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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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ON A CLASS OF NONLOCAL NONLINEAR ELLIPTIC PROBLEMS (*)

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Abstract — *In this paper we give a direct approach to the solvability of a class of nonlocal problems which admit a formulation in terms of quasi-variational inequalities We are motivated by nonlinear elliptic boundary value problems in which certain coefficients depend, in a rather general way, on the solution itself through global quantities like the total mass, the total flux or the total energy We illustrate the existence results with several applications, including an implicit Signorini problem for steady diffusion of biological populations and a class of operator equations in nonlinear mechanics We also discuss the non-uniqueness of the solutions*

Résumé — *Ce papier présente une approche directe de résolution de problèmes non locaux de type inéquations quasi variationnelles Notre motivation est la résolution de problèmes elliptiques non linéaires dont les coefficients dépendent de la solution au travers de quantités telles que la masse, le flux ou l'énergie totale Nous illustrons nos résultats d'existence par différentes applications dont un problème de Signorini implicite relatif à la diffusion de population en biologie et une classe de problèmes en mécanique non linéaire Nous nous intéressons également aux questions d'unicité*

1. A MODEL PROBLEM

Let Ω be a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary Γ. In the physical situation we have in mind Ω is, for instance, a pervious container of bacterias. Let u be the density of bacterias within this container and let f denote the supply of beings by external sources. Let us assume that the velocity of dispersion of this population is given by

$$\mathbf{v} = -a\left(\int_\Omega u\right) \nabla u , \tag{1.1}$$

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i.e., is proportional to the gradient of the density with a positive factor a depending on the entire population. Moreover, assume that death in this species is proportional to u by a factor $\lambda > 0$. A balance of population in space gives rise to the equation (see [GM]) :

$$- \operatorname{div} \left[a \left(\int_\Omega u \right) \nabla u \right] = f - \lambda u \quad \text{in } \Omega . \tag{1.2}$$

Moreover we assume that sample of bacterias is driven out of Ω by the total population through the factor a and by a dominant group living in some subdomain $\Omega' \subset \Omega$ through a factor γ. If we express this in terms of flux we are lead, for instance, to an equation of the type :

$$\mathbf{v} \cdot \mathbf{n} = a \left(\int_\Omega u \right) \gamma \left(\int_{\Omega'} u \right) \quad \text{on } \Gamma , \tag{1.3}$$

where $\mathbf{n}$ denotes the outward normal to Γ. Then, our problem becomes to find u such that

$$\begin{aligned} - a \left(\int_\Omega u \right) \Delta u + \lambda u = f & \quad \text{in } \Omega \\ \partial_n u + \gamma \left(\int_{\Omega'} u \right) = 0 & \quad \text{on } \Gamma . \end{aligned} \tag{1.4}$$

For convenience of notations we have dropped the Lebesgue measures in the integrals and denoted by $\partial_n u = \nabla u \cdot \mathbf{n}$ the normal derivative.

Our aim in this note is to show that this kind of problems is much easier to handle than the usual quasilinear equations. The difficulty in the local quasilinear case is due to the necessity of using a fixed point theorem in an infinite dimensional space like, for instance, the Schauder fixed point theorem (see [LL], [L], for instance). On the contrary, nonlocal problems such as (1.4) can be easily solved using Brouwer fixed point theorem.

To see that (1.4) requires, essentially, no more that so solve a nonlinear equation in a finite dimensional space let us consider it in the particular case when $\Omega' = \Omega$. More precisely, we assume that the factor γ also depends on the total population. Integrating the first equation over Ω we get, using Green's formula :

$$\int_\Omega f = - a \left(\int_\Omega u \right) \int_\Omega \Delta u + \lambda \int_\Omega u = - a \left(\int_\Omega u \right) \int_\Gamma \partial_n u + \lambda \int_\Omega u .$$

So, by the second equation of (1.4), with $\Omega' = \Omega$, if $|\Gamma|$ denotes the measure of Γ, we obtain

$$\int_\Omega f = |\Gamma| \, a \left(\int_\Omega u \right) \gamma \left(\int_\Omega u \right) + \lambda \int_\Omega u$$

and thus the total population

$$\sigma = \int_\Omega u$$

satisfies the equation

$$|\Gamma| \, a(\sigma) \, \gamma(\sigma) + \lambda \sigma = \int_\Omega f \, . \tag{1.5}$$

Conversely, let σ be a real solution of (1.5), and denote by $u = u(\sigma)$ the solution of

$$\begin{aligned} - a(\sigma) \, \Delta u + \lambda u = f \quad & \text{in } \Omega \\ \partial_n u + \gamma(\sigma) = 0 \quad & \text{on } \Gamma \end{aligned} \tag{1.6}$$

(we assume here, for instance, $a > 0$ and $f \in L^2(\Omega)$, so that the solution to this Neumann problem exists in a weak sense, as it is well-known). An easy computation yields $\sigma = \int_\Omega u$, so that u is a solution to (1.4), with $\Omega' = \Omega$. In this case this shows that (1.4) reduces to (1.5), which is an equation in $\mathbb{R}$! We can summarize our analysis in the following :

PROPOSITION 1 : *Let $a > 0$, $\gamma \geq 0$ two continuous functions from $\mathbb{R}$ into $\mathbb{R}^+$ and let $f \in L^2(\Omega)$. Then $u \in H^1(\Omega)$ is a weak solution to* (1.4), *with $\Omega' = \Omega$, iff u is a weak solution to* (1.6) *and σ satisfies* (1.5). *In particular, if*

$$\operatorname*{Inf}_{\tau \in \mathbb{R}} \{ |\Gamma| \, a(\tau) \, \gamma(\tau) + \lambda \tau \} < \int_\Omega f \tag{1.7}$$

then (1.4), *with $\Omega' = \Omega$, has at least one solution.*

Proof : The first part of the proposition is clear from our previous analysis. It remains to show the existence of a solution. The function $\phi(\tau) = |\Gamma| \, a(\tau) \, \gamma(\tau) + \lambda \tau$ tends to $+\infty$ when $\tau \to +\infty$. Thus by the assumption (1.7) and the intermediate values theorem it is immediate that the equation (1.5) has at least one root σ and the result follows.

Remark 1 : It should be noted that the existence of a solution depends on the data and, in particular on f (see (1.7)). If one assumes $0 < \alpha \leq a(\sigma) \leq \beta \;\forall \sigma$, and γ uniformly bounded, then it is clear that (1.7) is satisfied independently of f and then existence always holds. On the other hand, if γ grows too fast at $-\infty$, it is clear that (1.5), and thus also (1.4), may fail to have any solution.

Remark 2 : The physical situation we have considered requires $f \geqslant 0$. Then a natural assumption is, of course, $\gamma(0) = 0$ (no population implies no flux...) ; then one has

$$|\Gamma| \, a(0) \, \gamma(0) + \lambda 0 = 0 \leqslant \int_{\Omega} f \leqslant |\Gamma| \, a(\sigma) \, \gamma(\sigma) + \lambda \sigma ,$$

for σ large enough and existence holds in this case.

