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**ERROR ESTIMATES FOR EULER FORWARD SCHEME  
RELATED TO TWO-PHASE STEFAN PROBLEMS (\*)**

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*Abstract.* — In this paper, we establish error estimates related to the approximation of bidimensional Stefan problems with forced convection in the fluid phase. An enthalpy formulation of these problems is used. The considered discretization is based on Euler forward finite differences in time and  $C^0$  piecewise linear finite elements in space combined with a mass-lumping procedure. The proposed scheme is therefore easy to implement. Under some restrictions relative to the finite element mesh and to stability and non-degeneracy conditions, we prove a  $L^2$ -rate of convergence for temperature and a  $H^{-1}$ -rate for enthalpy both of order  $h^{2/3}$ .

*Résumé.* — Cette étude a pour objectif d'établir des estimations d'erreurs concernant l'approximation de problèmes bidimensionnels de type Stefan en présence de convection forcée dans la phase liquide. Une formulation en enthalpie de ces problèmes est considérée. Le schéma proposé repose sur l'utilisation d'une méthode d'Euler progressif dans la variable temporelle et d'éléments finis continus par morceaux en espace combinée avec une procédure de mass-lumping. Un tel algorithme est donc d'implantation aisée. Sous certaines restrictions relatives à la triangulation et sous des conditions de stabilité et de non-dégénérescence, nous démontrons des estimations d'erreur en norme  $L^2$  pour la température et en norme  $H^{-1}$  pour la variable enthalpie toutes deux d'ordre  $h^{2/3}$ .

**1. INTRODUCTION**

An important class of physical processes, such as phase-change phenomena, gives rise to parabolic free boundary problems of Stefan type. In this paper, we analyse the accuracy of a numerical approximation of bidimen-

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sional two-phase Stefan problems with a given convection term in the fluid phase, via the enthalpy formulation.

Due to their important applicative meanings, these problems have been extensively studied during recent years, both from theoretical and numerical points of view. For survey, we refer to Nochetto [21] and Danilyuk [7].

Within a numerical frame, various algorithms have been proposed in order to approximate the solution of such problems. Most of them are based on discretization by means of backward differences in time combined with finite elements (or finite differences) in space. Many results, including sharp error estimates, have been obtained for these algorithms (see Jerome-Rose [12], Nochetto [17, 18, 19], Elliott [9], Nochetto-Verdi [22]). Unfortunately, such schemes require, at each time step, the resolution of a nonlinear algebraic system which is quite expensive.

In order to avoid this difficulty, various linearized algorithms, essentially suggested by nonlinear semigroup theory, have been developed (see, e.g., Berger-Brezis-Rogers [2], Verdi [26], Magenes-Nochetto-Verdi [16], Nochetto-Verdi [23, 24], Amiez-Gremaud [1]).

In this paper, we consider an other linear scheme, introduced by Ciavaldini [6] for Stefan problems without convection, based on the approximation of the enthalpy formulation by Euler forward differences in time and  $C^0$  piecewise linear finite elements in space combined with numerical quadrature of the integrals. Such a scheme is, therefore, easy to implement. We establish here some error estimates in the case of Stefan problems with forced convection in the fluid phase.

### 1.1. Basic assumptions and notations

Let us first state the basic hypotheses we will use all along this paper. We assume that :

\*  $\Omega \subset \mathbb{R}^2$  is a convex bounded polygonal domain and we set :

$$Q = \Omega \times (0, T), \quad \text{where } T \text{ is a fixed positive number ;} \quad (1.1)$$

\*  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  is a piecewise  $C^1$  function such that :

$$\begin{aligned} \beta(0) = 0, \quad \beta(\xi) = T_f, \quad \forall \xi \in [u_1, u_2], \quad 0 < u_1 \leq u_2 \\ 0 < \mathcal{C} \leq \beta'(\xi) \leq B, \quad \forall \xi \notin [u_1, u_2], \end{aligned} \quad (1.2)$$

where  $T_f$  is a fixed positive constant (physically,  $T_f$  is related to the melting point, whereas the quantity  $L = u_2 - u_1$  corresponds to the latent heat) ;

