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COMPONENT MODE SYNTHESIS AND EIGENVALUES OF SECOND ORDER OPERATORS : DISCRETIZATION AND ALGORITHM (*)

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Abstract. — *Component mode synthesis belongs to the class of Galerkin methods and enables to compute the eigenpairs of a differential operator on a domain that can be subdivided into different subdomains on each of which part of the eigenpairs of the same operator are assumed to be known. Energy transfer between subdomains is achieved thanks to functions defined on the whole domain and called « static modes ». A new « fixed interface » method is presented, whose discretized version extends the Hurty [1965], Craig, and Bampton [1968] one. It is based on a special choice of the « static modes » as the eigenfunctions of the Poincaré-Steklov operator associated with the domain decomposition. Error bounds are derived in the case of the heat equation on a domain in $\mathbb{R}^n$, $n \geqslant 2$. A class of domain decomposition algorithms and several numerical tests are also presented.*

Keywords : Eigenvalues, domain decomposition, finite elements.

Résumé. — *La synthèse modale fait partie de la classe des méthodes de Galerkin et permet de calculer les éléments propres d'un opérateur différentiel sur un domaine que l'on peut décomposer en différents sous-domaines sur chacun desquels on suppose connue une partie des éléments propres du même opérateur. L'énergie s'échange entre les sous-domaines grâce à des fonctions définies sur le domaine entier et appelées « modes statiques ». On présente une nouvelle méthode « à interfaces fixes », dont la version discrétisée généralise celle de Hurty [1965], Craig et Bampton [1968]. Elle est fondée sur un choix spécial de « modes statiques » comme fonctions propres de l'opérateur de Poincaré-Steklov associé à la décomposition du domaine. On démontre des bornes d'erreur dans le cas de l'équation de la chaleur sur un domaine de $\mathbb{R}^n$, $n \geqslant 2$. On présente également une classe d'algorithmes de décomposition de domaine et quelques tests numériques.*

1. INTRODUCTION

The dynamics of a structure behaving linearly and subjected to an arbitrary variable load can be determined by its free vibrations and its initial state, thanks to the modal superposition principle (*cf.* Imbert [1979]), also called Fourier representation (*cf.* Dautray & Lions [1985]). But the computation of the needed eigenpairs gives rise to a number of practical difficulties especially if repeated analyses are required, when the structure is made of a large number of components connected together, which may differ among themselves, as far as thermomechanical or geometrical properties are concerned. For example, space vehicles can be viewed as a set of interconnected substructures like launchers, payloads, boosters, and appendages that are usually much smaller and lighter than the main body. In the same way, off-shore oil extraction facilities, as well as satellites are composed of trusses, plates... Modern slender bridges, that in addition make sometimes an essential use of cables, and industrial products such as cars or ships also share the same characteristics. From the mechanical point of view, the description of multi-structures often leads to somewhat mathematically complicated models (*cf.* Aufranc [1990], Bernadou *et al.* [1988], Bourquin & Ciarlet [1988], Bourquin & Ciarlet [1989], Ciarlet [1987], Ciarlet *et al.* [1987], Ciarlet *et al.* [1989], Le Dret [1987]). Their often clear decomposition in *much simpler components* suggests to start to analyse the latter under static or dynamic load, and then to take into account in the analysis of the entire structure the behavior of each component considered as isolated from the other ones. Component mode synthesis takes advantage of that idea, and thus belongs to the class of domain decomposition methods.

In order to describe the so-called « fixed interface method », we consider the following eigenvalue problem on a domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma \subset \mathbb{R}^n$ (*cf. fig.* 1) :

$$\begin{cases} -\Delta u + du = \lambda u & \text{in} \quad \Omega \,, \\ Du + N \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial \Omega \,, \end{cases} \qquad (1)$$

where d stands for a positive constant, and the pair (D, N) satisfies either $(D, N) = (1, 0)$, or $(D, N) = (0, 1)$. This formulation allows to treat simultaneously both homogeneous Dirichlet and Neumann boundary conditions.
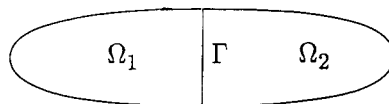

Figure 1. — The domain Ω, represented in $\mathbb{R}^2$.

This problem is well-posed, and admits a sequence of « global eigenpairs » $(\lambda_k, u_k)_{k=1}^{+\infty} \in \mathbb{R}^+ \times H^1(\Omega)$, arranged in such a way that $\lambda_{k+1} \geqslant \lambda_k$, $k \in \mathbb{N}$.

Let us assume that we know the N_i first eigenpairs $(\lambda_{ij}, u_{ij})_{j=1}^{N_i} \in \mathbb{R}^+ \times H^1(\Omega_i)$ of the same operator on each subdomain Ω_i, but associated with a *Dirichlet boundary condition on* Γ. They solve the boundary value problem

$$\begin{cases} -\Delta u + du = \lambda u & \text{in} \quad \Omega_i, \\ u = 0 & \text{on} \quad \Gamma, \\ Du + N \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega_i \backslash \Gamma, \end{cases} \tag{2}$$

which is also well-posed. The eigenfunctions u_{ij} are called « fixed interface modes ».

Let us assume that we also know N_Γ functions $(\tilde{u}_{\Gamma\ell})_{\ell=1}^{N_\Gamma} \in H^1(\Omega)$, called « static modes », such that the traces of the latter on Γ do not vanish. We may define the approximation space

$$V_N = \text{Span} \left\{ \bigoplus_{i=1,2} (u_{ij})_{j=1}^{N_i} \oplus (\tilde{u}_{\Gamma\ell})_{\ell=1}^{N_\Gamma} \right\}. \tag{3}$$

Roughly speaking, component mode synthesis consists in computing an approximation of a small number j_0 of eigenpairs $(\lambda_k, u_k)_{k=1}^{j_0}$ by a Galerkin method over the space V_N.

In fact, component modes u_{ij} and $\tilde{u}_{\Gamma\ell}$ have first to be computed by a finite element method, or any other suitable method. They may alternatively be measured. Close-form or exact analytical solutions are also accepted. Component mode synthesis thus results from two different approximations, mode truncation and discretization. Therefore, the true approximation space should be expressed as

$$V_N^h = \text{Span} \left\{ \bigoplus_{i=1,2} (u_{ij}^h)_{j=1}^{N_i} \oplus (\tilde{u}_{\Gamma\ell}^h)_{\ell=1}^{N_\Gamma} \right\}, \tag{4}$$

where the superscript h denotes a discretization parameter.

Component mode synthesis has become a very popular numerical tool in aerospace engineering in the last two decades, because it usually meets high standards of computational efficiency and versatility. It enables to perform structural optimization against vibrations or buckling (*cf.* Valid [1982]) by parametric studies, since geometrical or mechanical perturbations of one substructure do not affect the free vibrations of the other ones. Hence, repeated modal analyses of the entire structure remain very cheap.

Moreover, it fits in the modern managerial practice of large industrial projects where substructuring is naturally imposed by the multiplicity of contractors in charge of realizing specific parts of the structure.

There exists a lot of different methods depending on the type of boundary conditions on the interface Γ that are imposed to define the eigenpairs on the different subdomains, and on the way to couple the latter. For example, in problem (2), the Dirichlet condition imposed on Γ can be replaced by a Neumann or a Fourier condition. We refer to Craig [1985], Gibert [1988], Imbert [1979], Jezequel [1985], Meirovitch [1980], Morand [1977], and Valid [1977] for detailed reviews or analyses on this topic, to Morand & Ohayon [1979], and Valid & Ohayon [1974] for applications of the idea to fluid-structure interaction problems.

In Bourquin [1989, 1990b], we have proposed a new method which seems to be of wide applicability (*cf.* Bourquin [1991]). It differs from all existing ones, because the eigenfunctions of the Poincaré-Steklov operator related to the interface Γ, once lifted on each subdomain, are chosen as static modes. This strategy leads to a diagonal stiffness matrix. The method is well suited to error analysis in a continuous framework. Furthermore, its discretized version extends the classical Hurty [1965], Craig, and Bampton [1968] one, for which the static modes cannot be interpreted as discretized versions of continuous functions ; in this paper, both methods will be analysed and compared.

We follow the method advocated in Bourquin [1990a] and Bourquin [1991], by first focussing our attention on the mode truncation error : taking advantage of the error bounds derived for an « intermediate method » defined with continuous component modes that are assumed to be exactly computable, we now control the perturbation resulting from the discretization process by classical arguments relevant to the finite element theory.

This paper is organized as follows : section 2 is entirely devoted to background information regarding the intermediate method : our choice of static modes is explained and the main bounds are given for a model problem and without proof. Section 3 concentrates on the finite element approximation of the proposed component mode synthesis method. New error bounds related to this discretization are derived. In particular, the approximation of the eigenpairs of Poincaré-Steklov's operators is assessed. In section 4, error bounds are given for the continuous and the discretized versions of the Hurty, Craig, and Bampton fixed interface method. A natural algorithm is explained in section 5 and compared with the classical one. Other algorithms are also briefly sketched. Finally, numerical tests regarding the intermediate method and its discretized version are presented in section 6 ; they confirm the predicted rates of convergence as far as mode truncation error is concerned.

2. PRESENTATION OF THE INTERMEDIATE METHOD

In what follows, Ω denotes a domain in $\mathbb{R}^n$, $n \geq 2$, with piecewise regular boundary, separated in two parts Ω_1 and Ω_2 by a smooth surface Γ (*cf. fig.* 1). Let us denote by n_i the unit outer normal vector along the regular parts of the boundary $\partial \Omega_i$ of the domain Ω_i, and by n the unit outer normal vector along the regular parts of the boundary $\partial \Omega$ of the domain Ω. For the sake of simplicity, we assume the interface Γ to be part of a hyperplane. Most of the results stated in the sequel still hold when the domain Ω is decomposed into p subdomains, $p > 2$.

For any domain B, we denote by $H^1(B) = W^{1,2}(B)$ the Sobolev space of square integrable functions with square integrable derivatives. For a definition of the spaces $H^s(B)$, $s \in \mathbb{R}$, see Lions & Magenes [1968], and Grisvard [1985].

Without loss of generality, we may assume that the solution w of the elliptic homogeneous boundary value problem

$$\begin{cases} -\Delta w + dw = f & \text{in} \quad \Omega_i, \quad i = 1, 2, \quad f \in L^2(\Omega), \quad w = 0 \quad \text{on} \quad \Gamma, \\ Dw + N \dfrac{\partial w}{\partial n} = 0 & \text{on} \quad \partial \Omega_i \backslash \Gamma, \quad i = 1, 2, \end{cases} \tag{5}$$

satisfies $w|_{\Omega_i} \in H^{1+\alpha}(\Omega_i)$, $0 < \alpha \leq 1$, $\alpha \neq 1/2$, for each subdomain Ω_i. An application of the closed graph theorem yields $\left\| w|_{\Omega_i} \right\|_{H^{1+\alpha}(\Omega_i)} \leq C |f|_{L^2(\Omega)}$. The best exponent α can be assessed in various situations analysed by Kondrat'ev [1967], Grisvard [1985], and Dauge [1989] (see also Leguillon & Sanchez-Palencia [1987] for elasticity problems). Practical examples are given in Bourquin [1991].

Remark 1 : The results of this paper still hold if we only assume that $w|_{\Omega_i} \in H^{1+\alpha}(\Omega_i \cap W)$, where W denotes a neighbourhood of the interface Γ.

Let us introduce the spaces

$$\begin{cases} V = \{ v \in H^1(\Omega) ; \quad Dv = 0 \text{ on } \partial \Omega \} , \\ \mathring{V} = \{ v \in V ; \quad v = 0 \text{ on } \Gamma \} , \\ V_i = \{ v \in H^1(\Omega_i) ; \quad Dv = 0 \text{ on } \partial \Omega_i \backslash \Gamma \} , \\ \mathring{V}_i = \{ v \in V_i ; \quad v = 0 \text{ on } \Gamma \} . \end{cases} \tag{6}$$

We define the symmetric coercive continuous bilinear forms on the spaces $H^1(\Omega_i)$ and $H^1(\Omega)$ respectively by the following formulas :

$$\begin{cases} a_i(u, v) = \displaystyle\int_{\Omega_i} \nabla u \, \nabla v \, dx + d \int_{\Omega_i} uv \, dx \quad \forall u, v \in H^1(\Omega_i) \,, \\[4mm] a(u, v) = \displaystyle\sum_{i=1}^{2} a_i(u, v) \qquad\qquad\qquad\quad \forall u, v \in H^1(\Omega) \,, \end{cases} \tag{7}$$

where d stands for a strictly positive constant. We also define the bilinear forms

$$\begin{cases} (u, v)_i = \displaystyle\int_{\Omega_i} uv \, dx \quad \forall u, v \in L^2(\Omega_i) \,, \\[4mm] (u, v) = \displaystyle\sum_{i=1}^{2} (u, v)_i \quad \forall u, v \in L^2(\Omega) \,. \end{cases} \tag{8}$$

In this setting, problem (1) is equivalent to the variational eigenvalue problem

$$\begin{cases} \text{find } (\lambda, u) \in \mathbb{R} \times V \,, \quad \text{such that} \\ a(u, v) = \lambda (u, v) \qquad \forall v \in V \,. \end{cases} \tag{9}$$

In what follows, the spaces $H^1(\Omega)$, $H^1(\Omega_i)$, $L^2(\Omega)$, and $L^2(\Omega_i)$ will be endowed with the norms $\|v\|_a = \sqrt{a(v, v)}$, $\|v\|_{a_i} = \sqrt{a_i(v, v)}$, $|v|_{0, \Omega} = \sqrt{(v, v)}$, and $|v|_{0, \Omega_i} = \sqrt{(v, v)_i}$ respectively.