Remark 3 : It is clear that our data can be selected such that

$$|\Gamma| \, a(\sigma) \, \gamma(\sigma) + \lambda \sigma = \int_{\Omega} f ,$$

for σ in some interval I. This leads to an uncountable number of solutions for (1.4). We shall return to this question in Section 4.

The paper is organized as follows. First, in the next section, we show that a large class of problems, including (1.4) as an example, can be solved in a general abstract framework by direct methods of nonlinear analysis. Then in Section 3 we give some applications in nonlinear mechanics, including an obstacle problem for the extensible plate. This section can be dropped in a first reading. In Section 4 we come back to our original problem. More precisely, we solve an implicit Signorini problem for steady diffusion of populations extending (1.4). Finally we shall comment on the necessity of our assumptions and on non-uniqueness results.

2. ABSTRACT EXISTENCE RESULTS

Let V be a reflexive Banach space with dual V^*. For $\sigma \in \mathbb{R}^m$ consider $A(\sigma)$ a family of operators from V into V^*, $f(\sigma)$ a family of elements in V^* and $\mathbf{K}(\sigma)$ a family of nonempty closed convex sets in V. Denote by $\langle \, , \, \rangle$ the duality bracket between V^* and V, $\| \; \|$, $\| \; \|_*$ the norms in V and V^*, respectively. If μ is a mapping from V into $\mathbb{R}^m$, we consider quasi-variational inequalities of the type (see [L], [BC], [BL], [KS], [C], [R] for notation and definitions) :

$$u \in \mathbf{K}(\mu(u)) : \langle A(\mu(u)) \, u, v - u \rangle \geqslant \langle f(\mu(u)), v - u \rangle , \quad \forall v \in \mathbf{K}(\mu(u)) . \tag{2.1}$$

Although this type of implicit problems may be set in a more general framework for which abstract existence results are available (see [M], [BC], for instance), our aim in this section is to study a special class of data for which a direct and eventually simpler approach can be developped, based only on the Brouwer fixed point theorem.

On the operator A, we assume that, for each $\sigma \in \mathbb{R}^m$:

$$A(\sigma) \text{ is a strictly monotone, bounded, hemicontinuous operator from } V \text{ into } V^* \quad (2.2)$$

We recall that by strict monotonicity we mean :

$$\langle A(\sigma) v - A(\sigma) w, v - w \rangle > 0, \quad \forall v, w \in V, \quad v \neq w.$$

Moreover, we assume the following continuity properties on the mappings $\sigma \to A(\sigma)$, $\sigma \to \mathbf{K}(\sigma)$: for each $\sigma \in \mathbb{R}^m$ and for sequences $\sigma' \to \sigma$,

$$\forall v \in \mathbf{K}(\sigma), \quad \exists v_{\sigma'} \in \mathbf{K}(\sigma') : \quad v_{\sigma'} \to v \text{ in } V \text{ and } A(\sigma') v_{\sigma'} \to A(\sigma) v \text{ in } V^* \quad (2.3)$$

$$\forall \text{ subsequence } \sigma' \to \sigma, \quad v_{\sigma'} \in \mathbf{K}(\sigma') : \quad v_{\sigma'} \to v \text{ in } V\text{-weak} \Rightarrow v \in \mathbf{K}(\sigma). \quad (2.4)$$

The two above conditions on $\sigma \to \mathbf{K}(\sigma)$ define the well-known Mosco's convergence for convex sets. On f and on μ we assume, respectively,

$$f \text{ is a continuous mapping from } \mathbb{R}^m \text{ into } V^*$$

and

$$\exists C_f > 0 : \quad \|f(\sigma)\|_* \leq C_f, \quad \forall \sigma \in \mathbb{R}^m. \quad (2.5)$$

$$\mu \text{ is a continuous mapping from } V\text{-weak into } \mathbb{R}^m. \quad (2.6)$$

In the special case of our model problem (1.4) we have $m = 2$ and $\mu(u) = \left(\int_\Omega u, \int_{\Omega'} u \right)$. We shall use also the following uniform coerciveness assumption :

$$\exists v_0 \in \bigcap_{\sigma \in \mathbb{R}^m} \mathbf{K}(\sigma) \text{ such that } \forall M > 0, \quad \exists R(M) > 0 :$$

$$\|v - v_0\| > R(M) \Rightarrow \langle A(\sigma) v, v - v_0 \rangle > M \|v - v_0\|, \quad \forall \sigma \in \mathbb{R}^m. \quad (2.7)$$

Under the above assumptions we can prove the following result :

THEOREM 1 : *Assume that* (2.2)-(2.6) *hold. Then there exists at least one solution to* (2.1) *if one of the following conditions is satisfied :* i) (2.7) *holds ; or* ii) $\bigcup_{\sigma \in \mathbb{R}^m} \mathbf{K}(\sigma)$ *is a bounded set of* V.

Proof : For each $\sigma \in \mathbb{R}^m$, by the assumptions made, we know there exists a unique solution u_σ of the following variational inequality (see [L]) :

$$u_\sigma \in \mathbf{K}(\sigma) : \quad \langle A(\sigma) u_\sigma, v - u_\sigma \rangle \geq \langle f(\sigma), v - u_\sigma \rangle \quad \forall v \in \mathbf{K}(\sigma). \quad (2.8)$$

Step 1 : u_σ is bounded independently of σ.

Indeed, in case ii) there is nothing to prove. In case i) we take $v = v_0$ in (2.8) and we find

$$\langle A(\sigma) u_\sigma, u_\sigma - v_0 \rangle \leqslant \langle f(\sigma), u_\sigma - v_0 \rangle \leqslant \| f(\sigma) \|_* \, \| u_\sigma - v_0 \| \leqslant C_f \, \| u_\sigma - v_0 \| ,$$

and from (2.5) and (2.7) we conclude $\| u_\sigma - v_0 \| \leqslant R(C_f)$.

Step 2 : The map $\sigma \to u_\sigma$ is continuous from $\mathbb{R}^m$ into V-weak.

We have to show that $\sigma' \to \sigma \Rightarrow u_{\sigma'} \to u_\sigma$ in V-weak and, due to step 1 and the weak compactness of the closed balls in V, it is enough to show that u_σ is the only limit point of $u_{\sigma'}$. For that, let us assume for some subsequence, still denoted by σ', we have $u_{\sigma'} \to u$ in V-weak.

From (2.8) and the monotonicity of $A(\sigma')$ one has :

$$\langle A(\sigma) v_{\sigma'}, v_{\sigma'} - u_{\sigma'} \rangle \geqslant \langle f(\sigma'), v_{\sigma'} - u_{\sigma'} \rangle , \quad \forall v_{\sigma'} \in \mathbf{K}(\sigma') ; \quad (2.9)$$

owing to (2.4) and to (2.3), we can let $\sigma' \to \sigma$ in (2.9) and we obtain

$$\langle A(\sigma) v, v - u \rangle \geqslant \langle f(\sigma), v - u \rangle , \quad \forall v \in \mathbf{K}(\sigma) ;$$

since $u \in \mathbf{K}(\sigma)$, by (2.4), and using Minty's lemma and the uniqueness of a solution to (2.8) we conclude that $u = u_\sigma$, proving the required continuity property.