\*  $b : \mathbb{R} \rightarrow \mathbb{R}^2$  is a Lipschitz continuous function satisfying the condition :

$$b(\xi) = 0, \quad \forall \xi \in ]-\infty, T_f]; \quad (1.3)$$

\*  $u_0$  is a given function which satisfies :

$$\begin{aligned} \theta_0 &= \beta(u_0) \in W_0^{1,\infty}(\Omega), \\ F_0 &= \{x \in \Omega, \theta_0(x) = T_f\} \text{ is a Hölder continuous curve,} \\ \text{meas } \{x \in \Omega, T_f \leq \theta_0(x) \leq T_f + \varepsilon\} &= \mathcal{O}(\varepsilon). \end{aligned} \tag{1.4}$$

Besides, we point out that we will use the standard notations of functional analysis (see e.g. Brezis [3]) for the different spaces  $L^p(\Omega)$ ,  $H^m(\Omega)$ , ..., etc... In order to simplify the notations, we set :

$$H = H^1(\Omega); \quad V = H_0^1(\Omega); \quad V' = H^{-1}(\Omega).$$

Moreover,  $(\cdot, \cdot)$  will denote either the inner product in  $L^2(\Omega)$  or the duality product between  $V$  and  $V'$ , while the classical norms defined in  $L^2(\Omega)$ ,  $H$  or  $V'$  will be respectively represented by  $\|\cdot\|$ ,  $\|\cdot\|_H$  or  $\|\cdot\|_{V'}$ .

### 1.2. Statement of the problem

Our aim is to study the following nonlinear parabolic problem :

Problem  $(P)$  : Find

$$u \in H^1(0, T; V') \cap L^\infty(0, T; L^2(\Omega))$$

and  $\theta \in L^\infty(0, T; V) \cap H^1(0, T; L^2(\Omega))$  such that :

$$\begin{cases} \theta = \beta(u), & \text{a.e. in } Q = \Omega \times (0, T), & (1.5) \\ \left( \frac{\partial u}{\partial t}, \phi \right) + (\nabla \theta + b(\theta), \nabla \phi) = 0, & \forall \phi \in V, \text{ a.e. } t \in (0, T), & (1.6) \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \Omega. & (1.7) \end{cases}$$

Existence and uniqueness results related to problem  $(P)$  are already known. For details, we refer to Kamenomostskaya [13], Friedman [10] or Ladyzenskaya-Solonnikov-Ural'Ceva [14].

Within a solidification frame, problem  $(P)$  naturally arises, after applying a Kirchhoff transformation, from standard heat transfer theory with phase change when the fluid phase is allowed to move ( $b \neq 0$ ) (see e.g. Ladyzenskaya-Solonnikov-Ural'Ceva [14] or Lions [15]). In this case, the function  $\beta$  contains informations about thermal properties of the medium. The unknown  $u(x, t)$  represents the enthalpy variable whereas  $\theta(x, t)$  denotes the temperature at time  $t \in (0, T)$  and at point  $x \in \Omega$ .

It is well worth noting that, according to the shape of  $\beta$ , problem (1.5)-(1.7) models various other physical processes of relevant interest, such as, for instance, gas diffusion in porous media.

2. NUMERICAL SCHEME AND A PRIORI ESTIMATES

In this section, we will introduce a numerical scheme in order to approximate problem  $(P)$ . To this purpose, let us first define some notations and basic assumptions relative to the discretization.

2.1. Finite Elements

Let  $(\mathcal{T}_h)_{h>0}$  be a family of triangulation of  $\bar{\Omega}$  which are made up with triangles  $K \in (\mathcal{T}_h)$  of diameter  $h_k$ ; we set  $h = \max \{h_k, K \in (\mathcal{T}_h)\}$  and assume that :

$$*(\mathcal{T}_h) \text{ is regular and quasi-uniform (see Ciarlet [4]) ;} \tag{2.1}$$

$$*(\mathcal{T}_h) \text{ is acute, i.e., } \forall K \in (\mathcal{T}_h), \alpha_k \leq \pi/2, \tag{2.2}$$

where  $\alpha_k$  denotes the angles of the triangle  $K$ .

