \text{diam } (K_m) ,$$

$$h'_m = \begin{cases} h_m , & \text{for } n = 2 , \\ \text{the diameter of the inscribed circle in } K_m, & \text{for } n = 3 . \end{cases}$$

$$h = \max_{1 \leq m \leq M_h} (h_m) , \qquad h' = \min_{1 \leq m \leq M_h} (h'_m) .$$

We assume in addition that $\{C_h\}_h$ satisfies the « inverse assumption », i.e. there exists a positive constant d, such that $h/h' \leq d$, for any $C_h \in \{C_h\}_h$.

Let $k \geq 1$ be an integer. We denote by $\mathbb{P}_k$ the set of all the polynomials defined on $\mathbb{R}^{n-1}$ with degree $\leq k$. Define the subspaces for finite element approximations in non-periodic directions as follows :

$$X_h^k(Q) = \left\{ \eta / \eta \, |_{K_m} \in \mathbb{P}_k, \; 1 \leq m \leq M_h \right\} \cap H^1(Q) ,$$
$$X_{0,h}^k(Q) = X_h^k(Q) \cap H_0^1(Q) .$$

Let $N \geq 1$ be an integer, we define the subspaces for Fourier spectral approximations in periodic direction as follows :

$$S_N(I) = \left\{ \eta (x_n) = \sum_{|j| \leq N} \eta_j \, e^{ijx_n} / \eta_j = \bar{\eta}_{-j}, \; |j| \leq N \right\} .$$

Let $\delta = (h, N, k)$. By combining the above two kinds of approximations, we can approximate $H_p^1(\Omega)$ and $H_{0,p}^1(\Omega)$, by the following finite dimensional subspaces

$$V_\delta(\Omega) = X_h^k(Q) \otimes S_N(I) , \quad V_{0,\delta}(\Omega) = X_{0,h}^k(Q) \otimes S_N(I) .$$

Let τ be the mesh size of time, and $\Theta_\tau = \left\{ t = \ell \tau / 0 \leq \ell \leq \left[\frac{t_0}{\tau} \right] \right\}$. We approximate $\partial_t \eta (t)$ by its first order difference quotient, i.e.

$$\eta_t(t) = \frac{1}{\tau} \left(\eta (t + \tau) - \eta (t) \right) .$$

The fully discrete Fourier spectral-finite element schemes for solving (2.4) is to find the pair

$$(u_\delta(t), T_\delta(t), \varphi_\delta(t)) \in [V_{0,\delta}(\Omega)]^n \times [V_{0,\delta}(\Omega)] \times [V_\delta(\Omega)] \text{ for } t \in \Theta_\tau ,$$

satisfying

$$\begin{cases} (u_{\delta t}, v) + ([u_\delta . \nabla] u_\delta, v) + R(\nabla T_\delta, v) + R(T_\delta \nabla \varphi_\delta, v) + \\ + \sum_{m=1}^{3} J_m(T_\delta, \varphi_\delta, u_\delta, v) = (f, v), \quad \forall v \in [V_{0,\delta}(\Omega)]^n, \\ (T_{\delta t}, w) + ([u_\delta . \nabla] T_\delta, w) + J_4(T_\delta, \varphi_\delta, w) + \\ + J_5(T_\delta, \varphi_\delta, u_\delta, w) = 0, \quad \forall w \in V_{0,\delta}(\Omega), \\ (\varphi_{\delta t}, \chi) + ([u_\delta . \nabla] \varphi_\delta, \chi) + (\nabla . u_\delta, \chi) = 0, \quad \forall \chi \in V_\delta(\Omega). \end{cases} \tag{3.2}$$

We approximate the initial value condition (1.3) by its combined L^2-orthogonal projection in x_n direction and piecewise Lagrange's interpolation in other directions. More precisely, define P_N be the orthogonal projection operator from $L^2(I)$ onto $S_N(I)$, and define π_h^k from $C(\bar{Q})$ onto $X_h^k(Q)$ be the piecewise Lagrange's interpolation operator of order k, i.e., for any $\xi \in C(\bar{Q})$, $\pi_h^k \xi|_{K_m}$, $1 \leq m \leq M_h$, is the Lagrange's interpolation of order k of $\xi|_{K_m}$, and $\pi_h^k \xi$ is continuous in $\bar{Q}$. The composite operator $\mathscr{F}_\delta : L^2(I, C(\bar{Q})) - V_\delta(\Omega)$ is defined as $\mathscr{F}_\delta = P_N \circ \pi_h^k = \pi_h^k \circ P_N$, i.e., if $\eta(x) = \sum_{|j|=0}^{\infty} \eta_j(x_1, \dots, x_{n-1}) e^{ijx_n} \in \mathscr{L}^2(I; C(\bar{Q}))$, then

$$(\mathscr{F}_\delta \eta)(x) = \sum_{|j| \leq N} (\pi_h^k \eta_j)(x_1, \dots, x_{n-1}) e^{ijx_n}.$$

Thus we approximate (1.3) in the following way :

$$u_\delta(0) = \mathscr{F}_\delta U_0, \quad T_\delta(0) = \mathscr{F}_\delta T_0, \quad \varphi_\delta(0) = \mathscr{F}_\delta \varphi(0) = \mathscr{F}_\delta \ln \rho_0. \tag{3.3}$$

Remark 1 : We may approximate the initial condition by other methods, e.g., the interpolation over $x_1, \dots, x_{n-1}$ and x_n directions, the orthogonal projection with $L^2(\Omega)$-norm or $H^1(\Omega)$-norm, etc. Then the analysis below is still valid provided that these approximations are of the same convergence rates as $\mathscr{F}_\delta$.

IV. LEMMAS

LEMMA 1 [8, 12] : *If* $\alpha > \frac{n-1}{2}$, $\beta \geq 0$, $\bar{\alpha} = \min(\alpha, k+1)$, *then there exists a positive constant* C *independent of* h *and* N, *such that for all* $\eta \in H_p^{\alpha,\beta}(\Omega)$,

$$\| \eta - \mathscr{F}_\delta \eta \| \leq C (h^{\bar{\alpha}} + N^{-\beta}) \| \eta \|_{H^{\bar{\alpha},\beta}(\Omega)}.$$

LEMMA 2 [8, 12] : *If* $Nh \leqslant \mathrm{Cst.}$, $\alpha > \frac{n-1}{2}$, $\alpha \geqslant 1$, $\beta \geqslant 1$, $\bar{\alpha} = \min(\alpha, k+1)$, *then there exists a positive constant C independent of h and N, such that for all* $\eta \in Y_p^{\alpha,\beta}(\Omega)$,

$$\| \eta - \mathscr{F}_\delta \eta \|_1 \leqslant C (h^{\bar{\alpha}-1} + N^{1-\beta}) \| \eta \|_{Y^{\bar{\alpha},\beta}(\Omega)} .$$

LEMMA 3 : *If* $\alpha > \frac{n-1}{2}$, $\beta > 1/2$, *then there exists a positive constant C independent of h and N, such that for all* $\eta \in X_p^{\alpha,\beta}(\Omega)$,

$$\| \mathscr{F}_\delta \eta \|_{1,\infty} \leqslant C \| \eta \|_{X^{\alpha,\beta}(\Omega)} .$$