Step 3 : The map $\sigma \to \mu(u_\sigma)$ has a fixed point.

First due to step 2 and to (2.6), this map is continuous from $\mathbb{R}^m$ into itself. Moreover, $\mathbb{R}^m$ is carried out by $\sigma \to u_\sigma$ into some compact set of V-weak, which is carried out into some compact set of $\mathbb{R}^m$, i.e. into some closed ball $B_R \subset \mathbb{R}^m$. Then, clearly, $\sigma \to \mu(u_\sigma)$ is a continuous map from B_R into itself. Its results from the Brouwer fixed point theorem that there exists a solution σ^* to $\sigma = \mu(u_\sigma)$, to which the corresponding solution $u = u_{\sigma*}$ of (2.8) solves the quasi-variational inequality (2.1).

In the case of strong ellipticity it is possible to relax slightly the assumption (2.6) to

$$\mu \text{ is a continuous mapping from } V \text{ into } \mathbb{R}^m . \quad (2.10)$$

THEOREM 2 : *Assume, in addition to* (2.2)-(2.5) *and* (2.10), *that the following conditions hold*

$$\exists \rho > 1 , \quad \exists \nu > 0 : \quad \langle A(\sigma) v - A(\sigma) w, v - w \rangle \geqslant \nu \, \| v - w \|^\rho \quad \forall v, w \in V , \quad \forall \sigma \in \mathbb{R}^m . \quad (2.11)$$

$$\exists v_0 \in \bigcap_{\sigma \in \mathbb{R}^m} \mathbf{K}(\sigma) \quad \textit{such that} \quad \|A(\sigma) v_0\|_* \leq C_0 \quad \forall \sigma \in \mathbb{R}^m . \tag{2.12}$$

Then there exists at least one solution to (2.1).

Proof: For v_0 satisfying (2.12) we have

$$\begin{aligned} \langle A(\sigma) v, v - v_0 \rangle &= \\ &= \langle A(\sigma) v - A(\sigma) v_0, v - v_0 \rangle + \langle A(\sigma) v_0, v - v_0 \rangle \\ &\geq \nu \|v - v_0\|^p - \|A(\sigma) v_0\|_* \|v - v_0\| \geq \nu \|v - v_0\|^p - C_0 \|v - v_0\| , \end{aligned}$$

so that (2.7) holds. Then the arguments developed in the proof of the previous theorem apply here with the additional requirement that the convergence $u_{\sigma'} \to u_\sigma$, as $\sigma' \to \sigma$, in step 2, holds not only for the weak topology but also for the strong one. For that, we take in (2.8), for $\sigma = \sigma'$, $v_{\sigma'} \in \mathbf{K}(\sigma')$ such that, using (2.3), $v_{\sigma'} \to u_\sigma$ in V, and we find

$$\begin{aligned} \langle A(\sigma') v_{\sigma'}, v_{\sigma'} - u_{\sigma'} \rangle - \langle f(\sigma'), v_{\sigma'} - u_{\sigma'} \rangle &\geq \\ \langle A(\sigma') v_{\sigma'} - A(\sigma') u_{\sigma'}, v_{\sigma'} - u_{\sigma'} \rangle &\geq \nu \|v_{\sigma'} - u_{\sigma'}\|^p . \end{aligned}$$

Since $A(\sigma') v_{\sigma'} \to A(\sigma) u_\sigma$ in V^*, by (2.3), and $v_{\sigma'} - u_{\sigma'} \to 0$ in V-weak, the left hand term vanishes as $\sigma' \to \sigma$, and thus we get the strong convergence from

$$\|u_\sigma - u_{\sigma'}\| \leq \|u_\sigma - v_{\sigma'}\| + \|v_{\sigma'} - u_{\sigma'}\| \to 0 \quad \text{as} \quad \sigma' \to \sigma .$$

It is useful to consider the particular case where $\mathbf{K}(\sigma) \equiv \mathbf{K}$, $\forall \sigma \in \mathbb{R}^m$, i.e. $\mathbf{K}$ is a fixed nonempty closed convex set of V. We shall also consider the following sufficient condition for the continuity of $\sigma \to A(\sigma)$ in assumption (2.3) to be verified :

$$\forall v \in \mathbf{K} \text{ the map } \sigma \to A(\sigma) v \text{ is continuous from } \mathbb{R}^m \text{ into } V^* . \tag{2.13}$$

As special cases of Theorems 1 and 2, respectively, we can immediately establish sufficient conditions for the solvability of the following variational inequality.

$$u \in \mathbf{K} : \quad \langle A(\mu(u)) u, v - u \rangle \geq \langle f(\mu(u)), v - u \rangle \quad \forall v \in \mathbf{K} . \tag{2.14}$$

COROLLARY 1 : *Assume that* (2.2), (2.5), (2.6) *and* (2.13) *hold. Then if*

i) $\mathbf{K}$ *is a bounded nonempty closed convex set of* V *; or*

ii)

$$\exists v_0 \in \mathbf{K} : \|v - v_0\| \to +\infty \Rightarrow \langle A(\sigma) v, v - v_0 \rangle / \|v - v_0\| \to +\infty ,$$

uniformly in $\sigma \in \mathbb{R}^m$, *there exists at least one solution to the problem* (2.14).

In the case of strong coerciveness we also have :

COROLLARY 2 : *Assume that* (2.2), (2.5), (2.10), (2.11) *and* (2.13) *hold. Then if* $\mathbf{K}$ *is a closed convex set of* V *such that* $\exists v_0 \in \mathbf{K} : \|A(\sigma) v_0\|_* \leq C_0$, $\forall \sigma \in \mathbb{R}^m$, *there exists at least one solution to the problem* (2.14).

Remark 4 : We have made boundeness assumptions all along this section. Of course they can be relaxed in some cases at the expense of suitable growth conditions that we have not explored here.

Remark 5 : In (2.6) and in (2.10) we just need to define μ and to require the respective continuity properties on $\bigcup_{\sigma \in \mathbb{R}^m} \mathbf{K}(\sigma)$, which, of course, may be a proper convex subset of V.

3. APPLICATIONS TO BOUNDARY VALUE PROBLEMS

3.1. Let Ω be a bounded open set of $\mathbb{R}^n$. For $p > 1$ and $k \in N$. Let us denote by $W^{k,p}(\Omega)$ the usual Sobolev space of functions in $L^p(\Omega)$ whose derivatives D^α of all orders α, $0 \leq |\alpha| \leq k$, are in $L^p(\Omega)$, see [A]. We denote its norm by $\|v\|_{k,p} = \left(\sum_{|\alpha| \leq k} |D^\alpha v|_p^p \right)^{1/p}$ where $|\,.\,|_p$ is the usual L^p norm. If $\mathbf{K}$ is a nonempty closed convex set in $W^{k,p}(\Omega)$, let V be the closed subspace in $W^{k,p}(\Omega)$ spanned by $\mathbf{K} - \mathbf{K} = \{v - v' : v, v' \in \mathbf{K}\}$. We assume that $\mathbf{K} \subset V$.