For any function $v \in H^1(\Omega)$, we denote by $\mathrm{Tr}_\Gamma \, v$ its trace on the surface Γ. The trace operator Tr_Γ maps the space $H^1(\Omega)$ onto the space $H^{1/2}(\Gamma)$, and the space $H_0^1(\Omega)$ onto the space $H_{00}^{1/2}(\Gamma)$. Intrinsic definitions can be found in Ggaliardo [1957], or Lions & Magenes [1968], but we do not need them. We shall use that the imbeddings $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ and $H_{00}^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ are compact. Let us define the space

$$V_\Gamma = \mathrm{Tr}_\Gamma V \,, \tag{10}$$

in such a way that $V_\Gamma = H_{00}^{1/2}(\Gamma)$ if $(D, N) = (1, 0)$, and $V_\Gamma = H^{1/2}(\Gamma)$ otherwise.

The harmonic lifting operator

$$R : V_\Gamma \to V$$

$$v \to Rv = \tilde{v} \,,$$

where the function $\tilde{v}$ is defined by

$$\begin{cases} -\Delta\tilde{v} + d\tilde{v} = 0 \quad \text{in } \Omega_i, \quad i = 1, 2 \,, \\[2mm] D\tilde{v} + N \dfrac{\partial\tilde{v}}{\partial n} = 0 \quad \text{on } \partial\Omega \,, \\[2mm] \tilde{v} = v \qquad\qquad \text{on } \Gamma \,, \end{cases} \tag{11}$$

is obviously continuous.

For any complete family $(w_{\Gamma\ell})_{\ell=1}^{+\infty}$ in the space V_Γ, we set

$$\tilde{w}_{\Gamma\ell} = R w_{\Gamma\ell}\,. \tag{12}$$

The functions $\tilde{w}_{\Gamma\ell}$ are called « static modes ».

Let us recall that for each subdomain Ω_ι, the family of « fixed interface modes » $(u_{\iota j})_{j=1}^{+\infty}$ defined as the family of eigensolutions of problem (2) is complete and orthogonal in both spaces $\overset{\circ}{V}_\iota$ and $L^2(\Omega_\iota)$ (cf. Taylor [1958]). The functions $u_{\iota j}$ can be extended by zero to functions of the space V.

If we start with an arbitrary family of static modes $(\tilde{w}_{\Gamma\ell})_{\ell=1}^{+\infty}$, then the family

$$\mathcal{G} = \underset{\iota=1,2}{\oplus}\ (u_{\iota j})_{j=1}^{+\infty} \oplus (\tilde{w}_{\Gamma\ell})_{\ell=1}^{+\infty}\,, \tag{13}$$

is complete in the space V. Although it could be possible to define a Galerkin method based on the family $\mathcal{G}$ truncated at a certain range, we prefer to deal with static modes enjoying special orthogonality properties : to this end, let us introduce the symmetric bilinear form

$$b(u, v) = a(\tilde{u}, \tilde{v})\quad \forall u, v \in V_\Gamma\,, \tag{14}$$

the continuity and coerciveness of which are obvious consequences of the continuity of the lifting and trace operators R and Tr_Γ, and of the coerciveness of the bilinear form $a(\,.\,,\,.\,)$; thus, there exists an isomorphism $\mathcal{C} : V_\Gamma \to V'_\Gamma$, such that

$$b(u, v) = {}_{V'_\Gamma}\langle \mathcal{C}u, v\rangle_{V_\Gamma}\quad \forall u, v \in V_\Gamma\,. \tag{15}$$

In the sequel, we may denote this duality product by the symbol $\langle\,.\,,\,.\,\rangle$.

There exists also a self-adjoint compact operator T on $L^2(\Gamma)$ (cf. Taylor [1958]), such that its range lies in V_Γ and

$$b(Tu, v) = {}_{L^2(\Gamma)}(u, v)_{L^2(\Gamma)}\quad \forall u \in \mathcal{D}(T)\quad \forall v \in L^2(\Gamma)\,. \tag{16}$$

The operator $\mathcal{C}$ can be easily interpreted as a sum of two « inverse Calderon's operators » (cf. Grubb [1977], Calderon [1963]) : since we have

$$\langle \mathcal{C}u, v\rangle = \sum_{\iota=1}^{2} \int_{\Omega_\iota} \nabla\tilde{u}\,\nabla\tilde{v}\,dx + d\int_{\Omega_\iota}\tilde{u}\tilde{v}\,dx\quad \forall u, v \in V_\Gamma\,,$$

formal integration by parts yields immediately

$$\langle \mathcal{C}u, v\rangle = \sum_{\iota=1}^{2} \int_\Gamma \frac{\partial\tilde{u}|_{\Omega_\iota}}{\partial n_\iota}\,v\,d\Gamma\,, \tag{17}$$

where $d\Gamma$ denotes the Lebesgue measure on Γ. This identity does make sense without extra regularity assumptions on $\tilde{u}$, if the integrals in the right-hand side of (17) are understood as duality products : $\left\langle \dfrac{\partial \tilde{u}|_{\Omega_i}}{\partial n_i}, v \right\rangle$; in several classical papers (*cf.* Agoshkov [1988] and included references), the operator $\mathscr{C}$ is referred to as a Poincaré-Steklov operator. Here, we shall adopt this terminology for both operators $\mathscr{C}$ and T.

Let $\rho \in L^{\infty}(\Gamma)$ denote a positive function that is bounded away from zero : the spectral theory of self-adjoint compact operators (see Taylor [1958]) ensures the existence of *a family* $(u_{\Gamma\ell})_{\ell=1}^{+\infty} \in V_{\Gamma}$ *which forms an orthogonal basis of both spaces* V_{Γ} *and* $L^2(\Gamma)$, *and also of a sequence* $(\lambda_{\Gamma\ell})_{\ell=1}^{+\infty} \in \mathbb{R}$, *such that*

$$
\begin{cases}
0 < \lambda_{\Gamma 1} \leqslant \lambda_{\Gamma 2} \leqslant \dots , \qquad \lim_{\ell \to +\infty} \lambda_{\Gamma\ell} = +\infty , \\[2mm]
b(u_{\Gamma\ell}, v) = \lambda_{\Gamma\ell} \displaystyle\int_{\Gamma} u_{\Gamma\ell}\, v\rho\, d\Gamma \quad \forall v \in V_{\Gamma} .
\end{cases}
\tag{18}
$$

We let $\tilde{u}_{\Gamma_j} = R u_{\Gamma_j}$, for any integer j. Then *the family* $(\tilde{u}_{\Gamma\ell})_{\ell=1}^{+\infty}$ *associated with the spectral basis of the Poincaré-Steklov operator* T *will be chosen as basis of static modes. Thus, we have built a complete family in the space* V :

$$
\mathscr{F} = \bigoplus_{i=1,2} (u_{ij})_{j=1}^{+\infty} \oplus (\tilde{u}_{\Gamma\ell})_{\ell=1}^{+\infty} .
\tag{19}
$$

Remark 2 : The interest of those static modes seems to be partly technical (*cf.* Bourquin [1991]).

On the other hand, their definition is intrinsiquely tied to the problem under consideration no matter how many subdomains the whole domain is split in, and no matter how ramified the interface is. Notice that fixed interface modes also share the same property. Furthermore, our choice allows for a unified approach, because static and fixed interface modes are all solutions of eigenproblems derived from the main one.

From the mechanical point of view, the static modes correspond to the eigenvectors of the compliance matrix associated with the interface Γ. They represent the way the subdomains exchange energy. They also correspond to the free vibrations of the same elastic structure Ω but whose mass would be concentrated on the interface Γ. In this respect, the function ρ should be interpreted as the mass density of the structure along the interface Γ. It is introduced here as a purely numerical tool as in Destuynder [1989], and may be tuned in such a way as to speed up the convergence of the method. ■

The intermediate method corresponds to a presentation of the component mode synthesis in a continuous framework. It depends on a triple $N = (N_1, N_2, N_\Gamma)$ of control parameters which determine the interpolation space V_N defined by (3). It consists in computing the j_0 first solutions $(\lambda_k^N, u_k^N)_{k=1}^{j_0}$ of the finite-dimensional eigenvalue problem

$$\begin{cases} \text{find } (\lambda^N, u^N) \in \mathbb{R} \times V_N \quad \text{such that} \\ a(u^N, v^N) = \lambda^N (u^N, v^N) \quad \forall v^N \in V_N , \end{cases} \tag{20}$$

which admits a sequence of $N_T = N_1 + N_2 + N_\Gamma$ independent solutions according to the spectral theory of symmetric positive matrices. Of course, we impose the condition $j_0 \le N_T$. In the sequel, we assume that

$$\begin{cases} \displaystyle\int_\Gamma u_{\Gamma \ell} \, u_{\Gamma \ell} \, \rho \, d\Gamma = 1 , \quad \ell = 1, +\infty , \\[2mm] (u_{ij}, u_{ij})_i = 1 , \qquad j = 1, +\infty , \quad i = 1, 2 , \\[2mm] (u_k, u_k) = 1 , \qquad (u_k^N, u_k^N) = 1 . \end{cases} \tag{21}$$

Let us denote by P^N the orthogonal projection mapping on the space V_N with respect to the scalar product $a(., .)$.

The following result has been proved in Bourquin [1991, chap. 3] :

THEOREM 1 : *There exists functions* σ_k^N, $\varepsilon_i(s, k, N_i)$, $i = 1, 2$, $\varepsilon_\Gamma(\alpha, k, N_\Gamma)$ *and* $C(k)$, *such that*
i) *If* $\sigma_k^N < 1$, *then*

$$\begin{cases} \lambda_k \le \lambda_k^N \le \lambda_k (1 + \sigma_k^N) , \\[2mm] \| u_k - u_k^N \|_a^2 + | u_k - u_k^N |_{0, \Omega} \le C(k) \, \sigma_k^N . \end{cases} \tag{22}$$

ii) *The relative error* σ_k^N *satisfies*

$$0 \le \sigma_k^N \le C(k) \left[\sum_{i=1}^2 \frac{\varepsilon_i(s, k, N_i)}{\lambda_{iN_i}^{1+s}} + \frac{\varepsilon_\Gamma(\alpha, k, N_\Gamma)}{\lambda_{\Gamma N_\Gamma}^{2\alpha}} \right] , \quad 0 < s < \frac{1}{2}, \tag{23}$$

where

$$\lim_{N_i \to +\infty} \varepsilon_i(s, k, N_i) = \lim_{N_\Gamma \to +\infty} \varepsilon_\Gamma(\alpha, k, N_\Gamma) = 0 . \tag{24}$$

Moreover, there exists two positive bounded and increasing functions $C_1(N_1)$, $C_2(N_2)$ *and a constant* C_Γ *such that*

$$\sigma_k^N \le C(k) \left[\sum_{i=1}^2 \frac{\varepsilon_i(k, N_i)}{(C_i(N_i) \, N_i^{2/n})^{1+s}} + \frac{\varepsilon_\Gamma(\alpha, k, N_\Gamma)}{c_\Gamma \, N_\Gamma^{2\alpha/(n-1)}} \right] , \quad 0 < s < \frac{1}{2}. \tag{25}$$

3. FINITE ELEMENT APPROXIMATION

As mentioned in the introduction, all the modes u_{ij} and $u_{\Gamma\ell}$ have in practice to be computed by a Galerkin method for example. We shall analyse here the « true » component mode synthesis method and give similar error bounds as in theorem 1.