Proof : Suppose that $\eta(x) = \sum_{|j|=0}^{\infty} \eta_j(x_1, \ldots, x_{n-1}) \; e^{ijx_n} \in X_p^{\alpha,\beta}(\Omega)$. Since $\alpha > \frac{n-1}{2}$, we have from Sobolev's embedding theorem that $X_p^{\alpha,\beta}(\Omega) \hookrightarrow \mathscr{L}^2(I, C(\bar{Q}))$ and hence,

$$(\mathscr{F}_\delta \eta)(x) = \sum_{|j| \leqslant N} (\pi_h^k \eta_j)(x_1, \ldots, x_{n-1}) \, e^{ijx_n} .$$

Moreover, since $\{C_h\}_h$ is a regular family of triangulations of $\bar{Q}$, and satisfies the « inverse assumption », we have by the error estimates of function interpolations in finite element method that [16]

$$\| \pi_h^k \eta_j \|_{0,\infty,Q} \leqslant C \| \eta_j \|_{\alpha,Q} ,$$
$$| \pi_h^k \eta_j |_{1,\infty,Q} \leqslant C \| \eta_j \|_{1+\alpha,Q} .$$

Thus

$$\begin{aligned} \| \mathscr{F}_\delta \eta \|_{1,\infty} \leqslant & \\ & \leqslant \sum_{|j| \leqslant N} \left\{ (1 + |j|) \| \pi_h^k \eta_j \|_{0,\infty,Q} + | \pi_h^k \eta_j |_{1,\infty,Q} \right\} \\ & \leqslant C \sum_{|j| \leqslant N} \left\{ (1 + |j|) \| \eta_j \|_{\alpha,Q} + \| \eta_j \|_{1+\alpha,Q} \right\} \\ & \leqslant C \left\{ \sum_{|j| \leqslant N} [(1 + |j|^{2+2\beta}) \| \eta_j \|_{\alpha,Q}^2 + \right. \\ & \quad \left. + (1 + |j|^{2\beta}) \| \eta_j \|_{1+\alpha,Q}^2] \right\}^{1/2} \left\{ \sum_{|j| \leqslant N} (1 + |j|)^{-2\beta} \right\}^{1/2} \\ & \leqslant C \| \eta \|_{X^{\alpha,\beta}(\Omega)} . \end{aligned}$$

LEMMA 4 : *There exists a positive constant C_0 independent of h and N, such that for all $\eta \in V_\delta(\Omega)$,*

(i) $|\eta|_1^2 \leq (C_0 h^{-2} + N^2)\|\eta\|^2$,

(ii) $\|\eta\|_{0,\infty}^2 \leq C_0 h^{1-n} N \;\|\eta\|^2$,

(iii) $\|\eta\|_{0,\infty}^2 \leq C_0 \bar{D}(h, N)\|\eta\|_1^2$,

where

$$\bar{D}(h, N) = \begin{cases} \ln N, & \text{for } n = 2, \\ N|\ln h|, & \text{for } n = 3. \end{cases}$$

Proof : The proof of conclusion (i) may be found in [11, 12]. Now we prove conclusion (ii). Suppose

$$\eta(x) = \sum_{|j| \leq N} \eta_j(x_1, \ldots, x_{n-1})\, e^{ijx_n},$$

then $\eta_j \in X_h^k(Q) \subset H^1(Q)$. Since $\{C_h\}_h$ is regular and satisfies the « inverse assumption », we have from the inverse inequalities in finite element method that [16]

$$\|\eta_j\|_{0,\infty,Q} \leq C_0 h^{(1-n)/2}\|\eta_j\|_{0,2,Q}.$$

Hence, we can complete the proof of (ii) in the following way,

$$\begin{aligned}\|\eta\|_{0,\infty} &\leq \sum_{|j| \leq N} \|\eta_j\|_{0,\infty,Q} \leq C_0 h^{(1-n)/2} \sum_{|j| \leq N} \|\eta_j\|_{0,2,Q} \\ &\leq C_0 h^{(1-n)/2}\Big(\sum_{|j| \leq N} \|\eta_j\|_{0,2,Q}^2\Big)^{1/2}\Big(\sum_{|j| \leq N} 1\Big)^{1/2} \\ &\leq C_0 h^{(1-n)/2} N^{1/2}\|\eta\|.\end{aligned}$$

Next we prove conclusion (iii) for $n = 2$ and $n = 3$ separately. If $n = 2$, then we have clearly that

$$\|\eta_j\|_{0,\infty,Q} \leq C\,\|\eta_j\|_{0,2,Q}^{1/2}|\eta_j|_{1,2,Q}^{1/2}.$$

Thus

$$\begin{aligned}\|\eta\|_{0,\infty} &\leq \sum_{|j| \leq N} \|\eta_j\|_{0,\infty,Q} \leq C \sum_{|j| \leq N} \|\eta_j\|_{0,2,Q}^{1/2}\,|\eta_j|_{1,2,Q}^{1/2} \\ &\leq C\Big\{\sum_{|j| \leq N} (1+j^2)\|\eta_j\|_{0,2,Q}^2\Big\}^{1/4} \times \\ &\quad \times \Big\{\sum_{|j| \leq N} |\eta_j|_{1,2,Q}^2\Big\}^{1/4}\Big\{\sum_{|j| \leq N} (1+j^2)^{-1/2}\Big\}^{1/2} \\ &\leq C_0(\ln N)^{1/2}\,\|\eta\|_1.\end{aligned}$$

If $n = 3$, then by the inverse inequality on the subspaces $X_h^k(Q)$ (*cf.* Lemma 1.1 in [17]) that

$$\| \eta_j \|_{0, \infty, Q} \leqslant C \, |\ln h|^{1/2} \, \| \eta_j \|_{1, Q} \, .$$

Thus

$$\begin{aligned} \| \eta \|_{0, \infty} &\leqslant \sum_{|j| \leqslant N} \| \eta_j \|_{0, \infty, Q} \leqslant C \, |\ln h|^{1/2} \sum_{|j| \leqslant N} \| \eta_j \|_{1, Q} \\ &\leqslant C \, |\ln h|^{1/2} \Big(\sum_{|j| \leqslant N} \| \eta_j \|_{1, 2, Q}^2 \Big)^{1/2} \Big(\sum_{|j| \leqslant N} 1 \Big)^{1/2} \\ &\leqslant C_0 \, |\ln h|^{1/2} N^{1/2} \, \| \eta \|_1 \, . \end{aligned}$$

LEMMA 5 : *If* $\psi \in H^1(\Omega) \cap C_p(\Omega)$, ξ, $\eta \in V_{0, \delta}(\Omega)$, *then*

$$(\psi \, \partial_\ell \xi, \partial_j \eta) = (\psi \, \partial_j \xi, \partial_\ell \eta) - (\partial_j \psi \, \partial_\ell \xi - \partial_\ell \psi \, \partial_j \xi, \eta) \, , \quad \forall 1 \leqslant \ell, j \leqslant n \, .$$