If N denotes the number of derivations in x of order $0 \leq |\alpha| \leq k$, we assume

$$a_\alpha = a_\alpha(x, \sigma, \xi) : \Omega \times \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R} \quad \text{are Carathéodory functions}, \tag{3.1}$$

i.e., measurable in x for each (σ, ξ) and, for a.e. x, continuous in (σ, ξ), such that, for each α :

$$|a_\alpha(x, \sigma, \xi)| \leq C(\sigma)\, |\xi|^{p-1} + c(x)\,, \quad \forall \sigma \in \mathbb{R}^n\,, \tag{3.2}$$

for a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^n$, for some locally bounded function $C(\sigma) \geq 0$ and some function $c \in L^{p'}(\Omega)$, $p' = p/(p-1)$ is the conjugate exponent of p.

For u, $v \in W^{k,p}(\Omega)$ and $\sigma \in \mathbb{R}^n$, we define $A(\sigma)$ by

$$\langle A(\sigma) u, v \rangle = \sum_{|\alpha| \leq k} \int_\Omega a_\alpha(x, \sigma, Du)\, D^\alpha v\, dx\,, \tag{3.3}$$

where $Du = \{u, \nabla u, \ldots, D^\alpha u ; |\alpha| \leq k\}$. The assumption (3.2) implies that $a_\alpha(u, \sigma, Du) \in L^{p'}(\Omega)$ for each α, $|\alpha| \leq k$, and $A(\sigma)$ is well defined from V into V^*, dual of V endowed with the $W^{k,p}(\Omega)$-topology.

For a given map $\mu : \mathbf{K} \to \mathbb{R}^m$, we can consider the following form of (2.14):

$$u \in K : \sum_{|\alpha| \leq k} \int_{\Omega} a_\alpha(x, \mu(u), Du) D^\alpha(v-u) \geq 0 , \quad \forall u \in K \quad (3.4)$$

to which we can apply Corollary 1 and 2.

THEOREM 3 : *Assume that the coefficients a_α verify* (3.1), (3.2) *and lead to a strictly monotone operator $A(\sigma)$ through* (3.3). *Then, if one of the following conditions below holds, there exists at least one solution to* (3.4) :

i) $\exists v_0 \in \mathbf{K} : \langle A(\sigma) v, v - v_0 \rangle / \|v - v_0\|_{k,p} \to +\infty$, *uniformly in* $\sigma \in \mathbb{R}^m$ *when* $\|v - v_0\|_{k,p} \to +\infty$, *and* (2.6) *holds ; or* (3.5)

(ii) $\mathbf{K} \neq \varnothing$, (2.10) *holds and* $\exists q > 1$, $\exists \nu > 0$ *such that*

$$\langle A(\sigma) v - A(\sigma) w, v - w \rangle \geq \nu (\|v - w\|_{k,p})^q , \quad \forall v, w \in K , \quad \forall \sigma \in \mathbb{R}^m . \quad (3.6)$$

Proof : It is sufficient to observe that (3.1), (3.2) and the coerciveness assumptions imply that $A(\sigma)$ is an operator of the Leray-Lions type (see [LL] or [L], p. 182) verifying the condition (2.13).

Remark 6 : Let $\mathbf{K} = W_0^{k,p}(\Omega)$ and assume $\exists \eta > 0$:

$$\sum_{|\alpha| = k} a_\alpha(x, \sigma, \xi) \xi_\alpha \geq \eta \, |\xi|^p \quad \text{for} \quad |\xi| \text{ large} . \quad (3.7)$$

Then (3.5) holds provided η is sufficiently large, or *meas*(Ω) is sufficiently small, or the lower order terms are monotone, i.e., for a.e. $x \in \mathbb{R}$, $\forall \sigma \in \mathbb{R}^m$ and $\xi, \zeta \in \mathbb{R}^N$:

$$\sum_{|\beta| \leq k-1} [a_\beta(x, \sigma, \xi) - a_\beta(x, \sigma, \zeta)] [\xi_\beta - \zeta_\beta] \geq 0 . \quad (3.8)$$

In this case, Theorem 3 provides the solvability of the following Dirichlet problem

$$\sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha(a_\alpha(x, \mu(u), Du)) = 0 \quad \text{in} \quad \Omega \quad (3.9)$$

$$B_{k-1} u = 0 \quad \text{on} \quad \Gamma \quad (3.10)$$

where $B_{k-1} u = \{u, \partial_n u, \ldots, \partial_n^{k-1}\}$ and ∂_n denotes the normal derivative. Of course, the rigorous meaning of (3.10) requires some smoothness of Γ.

Remark 7 : As examples of $\mu(u) = (\mu_1[u], \ldots, \mu_m[u])$ verifying (2.6) we can choose several combinations of the following functionals

$$\mu_j[u] = \phi_j(\langle f_j, u \rangle), \quad j = 1, \ldots, m$$

for some $f_j \in V^*$ and for $\phi_j : \mathbb{R} \to \mathbb{R}$ any continuous function.

For instance, we may consider :

$$\langle f_{\Omega'}, u \rangle = \sum_{|\alpha| \leq k} \int_{\Omega'} f_\alpha D^\alpha u$$

where Ω' is any measurable subset of Ω and f_α are functions in $L^{p'}(\Omega)$;

$$\langle f_{\Gamma'}, u \rangle = \sum_{|\beta| \leq k-1} \int_{\Gamma'} g_\beta D^\beta u$$

where Γ' is any measurable subset of the sufficiently smooth boundary Γ of Ω, and $g_\beta \in L^q(\Gamma')$ where $q \geq 1$ is the conjugate exponent of the respective $L^b(\Gamma)$ where $D^\beta u$ lies by the trace and Sobolev imbeddings theorems ;

$$\langle f_0, u \rangle = \sum_{|\alpha| \leq \ell} c_\alpha D^\alpha u(x_0) \quad \text{for some fixed } x_0 \in \Omega \ ;$$

for some $c_\alpha \in \mathbb{R}$, provided $k > \ell + n/p$, by the Rellich-Kondratchov compact imbedding $W^{k,p} \subset C^\ell$.