Let h denote a small positive real parameter, and $(V^h)_{h>0}$ a family of finite-dimensional spaces such that $V^h \subset V$, $h > 0$, and for any function v in the space V, $P^h v \to v$ if $h \to 0$, where P^h denotes the orthogonal projection mapping on the space V^h with respect to the scalar product $a(., .)$. This choice thus excludes non-conforming finite element methods. We let

$$
\begin{cases}
\overset{\circ}{V}{}^h_i = V^h \cap \overset{\circ}{V}_i \,, \quad V^h_\Gamma = \mathrm{Tr}_\Gamma (V^h) \,, \\
\dim V^h = N^h \,, \quad \dim \overset{\circ}{V}{}^h_i = N^h_i \quad \text{and} \quad \dim V^h_\Gamma = N^h_\Gamma \,.
\end{cases}
\tag{26}
$$

In the sequel, $P^h_\Gamma : V_\Gamma \to V^h_\Gamma$ denotes the orthogonal projection mapping with respect to the norm $\| \, . \, \|_b$ on the space V^h_Γ.

For each subdomain Ω_i, let us define the approximate fixed interface modes $(\lambda^h_{ij}, u^h_{ij})^{N_i}_{j=1} \in \mathbb{R} \times \overset{\circ}{V}{}^h_i$ as the first N_i $(N_i \leqslant N^h_i)$ solutions of the variational eigenvalue problem

$$
\begin{cases}
\text{find } (\lambda^h, u^h) \in \mathbb{R} \times \overset{\circ}{V}{}^h_i \quad \text{such that} \\
a_i(u^h, v^h) = \lambda^h(u^h, v^h)_i \,, \quad \forall v^h \in \overset{\circ}{V}{}^h_i \,.
\end{cases}
\tag{27}
$$

We impose the functions u^h_{ij} to be normalized in the space $L^2(\Omega_i)$.

Let us introduce the spaces

$$
\begin{cases}
\overset{\circ}{V}{}^h = V^h \cap \overset{\circ}{V} \,; \\
\overset{\circ}{V}{}^h_{N_1, N_2} = \left\{ v^h \in \overset{\circ}{V}{}^h \,; \, v^h|_{\Omega_i} = \sum_{j=1}^{N_i} \beta_{ij} \, u^h_{ij} \,, \quad \beta_{ij} \in \mathbb{R} \right\} \,, \\
\widetilde{V}{}^h_N = \overset{\circ}{V}{}^h_{N_1, N_2} \oplus \mathrm{Span} \left\{ (\widetilde{u}_{\Gamma\ell})^{N_\Gamma}_{\ell=1} \right\} \,,
\end{cases}
\tag{28}
$$

and the corresponding orthogonal projection mappings with respect to the scalar product $a(., .)$ denoted by $\overset{\circ}{P}{}^h$, $\overset{\circ}{P}{}^h_{N_1, N_2}$, and $\widetilde{P}{}^h_N$ respectively. On the other hand, we call

$$
R^h : V^h_\Gamma \to V^h
$$

$$
v^h \to \widetilde{v}^{hh}
$$

the discrete harmonic lifting operator : the function $\tilde{v}^{hh}$ satisfies

$$\begin{cases} a(\tilde{v}^{hh}, w^h) = 0 & \forall w^h \in \overset{\circ}{V}{}^h , \\ \tilde{v}^{hh} = v^h & \text{on } \Gamma . \end{cases} \tag{29}$$

Of course, we do not have $R^h = P^h R$. In this setting, the bilinear form defined by

$$b_h(v^h, w^h) = a(R^h v^h, R^h w^h) \quad \forall v^h, w^h \in V^h_\Gamma \tag{30}$$

enjoys continuity and uniform coerciveness on the space V^h_Γ, endowed with the norm $\|v\|_b = b(v, v)^{1/2}$ of the space V_Γ. Since harmonic functions minimize the energy among all functions matching a prescribed value on the boundary, we have the more precise bound $b_h(v^h, v^h) \geq b(v^h, v^h)$. Therefore, *there exists a family of eigenvalues* $(\lambda^h_{\Gamma\ell})^{N^h_\Gamma}_{\ell=1} \in \mathbb{R}^+$ *and a family of associated eigenfunctions* $(u^h_{\Gamma\ell})^{N^h_\Gamma}_{\ell=1} \in V^h_\Gamma$ *satisfying*

$$\begin{cases} 0 < \lambda^h_{\Gamma 1} \leq \lambda^h_{\Gamma 2} \leq \cdots \leq \lambda^h_{\Gamma N^h_\Gamma} , \\ \lambda_{\Gamma\ell} \leq \lambda^h_{\Gamma\ell} \quad \forall \ell \leq N^h_\Gamma , \\ b_h(u^h_{\Gamma\ell}, v^h) = \lambda^h_{\Gamma\ell} \displaystyle\int_\Gamma u^h_{\Gamma\ell} v^h \rho \, d\Gamma \quad \forall v^h \in V^h_\Gamma . \end{cases} \tag{31}$$

In the sequel, the functions $\rho^{1/2} u^h_{\Gamma\ell}$ are assumed to be normalized in the space $L^2(\Gamma)$.

There exists also a self-adjoint compact operator $T^h : L^2(\Gamma) \to V_\Gamma$, such that its range lies in the space V^h_Γ and

$$b_h(T^h u, v^h) = \int_\Gamma u v^h \rho \, d\Gamma \quad \forall u \in L^2(\Gamma) , \quad \forall v^h \in V^h_\Gamma . \tag{32}$$

Of course the spectra of the bilinear form b_h and of the operator T^h coïncide.

Let us define the approximate static modes by the formula

$$\tilde{u}^{hh}_{\Gamma\ell} = R^h u^h_{\Gamma\ell} \quad \ell = 1, N_\Gamma, \ (N_\Gamma \leq N^h_\Gamma) . \tag{33}$$

The family

$$\mathscr{F}^h_N = \bigoplus_{i=1,2} (u^h_{ij})^{N_i}_{j=1} \oplus (\tilde{u}^{hh}_{\Gamma\ell})^{N_\Gamma}_{\ell=1} , \tag{34}$$

allows to define the approximation space

$$V^h_N = \text{Span } \mathscr{F}^h_N . \tag{35}$$

Let us denote by $P^{N, h}$ the orthogonal projection mapping on the space V_N^h.

It follows from the above definitions that the family $\mathscr{F}_N^h$ remains orthogonal in V^h for the scalar product $a(\,.\,,\,.\,)$. The space V_N^h appears as obtained from the space V_N through a rotation.

The real life component mode synthesis method consists in finding the j_0 first solutions $(\lambda_k^{N, h}, u_k^{N, h})_{k=1}^{j_0} \in \mathbb{R}^+ \times V_N^h$ of the well-posed eigenvalue problem

$$\begin{cases} \text{find } (\lambda^{N, h}, u^{N, h}) \in \mathbb{R} \times V_N^h \quad \text{such that} \\ a(u^{N, h}, v^{N, h}) = \lambda^{N, h}(u^{N, h}, v^{N, h}) \quad \forall v^{N, h} \in V_N^h \,. \end{cases} \tag{36}$$

It depends on four control parameters, N_1, N_2, N_Γ and h, and can be put in the following form :

$$\text{find } (\lambda, X) \in \mathbb{R} \times \mathbb{R}^{N_T}, \quad \text{such that } KX = \lambda MX, \tag{37}$$

where

$$K = \begin{bmatrix} \begin{bmatrix} \lambda_{11}^h & & & 0 \\ & \cdots & & \\ & & \lambda_{1N_1}^h & \\ & & & \lambda_{21}^h \\ & & & & \cdots \\ 0 & & & & \lambda_{2N_2}^h \end{bmatrix} & & 0 \\ & & \\ 0 & & \begin{bmatrix} \lambda_{\Gamma 1}^h & 0 \\ & \cdots & \\ 0 & & \lambda_{\Gamma N_\Gamma}^h \end{bmatrix} \end{bmatrix}, \tag{38}$$

and

$$M = \begin{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & \cdots & & \\ & & 1 & \\ & & & 1 \\ & & & & \cdots \\ 0 & & & & 1 \end{bmatrix} & [(u_{ij}^h, \tilde{u}_{\Gamma k}^{hh})] \\ [(\tilde{u}_{\Gamma \ell}^{hh}, u_{ij}^h)] & [(\tilde{u}_{\Gamma \ell}^{hh}, \tilde{u}_{\Gamma k}^{hh})] \end{bmatrix}. \tag{39}$$

Note that the stiffness matrix is diagonal, whereas the mass matrix possesses off-diagonal terms, whence this kind of method is said to act through mass coupling.

Of course, we assume that $j_0 \leqslant N_T$. Notice that for the sake of simplicity we have neglected the numerical integration.

Consider now the finite element approximation of the global eigenvalue problem, that is (9) with the choice $V = V^h$, and call $(\lambda_k^h, u_k^h)_{k=1}^{N_h} \in \mathbb{R} \times V^h$ the family of corresponding eigenpairs. Classical results in approximation theory (cf. Ciarlet et al. [1968], Strang & Fix [1973], see also Chatelin [1983], and Babuska & Osborn [1990]) allow to assess the error $\lambda_k^h - \lambda_k$, $k = 1, j_0$, in terms of the approximation error associated with the corresponding eigenfunction. Unfortunately, our attempt to obtain *directly* similar bounds of the error $\lambda_k^{N,h} - \lambda_k^h$ by the method developed in Bourquin [1990a] failed, except in the limit case where $N_\Gamma = N_\Gamma^h$, but this situation corresponds to the choice made by Hurty, Craig, and Bampton and will be analysed in section 4. As a matter of fact, we need the vibration problems to be posed in non-variational form as well as in variational form. Variational approximations forbid such a possibility. Nevertheless, the error analysis regarding the real method (36) leads to the following results :

THEOREM 2 : i) *The following inequalities hold :*

$$\lambda_k \leqslant \lambda_k^h \leqslant \lambda_k^{N,h} \quad \forall k \leqslant j_0 . \tag{40}$$

ii) *Moreover, there exists functions* $\sigma_k^{N,h}$, $\varepsilon_i(s, k, N_i)$, $\varepsilon_\Gamma(\alpha, k, N_\Gamma)$, $\varepsilon_h(k, N)$, $C_i(N_i)$ and $C(k)$, such that

$$\begin{cases} \lambda_k^{N,h} - \lambda_k \leqslant \lambda_k \, \sigma_k^{N,h} , \\ \|u_k - u_k^{N,h}\|_a^2 + \|u_k - u_k^{N,h}\|_{L^2(\Omega)} \leqslant C(k) \, \sigma_k^{N,h} , \end{cases} \quad \text{if } \sigma_k^{N,k} < 1 , \tag{41}$$

and

$$\sigma_k^{N,h} \leqslant C(k) \, \lambda_k^2 \left[\sum_{i=1}^{2} \frac{\varepsilon_i(s, k, N_i)}{\lambda_{iN_i}^{1+s}} + \frac{\varepsilon_\Gamma(\alpha, k, N_\Gamma)}{\lambda_{\Gamma N_\Gamma}^{2\alpha}} \right] + \varepsilon_h(k, N) , \tag{42}$$

or equivalently

$$\sigma_k^{N,h} \leqslant C(k) \left[\sum_{i=1}^{2} \frac{\varepsilon_i(s, k, N_i)}{\left(C_i(N_i) N_i^{\frac{2}{n}} \right)^{1+s}} + \frac{\varepsilon_\Gamma(\alpha, k, N_\Gamma)}{N_\Gamma^{\frac{2\alpha}{n-1}}} \right] + \varepsilon_h(k, N) , \tag{43}$$

where the functions ε_i *and* ε_Γ *satisfy (24) ; the functions* $C_i(N_i)$ *enjoy the same properties as in theorem 1.*

The discretization error $\varepsilon_h(k, N)$ *is bounded as follows :*

$$\varepsilon_h(k, N) \leqslant \sum_{i=1}^{2} \sum_{j=1}^{N_i} C_{ij} \|(I - \mathring{P}^h) u_{ij}\|_a^2 + \sum_{\ell=1}^{N_\Gamma} C_\ell \|(I - P^h) \tilde{u}_{\Gamma\ell}\|_a^2 . \tag{44}$$

∎

Some comments are necessary :

i) First of all, the discretization error $\varepsilon_h(k, N)$ tends to zero with h, since we deal with a convergent sequence of approximation spaces $(P^h \to I$ pointwise).

ii) Inequality (40) means in particular that component mode synthesis cannot enhance the accuracy of the finite element solution.

iii) As expected, mode truncation error and discretization error add to one another. This feature essentially reflects the method of proof. The main difference with the result of theorem 1 lies in the emergence of a possible balance between both types of error. Taking into account a large number of modes deteriorates the discretization error inasmuch as the approximation errors of all component modes add to one another, and more especially as the latter may grow rather fast with the eigenvalue reference number j or ℓ. On the other hand, mode truncation error remains insensitive to the discretization parameter h. Thus, both errors play a different role from each other. In practice, the number of modes N should be chosen first, and then, for that value of N, the parameter h should be adjusted to a suitable value insuring the discretization error to stay of the order of magnitude of the prescribed accuracy. Furthermore, if numerical results need be improved, the parameters N and h have to be modified simultaneously in such a way that the errors of the different types remain of the same order of magnitude. Nevertheless, from Poincaré's [1890] Min-Max principle, fixing the discretization parameter h and increasing the number of modes will improve the approximation, but this may not be always the best strategy. Note that a similar balance of error has been observed by Door [1989], who suggests to enforce interdomain coupling by means of Lagrange multipliers. This strategy leads to a stability condition limiting the number of Lagrange multipliers with respect to the mesh size.

iv) As a consequence of the above remark, and assuming theorem 3 to state optimal bounds, discretization may impair the accuracy of the proposed method, except if the mode truncation error behaves nicely : in that case, a very small number of modes will be sufficient to control the process, keeping the discretization error at a low magnitude for a reasonable value of the mesh size. This is why our basic numerical test will concern the intermediate method, which allows discretization effects to be engineered away, and for which we shall give in section 6 an example of application.