Proof : Denote by $\vec{n} = (n_1, \ldots, n_n)$ the unit outward normal vector of $\partial(K_m \times I) = (\partial K_m \times I) \, U (K_m \times \partial I)$ (see *fig. 1*). Then we have by integration by parts that

$$\begin{aligned} (\psi \, \partial_\ell \xi, \partial_j \eta) &= \sum_{m=1}^{M_h} \int_{K_m \times I} \psi(x) \, \partial_\ell \xi(x) \, \partial_j \eta(x) \, dx \\ &= \sum_{m=1}^{M_h} \Big\{ \psi \eta \, \partial_\ell \xi \, . \, n_j |_{\partial(K_m \times I)} \\ &\quad - \int_{K_m \times I} \eta(x) (\psi(x) \, \partial_j \, \partial_\ell \xi(x) + \partial_j \psi(x) \, \partial_\ell \xi(x)) \, dx \Big\} \\ &= \sum_{m=1}^{M_h} \Big\{ \psi \eta (\partial_\ell \xi \, . \, n_j - \partial_j \xi \, . \, n_\ell) |_{\partial(K_m \times I)} + \int_{K_m \times I} \psi(x) \, \partial_j \xi(x) \, \partial_\ell \eta(x) \, dx \\ &\quad - \int_{K_m \times I} \eta(x) (\partial_j \psi(x) \, \partial_\ell \xi(x) - \partial_\ell \psi(x) \, \partial_j \xi(x)) \, dx \Big\} \, . \end{aligned}$$

Clearly, if the following two equalities are proved,

$$A_1 = \sum_{m=1}^{M_h} \{ \psi \eta (\partial_\ell \xi \, . \, n_j - \partial_j \xi \, . \, n_\ell) \} |_{K_m \times \partial I} = 0 \, ,$$

$$A_2 = \sum_{m=1}^{M_n} \{ \psi \eta (\partial_\ell \xi \, . \, n_j - \partial_j \xi \, . \, n_\ell) \} |_{\partial K_m \times I} = 0 \, ,$$

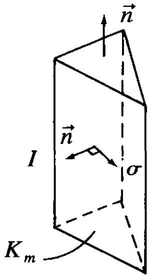

Figure 1.

then this lemma can be easily seen. By the periodicity of ψ, ξ and η, we have clearly that $A_1 = 0$. Now we prove $A_2 = 0$. Firstly, if $\ell = j$, then we have trivially $A_2 = 0$; Next, if $\ell \neq j$, but one of them is equal to n, say $j = n$, we have in this case that n_j vanishes on $\partial K_m \times I$. Thus

$$\psi\eta\ \partial_\ell\ \xi\ .\ n_j|_{\partial K_m \times I} = 0\ , \quad 1 \leqslant m \leqslant M_h\ .$$

On the other hand, since $\xi \in V_{0,\delta}(\Omega)$ and $j = n$, we have $\partial_j\xi \in V_{0,\delta}(\Omega)$. Therefore $\partial_j\xi \in C(\bar{Q})$, and $\partial_j\xi|_{\partial Q \times I} = 0$. Hence

$$\sum_{m=1}^{M_h} \psi\eta\ \partial_j\xi\ .\ n_\ell|_{\partial K_m \times I} = \psi\eta\ \partial_j\xi\ .\ n_\ell|_{\partial Q \times I} = 0\ .$$

Thus we have also $A_2 = 0$; Finally, if $\ell \neq j$, and neither ℓ nor j equals to n, then there must be in this case $n = 3$, $\ell = 1$ and $j = 2$, or $n = 3$, $j = 1$ and $\ell = 2$, say the former. Then we have in this case

$$(\partial_1\xi\ .\ n_2 - \partial_2\xi\ .\ n_1)|_{\partial K_m \times I} = \frac{\partial\xi}{\partial\sigma}\bigg|_{\partial K_m \times I}$$

where $\sigma = (n_2, -n_1, 0)$ is the unit tangential vector on $\partial K_m \times I$. Since ψ, ξ and η are continuous in $\bar{\Omega}$, and ξ vanishes on $\partial Q \times I$, we have

$$\sum_{m=1}^{M_h} \psi\eta\,(\partial_1\xi\ .\ n_2 - \partial_2\xi\ .\ n_1)|_{\partial K_m \times I} = \psi\eta\ \frac{\partial\xi}{\partial\sigma}\bigg|_{\partial K_m \times I} = 0\ .$$

By combining the above arguments, we complete the proof of this lemma.

LEMMA 6 [3] : *Suppose the following conditions are fulfilled :*

(i) η *is a non-negative function defined on* Θ_τ, ρ_0, B_0, $a_\ell(h, N)$ *and* M_ℓ, $1 \leqslant \ell \leqslant m$, *are non-negative constants ;*

(ii) $\rho(t) = \rho(\eta(0), \eta(\tau), \ldots, \eta(t-\tau))$ *satisfies that* $\rho(t) \leqslant \rho_0$ *for all* $\eta(t') \leqslant M_0/a_0(h, N)$;

(iii) $H_\eta(t) = \eta(t)\,[M(\eta(t)) + a_0(h, N)\,B(\eta(t))\,\eta(t)] + \sum_{\ell=1}^{m} \xi_\ell(\eta(t))$,

where $M(\eta(t)) \leq M_0$ *and* $B(\eta(t)) \leq B_0$ *for all* $\eta(t) \leq M_0/a_0(h, N)$; *and* $\xi_\ell(\eta(t)) \leq 0$ *for all* $\eta(t) \leq M_\ell/a_\ell(h, N)$, $1 \leq \ell \leq m$;

(iv) $G_\eta(t) = G(\eta(t), \eta(t-\tau)) \geq \eta(t)$;

(v) $\eta(0) \leq \rho(0) \leq \rho_0$, *and*

$$G_\eta(t) \leq \rho(t) + \tau \sum_{t'=0}^{t-\tau} H_\eta(t'), \quad \forall t \in \Theta_\tau ;$$

(vi) $\rho_0\, e^{(1+B_0)M_0 t_1} \leq \min_{0 \leq \ell \leq m} (M_\ell/a_\ell(h, N))$;

Then we have for all $t \in \Theta_\tau$ *and* $t \leq t_1$ *that*

$$\eta(t) \leq \rho_0\, e^{(1+B_0)M_0 t}.$$

V. ERROR ESTIMATIONS

Let (U, T, φ) be the solution of (3.1). Define $u_* = \mathcal{F}_\delta U$, $T_* = \mathcal{F}_\delta T$, $\varphi_* = \mathcal{F}_\delta \varphi$. Then we derive from (3.1) that

$$\begin{cases} (u_{*t}, v) + ([u_* . \nabla]\, u_*, v) + R(\nabla T_*, v) + R(T_* \nabla\varphi_*, v) + \\ \qquad + \sum_{m=1}^{3} J_m(T_*, \varphi_*, u_*, v) \\ \qquad = (f_1, v) + (\tilde f_1, v) + \sum_{m=1}^{3} \tilde H_m(v), \quad \forall v \in [V_{0,\delta}(\Omega)]^n, \\ (T_{*t}, \omega) + ([u_* . \nabla]\, T_*, \omega) + J_4(T_*, \varphi_*, \omega) + J_5(T_*, \varphi_*, u_*, \omega) \\ \qquad = (\tilde f_2, \omega) + \sum_{m=4}^{5} \tilde H_m(\omega), \quad \forall \omega \in V_{0,\delta}(\Omega), \\ (\varphi_{*t}, \chi) + ([u_* . \nabla]\, \varphi_*, \chi) + (\nabla . u_*, \chi) = (\tilde f_3, \chi), \\ \qquad \forall \chi \in V_\delta(\Omega), \end{cases} \tag{5.1}$$

where

$$\tilde f_1 = u_{*t} - \partial_t U + [u_* . \nabla]\, u_* - [U . \nabla]\, U + R[\nabla(T_* - T) + T_* \nabla\varphi_* - T\nabla\varphi],$$

$$\tilde f_2 = T_{*t} - \partial_t T + [u_* . \nabla]\, T_* - [U . \nabla]\, T,$$

$$\tilde f_3 = \varphi_{*t} - \partial_t \varphi + [u_* . \nabla]\, \varphi_* - [U . \nabla]\, \varphi + \nabla . (u_* - U),$$

$$\tilde{H}_m(v) = J_m(T_*, \varphi_*, u_*, v) - J_m(T, \varphi, u, v), \qquad m = 1, 2, 3,$$

$$\tilde{H}_4(\omega) = J_4(T_*, \varphi_*, \omega) - J_4(T, \varphi, \omega),$$

$$\tilde{H}_5(\omega) = J_5(T_*, \varphi_*, u_*, \omega) - J_5(T, \varphi, u, \omega).$$