Remark 8 : Of course, the condition (2.6) implies (2.10). In addition to the examples of the preceding remark, in order to give examples of μ satisfying (2.10) we can consider functions of the type

$$\mu[u] = \int_{\Omega'} \phi(Du) \quad \text{or} \quad \mu[u] = \int_{\Gamma'} \psi(Du),$$

provided ϕ and ψ are continuous functions such that, combining the compact imbeddings and trace theorems with the growth conditions of ϕ and ψ, we have $\phi(Du) \in L^1(\Omega')$ and $\psi(Du) \in L^1(\Gamma')$ for every $u \in W^{k,p}(\Omega)$. For instance, we can consider continuous functions of the « energy functional »

$$\sum_{|\alpha| = k} \int_\Omega |D^\alpha u|^p .$$

3.2. Now we develop two special examples arising in nonlinear mechanics. First let $k = 1$, $p \geqslant 2$ and define for $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2$:

$$\langle A(\sigma) u, v \rangle = \phi_1(\sigma_1) \int_\Omega |\partial_i u|^{p-2} \partial_i u \, \partial_i v + \phi_2(\sigma_2) \int_\Omega |u|^{p-2} uv - \int_\Omega fv \quad (3.11)$$

for $u, v \in W^{1,p}(\Omega)$, $f \in W^{-1,p'}(\Omega)$. Where we use the sommation convention on $i = 1, \dots, n$, we denote $\partial_i = \partial/\partial x_i$ and we assume that the given functions ϕ_j, for $j = 1, 2$, are such that :

$$\phi_j : \mathbb{R} \to \mathbb{R}^+ \text{ are continuous and } \phi_j(\tau) \geqslant \nu_j \quad \text{for} \quad \forall \tau \in \mathbb{R} . \quad (3.12)$$

Now let Γ_0 be a subset of Γ. In order to cover several possibilities we allow :

Case (D) : $\Gamma_0 = \Gamma$ (Dirichlet problem) ;
Case (N) : $\Gamma_0 = \varnothing$ (Neumann problem) ;
Case (M) : $\Gamma_0 \neq \varnothing$ and $\Gamma_1 = \Gamma \setminus \Gamma_0 \neq \varnothing$ (Mixed problem).

In case (D) we set $V = W_0^{1,p}(\Omega)$, in case (N) $V = W^{1,p}(\Omega)$, and in case (M) we introduce

$$V = \{ v \in W^{1,p}(\Omega) : v = 0 \text{ on } \Gamma_0 \}$$

and we assume $|\Gamma_0| > 0$, so that the Poincaré inequality $|v|_p \leqslant c \, \|\nabla v\|_p$ holds also in case (M). If we define $\mu : V \to \mathbb{R}^2$, by

$$\mu(v) = \left(\phi_1 \left(\int_\Omega |\nabla v|^p \right), \phi_2 \left(\int_\Omega |v|^p \right) \right) \quad (3.13)$$

we have the following Corollary of Theorem 3.ii) :

PROPOSITION 2 : *Let* $f \in W^{-1,p'}(\Omega)$, $p \geqslant 2$ *and* (3.12) *hold with* $\nu_1 > 0$, $\nu_2 \geqslant 0$ *in cases* (D) *and* (M) *and* $\nu_2 > 0$ *in case* (N). *Then there exists at least one solution* $u \in V$ *to the nonlocal problem :*

$$\phi_1 \left(\int_\Omega |\nabla u|^p \right) \int_\Omega |\partial_i u|^{p-2} \partial_i u \, \partial_i v + \phi_2 \left(\int_\Omega |u|^p \right) \int_\Omega |u|^{p-2} uv = \int_\Omega fv , \quad \forall v \in V . \quad (3.14)$$

Proof : It is sufficient to remark that the condition (3.6) holds with $q = p$ and $\nu = k_p \inf(\nu_1, \nu_2)$ in case (N) and $\nu = k_p c^p \nu_1$, in cases (D) and

(M), where $c > 0$ is the constant in the respective Poincaré inequality and $k_p > 0$ is a constant depending only on $p \geq 2$.

Remark 9 : This variational problem, which may be used to describe the flow of a class of non-newtonian fluids with viscosities depending on the total energy, corresponds formally to the Dirichlet, Neumann or Mixed boundary value problem :

$$- \phi_1\left(\int_\Omega |\nabla u|^p\right) \partial_i(|\partial_i u|^{p-2} \partial_i u) + \phi_2\left(\int_\Omega |u|^p\right) |u|^{p-2} u = f \quad \text{in } \Omega ,$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad |\partial_i u|^{p-2} \partial_i u \, n_i = 0 \quad \text{on} \quad \Gamma_1 .$$

Remark 10 : Consider $\phi_j(\tau) = \int_0 \tau\phi_j(\sigma)\, d\sigma$. Then any solution to (3.14) corresponds to a stationary point of the functional

$$J(v) = 1/p\phi_1\left(\int_\Omega |\nabla v|^p\right) + 1/p\phi_2\left(\int_\Omega |v|^p\right) - \int_\Omega fv , \quad v \in V .$$

Of course, if ϕ_j are monotone nondecreasing functions then J is a convex functional and the existence of a solution is guaranteed by convex analysis. If, in addition, ϕ_1 is increasing then the Dirichlet and the mixed problems have a unique solution. If ϕ_2 is also increasing then the Neumann problem is also uniquely solvable.

3.3. Consider now the case of the plate operator with a nonlinearity of nonlocal type, by introducing

$$\langle A(\sigma) u, v\rangle = \phi_1(\sigma_1) \int_\Omega \Delta u \,\Delta v + \phi_2(\sigma_2) \int_\Omega \nabla u \,.\, \nabla v = \int_\Omega f(x, \sigma_3)\, v , \tag{3.15}$$

for $u, v \in H^2(\Omega) = W^{2,2}(\Omega)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$. Here Ω is a bounded domain of $\mathbb{R}^2$.

Let $\psi : \Omega \to [-\infty, +\infty[$ be a measurable function, representing an obstacle below the plate, such that

$$\mathbf{K}_\psi = \{v \in V : v \geq \psi \text{ a.e. in } \Omega \} \neq \emptyset \tag{3.16}$$

where we allow two possibilities for the subspace $V \subset H^2(\Omega)$:

$$V = H_0^2(\Omega) \quad \text{(clamped plate)} \tag{3.17}$$

$$V = H^2(\Omega) \cap H_0^1(\Omega) \quad \text{(simply supported plate) .} \tag{3.18}$$

We recall that by the Sobolev imbedding theorem the functions of $H^2(\Omega)$ are continuous in Ω. Now, if D is any closed subset of Ω, if $\sup_D u$ denotes the supremum of u on D, we can define a continuous map $\mu : H^2(\Omega) \to \mathbb{R}^3$ by setting

$$\mu(u) = \left(\int_\Omega |\Delta u|^2, \int_\Omega |\nabla u|^2, \sup_D u \right).$$

For the forcing term $f : \Omega \times \mathbb{R} \to \mathbb{R}$, we assume $f = f(x, \tau)$ is a Carathéodory function (measurable in x, $\forall \tau$, and continuous in τ for a.e. x) such that

$$\exists f_0 \in L^2(\Omega) : \quad |f(x, \tau)| \leqslant f_0(x), \quad \text{a.e. } x \in \Omega, \quad \forall \tau \in \mathbb{R}. \tag{3.19}$$