Let us now consider the following finite element space :

$$* V_h = \left\{ \phi \in C^0(\bar{\Omega}), \phi|_K \in \mathbb{P}^1(K), \forall K \in (\mathcal{T}_h) \right. \\ \left. \text{and } \phi = 0 \text{ on } \partial\Omega \right\} \subset V, \tag{2.3}$$

as well as the discrete inner product  $(\cdot, \cdot)_h$  and its corresponding norm  $\|\cdot\|_h$  defined for any continuous functions  $v$  and  $\phi$  by :

$$(v, \phi)_h = \sum_{K \in \mathcal{T}_h} \int_K r_h(v\phi) dx; \quad \|v\|_h = (v, v)_h^{1/2}, \tag{2.4}$$

where  $r_h$  is the Lagrange interpolation operator related to  $V_h$ . Notice that the integrals  $(v, \phi)_h$  can be calculated by means of the vertex quadrature rule.

It is well-known that  $(\cdot, \cdot)_h$  is an inner product on  $V_h$  equivalent to the one induced by the classical  $L^2$  topology, i.e. :

$$\|v\| \leq \|v\|_h \leq C \|v\|, \quad \forall v \in V_h, \tag{2.5}$$

where  $C$  is a positive constant independent of  $h$  (see e.g. Raviart [25, p. 250]). In addition, for any  $v, \phi \in V_h$ , we have :

$$|(v, \phi) - (v, \phi)_h| \leq Ch^2 \|v\|_H \|\phi\|_H, \tag{2.6}$$

where  $C$  is a constant independent of  $h$  (see Ciarlet-Raviart [5]).

Besides, the quasi-uniform assumption (2.1) entails the following inverse inequality (see Ciarlet [4, p. 142]) :

$$\|\nabla u\| \leq \frac{\tilde{C}}{h} \|u\|_h, \quad \forall u \in V_h, \tag{2.7}$$

where  $\tilde{C}$  is a constant independent of  $h$ .

Finally, let  $\tau = T/N$  be the size of a uniform partition of the interval  $[0, T]$ , for  $N \in \mathbb{N}^*$  arbitrary. For any continuous function  $v$  (resp. integrable) in time, defined in  $Q$ , we set :

$$v^n(\cdot) = v(\cdot, n\tau); \partial v^{n+1} = (v^{n+1} - v^n)/\tau; \bar{v}^{n+1} = \frac{1}{\tau} \int_{n\tau}^{(n+1)\tau} v(\cdot, t) dt.$$

With these notations, we can state the following discrete scheme.

**2.2. The numerical scheme**

Problem  $(P_{h\tau})$  : For  $n = 0, 1, \dots, N - 1$ , find  $U^{n+1} \in V_h$  such that :

$$(\partial U^{n+1}, \phi)_h + (\nabla \Theta^n + B^n, \nabla \phi) = 0, \quad \forall \phi \in V_h, \tag{2.8}$$

where  $\Theta^n$  and  $B^n$  ( $n \geq 1$ ) denote respectively the discrete functions :

$$\Theta^n = r_h \beta(U^n), \quad B^n = r_h b(\Theta^n), \tag{2.9}$$

and where the initial data  $U^0, \Theta^0$  and  $B^0$  are given by :

$$\begin{aligned} \Theta^0 &= r_h \beta(u_0) = r_h \theta_0, \quad B^0 = r_h b(\Theta^0), \\ U^0(A_i) &\in \beta^{-1}(\Theta^0(A_i)), \quad \forall A_i \text{ vertex of } K \in (\mathcal{T}_h). \end{aligned} \tag{2.10}$$

*Remark 2.1* : The discrete problem  $(P_{h\tau})$  is a system of linear algebraic equations which clearly admits a unique solution for each  $n, 0 \leq n \leq N - 1$ . Moreover, according to relations (2.10) and (1.4), we get :

$$\Theta^0 = r_h \beta(U^0) \quad \text{and} \quad \|\nabla \Theta^0\| \leq \|\nabla \theta_0\| \leq C,$$

where  $C$  is a constant independent of  $h$ .

**2.3. A priori estimates**

In order to establish a priori estimates for the considered scheme, let us first recall an elementary property proved by Ciavaldini [6].