Let $(u_\delta, T_\delta, \varphi_\delta)$ be the solution of scheme (3.2)-(3.3). Define $\tilde{u} = u_\delta - u_*$, $\tilde{T} = T_\delta - T_*$, $\tilde{\varphi} = \varphi_\delta - \varphi_*$. Then we have by (3.3) that $\tilde{u}(0) = 0$ and $\tilde{T}(0) = \tilde{\varphi}(0) = 0$. By subtract the formulas in (5.1) from the corresponding ones in (3.2), we obtain that

$$\begin{cases} (\tilde{u}_t, v) + \sum_{m=1}^{3} F_m(v) + \sum_{m=1}^{3} \tilde{J}_m(v) = -(\tilde{f}_1, v) - \sum_{m=1}^{3} \tilde{H}_m(v), \\ \qquad\qquad\qquad\qquad \forall v \in [V_{0,\delta}(\Omega)]^n, \\ (\tilde{T}_t, \omega) + F_4(\omega) + \sum_{m=4}^{5} J_m(\omega) = -(\tilde{f}_2, \omega) - \sum_{m=4}^{5} \tilde{H}_m(\omega), \\ \qquad\qquad\qquad\qquad \forall \omega \in V_{0,\delta}(\Omega)^n, \\ (\tilde{\varphi}_t, \chi) + \sum_{m=5}^{6} F_m(\chi) = -(\tilde{f}_3, \chi), \quad \forall \chi \in V_\delta(\Omega), \end{cases} \tag{5.2}$$

where

$$F_1(v) = ([\tilde{u} \cdot \nabla] u_*, v) + ([(u_* + \tilde{u}) \cdot \nabla] \tilde{u}, v),$$

$$F_2(v) = R(\nabla \tilde{T}, v)$$

$$F_3(v) = R(\tilde{T} \cdot \nabla \varphi_*, v) + R((T_* + \tilde{T}) \cdot \nabla \tilde{\varphi}, v),$$

$$F_4(\omega) = ([\tilde{u} \cdot \nabla] T_*, \omega) + ([(u_* + \tilde{u}) \cdot \nabla] \tilde{T}, \omega),$$

$$F_5(\chi) = ([\tilde{u} \cdot \nabla] \varphi_*, \chi) + ([(u_* + \tilde{u}) \cdot \nabla] \tilde{\varphi}, \chi),$$

$$F_6(\chi) = (\nabla \cdot \tilde{u}, \chi),$$

$$\tilde{J}_m(v) = J_m(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}, u_* + \tilde{u}, v) - J_m(T_*, \varphi_*, u_*, v), \quad m = 1, 2, 3,$$

$$\tilde{J}_4(\omega) = J_4(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}, \omega) - J_4(T_*, \varphi_*, \omega),$$

$$\tilde{J}_5(\omega) = J_5(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}, u_* + \tilde{u}, \omega) - J_5(T_*, \varphi_*, u_*, \omega).$$

We put $v = \tilde{u} + \tau \tilde{u}_t$, $\omega = \tilde{T} + \tau \tilde{T}_t$, $\chi = \tilde{\varphi} + \tau \tilde{\varphi}_t$ in (5.2), and add up the three equalities resulted. Note the following identity [3]

$$2(\tilde{\eta}_t, \tilde{\eta}) = (\|\tilde{\eta}\|^2)_t - \tau \|\tilde{\eta}_t\|^2,$$

we get that

$$(\|\tilde{u}\|^2 + \|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)_t + \tau(1-\varepsilon)(\|\tilde{u}_t\|^2 + \|\tilde{T}_t\|^2 + \|\tilde{\varphi}_t\|^2)$$
$$+ 2\sum_{m=1}^{3} F_m(\tilde{u} + \tau\tilde{u}_t) + 2F_4(\tilde{T} + \tau\tilde{T}_t) + 2\sum_{m=5}^{6} F_m(\tilde{\varphi} + \tau\tilde{\varphi}_t)$$
$$+ 2\sum_{m=1}^{3} \tilde{J}_m(\tilde{u} + \tau\tilde{u}_t) + 2\sum_{m=4}^{5} \tilde{J}_m(\tilde{T} + \tau\tilde{T}_t)$$
$$\leqslant \|\tilde{u}\|^2 + \|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2 + \left(1 + \frac{\tau}{\varepsilon}\right)\sum_{m=1}^{3}\|\tilde{f}_m\|^2$$
$$+ 2\sum_{m=1}^{3} \tilde{H}_m(\tilde{u} + \tau\tilde{u}_t) + 2\sum_{m=4}^{5} \tilde{H}_m(\tilde{T} + \tau\tilde{T}_t)\,, \tag{5.3}$$

where $\varepsilon > 0$ is a suitably small constant. Next we estimate the terms F_m, $\tilde{J}_m$ and $\tilde{H}_m$ in (5.3) separately. We assume that u, T and φ are sufficiently smooth, $B_0 < T < B_1$ and $|\varphi| < B_2$, then we have $B_0 < T_* < B_1$ and $|\varphi_*| < B_2$, if h^{-1} and N are large enough. Thus we conclude from (ii) of Lemma 4 that there exist a suitably small constant $\tilde{B} > 0$, such that $B_0 < T_* + \tilde{T} < B_1$ and $|\varphi_* + \tilde{\varphi}| < B_2$, for all $\tilde{T}$ and $\tilde{\varphi}$ satisfying that

$$\|\tilde{T}\| \leqslant \tilde{B}h^{(n-1)/2}N^{-1/2}, \quad \|\tilde{\varphi}\| \leqslant \tilde{B}h^{(n-1)/2}N^{-1/2}. \tag{5.4}$$