With these definitions the problem (3.4) reads $u \in \mathbf{K}_\psi$:

$$\phi_1\left(\int_\Omega |\Delta u|^2 \right) \int_\Omega \Delta u \, \Delta(v - u) + \phi_2\left(\int_\Omega |\nabla u|^2 \right) \int_\Omega \nabla u \, . \nabla (v - u)$$
$$\geqslant \int_\Omega f(s, \sup_D u)(v - u), \quad \nabla v \in \mathbf{K}_\psi. \tag{3.20}$$

A solution u to this problem can be interpreted as the equilibrium deflection of a nonlinear plate, which is forced to lie above an obstacle by some force depending on the maximum deflection in some domain D and is clamped at the edges (case (3.17)) or simply supported (case (3.18)). In the case (3.18) we assume the boundary Γ of Ω is smooth, say of class $C^{1,1}$, or else Ω is a convex domain. Then we recall that in both cases $v \to |\Delta v|_2$ is a norm in V equivalent to the one induced by $\| \cdot \|_{2,2}$ (see [R], for instance). As an immediate consequence of Theorem 3.ii) we have the solvability of (3.20).

PROPOSITION 3 : *Let the conditions* (3.16), (3.17) *or* (3.18), (3.19) *and* (3.12), *with* $\nu_1 > 0$ *and* $\nu_2 \geqslant 0$, *hold. Then there exists at last one solution to* (3.20).

Remark 11 : As in Remark 10, the associated energy functional, in the case $f \equiv f(x)$, is now given by

$$E(v) = 1/2 \, \phi_1\left(\int_\Omega |\Delta v|^2 \right) + 1/2 \, \phi_2\left(\int_\Omega |\nabla v|^2 \right) - \int_\Omega f v, \quad v \in \mathbf{K}_\psi,$$

and similar observations can be made regarding the uniqueness of solutions.

Remark 12 : We can consider the case $\psi \equiv -\infty$, i.e., $\mathbf{K}_\psi = V$, as covered by Proposition 3. This corresponds formally to solve the equation

$$\phi_1\left(\int_\Omega |\Delta u|^2 \right) \Delta^2 u - \phi_2\left(\int_\Omega |\nabla u|^2 \right) \Delta u = f(x, \sup_D u) \quad \text{in} \quad \Omega$$

with the boundary values

$$u = \partial_n u = 0 \quad \text{on} \quad \Gamma ,$$

for the clamped plate (3.17), or with

$$u = \Delta u = 0 \quad \text{on} \quad \Gamma ,$$

for the simply supported plate (3.18).

Remark 13 : The special case where $\phi_1 \equiv 1$ and

$$\phi_2(\tau) = \alpha + \tau , \quad \tau \in \mathbb{R} , \quad \alpha \geq 0$$

has been proposed as a model for a class of a nonlinear extensible plates (see [E]). In the one-dimensional case it arises also in the study of hinged extensible beams. The corresponding dynamic problems (without obstacle) have been considered by several authors (see [M], [Y] for instance).

Remark 14 : We can also consider the obstacle problem for the membrane, by letting in (3.20) $\phi_1 \equiv 0$, $V = H_0^1(\Omega)$ in (3.16) and by assuming $\phi_2 \geq \nu_2 > 0$ and replacing the forcing term, for instance by $f = f\left(x, \int_D u\right)$.

4. APPLICATION TO POPULATION DISPERSION WITH IMPLICIT CONSTRAINT

Let the notation be as in the introduction and consider again the model equation (1.2), but now suppose that it has been observed that our species is stable if its density of population is, at every point of the boundary, greater than some critical threshold depending on the total flux entering across the entire boundary, i.e.,

$$u(x) \geq \psi\left(x, -\int_\Gamma \mathbf{v} \cdot \mathbf{n}\right) = \psi\left(x, a\left(\int_\Omega u\right) \int_\Gamma \partial_n u\right) \equiv \psi[u](x) \quad \text{for} \quad x \in \Gamma , \tag{4.1}$$

and the flux boundary condition of the type (1.3) holds only at points where $u > \psi$.

Instead of the boundary value problem (1.4) we are now led to the following implicit Signorini type problem :

$$-a\left(\int_\Omega u\right) \Delta u + \lambda u = f \quad \text{in} \quad \Omega \tag{4.2}$$

$$u \geq \psi[u] , \quad \partial_n u + \gamma\left(\int_{\Omega'} u, \int_{\Gamma'} u\right) \geq 0 , \quad \text{and} \tag{4.3}$$

$$\{u - \psi[u]\} \left\{\partial_n u + \gamma\left(\int_{\Omega'} u, \int_{\Gamma'} u\right)\right\} = 0 \quad \text{on} \quad \Gamma . \tag{4.4}$$

Here we assume $\lambda > 0$, $\Omega' \subset \Omega$ and $\Gamma' \subset \Gamma$ are relatively open subsets and $a : \mathbb{R} \to \mathbb{R}^+$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that, for some constants ω, M :

$$0 < \omega \leq a(\sigma_1) \leq M, \ \left|\gamma(\sigma_2, \sigma_3)\right| \leq M, \quad \forall \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}. \quad (4.5)$$

Letting $V = H^1(\Omega)$ with norm $\|v\| = \|v\|_{1,2}$, for each $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{R}^4$, we introduce the following monotone and continuous operator A: $V \to V^*$, given by

$$\langle A(\sigma)u, v\rangle = \int_\Omega a(\sigma_1)\nabla u \cdot \nabla v + \lambda \int_\Omega uv + \\ + a(\sigma_1)\gamma(\sigma_2, \sigma_3)\int_\Gamma v, \quad \forall u, v \in H^1(\Omega). \quad (4.6)$$

From (4.5) it is clear that (2.11) holds with $\rho = 2$ and $\nu = \min(\omega, \lambda)$. For $f \in L^2(\Omega)$ we set

$$\langle f, v\rangle = \int_\Omega fv, \quad \forall v \in H^1(\Omega).$$

The linear map μ given by

$$\mu(u) = \left(\int_\Omega u, \int_{\Omega'} u, \int_{\Gamma'} u, \int_\Gamma \partial_n u\right) \quad (4.7)$$

is not defined on the entire $H^1(\Omega)$, but only in its subspace

$$H_\Delta(\Omega) = \{v \in H^1(\Omega) : \Delta v \in L^2(\Omega)\}.$$

As it is well-known (see [BC], [M] or [R], for instance), when Γ is a Lipschitz boundary, the normal derivative $\partial_n : H_\Delta(\Omega) \to H^{-1/2}(\Gamma) = [H^{1/2}(\Gamma)]'$ is a linear operator satisfying, for some constant $C > 0$,

$$\|\partial_n u\|_{H^{-1/2}(\Gamma)} \leq C\|u\|_\Delta \equiv C\left\{|u|_2^2 + |\nabla u|_2^2 + |\Delta u|_2^2\right\}^{1/2}. \quad (4.8)$$

In the last argument of (4.7), we set $\int_\Gamma \partial_n u = \langle \partial_n u, 1\rangle_\Gamma$, where $\langle\ ,\ \rangle_\Gamma$ denotes the duality brackets between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, and $H^{1/2}(\Gamma)$ denotes the space of traces of $H^1(\Omega)$ functions.