However we can refine in some sense the results given in theorem 2 :

THEOREM 3 : *Define the function* $w_k \in \overset{\circ}{V}$ *solution of problem* (5) *with* $f = \lambda_k u_k$. *Then there exists functions* $\sigma_k^{N,\,h}$, $\varepsilon_\Gamma(\alpha, \ k, \ N_\Gamma)$, $\varepsilon_2(k, h)$, $\varepsilon_3(k, \ h, \ N_\Gamma)$, $C(k)$, $C_\imath(N_\imath)$, $\imath = 1, 2$, *and constants* C_ℓ *such that* (41) *holds and*

$$\sigma_k^{N,\,h} \leqslant C\,(k) \left[\sum_{\iota=1}^{2} \frac{1}{\lambda_{\iota N_\iota}^{h}} + \frac{\varepsilon_\Gamma(\alpha,\,k,\,N_\Gamma)}{\lambda_{\Gamma N_\Gamma}^{2\alpha}} \right] + \varepsilon_2(k,\,h) + \varepsilon_3(k,\,h,\,N_\Gamma),\quad (45)$$

and

$$\sigma_k^{N,\,h} \leqslant C\,(k) \left[\sum_{\iota=1}^{2} \frac{1}{C_\iota(N_\iota)\,N_\iota^{\frac{2}{n}}} + \frac{\varepsilon_\Gamma(\alpha,\,k,\,N_\Gamma)}{C_\Gamma\,N_\Gamma^{\frac{2\alpha}{n-1}}} \right] + \varepsilon_2(k,\,h) +$$

$$+ \varepsilon_3(k,\,h,\,N_\Gamma).\quad (46)$$

where the function ε_Γ satisfies (23) and

$$\begin{cases} \varepsilon_2(k,\,h) \leqslant C\,(k) \left\| (I - \overset{\circ}{P}{}^{h})\,w_k \right\|_a^2, \\[2mm] \varepsilon_3(k,\,h,\,N_\Gamma) \leqslant C\,(k) \sum_{\ell=1}^{N_\Gamma} C_\ell \left\| (I - P^{h})\,\tilde{u}_{\Gamma\ell} \right\|_a^2. \end{cases} \quad (47)$$

∎

Let us briefly comment on those results :

i) Notice that we have $\lim\limits_{h\to 0} \varepsilon_2(k,\,h) = \lim\limits_{h\to 0} \varepsilon_3(k,\,h,\,N_\Gamma) = 0$, since we deal with a convergent family of approximation spaces.

ii) Theorem 2 is a priori not better than theorem 1, even if $\lambda_{\iota N_\iota}^{h} \geqslant \lambda_{\iota N_\iota}$, but brings an enhancement as far as *discretization error* is concerned : by comparing (44) and (46), this error does not depend any more on the number N_ι of fixed interface modes taken into account ; therefore, mode truncation error and discretization error tend to decouple. However, they remain coupled through the static modes $\tilde{u}_{\Gamma\ell}$.

Proof of theorem 2 : Inequalities (36) follow directly from the Min-Max principle. Proof of inequality (38) is broken in two steps.

Step 1 : error splitting.

Let $V\,(k)$ denote the direct sum of the first eigenspaces associated respectively with λ_1, λ_2, ..., λ_k. We recall without proof the following result (*cf.* Ciarlet *et al.* [1968], Strang & Fix [1973], see also Chatelin [1983], and Babuska & Osborn [1990]), which holds for any convergent sequence of approximation spaces V_N.

LEMMA 1 : *There exists functions* σ_k^N, $C\,(k)$, $C'\,(k)$ *such that*

$$\begin{cases} \lambda_k \leqslant \lambda_k^N \leqslant \lambda_k(1 + \sigma_k^N) & \text{if } \sigma_k^N < 1, \\[2mm] \left\| u_k - u_k^N \right\|_{L^2(\Omega)} + \left\| u_k - u_k^N \right\|_a^2 \leqslant C'\,(k)\,\sigma_k^N & \text{if } \sigma_k^N < 1, \\[2mm] \sigma_k^N \leqslant C\,(k) \sup\limits_{\substack{v\in V(k) \\ (v,\,v)=1}} \left\| v - P^N v \right\|_a^2. \end{cases} \quad (48)$$

∎

From lemma 1 applied to the sequence of approximation spaces V_N^h, the relative error $\sigma_k^{N,h}$ is controlled by the approximation error

$$E^{N,h,k} = \left\| u_k - P^{N,h} u_k \right\|_a^2. \tag{49}$$

Using triangular and Schwarz's inequality, together with definition (35) and normalization, we obtain

$$E^{N,h,k} \leqslant E^{N,k} + C(k) \left(\sum_{\iota=1}^{2} \sum_{j=1}^{N_\iota} \left\| u_{\iota j} - u_{\iota j}^h \right\|_a^2 + \sum_{\ell=1}^{N_\Gamma} \left\| \tilde{u}_{\Gamma \ell} - \tilde{u}_{\Gamma \ell}^{hh} \right\|_a^2 \right), \tag{50}$$

where

$$E^{N,k} = \left\| u_k - P^N u_k \right\|_a^2, \tag{51}$$

and

$$C(k) = \left[\sum_{\iota=1}^{2} \sum_{j=1}^{N_\iota} ({}^k\alpha_{\iota j})^2 + \sum_{\ell=1}^{N_\Gamma} ({}^k\alpha_{\Gamma \ell})^2 \right], \tag{52}$$

if we have the expansion $u_k = \sum_{\iota=1}^{2} \sum_{j=1}^{+\infty} {}^k\alpha_{\iota j} u_{\iota j} \sum_{\ell=1}^{+\infty} {}^k\alpha_{\Gamma \ell} \tilde{u}_{\Gamma \ell}$. Again, normalization yields $C(k) = \lambda_k \max(\lambda_{11}^{-1}, \lambda_{21}^{-1}, \lambda_{\Gamma 1}^{-1})$.

Step 2 : estimate of the gap between the spaces V_N^h and V_N.

Applying lemma 1 to problem (27), there exists constants $C_{\iota j}$ such that

$$\left\| u_{\iota j} - u_{\iota j}^h \right\|_a^2 \leqslant C_{\iota j} \left\| (I - \mathring{P}^h) u_{\iota j} \right\|_a^2; \tag{53}$$

it remains to evaluate the last terms of (50) by similar techniques. The following lemma indicates the expected accuracy of the eigenpairs related to the Poincaré-Steklov operator when the latter are computed by a Galerkin method as described in (31) and (33).

LEMMA 2 : *There exists a constant C_ℓ such that*

$$\begin{cases} \lambda_{\Gamma \ell} \leqslant \lambda_{\Gamma \ell}^h \leqslant \lambda_{\Gamma \ell}(1 + \sigma^{\ell,h}), \\ \left\| \tilde{u}_{\Gamma \ell} - \tilde{u}_{\Gamma \ell}^{hh} \right\|_a \leqslant \sigma^{\ell,h}, \\ \sigma^{\ell,h} = C_\ell \left\| (I - P^h) \tilde{u}_{\Gamma \ell} \right\|_a. \end{cases} \tag{54}$$

■

That the order of convergence is the same for the eigenvalues and the corresponding eigenfunctions follows from the fact that we do not have $T^h = P_\Gamma^h T$, where T and T^h are defined by (16) and (32) respectively.

Proof : Let $I^{1/2} : V_\Gamma \to L^2(\Gamma)$ denote the canonical injection mapping. In order to apply Chatelin [1983, theorem 6.7], it suffices to prove that

$$T^h \circ I^{1/2} \to T \circ I^{1/2} \quad \text{in operator norm ,} \quad (55)$$

or that $T^h \to T$ pointwise, since the operator $I^{1/2}$ is compact.

For arbitrary functions $u \in L^2(\Gamma)$ and $v \in V_\Gamma$, we have

$$b((T^h - T) u, v) = b((T^h - T) u, P^h_\Gamma v) - b((I - P^h_\Gamma) Tu, v) ; \quad (56)$$

on the other hand, for every function $v^h \in V^h_\Gamma$ the following identity follows from definition (14) :

$$b((T^h - T) u, v^h) = a(RTr_\Gamma (R^h T^h - RT) u, Rv^h) ;$$

since the function Rv^h is harmonic, we obtain

$$b((T^h - T) u, v^h) = a((R^h T^h - RT) u, Rv^h) , \quad (57)$$

or equivalently

$$b((T^h - T) u, v^h) = a((R^h T^h - RT) u, P^h Rv^h) - a((I - P^h) RTu, Rv^h) . \quad (58)$$

But for every function $w^h \in V^h$, the following identities hold :

$$\begin{cases} a(R^h T^h u, w^h) = a(R^h T^h u, R^h \text{Tr}_\Gamma w^h) , \\ a(RTu, w^h) = a(RTu, R \text{Tr}_\Gamma w^h) . \end{cases} \quad (59)$$

Hence, from the definitions (14), (16), (30) and (32), it follows that

$$a((R^h T^h - RT) u, w^h) = 0 . \quad (60)$$

Combining (56), (58) and (60), we obtain

$$\| (T^h - T) u \|_b \leq C (\| (I - P^h) RTu \|_a + \| (I - P^h_\Gamma) Tu \|_b) . \quad (61)$$

Since we have $\| (I - P^h_\Gamma) v \|_b \leq \| v - v^h \|_b$ for every function v^h in V^h_Γ, the choice $v^h = \text{Tr}_\Gamma P^h Rv$ in this inequality leads to the bound

$$\| (I - P^h_\Gamma) v \|_b \leq C \| (I - P^h) Rv \|_a . \quad (62)$$

Thus, estimates (61) and (62) yield

$$\| (T - T^h) u \|_b \leq C \| (I - P^h) RTu \|_a , \quad (63)$$

and (55) holds, since $P^h \to I$ pointwise.