Under these assumptions, we have the boundedness of the variables in (1.5) valid for all $T_* + \tilde{T}$ ands $\varphi_* + \tilde{\varphi}$. For simplicity, we define $D(h, N) = C_0 \cdot \bar{D}(h, N)(C_0 h^2 + N^{-2})$ (*cf.* Lemma 4), and denote by M a positive constant independent of h, N and τ, but may depend on ε, R, $\|\eta\|_{X^{\alpha', \beta'}_{(\Omega)}}$ $\left(\alpha' > \frac{n-1}{2},\ \beta' > \frac{1}{2},\ \eta = U,\ T,\ \varphi\right)$, $\|U\|_2$, $\|T\|_2$ and k_1, ν_1, Φ_1, s_1, etc., and may be of different values in different expressions. At first, we have from Lemma 3 and Lemma 4 that

$$|F_1(\tilde{u})| \leqslant |u_*|_{1,\infty}\|\tilde{u}\|^2 + \|u_* + \tilde{u}\|_{\infty}|\tilde{u}|_1\|\tilde{u}\|$$
$$\leqslant \varepsilon|\tilde{u}|^2 + M(1 + \|\tilde{u}\|^2_{0,\infty})\|\tilde{u}\|^2$$
$$\leqslant \varepsilon|\tilde{u}|^2 + M(1 + D(h, N)\|\tilde{u}\|^2)\|\tilde{u}\|^2\,,$$

$$|F_2(\tilde{u})| \leqslant \varepsilon|\tilde{T}|_1^2 + M\|\tilde{u}\|^2\,,$$

$$|F_3(\tilde{u})| \leqslant M\Big\{(\|\tilde{T}\| + \|\tilde{\varphi}\|)\|\tilde{u}\| + \|\tilde{\varphi}\|\,|\tilde{u}|_1$$
$$+ \|\tilde{\varphi}\|_{0,\infty}(\|\tilde{u}\|\,|\tilde{T}|_1 + \|\tilde{T}\|\,|\tilde{u}|_1)\Big\}$$

$$\leqslant \varepsilon |\tilde{u}|_1^2 + \varepsilon |\tilde{T}|_1^2 + M(1 + D(h, N)\|\tilde{\varphi}\|^2) \times (\|\tilde{u}\|^2 + \|\tilde{T}\|^2) + M\|\tilde{\varphi}\|^2 .$$

and similarly,

$$|F_4(\tilde{T})| \leqslant \varepsilon |\tilde{T}|_1^2 + M(\|\tilde{u}\|^2 + \|\tilde{T}\|^2 + D(h, N)\|\tilde{u}\|^2 \|\tilde{T}\|^2),$$

$$\begin{aligned} |F_5(\tilde{\varphi})| &\leqslant |([\tilde{u} \cdot \nabla]\varphi_*, \tilde{\varphi})| + |(\nabla(\tilde{\varphi}^2/2), u_* + \tilde{u})| \\ &= |([\tilde{u} \cdot \nabla]\varphi_*, \tilde{\varphi})| + |(\tilde{\varphi}^2/2, \nabla \cdot (u_* + \tilde{u}))| \\ &\leqslant M(\|\tilde{u}\|^2 + \|\tilde{\varphi}\|^2) + MD(h, N)\|\tilde{\varphi}\|^2 |\tilde{u}|_1^2 , \end{aligned}$$

$$|F_6(\tilde{\varphi})| \leqslant \varepsilon |\tilde{u}|_1^2 + M\|\tilde{\varphi}\|^2 .$$

Next, we estimate $\tilde{J}_m(\tilde{u})$, $1 \leqslant m \leqslant 3$, and $\tilde{J}_4(\tilde{T})$. It is not difficult to verify that

$$\tilde{J}_1(\tilde{u}) = (e^{-\varphi_* - \tilde{\varphi}} \kappa(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}), (\nabla \cdot \tilde{u})^2) + \sum_{q=1}^{5} \tilde{J}_{1,q}(\tilde{u}), \quad (5.5)$$

where

$$\tilde{J}_{1,1}(\tilde{u}) = ([e^{-\varphi_* - \tilde{\varphi}} \kappa(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}) - e^{-\varphi_*} \kappa(T_*, \varphi_*)] \cdot \nabla u_*, \nabla \cdot \tilde{u}),$$

$$\tilde{J}_{1,2}(\tilde{u}) = -([e^{-\varphi_* - \tilde{\varphi}} \kappa(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}) - e^{-\varphi_*} \kappa(T_*, \varphi_*)] \cdot (\nabla \cdot u_*)(\nabla \varphi_*), \tilde{u}),$$

$$\tilde{J}_{1,3}(\tilde{u}) = -(e^{-\varphi_* - \tilde{\varphi}} \kappa(T_* + \tilde{T}, \varphi_* + \tilde{\varphi})(\nabla \cdot \tilde{u})(\nabla \varphi_*), \tilde{u}),$$

$$\tilde{J}_{1,4}(\tilde{u}) = -(e^{-\varphi_* - \tilde{\varphi}} \kappa(T_* + \tilde{T}, \varphi_* + \tilde{\varphi})(\nabla \cdot \tilde{u})(\nabla \varphi), \tilde{u}),$$

$$\tilde{J}_{1,5}(\tilde{u}) = -(e^{-\varphi_* - \tilde{\varphi}} \kappa(T_* + \tilde{T}, \varphi_* + \tilde{\varphi})(\nabla \cdot u_*)(\nabla \tilde{\varphi}), \tilde{u}).$$

We have from Lemma 3, Lemma 4 and (1.5) that

$$|\tilde{J}_{1,1}(\tilde{u})| \leqslant \varepsilon |\tilde{u}|_1^2 + M(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2),$$

$$|\tilde{J}_{1,2}(\tilde{u})| \leqslant M(\|\tilde{u}\|^2 + \|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2),$$

$$|\tilde{J}_{1,3}(\tilde{u})| \leqslant \varepsilon |\tilde{u}|_1^2 + M\|\tilde{u}\|^2,$$

$$\begin{aligned} |\tilde{J}_{1,4}(\tilde{u})| &\leqslant M|\tilde{u}|_{0,\infty} |\tilde{u}|_1 |\tilde{\varphi}|_1 \\ &\leqslant \varepsilon |\tilde{u}|_1^2 + MD(h, N)\|\tilde{\varphi}\|^2 |\tilde{u}|_1^2 . \end{aligned}$$

If we assume $u \in [C(0, t; H_p^2(\Omega))]^n$, then we have by integeration by parts that

$$\tilde{J}_{1,5}(\tilde{u}) = \sum_{q=1}^{3} z_q(\tilde{u}),$$

where

$$Z_1(\tilde{u}) = -([e^{-\varphi_* - \tilde{\varphi}} \kappa(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}) - e^{-\varphi_*} \kappa(T_*, \varphi_*)] \times \\ \times (\nabla . u_*)(\nabla \tilde{\varphi}), \tilde{u}),$$

$$Z_2(\tilde{u}) = -(e^{-\varphi_*} \kappa(T_*, \varphi_*)[\nabla . (u_* - U)](\nabla \tilde{\varphi}), \tilde{u}),$$

$$Z_3(\tilde{u}) = -(e^{-\varphi_*} \kappa(T_*, \varphi_*)(\nabla . U)(\nabla \tilde{\varphi}), \tilde{u}) \\ = (\tilde{\varphi}, \nabla . [e^{-\varphi_*} \kappa(T_*, \varphi_*)(\nabla . U)\tilde{u}]).$$

It is easy to verify that

$$|Z_1(\tilde{u})| \leqslant \varepsilon |\tilde{u}|_1^2 + MD(h, N)(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)\|\tilde{\varphi}\|^2,$$

$$|Z_2(\tilde{u})| \leqslant M|u_* - U|_1^2 + MD(h, N)\|\tilde{\varphi}\|^2 |\tilde{u}|_1^2,$$

$$|Z_3(\tilde{u})| \leqslant \varepsilon |\tilde{u}|_1^2 + M(\|\tilde{u}\|^2 + \|\tilde{\varphi}\|^2).$$