Now let $\psi : \Gamma \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that the mapping

$$\tau \to \psi(\cdot, \tau) \text{ is continuous from } \mathbb{R} \text{ into } H^{1/2}(\Gamma). \quad (4.9)$$

Hence, for each $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, we can define the convex subset

$$\mathbf{K}(\sigma) = \mathbf{K}(\sigma_1, \sigma_4) = \{ v \in H^1(\Omega) : v(x) \geq \psi(x, a(\sigma_1)\,\sigma_4) \text{ a.e. } x \in \Gamma \} \tag{4.10}$$

and if we assume

$$\exists h \in H^{1/2}(\Gamma) : h \geq \psi(x, \tau), \quad \forall \tau \in \mathbb{R} \tag{4.11}$$

it is clear that (2.12) holds for instance, by letting v_0 be the harmonic function in Ω with trace on Γ equal to h.

For any $\sigma \in \mathbb{R}^4$ and $f \in L^2(\Omega)$, there exists a unique u_σ which solves (2.8) with $A(\sigma)$ and $\mathbf{K}(\sigma)$ given by (4.6) and (4.10), respectively. It is immediate that u_σ is bounded in $V = H^1(\Omega)$, independently of $\sigma \in \mathbb{R}^4$, and that the continuity of $\sigma \to A(\sigma)$ in the sense of (2.3) holds. To verify the continuity of $\sigma \to \mathbf{K}(\sigma)$ in the Mosco sense, we proceed as follows : for any $v \in \mathbf{K}(\sigma)$, we can take the sequence of functions $v' \in \mathbf{K}(\sigma')$, defined by $v' = v + w' - w$, where w' and w are the harmonic functions whose traces $w'|_\Gamma = \psi(\cdot, \sigma') \to w|_\Gamma = \psi(\cdot, \sigma)$ in $H^{1/2}(\Gamma)$; this implies $v' \to v$ in $H^1(\Omega)$ as $\sigma' \to \sigma$; on the other hand, if $v' \in \mathbf{K}(\sigma')$ and $v' \to v$ in $H^1(\Omega)$-weak as $\sigma' \to \sigma$, then, by the continuity of the trace operator and the assumption (4.9) we have $v|_\Gamma \geq \psi(\cdot, \sigma)$ and $v \in \mathbf{K}(\sigma)$.

Consequently, arguing as in the second part of the proof of Theorem 2, we conclude that the map $\sigma \to u_\sigma$ is continuous from $\mathbb{R}^4$ into $H^1(\Omega)$. By taking $v = u_\sigma \pm \phi$ in (2.8) for this case, with an arbitrary $\phi \in \mathcal{D}(\Omega)$, we find

$$-a(\sigma_1)\,\Delta u_\sigma + \lambda u_\sigma = f \quad \text{a.e. in } \Omega . \tag{4.12}$$

Since u_σ is bounded in $H^1(\Omega)$ and a verifies (4.5), we have

$$\| u_\sigma \|_\Delta \leq C \quad \text{independently of } \sigma \in \mathbb{R}^4 . \tag{4.13}$$

Then, by (4.12) and (4.13), the continuity of $\sigma \to u_\sigma$ also holds from $\mathbb{R}^4$ into $H_\Delta(\Omega)$-strong, for the topology of the norm $\| \cdot \|_\Delta$, and by (4.8), $\sigma \to \partial_n u_\sigma$ is continuous in $H^{-1/2}(\Gamma)$.

Therefore the map $\sigma \to \mu(u_\sigma)$ is continuous and has a bounded range. So, by the Brouwer fixed point theorem it has a fixed point, which provides a solution to (2.1) corresponding to the definitions (4.6), (4.7) and (4.10).

We note that in the present case the mapping

$$v \to \mathbf{K}[v] \equiv \mathbf{K}\left(\int_\Omega v, \int_\Gamma \partial_n v \right)$$

is not defined for every $v \in V = H^1(\Omega)$, but only for the subspace $H_\Delta(\Omega)$. This is the reason why the Theorem 1 or 2 cannot be applied directly to the solvability of the following quasi-variational inequality corresponding to (4.2)-(4.4) :

$$u \in H_{\Delta}(\Omega) \cup \mathbf{K}[u] ,$$

$$\int_{\Omega} \left\{ a\left(\int_{\Omega} u \right) \nabla u \,.\, \nabla (v - u) + (\lambda u - f)(v - u) \right\} +$$
$$+ a\left(\int_{\Omega} u \right) \gamma \left(\int_{\Omega'} u, \int_{\Gamma'} u \right) \int_{\Gamma} (v - u) \geqslant 0 , \quad \forall v \in \mathbf{K}[u] . \quad (4.14)$$

Nevertheless the preceding discussion has proved the following existence result.

THEOREM 4 : *Let the above assumptions hold, in particular assume* (4.5), (4.9) *and* (4.11). *Then for any* $f \in L^2(\Omega)$ *there exists at least one solution to* (4.14).

Remark 15 : The solution u to (4.14) satisfies the equation (4.2) a.e. in Ω, the first inequality of (4.3) a.e. on Γ, its second one in the distributional sense in $H^{-1/2}(\Gamma)$, but the product in the condition (4.4) must be interpreted with the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, i.e.,

$$\left\langle \partial_n u + \gamma \left(\int_{\Omega'} u, \int_{\Gamma'} u \right), u - \psi[u] \right\rangle_{\Gamma} = 0 .$$

In order that (4.4) to be valid a.e. in Γ, the additional regularity $\partial_n u \in L^2(\Gamma)$ would be required.

Remark 16 : We could replace the total flux $\int_{\Gamma} \partial_n u$ in (4.7) by any other continuous functional from $H^{-1/2}(\Gamma)$ into $\mathbb{R}$. For instance, the special case $a \equiv 1$, $\gamma \equiv 0$ and

$$\psi[u](x) = h(x) - \langle \partial_n u, \eta \rangle_{\Gamma}$$

for given functions $h \in H^{1/2}(\Gamma)$ and $\eta \in H^{1/2}(\Gamma)$, $\eta \geqslant 0$, have been solved in [M] and in [JM] with a more abstract framework. This provides a new and more direct proof of that result of [JM].

Remark 17 : If we take in (4.10), $\psi \in H^{1/2}(\Gamma)$ independent of σ, the convex set $\mathbf{K}$ is fixed and well defined in $H^1(\Omega)$ and the solvability of the nonlocal Signorini problem is a direct consequence of Corollary 2, taking $\mu(u) = \left(\int_{\Omega} u, \int_{\Omega'} u, \int_{\Gamma'} u \right) \in \mathbb{R}^3$.