Therefore the family $T^h \circ I^{1/2}$ forms a strongly stable approximation of the compact operator $T \circ I^{1/2}$. Hence, estimates indicated in (54) follow directly from Chatelin [1983, theorem 6.7] and from inequalities (31) and (63). Thus lemma 4 is proved. ∎

From (50), (53), and (54), one can derive the following estimate :

$$E^{N,h,k} \leqslant E^{N,k} + C(k) \times$$

$$\times \left[\sum_{i=1}^{2} \sum_{j=1}^{N_i} C_{ij} \left\| (I - \overset{\circ}{P}{}^h) u_{ij} \right\|_a^2 + \sum_{\ell=1}^{N_\Gamma} C_\ell \left\| (I - P^h) \tilde{u}_{\Gamma\ell} \right\|_a^2 \right] . \quad (64)$$

Recalling that the approximation error $E^{N,k}$ satisfies the same inequalities as σ_k^N (cf. Bourquin [1991, chap. 3]), namely (23) and (25), inequality (64) yields (42), (43) and (44) and complete the proof of theorem 2. ∎

Proof of theorem 3 : Again, we have to estimate the error $E^{N,h,k}$ defined by (49). We obtain immediately

$$E^{N,h,k} \leqslant \tilde{E}^{N,h,k} + \inf_{v \in V_N^h} \left\| \tilde{P}^{N,h} u_k - v \right\|_a^2 , \quad (65)$$

where the projection mapping $\tilde{P}^{N,h}$ is defined after (28) and

$$\tilde{E}^{N,h,k} = \left\| (I - \tilde{P}^{N,h}) u_k \right\|_a^2 . \quad (66)$$

On the other hand, definition (28) leads to the identity

$$V = \overset{\circ}{V}{}^h_{N_1, N_2} \oplus (\overset{\circ}{V}{}^h_{N_1, N_2})^{\perp \overset{\circ}{V}{}^h} \oplus (\overset{\circ}{V}{}^h)^{\perp \overset{\circ}{V}} \oplus \mathrm{Span} \left\{ (\tilde{u}_{\Gamma\ell})_{\ell=1}^{+\infty} \right\} , \quad (67)$$

and we can decompose the function u_k according to (67) ; it follows that

$$u_k = (I - \overset{\circ}{P}{}^h)(u_k - R \, \mathrm{Tr}_\Gamma \, u_k) + \sum_{i=1,2}^{N_i^h} \sum_{j=1}^{N_i^h} {}^k \alpha_{ij}^h u_{ij}^h + \sum_{\ell=1}^{+\infty} {}^k \alpha_{\Gamma\ell} \tilde{u}_{\Gamma\ell} . \quad (68)$$

Therefore, by obvious orthogonality properties, we have

$$\tilde{P}^{N,h} u_k = \sum_{i=1,2}^{N_i} \sum_{j=1}^{N_i} {}^k \alpha_{ij}^h u_{ij}^h + \sum_{\ell=1}^{N_\Gamma} {}^k \alpha_{\Gamma\ell} \tilde{u}_{\Gamma\ell} , \quad (69)$$

and the approximation error (66) can be rewritten

$$\tilde{E}^{N,h,k} = \sum_{i=1,2} \sum_{j=1+N_i}^{N_i^h} \lambda_{ij}^h ({}^k \alpha_{ij}^h)^2 + \left\| (I - \overset{\circ}{P}{}^h) w_k \right\|_a^2 +$$

$$+ \sum_{\ell=1+N_\Gamma}^{+\infty} \lambda_{\Gamma\ell} ({}^k \alpha_{\Gamma\ell})^2 , \quad (70)$$

since the function w_k satisfies $w_k = u_k - R \, \mathrm{Tr}_\Gamma \, u_k$.

It remains to mimick the proof of theorem 1 given in Bourquin [1991, chap. 3] : let us first compute the coefficients ${}^{k}\alpha_{ij}^{h}$ and ${}^{k}\alpha_{\Gamma\ell}$: from (9) with the choice $v = u_{ij}^{h}$ and orthogonality properties of the functions u_{ij}^{h}, we obtain

$$ {}^{k}\alpha_{ij}^{h} \, \lambda_{ij}^{h} = \lambda_{k}(u_{k}, u_{ij}^{h}) \, . \tag{71} $$

In the same way, we get

$$ {}^{k}\alpha_{\Gamma\ell} \, \lambda_{\Gamma\ell} = \lambda_{k}(u_{k}, \tilde{u}_{\Gamma\ell}) \, . \tag{72} $$

Since the family $(u_{ij}^{h})_{j=1}^{N_{i}^{h}}$ is orthonormalized in the space $L^{2}(\Omega_{i})$, normalization condition $(u_{k}, u_{k}) = 1$, together with (70), (71), (72), and Bourquin [1991, chap. 3, lemma 5] yields

$$ \tilde{E}^{N, h, k} \leqslant \lambda_{k}^{2} \left[\sum_{i=1,2} \frac{1}{\lambda_{iN_{i}}^{h}} + \frac{\varepsilon_{\Gamma}(\alpha, k, N_{\Gamma})}{\lambda_{\Gamma N_{\Gamma}}^{2\alpha}} \right] + \left\| (I - \overset{\circ}{P}{}^{h}) \, w_{k} \right\|_{a}^{2} , \tag{73} $$

where the function $\varepsilon_{\Gamma}(\alpha, k, N_{\Gamma})$ satisfies (24).

On the other hand, the last term in (65) can be estimated thanks to (69) and lemma 2, as follows :

$$ \inf_{v \in V_{N}^{h}} \left\| \tilde{P}^{N, h} u_{k} - v \right\|_{a}^{2} \leqslant \frac{\lambda_{k}}{\lambda_{\Gamma 1}} \sum_{\ell=1}^{N_{\Gamma}} C_{\ell} \left\| (I - P^{h}) \, \tilde{u}_{\Gamma\ell} \right\|_{a}^{2} . \tag{74} $$

Lemma 1, inequalities (65), (73) and (74) complete the proof of theorem 3.
∎

So far, the type of approximation space V^{h} remains arbitrary. In nowadays practice, conforming finite elements are widely used for this kind of analysis. If the parameter h represents the typical size of a uniform mesh, the convergence rate of the functions $\varepsilon_{h}(k, N)$, $\varepsilon_{2}(k, h)$, and $\varepsilon_{3}(k, h, N_{\Gamma})$ depends on the estimated regularity of the functions to be approximated, presently u_{ij} and $\tilde{u}_{\Gamma\ell}$. By hypothesis, we already know that $u_{ij} \in H^{1+\alpha}(\Omega_{i})$, $\alpha > 0$. On the other hand, it is possible to predict the minimal regularity of the static modes $\tilde{u}_{\Gamma\ell}$ in the case of model problems (see Lemrabet [1977, 1978]). One can prove that the function $u_{\Gamma\ell}$ is smooth only in the interior of the interface Γ. In any case, if there exists a real number $\sigma > 0$ such that all the functions w_{k}, u_{ij} and $\tilde{u}_{\Gamma\ell}$ belong to the space $\hat{H}^{1+\sigma}(\Omega)$, where $\hat{H}^{s}(\Omega)$ denotes the usual Besov space defined by the K-method (cf. Babuska & Osborn [1990]), then we obtain

$$ \varepsilon_{h}(k, N) + \varepsilon_{2}(k, h) + \varepsilon_{3}(k, h, N_{\Gamma}) = 0(h^{\sigma}) \, . \tag{75} $$
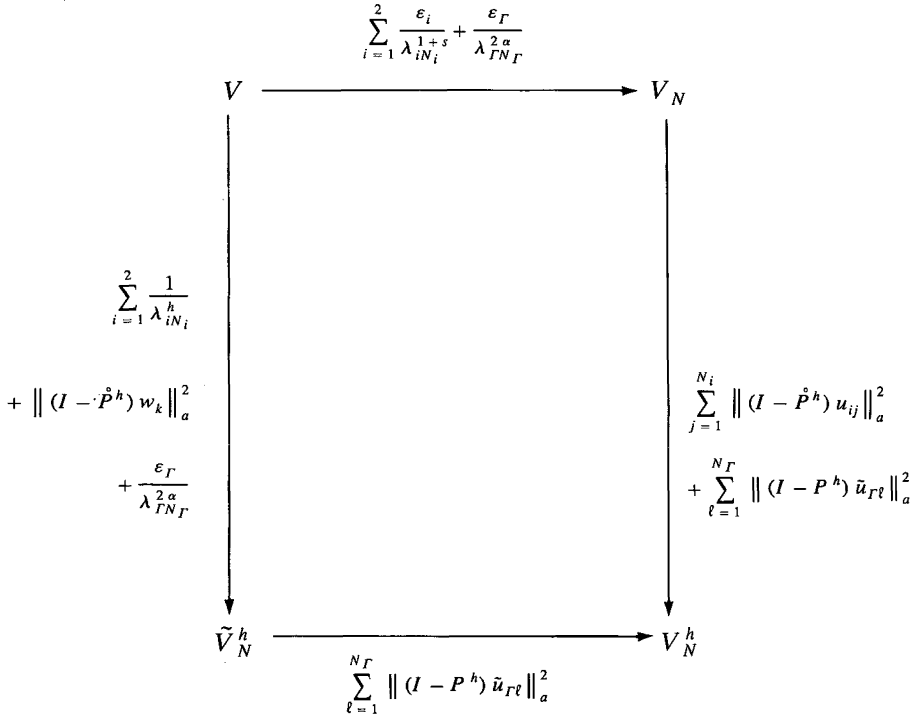
This a priori poor convergence rate can be strikingly enhanced by means of Babuska's *h-p* version of the Finite Element Method with a non-uniform mesh, designed in such a way as to capture the singularities (*cf.* Babuska & Osborn [1990] and included references). We shall not go further in that direction.

However, in the case where all the functions involved in the approximation process are regular on each subdomain, more usual finite element theory (*cf.* Ciarlet [1978]) yields

$$\varepsilon_h(k, N) + \varepsilon_2(k, h) + \varepsilon_3(k, h, N_\Gamma) \leq$$

$$\leq C h^{m-1} \max_{i,j,\ell} (\|w_k\|_{m, \Omega_i}, \|u_{ij}\|_{m, \Omega_i}, \|\tilde{u}_{\Gamma\ell}\|_{m, \Omega_i}), \quad (76)$$

for elements of sufficiently high order, where $\| \ \|_{m, \Omega_i}$ denotes the norm in the space $H^m(\Omega_i)$.

In order to summarize the results of theorems 1, 2 and 3, let us draw a chart indicating the different approximation spaces involved, as well as the approximation or discretization errors that add to one another, thus yielding the error associated with component mode synthesis method :

$$\sum_{i=1}^{2} \frac{\varepsilon_i}{\lambda_{iN_i}^{1+s}} + \frac{\varepsilon_\Gamma}{\lambda_{\Gamma N_\Gamma}^{2\alpha}}$$

$$V \longrightarrow V_N$$

$$\sum_{i=1}^{2} \frac{1}{\lambda_{iN_i}^h}$$

$$+ \|(I - \mathring{P}^h) w_k\|_a^2$$

$$+ \frac{\varepsilon_\Gamma}{\lambda_{\Gamma N_\Gamma}^{2\alpha}}$$

$$\sum_{j=1}^{N_i} \|(I - \mathring{P}^h) u_{ij}\|_a^2$$

$$+ \sum_{\ell=1}^{N_\Gamma} \|(I - P^h) \tilde{u}_{\Gamma\ell}\|_a^2$$

$$\tilde{V}_N^h \longrightarrow V_N^h$$

$$\sum_{\ell=1}^{N_\Gamma} \|(I - P^h) \tilde{u}_{\Gamma\ell}\|_a^2$$

4. ERROR BOUNDS FOR THE HURTY, CRAIG, AND BAMPTON METHOD

Let us now turn to the method developed by Hurty [1965] Craig and Bampton [1968]. It is based on a simple but blind choice of static modes ; the latter are obtained through the lifting in Ω of all the basis functions that span the space V_Γ (or V_Γ^h). Although it has become a popular engineering tool in the last decades, and despite of its already long career, we give and prove in this section the corresponding error bounds, that we could not find in the litterature.

Let us start with a formulation in a continuous framework : we consider here an arbitrary Riesz basis $(w_{\Gamma \ell})_{\ell=1}^{+\infty}$ of the space V_Γ. Then the family $\mathcal{G} = \underset{i=1,2}{\oplus} (u_{ij})_{j=1}^{+\infty} \oplus (\tilde{w}_{\Gamma \ell})_{\ell=1}^{+\infty}$ forms a Riesz basis of the space V (cf. Bourquin [1991, chap. 3]). Introducing the space

$$\tilde{V}_{N_1, N_2} = \underset{i=1,2}{\oplus} \text{Span} \left\{ (u_{ij})_{j=1}^{N_i} \right\} \oplus RV_\Gamma \, , \qquad (77)$$

where, of course $RV_\Gamma = \text{Span} \left\{ (\tilde{w}_{\Gamma \ell})_{\ell=1}^{+\infty} \right\}$, the intermediate method would consist in computing the j_0 first eigenpairs $(\tilde{\lambda}_k^{N_1, N_2}, \tilde{u}_k^{N_1, N_2})_{k=1}^{j_0}$ of the variational problem (9) with $V = \tilde{V}_{N_1, N_2}$. The approximation space $\tilde{V}_{N_1, N_2}$ is infinite-dimensional, but a finite element (or any type of) discretization restaures a finite number of static modes, and therefore of degrees of freedom.

Let $(w_{\Gamma \ell}^h)_{\ell=1}^{N_\Gamma^h}$ denote any basis of the space V_Γ^h. We may introduce the space

$$\tilde{V}_{N_1, N_2}^h = \underset{i=1,2}{\oplus} \text{Span} \left\{ (u_{ij}^h)_{j=1}^{N_i} \right\} \oplus R^h(V_\Gamma^h) \, , \qquad (78)$$

where $R^h(V_\Gamma^h) = \text{Span} \left\{ (\tilde{w}_{\Gamma \ell}^{hh})_{\ell=1}^{N_\Gamma^h} \right\}$; we have $\dim \tilde{V}_{N_1, N_2}^h = N_1 + N_2 + N_\Gamma^h$. This expression depends on h, or more precisely on the number of degrees of freedom associated with the interface.