By substitute the above estimates into (5.5), we obtain that

$$\left|\tilde{J}_1(\tilde{u}) - (e^{-\varphi_* - \tilde{\varphi}} \kappa(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}), (\nabla . \tilde{u})^2)\right| \leqslant \alpha(\tilde{u}, \tilde{T}, \tilde{\varphi}, h, N), \tag{5.6}$$

where

$$\alpha(\tilde{u}, \tilde{T}, \tilde{\varphi}, h, N) = \varepsilon |\tilde{u}|_1^2 + \\ + M(1 + D(h, N)\|\tilde{\varphi}\|^2)(\|\tilde{u}\|^2 + \|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2) \\ + MD(h, N)(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)|\tilde{u}|_1^2 + M|u_* - U|_1^2.$$

Similarly, we have

$$\left|\tilde{J}_2(\tilde{u}) - \left(e^{-\varphi_* - \tilde{\varphi}} \nu(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}), \sum_{\ell, j=1}^{m} (\partial_j \tilde{u}^{(\ell)})^2\right)\right| \leqslant \\ \leqslant \alpha(\tilde{u}, \tilde{T}, \tilde{\varphi}, h, N).$$

By Lemma 5, we have in the same way that

$$\left|\tilde{J}_3(\tilde{u}) - (e^{-\varphi_* - \tilde{\varphi}} \nu(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}), (\nabla . \tilde{u})^2)\right| \leqslant \alpha(\tilde{u}, \tilde{T}, \tilde{\varphi}, h, N).$$

Still by the analogy to the proof of (5.6), we can get that

$$\Big|\tilde{J}_4(\tilde{T}) - \Big(e^{-\varphi_* - \tilde{\varphi}}\, U(T_* + \tilde{T}, \varphi_* + \tilde{\varphi})(T_* + \tilde{T})^{-1} \times$$

$$\times S_T^{-1}(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}), \sum_{j=1}^{n} (\partial_j \tilde{T})^2\Big)\Big|$$

$$\leqslant \varepsilon \big|\tilde{T}\big|_1^2 + M(1 + D(h, N)\|\tilde{\varphi}\|^2)(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)$$

$$+ MD(h, N)\|\tilde{\varphi}\|^2 \,\big|\tilde{T}\big|_1^2 + M\,|T_* - T|_1^2 .$$

Now we estimate $\tilde{J}_m(2\,\tau\tilde{u}_t)$, $1 \leqslant m \leqslant 3$, and $\tilde{J}_4(2\,\tau\tilde{T}_t)$. Let $\lambda = \tau(C_0 h^{-2} + N^2)$, and assume that $\lambda \leqslant \lambda^* = \text{Cst}$. Then we have from (i) of Lemma 4 that

$$\tau\,|\eta_t|_1^2 \leqslant \lambda\,\|\eta_t\|^2, \quad \eta = \tilde{u}, \tilde{T}, \tilde{\varphi} .$$

Since

$$\|\nabla \cdot \tilde{u}\|^2 \leqslant n\,|\tilde{u}|_1^2 ,$$

we have from (1.5) that

$$\Big|\big(e^{-\varphi_* - \tilde{\varphi}}\,\kappa(T_* + \tilde{T}, \varphi_* + \tilde{\varphi})\,\nabla \cdot \tilde{u}, 2\,\tau\,\nabla \cdot \tilde{u}_t\big)\Big| \leqslant$$

$$\leqslant 2\,\tau \Phi_1\,\kappa_1 \|\nabla \cdot \tilde{u}\|\,\|\nabla \cdot \tilde{u}_t\| \leqslant \frac{\tau}{8}\,\|\tilde{u}_t\|^2 + 8\,\lambda n^2\,\Phi_1^2 K_1^2 |\tilde{u}|_1^2$$

and we have furthermore that

$$\big|\tilde{J}_1(2\,\tau\tilde{u}_t)\big| \leqslant \frac{\tau}{8}\,\|\tilde{u}_t\|^2 + 8\,\lambda n^2\,\Phi_1^2\,\kappa_1^2 |\tilde{u}|_1^2 + \beta(\tilde{u}, \tilde{T}, \tilde{\varphi}, h, N),$$

where

$$\beta(\tilde{u}, \tilde{T}, \tilde{\varphi}, h, N) = \varepsilon\tau\,\|\tilde{u}_t\|^2 +$$

$$+ M(\|\tilde{u}\|^2 + \|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2) + MD(h, N)\|\tilde{\varphi}\|^2\,|\tilde{u}|_1^2 .$$

Analogously, we have

$$\big|\tilde{J}_2(2\,\tau\tilde{u}_t)\big| \leqslant \frac{\tau}{8}\,\|\tilde{u}_t\|^2 + 8\,\lambda\,\Phi_1^2\,\nu_1^2 |\tilde{u}|_1^2 + \beta(\tilde{u}, \tilde{T}, \tilde{\varphi}, h, N),$$

$$\big|\tilde{J}_3(2\,\tau\tilde{u}_t)\big| \leqslant \frac{\tau}{8}\,\|\tilde{u}_t\|^2 + 8\,\lambda n^2\,\Phi_1^2\,\nu_1^2 |\tilde{u}|_1^2 + \beta(\tilde{u}, \tilde{T}, \tilde{\varphi}, h, N),$$

$$\big|\tilde{J}_4(2\,\tau\tilde{T}_t)\big| \leqslant \frac{\tau}{2}\,\|\tilde{T}_t\|^2 + 2\,\lambda\,\Phi_1^2\,\mu_1^2 B_0^{-2} S_0^{-2} |\tilde{T}|_1^2 +$$

$$+ \varepsilon\tau\,\|\tilde{T}_t\|^2 + M(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2) + MD(h, N)\|\tilde{\varphi}\|^2\,|\tilde{T}|_1^2 ,$$

Moreover, it is not difficult to check that

$$\left|2\,\tilde{J}_5(\tilde{T}+\tau\tilde{T}_t)\right| \leqslant \varepsilon\tau\left\|\tilde{T}_t\right\|^2+\varepsilon\left|\tilde{T}\right|_1^2+M(\|\tilde{u}\|^2+\left\|\tilde{T}\right\|^2+\|\tilde{\varphi}\|^2)\,.$$

Finally, analogously to the estimates of $\tilde{J}_m$, we can bound $\tilde{H}_m(\tilde{u}+\tau\tilde{u}_t)$, $1\leqslant m\leqslant 3$, and $\tilde{H}_m(\tilde{T}+\tau\tilde{T}_t)$, $4\leqslant m\leqslant 5$, as follows

$$2\sum_{m=1}^{3}\left|\tilde{H}_m(\tilde{u}+\tau\tilde{u}_t)\right|+2\sum_{m=4}^{5}\left|\tilde{H}_m(\tilde{T}+\tau\tilde{T}_t)\right|\leqslant$$

$$\leqslant \varepsilon\tau(\left\|\tilde{u}_t\right\|^2+\left\|\tilde{T}_t\right\|^2)+\varepsilon|\tilde{u}|_1^2+\varepsilon\left|\tilde{T}\right|_1^2+M(\|\tilde{u}\|^2+\left\|\tilde{T}\right\|^2+\|\tilde{\varphi}\|^2)+$$
$$M(\|u_*-U\|_1^2+\|T_*-T\|_1^2+\|\varphi_*-\varphi\|_1^2)\,.$$

Define

$$\tilde{A}(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi})=\min\,(n\kappa(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi})+$$
$$+(n+1)\,\nu(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi}),\ \nu(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi}))$$

$$\tilde{F}(\mathscr{D})=(e^{-\varphi_*-\tilde{\varphi}}[\kappa(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi})+\nu(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi})],\ (\nabla\,.\,\tilde{u})^2)_{\mathscr{D}}$$
$$+\left(e^{-\varphi_*-\tilde{\varphi}}\,\nu(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi}),\ \sum_{\ell,j=1}^{n}(\partial_j\tilde{u}^{(\ell)})^2\right)_{\mathscr{D}},$$

$$\Omega^+=\left\{x\in\Omega/\kappa(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi})+\nu(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi})>0\right\},$$
$$\Omega^-=\Omega\backslash\Omega^+\,.$$