Remark 18 : If we suppose $f \geqslant 0$, $\gamma \leqslant 0$, then by letting $v = u^+ = \sup(u, 0) \in \mathbf{K}[u]$ in (4.14), we easily find that $u \geqslant 0$ a.e. in

Ω, which is a natural physical assumption. In particular, if we let $\psi < 0$ on Γ, we have $u > \psi[u]$ a.e. on Γ and u solves (4.2) with the Neumann boundary condition

$$\partial_n u + \gamma\left(\int_{\Omega'} u, \int_{\Gamma'} u\right) = 0 \text{ on } \Gamma\,, \tag{4.15}$$

which includes the introductory model problem (1.4) as a special case.

In addition to the preceding remark, if we choose $\mathbf{K} = V = H^1(\Omega)$, A given by (4.6) and $\mu(u) = \left(\int_{\Omega} u, \int_{\Omega'} u, \int_{\Gamma'} u\right)$ a direct consequence of Corollary 2 yields the following result.

PROPOSITION 4 : *Let* (4.5) *hold and* $f \in L^2(\Omega)$. *Then there exists at least one solution to the nonlocal Neumann problem* (4.2), (4.15) *and, in particular, also for the introductory model problem.*

If u is the density of some population, u should be a nonnegative function. At points where $u = 0$ there are no members of that population, so that the equilibrium equation (4.2) would only be verified in the subset $\{u > 0\}$ of Ω, as well as the flux condition (4.15) would hold only on $\Gamma \cap \{u > 0\}$. Therefore, if we want to consider the equilibrium of a population which does not fill completely the container Ω and we do not know a priori where it lies, we should formulate this free boundary problem as the following unilateral problem

$$u \geq 0,\ -a\left(\int_{\Omega} u\right)\Delta u + \lambda u \geq f,\quad u\left\{-a\left(\int_{\Omega} u\right)\Delta u + \lambda u - f\right\} = 0 \text{ in } \Omega$$

$$u \geq 0,\ \partial_n u + \gamma\left(\int_{\Omega'} u, \int_{\Gamma'} u\right) \geq 0,\quad u\left\{\partial_n u + \gamma\left(\int_{\Omega'} u, \int_{\Gamma'} u\right)\right\} = 0 \text{ on } \Gamma\,.$$

This problem is solvable directly as an immediate consequence of Corollary 2, in the following form

$$u \in \mathbf{K} \equiv \{v \in H^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$$

$$\int_{\Omega}\left\{a\left(\int_{\Omega} u\right)\nabla u \,.\, \nabla(v-u) + (\lambda u - f)(v-u)\right\} +$$
$$+\, a\left(\int_{\Omega} u\right)\gamma\left(\int_{\Omega'} u, \int_{\Gamma'} u\right)\int_{\Gamma}(v-u) \geq 0\,,\quad \forall v \in \mathbf{K}\,. \tag{4.16}$$

PROPOSITION 5 : *If* (4.5) *holds, for any* $f \in L^2(\Omega)$ *the nonlocal variational inequality* (4.16) *has at least one solution.*

5. SOME ADDITIONAL REMARKS

5.1. First, let us show again that uniqueness or comparison properties fail in general for nonlocal problems. This is an essential difference with respect to the local nonlinear problems (see, for instance [GT] or [CM]), as we have already noted in Remark 3. Indeed consider, for $f \in L^2(\Omega)$ and for some continuous function a satisfying $a(\sigma) \geq \nu > 0$, $\forall \sigma \in \mathbb{R}$, the following nonlocal problem, which we know is solvable :

$$- a\left(\int_\Omega u\right) \Delta u = f \text{ in } \Omega \,, \quad u = 0 \text{ on } \Gamma \,. \tag{5.1}$$

Let us now introduce u_0 the solution of

$$- \Delta u_0 = f \text{ in } \Omega \,, \quad u_0 = 0 \text{ on } \Gamma \,. \tag{5.2}$$

For simplicity set $\mu(u) = \int_\Omega u$. Then, clearly, u is a solution to (5.1) if and only if

$$u = u_0 / a(\mu(u)) \,.$$

Integrating over Ω, this implies

$$\mu(u) = \mu(u_0)/a(\mu(u)) \,.$$

Conversely, if σ is a solution to

$$\sigma = \mu(u_0)/a(\sigma) \,. \tag{5.3}$$

then

$$u = \sigma u_0 / \mu(u_0) \tag{5.4}$$

solves (5.1). Indeed (5.4) implies that $\mu(u) = \sigma$ and (5.3), (5.4) with (5.2) yields $-\Delta u = f/a(\sigma)$ in Ω. Thus if we fix, for instance, f such that $u_0 > 0$ in Ω and if we define

$$a(\sigma) = \mu(u_0)/\sigma \tag{5.5}$$

on some interval of $\mathbb{R}^+$, then (5.1) corresponding to this $a(\sigma)$ would have infinitely many solutions (even an uncountable set of solutions !). Again, since the solvability of (5.1) reduces to solve (5.3), it should be noted that the existence and the number of solutions will depend on f.

Remark 19 : The situation may be entirely different in the evolution case. For instance, if one considers the parabolic problem associated with (5.1), then it can be shown that it has a unique solution (see [CR]).

5.2. We have always assumed a continuity hypothesis on the nonlocal nonlinearity. Let us now show that existence can fail if, for instance, in (5.1) the function a is not assumed to be continuous.

Let $f \in L^2(\Omega)$, $f < 0$ in Ω. Assume as above that $\mu(u) = \int_\Omega u$. Then for two real numbers a, b $(0 < b < a)$, denote by u_a and u_b, respectively, the solutions of

$$- a \, \Delta u_a = f \text{ in } \Omega \,, \quad u_a = 0 \text{ on } \Gamma \,,$$

and

$$- b \, \Delta u_b = f \text{ in } \Omega \,, \quad u_b = 0 \text{ on } \Gamma \,.$$

Since f is positive one has $- \Delta u_a = f/a < f/b = - \Delta u_b$. Hence by the strong maximum principle we conclude $u_a < u_b$ in Ω, and

$$\mu(u_a) = \int_\Omega u_a(x) \, dx < \mu(u_b) = \int_\Omega u_b(x) \, dx \,.$$

For $m = (\mu(u_a) + \mu(u_b))/2$, set

$$a(\sigma) = b \quad \text{if} \quad s \leqslant m \quad \text{and} \quad a(\sigma) = a \quad \text{if} \quad s > m \,.$$

The problem (5.1) corresponding to this function $a(\sigma)$ has no solution. Indeed assume that u is a solution of (5.1). If $\mu(u) \leqslant m$ then $a(\mu(u)) = b$ and $u = u_b$ which is impossible since we should have $\mu(u) = \mu(u_b) > m$. Then $a(\mu(u)) = a$ and $u = u_a$, which is again impossible since we should have $\mu(u) = \mu(u_a) < m$. This completes the proof of the nonexistence of solution to (5.1) for the case of a discontinuous coefficient as above.

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