The real method consists in computing the j_0 first eigenpairs $(\tilde{\lambda}_k^{N_1, N_2, h}, \tilde{u}_k^{N_1, N_2, h})_{k=1}^{j_0}$ of problem (9) with $V = \tilde{V}_{N_1, N_2}^h$. This problem can be formulated thanks to matrices :

$$\text{find } (\lambda, X) \in \mathbb{R}x^{N_1 + N_2 + N_\Gamma^h} \text{ such that } KX - \lambda MX = 0 \, , \qquad (79)$$

where

$$K = \begin{bmatrix} \begin{bmatrix} \lambda^h_{11} & & & & 0 \\ & \ddots & & & \\ & & \lambda^h_{1N_1} & & \\ & & & \lambda^h_{21} & \\ & & & & \ddots \\ 0 & & & & \lambda^h_{2N_2} \end{bmatrix} & & 0 \\ & & \\ & 0 & & [a(\tilde{w}^{hh}_{w\Gamma\ell}, \tilde{w}^{hh}_{\Gamma k})] \end{bmatrix}, \quad (80)$$

and

$$M = \begin{bmatrix} \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ 0 & & & & 1 \end{bmatrix} & [(u^h_{ij}, \tilde{w}^{hh}_{\Gamma k})] \\ & \\ [(\tilde{w}^{hh}_{\Gamma\ell}, u^h_{ij})] & [(\tilde{w}^{hh}_{\Gamma\ell}, \tilde{w}^{hh}_{\Gamma k})] \end{bmatrix}. \quad (81)$$

None of those matrices is diagonal, but since the stiffness matrix K exhibits a clearer sparsity than the mass matrix, this method is said to act through mass coupling. As a matter of fact, the « internal degrees of freedom » (or fixed interface modes) of both subdomains are not coupled through stiffness.

With these notations, the following bounds can be derived :

THEOREM 4 :

i) *For any set of parameters* $N = (N_1, N_2, N_\Gamma)$, *we have*

$$\lambda_k \le \tilde{\lambda}^{N_1, N_2}_k \le \lambda^N_k . \quad (82)$$

ii) *Moreover, there exists functions* $\tilde{\sigma}^{N_1, N_2}_k$, $C(k)$, *such that if* $\tilde{\sigma}^{N_1, N_2}_k < 1$, *then*

$$\begin{cases} \lambda_k \le \tilde{\lambda}^{N_1, N_2}_k \le \lambda_k(1 + \tilde{\sigma}^{N_1, N_2}_k) , \\ \left\| u_k - \tilde{u}^{N_1, N_2}_k \right\|^2_a + \left\| u_k - \tilde{u}^{N_1, N_2}_k \right\|_{0, \Omega} \le C(k) \tilde{\sigma}^{N_1, N_2}_k ; \end{cases} \quad (83)$$

furthermore, the relative error $\tilde{\sigma}^{N_1, N_2}_k$ *satisfies*

$$\tilde{\sigma}^{N_1, N_2}_k \le C(k) \sum_{i=1}^{2} \frac{\varepsilon_i(s, k, N_i)}{\lambda^{1+s}_{iN_i}} , \quad 0 < s < \frac{1}{2} , \quad (84)$$

where $\lim\limits_{j \to +\infty} \varepsilon_i(s, k, j) = 0$, $i = 1, 2$, *and there exists functions* $C_i(N_i)$

satisfying the same property as in theorem 1, such that

$$\tilde{\sigma}_k^{N_1, N_2} \leqslant C(k) \sum_{i=1}^{2} \frac{\varepsilon_i(k, N_i)}{C_i(N_i) N_i^{\frac{2(1+s)}{n}}}, \qquad 0 < s < \frac{1}{2}. \tag{85}$$

∎

In the same way, the following result can be stated regarding the discretized version of the Hurty, Craig and Bampton method:

THEOREM 5 :
i) *For every set of parameters* $N = \{(N_1, N_2, N_\Gamma), h\}$, *we have*

$$\lambda_k \leqslant \lambda_k^h \leqslant \tilde{\lambda}_k^{N_1, N_2, h} \leqslant \lambda_k^{N, h}. \tag{86}$$

ii) *Moreover, there exists functions* $\tilde{\sigma}_k^{N_1, N_2, h}$, $C(k)$, *such that, if* $\tilde{\sigma}_k^{N_1, N_2, h} < 1$, *then*

$$\begin{cases} \lambda_k \leqslant \tilde{\lambda}_k^{N_1, N_2, h} \leqslant \lambda_k (1 + \tilde{\sigma}_k^{N_1, N_2, h}), \\ \left\| u_k - \tilde{u}_k^{N_1, N_2, h} \right\|_a^2 + \left\| u_k - \tilde{u}_k^{N_1, N_2, h} \right\|_{0, \Omega} \leqslant C(k) \, \tilde{\sigma}_k^{N_1, N_2, h}; \end{cases} \tag{87}$$

furthermore, the relative error $\tilde{\sigma}_k^{N_1, N_2, h}$ *satisfies*

$$\tilde{\sigma}_k^{N_1, N_2, h} \leqslant C(k) \left[\sum_{i=1}^{2} \frac{1}{\lambda_{iN_i}^h} + \left\| (I - P^h) u_k \right\|_a^2 \right], \tag{88}$$

and

$$\tilde{\sigma}_k^{N_1, N_2, h} \leqslant C(k) \left[\sum_{i=1,2} \frac{1}{C_i(N_i) N_i^{\frac{2}{n}}} + \left\| (I - P^h) u_k \right\|_a^2 \right], \tag{89}$$

where the functions $C_i(N_i)$ *satisfy the same properties as in theorem 1.*

∎

Let us now briefly comment on those results :

i) Inequalities (82) and (86) mean in particular that the Hurty, Craig, and Bampton method is a priori at least as accurate as the method using the eigenpairs of the Poincaré-Steklov operator, but it uses a much larger number of basis functions.

ii) It turns out that mode truncation and discretization errors totally decouple for the Hurty, Craig, and Bampton method. But this nice feature

hides the fact that the number of basis functions $N_1 + N_2 + N_\Gamma^h$ depends on the discretization parameter; in the case of a continuous formulation, the space $\tilde{V}_N$ is even infinite-dimensional, thus forbidding any computational work.

Proof of theorems 4 and 5 : It appeals to the technique already used in the proof of theorems 1, 2 and 3. Inequalities (82) and (86) are standard applications of Poincaré's [1890] Min-Max principle. We briefly sketch the rest of the proof, starting with theorem 4 : from lemma 1, it suffices to assess the approximation error

$$\tilde{E}^{N_1, N_2, k} = \left\| (I - \tilde{P}^{N_1, N_2}) u_k \right\|_a^2, \tag{90}$$

where $\tilde{P}^{N_1, N_2}$ denotes the orthogonal projection mapping on the space $\tilde{V}_{N_1, N_2}$ defined by (77). Since the following decomposition holds :

$$u_k = R \operatorname{Tr}_\Gamma u_k + \sum_{i=1,2} \sum_{j=1}^{+\infty} {}^k\alpha_{ij}\, u_{ij}, \tag{91}$$

so that

$$(I - \tilde{P}^{N_1, N_2}) u_k = \sum_{i=1,2} \sum_{j > N_i} {}^k\alpha_{ij}\, u_{ij}, \tag{92}$$

the error (90) can be written

$$\tilde{E}^{N_1, N_2, k} = \sum_{i=1,2} \sum_{j > N_i} \lambda_{ij} ({}^k\alpha_{ij})^2. \tag{93}$$

The coefficients ${}^k\alpha_{ij}$ are computed in the same way as in theorem 1 (*cf.* Bourquin [1991, chap. 3]), thus we obtain

$$\tilde{E}^{N_1, N_2, k} \leqslant \lambda_k^2 \sum_{i=1,2} \frac{\varepsilon_i(s, k, N_i)}{\lambda_{iN_i}^{1+s}}, \quad 0 < s < \frac{1}{2}, \tag{94}$$

where

$$\varepsilon_i(s, k, N_i) = \sum_{j > N_i} (u_k, u_{ij})_i^2\, \lambda_{ij}^s.$$

Those inequalities, together with lemma 1 and orthogonality properties of the functions u_{ij} yield (82), (83) and (84). Inequality (85) is a direct consequence of (84), of the Min-Max principle, and of Weyl's formula which states that the *j*-th eigenvalue μ_j of a self-adjoint elliptic operator of order $2m$ on a bounded domain in $\mathbb{R}^n$ satisfies

$$\mu_j \sim C j^{\frac{2m}{n}} \quad \text{when} \quad j \to +\infty. \tag{95}$$

■

The proof of theorem 5 relies on similar arguments, except that we have to take advantage of the following orthogonal decompositions :

$$u_k = (I - P^h) u_k + R^h \operatorname{Tr}_\Gamma P^h u_k + \sum_{i=1,2} \sum_{j=1}^{N_i^h} {}^k\alpha_{ij}^h u_{ij}^h, \qquad (96)$$

and consequently

$$(I - \tilde{P}^{N_1, N_2, h}) u_k = (I - P^h) u_k + \sum_{i=1,2} \sum_{j=1+N_i}^{N_i^h} {}^k\alpha_{ij}^h u_{ij}^h,$$

according to (78).   ∎

Remark 3 : Theorems 4 and 5 extend immediately to any self-adjoint coercive eigenvalue problem in variational form. Therefore, *these results apply to the case of non-homogeneous three-dimensional linearized elasticity.*

5. ALGORITHMS

The algorithm of the component mode synthesis method presented in section 3, equation (36), consists basically of five parts :

i) once the control parameters N and h are set up, compute the fixed interface modes $(\lambda_{ij}^h, u_{ij}^h)$ by solving problem (27) ;

ii) compute the static modes, solutions of problems (31) and (33) ;

iii) form the global stiffness and mass matrices K and M defined by (38) and (39) ;

iv) solve the small-scale eigenvalue problem (37), of size $N_T = N_1 + N_2 + N_\Gamma$, which yields the global eigenvalues $\lambda_k^{N, h}$ and the decomposition of the associated eigenfunctions on the family $\mathscr{F}^{N, h}$. Notice that the mass matrix is full, whereas the stiffness matrix is diagonal ;

v) restaure the mode shapes u_k everywhere in the domain Ω.

Steps i), iii), iv) and v) appeal to standard finite element techniques. Steps i) and sometimes ii) and iii) are the most time-consuming ones in such kind of analysis.

The computation of the fixed interface modes is amenable to a Lanczos or a subspace inverse power iteration method (*cf.* Lascaux & Theodor [1986]). Since we expect subdomains to exhibit simple geometrical characteristics, fast solvers (FFT, multigrid...) may be used, and a clever initial guess for the N_i eigenvectors may be proposed, instead of a blind one. On the other hand, any uniform change of the thermomechanical properties in one subdomain leaves the corresponding fixed interface modes unchanged. In

addition, local geometrical modifications of a given subdomain does not alter the modes of the other ones. Thus repeated modal analysis are made possible at a reasonable cost, allowing for structural optimization against vibrations or buckling (*cf.* Valid [1982]).

Step ii) may be processed by a Lanczos method or an inverse power method using repeatedly the resolution of the source problem $(T^h)^{-1} u^h = f^h$, which can be treated by various iterative techniques (*cf.* Agoshkov [1989]) that usually take advantage of the natural decomposition of the operator $(T^h)^{-1}$ in a sum of two (or more) inverse Calderon operators. This feature allows to exploite the coarse granularity of modern parallel computer architectures. Among current researches in the field of domain decomposition (*cf.* for example Bjorstad & Widlund [1986], Bramble *et al.* [1986]) a new preconditioned conjugate gradient method (*cf.* Bourgat *et al.* [1988]) seems to yield rather promising performances (*cf.* Bourquin & d'Hennezel [1992]).

An alternative approach would consist in expressing the Poincaré-Steklov operator as a singular integral operator by using a single or double layer potential representation ; inverse power techniques would basically amount to compute repeatedly the action of such an operator on a function defined on the interface. Therefore, the determination of the spectrum of Poincaré-Steklov's operators is amenable to the algorithm recently proposed by Beylkin *et al.* [1989], and based on the « fast wavelet transform ». However, the presence of corners in the subdomains may deteriorate the speed of the algorithm, according to Coifman [1989].