We have by (5.4) and (1.5) that $\tilde{A}(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi})>0$. On the other hand, it is obvious that

$$\tilde{F}(\Omega^+)\geqslant\left(e^{-\varphi_*-\tilde{\varphi}}\tilde{A}(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi}),\ \sum_{\ell,j=1}^{n}(\partial_j\tilde{u}^{(\ell)})^2\right)_{\Omega^+}. \tag{5.7}$$

By the following inequality

$$(\nabla\,.\,\tilde{u})^2\leqslant n\sum_{j=1}^{n}(\partial_j\tilde{u}^{(j)})^2\,,$$

we can easily check that (5.7) is also valid for Ω^-. Thus we get

$$(e^{-\varphi_*-\tilde{\varphi}}[\kappa(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi})+\nu(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi})],\ (\nabla\,.\,\tilde{u})^2)+$$
$$+\left(e^{-\varphi_*-\tilde{\varphi}}\,\nu,\ (T_*+\tilde{T},\,\varphi_*+\tilde{\varphi}),\ \sum_{i,j=1}^{n}(\partial_j\tilde{u}^{(\ell)})^2\right)$$
$$=\tilde{F}(\Omega^+)+\tilde{F}(\Omega^-)$$
$$\geqslant\left(e^{-\varphi_*-\tilde{\varphi}}\tilde{A}(T_*+\tilde{T},\,\varphi_*+\tilde{\varphi}),\ \sum_{i,j=1}^{n}(\partial_j\tilde{u}^{(\ell)})^2\right).$$

By combining the above estimates for F_m, J_m and H_m, we derive from (5.3) that

$$(\|\tilde{u}\|^2 + \|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)_t + \tau(5/8 - 3\varepsilon)(\|\tilde{u}_t\|^2 + \|\tilde{T}_t\|^2 + \|\tilde{\varphi}_t\|^2) +$$
$$+ (2 e^{-\varphi_* - \tilde{\varphi}} \tilde{A}(T_* + \tilde{T}, \varphi_* + \tilde{\varphi})$$
$$- 12\varepsilon - MD(h, N)(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2), \sum_{i,j=1}^{n} (\partial_j \tilde{u}^{(\ell)})^2)$$
$$+ (2 e^{-\varphi_* - \tilde{\varphi}}(T_* + \tilde{T})^{-1} S_T^{-1}(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}) \tag{5.8}$$
$$\times \mu(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}) - 6\varepsilon - MD(h, N)\|\tilde{\varphi}\|^2, \sum_{j=1}^{n} (\partial_j \tilde{T})^2)$$
$$- 8\lambda \Phi_1^2(n^2 \kappa_1^2 + (n^2 + 1)\nu_1^2)$$
$$\times |\tilde{u}|_1^2 - 2\lambda \Phi_1^2 \mu_1^2 B_0^{-2} S_0^{-2} |\tilde{T}|_1^2 \leqslant \tilde{R}(\tilde{u}, \tilde{T}, \tilde{\varphi}) + \tilde{Z},$$

where

$$\tilde{R}(\tilde{u}, \tilde{T}, \tilde{\varphi}) = M[1 + D(h, N)(\|\tilde{u}\|^2 + \|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)] \times$$
$$\times (\|\tilde{u}\|^2 + \|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2)],$$
$$\tilde{Z} = M(\|u_* - U\|_1^2 + \|T_* - T\|_1^2 + \|\varphi_* - \varphi\|_1^2) + \left(1 + \frac{\tau}{\varepsilon}\right) \sum_{m=1}^{3} \|\tilde{f}_m\|^2 .$$

Suppose a be a non-negative number, and

$$\lambda < \min\left(\frac{A_0 \Phi_0}{8 \Phi_1^2[n^2 \kappa_1^2 + (n^2 + 1)\nu_1^2]}, \frac{\mu_0 \Phi_0 B_0^2 S_0^2}{2 \mu_1^2 \Phi_1^2 B_1 S_1}\right). \tag{5.9}$$

Define

$$\tilde{E}_a(\tilde{\eta}, t) = \|\tilde{\eta}(t)\|^2 + \tau \sum_{t'=0}^{t-\tau} \left(a, \sum_{j=1}^{n} (\partial_j \tilde{\eta}(t'))^2\right) + \frac{\tau^2}{2} \sum_{t'=0}^{t-\tau} \|\tilde{\eta}_t(t')\|^2 ,$$
$$\tilde{G}(t) = \tilde{E}_{\Phi_0 A_0/2}(\tilde{u}, t) + \tilde{E}_{\Phi_0 A_0 B_1^{-1} S_1^{-1}/2}(\tilde{T}, t) + \tilde{E}_0(\tilde{\varphi}, t) .$$

By summerizing (5.8) for all $t \leqslant t - \tau$, and $t \in \Theta_\tau$, we obtain that

$$\tilde{G}(t) \leqslant \rho(t) + \tau \sum_{t'=0}^{t-\tau} \left\{\tilde{R}(\tilde{u}(t'), \tilde{T}(t'), \tilde{\varphi}(t')) + \right.$$
$$\left. + \sum_{m=1}^{2} \xi_m(\tilde{u}(t'), \tilde{T}(t'), \tilde{\varphi}(t'))\right\},$$

where

$$\rho(t) = \tau \sum_{t'=0}^{t-\tau} \tilde{Z}(t'),$$

$$\xi_1(\tilde{u}, \tilde{T}, \tilde{\varphi}) = -(2 e^{-\varphi_* - \tilde{\varphi}} \tilde{A}(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}) - \frac{1}{2} \Phi_0 A_0 -$$

$$- 8 \lambda \Phi_1^2 (n^2 \kappa_1^2 + (n^2 + 1) \nu_1^2) - 12 \varepsilon$$

$$- MD(h, N)(\|\tilde{T}\|^2 + \|\tilde{\varphi}\|^2), \sum_{\ell, j = 1}^{n} (\partial_j \tilde{u}^{(\ell)})^2),$$

$$\xi_2(\tilde{u}, \tilde{T}, \tilde{\varphi}) = -(2 e^{-\varphi_* - \tilde{\varphi}} (T_* + \tilde{T})^{-1} S_T^{-1} (T_* + \tilde{T}, \varphi_* + \tilde{\varphi}) \times$$

$$\times \mu(T_* + \tilde{T}, \varphi_* + \tilde{\varphi}) - \frac{1}{2} \Phi_0 \mu_0 B_1^{-1} S_1^{-1}$$

$$- 2 \lambda \Phi_1^2 \mu_1^2 B_0^{-2} S_0^{-2} - 6 \varepsilon - MD(h, N) \|\tilde{\varphi}\|^2, \sum_{j=1}^{n} (\partial_j \tilde{T})^2).$$

By applying Lemma 6, with $a_0(h, N) = C_0 h^{-2} + N^2$, $a_1(h, N) = a_2(h, N) = D(h, N)$, we conclude that if (5.4) holds, and there exists $t_1 \in \Theta_\tau$, such that

$$\rho(t_1) \leqslant M_1 \min(1/D(h, N), 1/(C_0 h^{-2} + N^2)), \tag{5.10}$$

where $M_1 > 0$ is a suitably small constant, independing on h, N and τ, then there exist positive constants M_2 and M_3, such that for all $t \leqslant t_1$ and $t \in \Theta_\tau$,

$$\tilde{G}(t) \leqslant M_2 \rho(t) e^{M_3 t}. \tag{5.11}$$

Since (5.10) implies (5.4), so we need only to estimate $\rho(t)$ and to examine condition (5.10), to obtain the convergence rate.