Of course, a boundary element method remains a more traditional way of tackling the problem.

Let us now turn to the method developed by Hurty [1965], Craig, and Bampton [1968]. It is based on a different choice of static modes ; the latter are obtained through the lifting in Ω of all the N_Γ^h basis functions that span the space V_Γ^h ; in the case of conforming Lagrange elements, this strategy amounts to compute the functions $(R^h w_{\Gamma\ell}^h)_{\ell=1}^{N_\Gamma^h}$, where the function $w_{\Gamma\ell}^h$ matches a unit value at node ℓ and vanishes at all other nodes. This method avoids the resolution of the eigenvalue problem (31) and step ii) becomes very simple from the algorithmic point of view. It depends on three parameters N_1, N_2 and h ; mode truncation remains tied to the discretization. It is known to give accurate results and to fit easily in most finite element codes (*cf.* Imbert [1979]). Nevertheless, it does not allow for mesh refinement in large-scale computations, because the *number of unknowns $N_T = N_1 + N_2 + N_\Gamma^h$ of problem* (37) *may rapidly become large and the mass matrix remains full.* In addition, the stiffness matrix is far from being diagonal as in our method. Furthermore, when the number of degrees of freedom on the interface Γ increases, computing the static modes amounts

to solve $2 N_\Gamma^h$ Dirichlet problems, and forming the mass and stiffness matrices (80) and (81) involves the computation of at least $2(N_\Gamma^h)^2$ scalar products as well as $2(N_\Gamma^h)$ matrix vector products. *In this situation, the time-consuming steps iii) and iv) deteriorate the computational efficiency of the method.*

On the contrary, our method leads to a more sophisticated algorithm (because of step ii) but the latter does not deteriorate under mesh refinement : the size N_T of the final eigenvalue problem remains independent of the mesh size. It will be shown numerically in the next section that only few static modes are necessary to achieve a stringent accuracy, such that *steps iii) and iv)* should *remain very cheap, even in large-scale computations.* Properly preconditioned conjugate gradient algorithms may keep the number of necessary iterations in step ii) independent of the mesh size.

6. NUMERICAL TESTS

We assess here in relatively simple cases the quality of the approximation rather than the speed of the algorithm that involves the various ways of using component mode synthesis. This problem will be addressed somewhere else.

6.1. The intermediate method

In this section, numerical tests of the intermediate method are presented. They allow first of all to discuss the sharpness of the error bounds given in theorems 1 or 2 and second of all to determine how many fixed interface and static modes are necessary to achieve a reasonable accuracy.

For that purpose, let us consider the eigenvalue problem associated with heat diffusion through an heterogeneous medium : we define the bilinear forms

$$a_i(u, v) = k_i \int_{\Omega_i} \nabla u \, \nabla v \, dx \quad \forall u, v \in H^1(\Omega_i), \quad i = 1, 2, \quad (97)$$

and define the form $a(u, v)$ as in (5). Let us impose homogeneous Dirichlet boundary conditions on $\partial\Omega$ and assume that $k_1 \neq k_2$. The above theory applies to the eigenvalue problem (9). If Ω_1 and Ω_2 are congruent to the unit square in $\mathbb{R}^2$ (*cf. fig.* 2), the different modes of interest for the intermediate method are given by the following explicit formulas :

$$\begin{cases} j = (n_1, n_2) \in \mathbb{N}^2, \quad \lambda_{2j} = k_2 \, \Pi^2(n_1^2 + n_2^2), \quad \lambda_{1j} = \dfrac{k_1}{k_2} \lambda_{2j}, \\ u_{2j} = 2 \sin n_1 \, \Pi x \sin n_2 \, \Pi y \quad \text{in } \Omega_2, \quad u_{1j}(x, y) = u_{2j}(-x, y), \end{cases} \quad (98)$$

and

$$\begin{cases} \lambda_{\Gamma\ell} = (k_1 + k_2)\,\ell\Pi \coth \ell\Pi\,, \\ \tilde{u}_{\Gamma\ell} = \operatorname{sh} \ell\Pi(1-x)\sin \ell\Pi y \quad \text{in } \Omega_2\,, \\ \tilde{u}_{\Gamma\ell}|_{\Omega_1}(x,y) = \tilde{u}_{\Gamma\ell}|_{\Omega_2}(-x,y)\,. \end{cases} \tag{99}$$

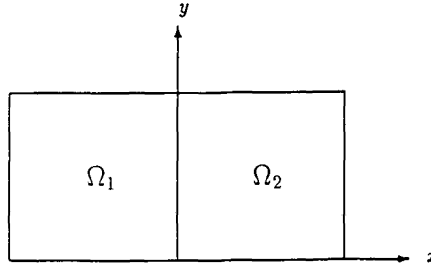Expressions of system (99) stem from the resolution of problem (18) by separation of variables.

Figure 2. — The tested domain.

The mass and stiffness matrices can be computed explicitely and problem (9) with $V = V_N$ solved very easily. The approximate eigenvalues and corresponding mode shapes are compared with the eigenpairs computed « blindly » by means of the finite element code CESAR using about 2 000 nodes of Pi-Lagrange triangles. We let the parameters N_1, N_Γ vary and fix $k_1 = 1$ and $N_2 = 30$.

Before to elaborate on the results, let us point out that the model problem presented above allows component mode synthesis method to exhibit some interesting features : when the diffusivities k_1 and k_2 are different, the global eigenvalue problem (9) with $V = H_0^1(\Omega)$ does not possess explicit solutions to our knowledge, whereas the local eigenvalue problems (2) do, as well as (18). Furthermore, the transmission condition $k_1 \dfrac{\partial u}{\partial n_1} + k_2 \dfrac{\partial u}{\partial n_2} = 0$ on Γ forbids any high-order global regularity, but the fixed interface and static modes are smooth up to the boundary in each subdomain.

Formula (99) proves the sharpness of the lower bounds associated with the eigenvalues $(\lambda_{\Gamma\ell})_{\ell=1}^\infty$ (cf. Bourquin [1991, chap. 3]). Moreover, the functions $u_{\Gamma\ell}$ are regular in $\bar{\Gamma}$. Their harmonic liftings are also regular in each subdomain.

Let us now comment on the numerical results. We have represented the two first eigenfunctions (cf. fig. 3 and 4) computed with our method (fig. 3b and 4b) for the choice $N_1 = 8$, $N_2 = 6$, $N_\Gamma = 3$, and with the finite element method (fig. 3a and 4a). Since eigenfunctions obtained via both methods have not been normalized in the same way, the reader's attention should be

focused on their general shape. Notice that their level lines possess a corner point when they cross the interface Γ; this is due to the transmission condition that implies a discontinuity of the normal derivative at this location. *The discontinuity is very well captured by component mode synthesis,* but this feature does not seem to be generic.

By looking at the « Fourier coefficients » ${}^k\alpha_{ij}^N$ and ${}^k\alpha_{\Gamma\ell}^N$ weighting the decomposition of the function u_k^N on the family $\mathscr{F}$, it is possible to know which component modes are effectively tuned. In the following table, ε denotes various numbers inferior to 10^{-7}.

Ω_1

	u_1^N	u_2^N
1	1.1	2.6×10^{-2}
2	ε	1.5×10^{-6}
3	4×10^{-2}	6.1×10^{-1}
4	ε	ε
5	ε	ε
6	1.2×10^{-2}	6×10^{-2}
7	ε	ε
8	ε	ε

Ω_2

	u_1^N	u_2^N
1	1.9×10^{-2}	10^{-1}
2	ε	ε
3	5.5×10^{-3}	2.2×10^{-1}
4	ε	ε
5	ε	ε
6	2×10^{-3}	7.6×10^{-3}

Γ

	u_1^N	u_2^N
1	3×10^{-1}	4.9×10^{-1}
2	ε	ε
3	ε	ε

It is noteworthy that only the first static mode is tuned for the first two global eigenfunctions u_1^N and u_2^N; the computation of higher order eigenfunctions (up to the seventh) involves only three of them.

Eigenvalues also behave nicely; first of all we fix the values of N_Γ and N_2 and let N_1 vary; the error is plotted in figure 5 as well as an average slope in log-log scale which represents the observed numerical convergence rate with respect to the number of fixed interface modes, that is the value of β in some fitting curve $N_\Gamma \to a + \dfrac{b}{(N_\Gamma)^\beta}$. The error decays *exactly as predicted* since the value of β is about 1.5. Based on other tests, we can notice that the *numerical convergence rate does not depend on the value of the diffusivities.*

We also fix the value of N_1 and N_2, let N_Γ vary, and plot the analogous quantities. The *error decays very fast,* since in general two or three of them are sufficient to yield a stringent accuracy, as shown on figure 6 (for $k_1 = 1$, $k_2 = 5$).
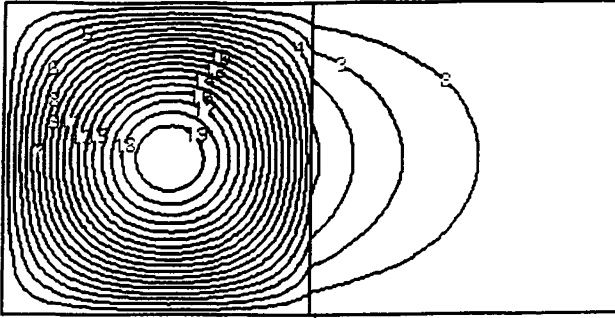
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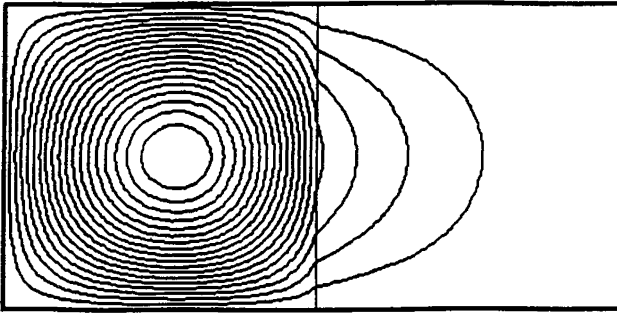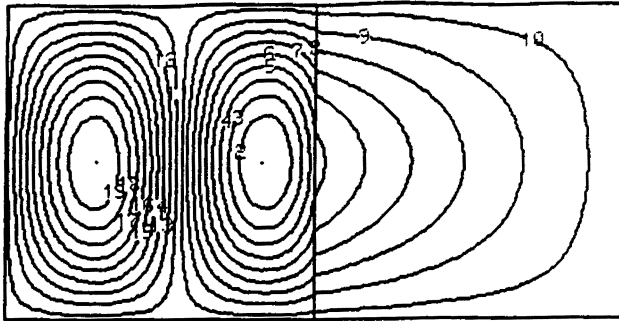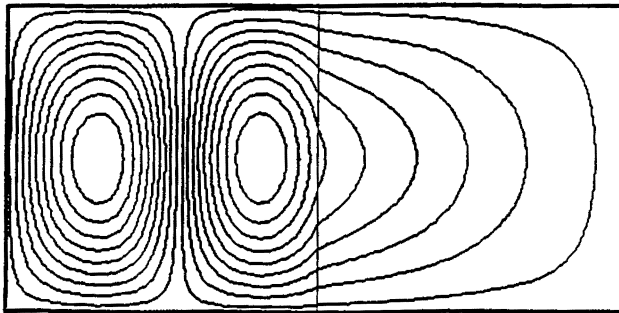Figure 3*a*. — Finite Element solution (CESAR).

Figure 3*b*. — Component Mode Synthesis Solution.

Figure 3. — Level lines of the first eigenfunction (homogeneous Dirichlet conditions).
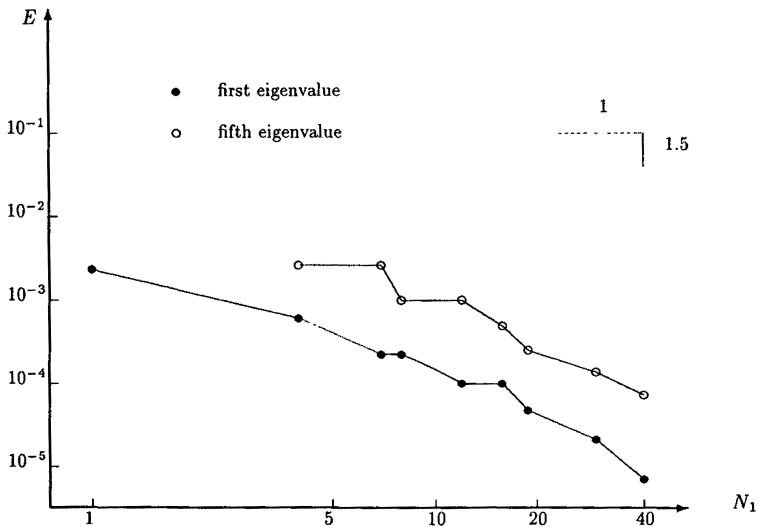
Therefore, static mode and fixed interface mode truncation errors behave at least as good as predicted, and may exhibit an interesting superconvergence phenomenon, which corresponds to the more or less rapid decay of the functions ε_i and ε_Γ. In this respect, the function ε_Γ seems to decay really fast. In this example, the function w_k defined in theorem 3 can be shown to satisfy at least $\dfrac{\partial w_k}{\partial n_i} \in H_{00}^{1/2}(\Gamma) = \mathscr{D}(T^{1/2})$; therefore, according to Bourquin [1990*a*, Remark 10], the error bound associated with the static modes reads $\dfrac{\varepsilon_\Gamma(k, N_\Gamma)}{N_\Gamma^2}$ in theorem 1 instead of $\dfrac{\varepsilon_\Gamma(\alpha, k, N_\Gamma)}{N_\Gamma^{2\alpha}}$, $\alpha < 1$. A more detailed analysis of the singularities related to this problem might yield a better convergence result.
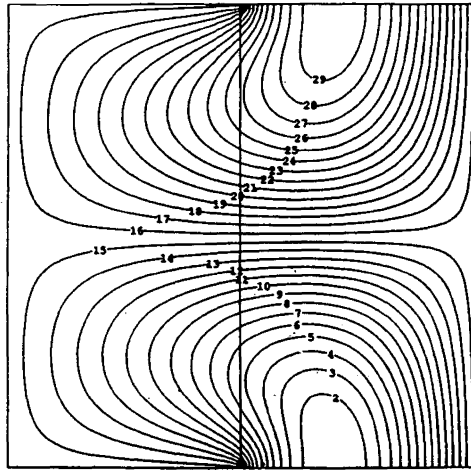
Figure 4a. — Finite Element solution (CESAR).

Figure 4b. — Component Mode Synthesis Solution.

Figure 4. — Level lines of the second eigenfunction (homogeneous Dirichlet conditions).

Figure 5. — Error decay for the first and the fifth eigenvalue.
E : error. N_1 : Number of fixed interface modes on Ω_1.
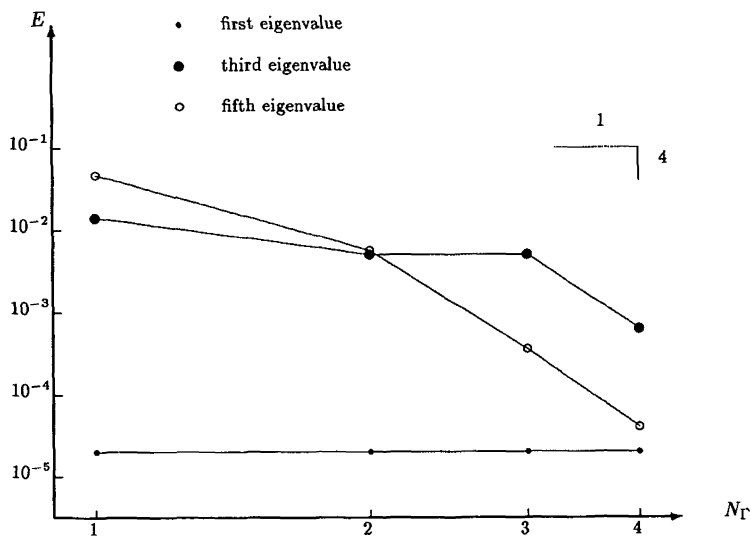
Figure 6. — Error decay for the first, third, and fifth eigenvalue.
E : error. N_Γ : Number of static modes.

6.2. The discretized version of the method

We intend here to test the method with component modes computed with a finite element code. Again, we do not focus on the algorithm but rather on the quality of the approximation. We consider the Laplace operator on a unit square divided in two parts as in figure 2 ; an homogeneous Dirichlet boundary condition is imposed everywhere except on the upper and lower boundaries of the right subdomain, where an homogeneous Neumann boundary condition is chosen. To our knowledge, there is no explicit solution for any kind of mode except for the fixed interface ones. Moreover, a singularity is expected to appear at the corners separating two boundaries of different type. Various other cases will be presented in Bourquin & d'Hennezel [1992].

The approximate static modes are computed in the framework of the program library MODULEF thanks to a Lanczos method applied to the schur complement matrix associated with the interface. A consistent mass matrix involving only the interface is used. The eigenpairs are computed first globally with the finite element code CESAR (*fig.* 7a, and 8a), and then by component mode synthesis (*fig.* 7b, and 8b) for the choice $N_1 = N_2 = 8$, $N_\Gamma = 3$. The plots confirm the good agreement between the frequencies and mode shapes of both computations.
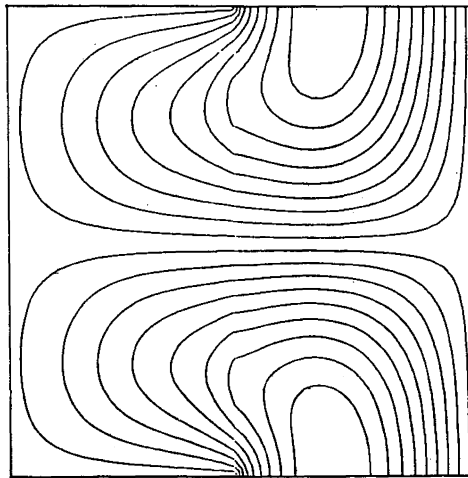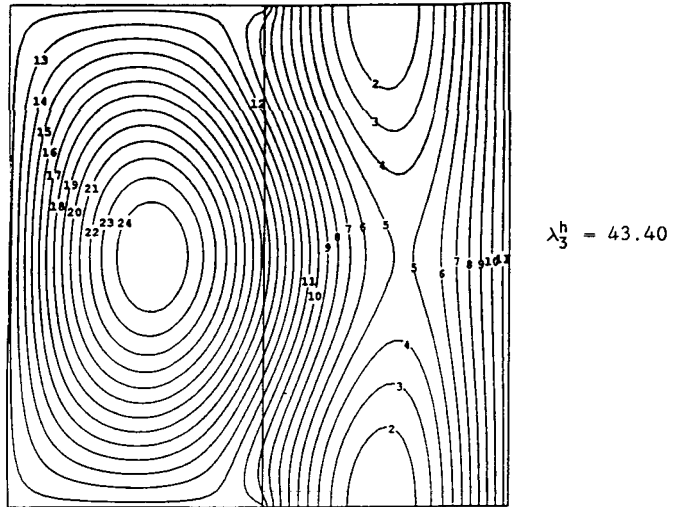
Figure 7*a*. — Finite Element solution (CESAR).
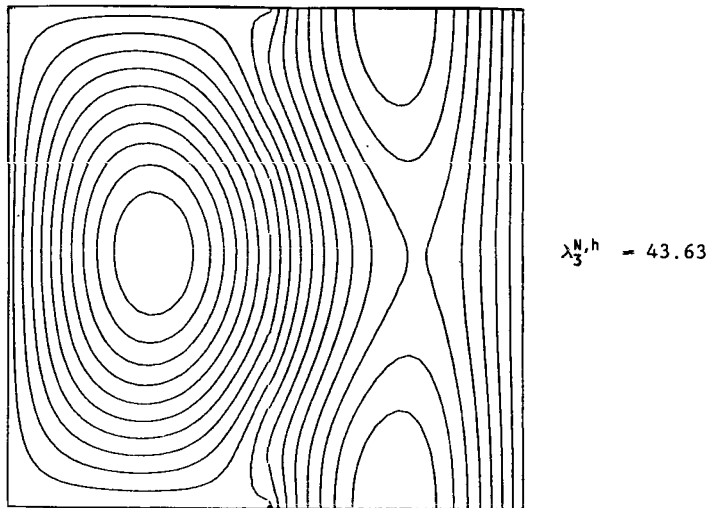

Figure 7*b*. — Component Mode Synthesis Solution (MODULEF).

Figure 7. — Level lines of the second eigenfunction (jump of boundary conditions).

7. CONCLUDING REMARKS

In this paper, the problem of deriving error bounds for the component mode synthesis method on arbitrary domains is addressed. This attempt has suggested to introduce a new method, amenable to numerical analysis in a natural way ; the method can be expressed in a continuous framework, and

$$\lambda_3^h = 43.40$$

Figure 8a. — Finite Element solution (CESAR).

$$\lambda_3^{N,h} = 43.63$$

Figure 8b. — Component Mode Synthesis Solution (MODULEF).

Figure 8. — Level lines of the third eigenfunction (jump of boundary conditions).

extends, once discretized, the classical Hurty, Craig and Bampton one, that is also analysed here. Error bounds dictate a strategy to choose the different component modes to be taken into account. Apart from those advantages regarding the component mode synthesis theory, the proposed method appeals to a number of modes independent of the discretization parameter

and proves numerically attractive. Some specific aspects of the preceding analysis as well as open questions are worth to be highlighted :

   i) The properties of the partial differential equation governing the evolution of the structure are deeply involved, in particular through Weyl's formula, spectral properties of Poincaré-Steklov's operators, and the theory of regularity up to the boundary in the subdomains, which obviously lack of smoothness. Therefore, it seems necessary to introduce an intermediate method. This approach has been adopted in several contributions to that field (*cf.* Jezequel [1985], Morand [1977], Morand & Ohayon [1979], Valid [1982]). *However, it is possible to derive error bounds directly for the classical Hurty* [1965], *Craig, and Bampton* [1968] *method in its discretized version.*

   ii) On the other hand, it seems very important for the potential applicability of this kind of method, that *the definition and the number of the different component modes, and consequently the mode truncation error, do not depend on the mesh size,* or more generally on the finite element or Galerkin method used in the practical computation. As a matter of fact, this feature allows for an intrinsic definition of the intermediate component mode synthesis method, and thus for a large versatility of the real method : various algorithms and approximation methods may be used. In particular, *mesh refinements are made possible* in the vicinity of the interfaces, whereas they are forbidden when using the classical fixed interface method, that would rapidly become to expensive (*cf.* Imbert [1979]).

   iii) Most domain decomposition methods reduce a global problem posed over a domain Ω to a problem posed on the interface and that can be solved either directly by inverting the Schur complement matrix or iteratively thanks to repeated computations involving each time only one subdomain. In any case, the problem solved is equivalent to the initial one. By contrast, a component mode synthesis method appears more clearly as an approximation process, eventhough it leads to an algorithm that takes advantage of the domain decomposition. Thus it should be emphasized that the final small-scale eigenvalue problem is not equivalent to the global one, but theorems 1, 2, 3, 4, and 5 enable to control the approximation error. Of course, both types of algorithm are amenable to parallel computations.

   iv) The method also applies to the eigenvalue problems related to non-homogeneous elastic structures described by various models (*cf.* Bourquin [1991]).

   v) Numerical tests indicate that the decay rate of the functions $\varepsilon_\Gamma(\alpha, k, N_\Gamma)$ may be rather high. The question of a priori estimating those quantities remains open. In particular, it would be interesting to *characterize the domains for which a superconvergence phenomenon regarding the static modes does occur.*

vi) Our choice of static modes has been introduced mainly for technical reasons. Is *it an optimal choice* with respect to the number of modes to be taken into account ? This problem could be tackled in the light of optimal Galerkin approximation (*cf.* Aubin [1972]).

vii) Component mode synthesis relies on the basic assumption that global vibrations of a structure should behave just like local ones, thus leading to the idea of performing a local (non-harmonic) Fourier analysis in the space variable. Therefore, Hurty's method bears a deep ressemblance to the wavelet analysis, that has been introduced many years thereafter in signal processing.

viii) Non-conforming extensions can be thought of : for example, the eigenfunctions of each inverse Calderon's operator $\mathscr{C}_i$, that are easy to compute, could be lifted in the corresponding subdomain Ω_i, and used as basis of static modes. The coupling between substructures should be achieved through some variational identity acting on the traces of the test functions (*cf.* Bernardi *et al.* [1989*a*, *b*]), in the spirit of minimum weighted residual techniques. The method of « intermediate problems » (*cf.* Weinstein & Stenger [1972]) should be advocated because it constitutes an interesting mathematical background of non-conforming component mode synthesis methods (*cf.* Jezequel [1985]). The numerical analysis of the latter may appeal to the abstract theory developed in Mercier *et al.* [1981], but lies beyond the scope of this paper.

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