Analogously to the estimates for $|F_m|$, $1 \leqslant m \leqslant 6$, we get that

$$\sum_{m=1}^{3} \|\tilde{f}_m\|^2 \leqslant \|u_{*t} - \partial_t U\|^2 + \|T_{*t} - \partial_t T\|^2 + \|\varphi_{*t} - \partial_t \varphi\|^2$$

$$+ M(\|u_* - U\|_1^2 + \|T_* - T\|_1^2 + \|\varphi_* - \varphi\|_1^2).$$

By Taylor's formula of expansion and Lemma 1, we have

$$\|\eta_{*t} - \partial_t \eta\| \leqslant \|\eta_{*t} - \eta_t\| + \|\eta_t - \partial_t \eta\|$$

$$\leqslant \frac{1}{\tau} \left\| \int_t^{t+\tau} \left[\mathscr{F}_\delta\left(\frac{\partial \eta}{\partial t}(t') \right) - \frac{\partial \eta}{\partial t}(t') \right] dt' \right\| +$$

$$+ \frac{1}{\tau} \left\| \int_t^{t+\tau} (t + \tau - t') \frac{\partial^2 \eta}{\partial t^2}(t')\, dt' \right\|$$

$$\leq \frac{1}{\tau}\int_t^{t+\tau}\left\|\mathscr{F}_\delta\left(\frac{\partial \eta}{\partial t}(t')\right)-\frac{\partial \eta}{\partial t}(t')\right\| dt' + \int_t^{t+\tau}\left\|\frac{\partial^2 \eta}{\partial t^2}(t')\right\| dt'$$

$$\leq M\tau^{-1/2}(h^{\bar{\alpha}-1}+N^{1-\beta})\left[\int_t^{t+\tau}\left\|\frac{\partial \eta}{\partial t}(t')\right\|^2_{H^{\alpha-1,\beta-1}(\Omega)} dt'\right]^{1/2} +$$

$$+ M\tau^{1/2}\left[\int_t^{t+\tau}\left\|\frac{\partial^2 \eta}{\partial t^2}(t')\right\|^2 dt'\right]^{1/2}$$

where $\alpha > \frac{n+1}{2}$, $\beta \geq 1$, $\bar{\alpha} = \min(\alpha, k+1)$, $\eta = U, T, \varphi$. Moreover, we have from Lemma 2 that

$$\|\eta_* - \eta\|_1 \leq M(h^{\bar{\alpha}-1} + N^{1-\beta})\|\eta\|_{Y^{\bar{\alpha},\beta}(\Omega)}, \quad \eta = U, T, \varphi .$$

Hence,

$$\rho(t) \leq M(t)(\tau^2 + h^{2(\bar{\alpha}-1)} + N^{2(1-\beta)}),$$

where $M(t) > 0$ is depending only on $\|\eta\|_{C(0,t;Y^{\bar{\alpha},\beta}(\Omega)\cap X^{\alpha',\beta'}(\Omega))}$, $\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(0,t;H^{\alpha-1,\beta-1}(\Omega))}$, $\left\|\frac{\partial^2 \eta}{\partial t^2}\right\|_{L^2(0,t;L^2(\Omega))}$, $\eta = U, T, \varphi$, $\|U\|_{C(0,t;H^2(\Omega))}$, $\|T\|_{C(0,t;H^2(\Omega))}$, R, ε, etc.

We assume that $h = N^{-a}$, $a \geq 1$, and that the following condition holds,

$$\begin{cases} \bar{\alpha} > 2 - \frac{1}{a} - \frac{n}{2a}, \\ \beta > a + \frac{n}{2}, \end{cases} \tag{5.12}$$

then we have $\rho(t) = o(1/D(h, N))$. Thus we have (5.10) valid for $t = t_0$, provided h^{-1} and N are sufficiently large.

By all the above arguments, together with the triangle inequalities,

$$\|\eta_\delta - \eta\| \leq \|\eta_* - \eta\| + \|\tilde{\eta}\|, \quad \eta = U, T, \varphi,$$

we obtain the following theorem :

THEOREM 1 : *Assume the solution* (U, T, φ) *of* (3.1) *satisfies that*

$$U \in C(0, t_0; [Y^{\alpha,\beta}_{0,p}(\Omega) \cap X^{\alpha',\beta'}_p(\Omega) \cap H^2(\Omega)]^n)$$
$$\cap H^1(0, t_0; [H^{\alpha-1,\beta-1}_p(\Omega)]^n)$$
$$\cap H^2(0, t_0; [L^2(\Omega)]^n),$$

$$T \in C\left(0, t_0 ; Y_{0,p}^{\alpha,\beta}(\Omega) \cap X_p^{\alpha',\beta'}(\Omega) \cap H^2(\Omega)\right)$$
$$\cap H^1(0, t_0 ; H_p^{\alpha-1,\beta-1}(\Omega))$$
$$\cap H^2(0, t_0 ; L^2(\Omega)),$$
$$\varphi \in C\left(0, t_0 ; Y_p^{\alpha,\beta}(\Omega) \cap X_p^{\alpha',\beta'}(\Omega)\right) \cap H^1(0, t_0 ; H_p^{\alpha-1,\beta-1}(\Omega))$$
$$\cap H^2(0, t_0 ; L^2(\Omega)),$$

with $\alpha > \frac{n+1}{2}$, $\beta \geq 1$, $\alpha' > \frac{n-1}{2}$, $\beta' > \frac{1}{2}$, $(u_\delta, T_\delta, \varphi_\delta)$ *is the solution of scheme* (3.2)-(3.3), $\bar{\alpha} = \min(\alpha, k+1)$. *Suppose the following two conditions are fulfilled*

(i) $B_0 < T < B_1$, $|\varphi| < B_2$, *and* (1.5) *holds :*

(ii) $h = N^{-a}$, *with* $a \geq 1$, $\lambda = \tau(C_0 h^{-2} + N^2)$, *and* (5.9), (5.12) *hold ; then there exist two positive constants* M_4 *and* M_5, *independent of* h, N *and* τ, *such that when* h, N^{-1}, *and* τ *are small enough, we have for all* $t \in \Theta_\tau$ *that*

$$\|u_\delta(t) - U(t)\|^2 + \|T_\delta(t) - T(t)\|^2 + \|\varphi_\delta(t) - \varphi(t)\|^2 \leq$$
$$\leq M_4 e^{M_5 t}(\tau^2 + h^{2(\bar{\alpha}-1)} + N^{2(1-\beta)}).$$

Remark 5.1 : Generally, the solution of (3.1) possesses a good regularly in the periodic direction, then the Fourier spectral method adopted in this direction usually has a higher ability of resolution. Thus the step size N^{-1} in x_n direction can be set larger than that in the other directions, to save computations. Hence the assumption $a \geq 1$ is reasonable. Numerical results available also give positive evidences to this observation (*cf.* e.g. [8, 11]).

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