LEMMA 2.1 : For any functions  $(U^n)$  and  $(\Theta^n)$  satisfying relation (2.9), the following property holds :

$$(U^{n+1} - U^n, \Theta^{n+1} - \Theta^n)_h \geq \frac{1}{B} \|\Theta^{n+1} - \Theta^n\|_h^2, \tag{2.11}$$

where  $B$  is defined in (1.2). Moreover, if the triangulation  $(\mathcal{T}_h)$  satisfies assumption (2.2), we get :

$$(\nabla U^n, \nabla \Theta^n) \geq \frac{1}{B} \|\nabla \Theta^n\|^2. \tag{2.12}$$

We are now able to establish the main a priori estimates results.

PROPOSITION 2.1 : *For any triangulation  $(\mathcal{T}_h)$  satisfying assumptions (2.1), (2.2) and under the following stability condition : there exists  $\gamma < 1$  such that :*

$$\frac{\tau}{h^2} < \frac{2\gamma}{B\tilde{C}^2}, \quad (2.13)$$

where  $B$  and  $\tilde{C}$  are respectively defined in (1.2) and (2.7), the discrete scheme (2.8)-(2.10) is  $L^2$ -stable, i.e., there exists a constant  $C$ , independent of  $\tau$  and  $h$ , such that :

$$\max_{0 \leq n \leq N} \|\nabla \Theta^n\| + \frac{1}{\tau} \sum_{n=0}^{N-1} \|\Theta^{n+1} - \Theta^n\|_h^2 \leq C, \quad (2.14)$$

$$\max_{0 \leq n \leq N} \|U^n\|_h + \sum_{n=0}^{N-1} \|U^{n+1} - U^n\|_h^2 \leq C. \quad (2.15)$$

*Proof :* Let us choose  $\phi = \tau \partial \Theta^{n+1} \in V_h$  in (2.8). We then get :

$$\tau (\partial U^{n+1}, \partial \Theta^{n+1})_h + \tau (\nabla \Theta^n, \nabla \partial \Theta^{n+1}) - \tau (\nabla B^n, \partial \Theta^{n+1}) = 0. \quad (2.16)$$

Using relation (2.7) as well as the following elementary equality :

$$p(p - q) = \frac{1}{2} (p^2 - q^2 + (p - q)^2), \quad \forall (p, q) \in \mathbb{R}^2,$$

we first notice that :

$$\tau (\nabla \Theta^n, \nabla \partial \Theta^{n+1}) \geq -\frac{1}{2} \left( \|\nabla \Theta^n\|^2 - \|\nabla \Theta^{n+1}\|^2 + \frac{\tilde{C}^2}{h^2} \tau^2 \|\partial \Theta^{n+1}\|_h^2 \right).$$

Moreover, applying Young inequality to the last term of (2.16) yields :

$$|\tau (\nabla B^n, \partial \Theta^{n+1})| \leq \frac{\tau}{2\nu} \|\nabla B^n\|^2 + \tau \frac{\nu}{2} \|\partial \Theta^{n+1}\|_h^2, \quad \forall \nu > 0.$$

Using the Lipschitz-continuity of  $b$ , we have :

$$\|\nabla B^n\| = \|\nabla r_h b(\Theta^n)\| \leq C \|\nabla \Theta^n\|. \quad (2.17)$$

Therefore, choosing  $\nu = (1 - \gamma)/B$  and applying relation (2.11) to the first term of equality (2.16), we get :

$$\tau \left( \frac{1 + \gamma}{B} - \frac{\tau \tilde{C}^2}{h^2} \right) \|\partial \Theta^{n+1}\|_h^2 - \left( 1 + \tau \frac{C^2 B}{1 - \gamma} \right) \|\nabla \Theta^n\|^2 + \|\nabla \Theta^{n+1}\|^2 \leq 0.$$

If we sum this last relation over  $n$ ,  $0 \leq n \leq m - 1$ ,  $1 \leq m \leq N$ , we obtain :

$$\begin{aligned} \left( \frac{1 + \gamma}{B} - \frac{\tau \tilde{C}^2}{h^2} \right) \sum_{n=0}^{m-1} \tau \|\partial \Theta^{n+1}\|_h^2 + \left( 1 + \tau \frac{C^2 B}{1 - \gamma} \right) \|\nabla \Theta^m\|^2 &\leq \\ &\leq \frac{C^2 B}{1 - \gamma} \sum_{n=1}^m \tau \|\nabla \Theta^n\|^2 + \left( 1 + \tau \frac{C^2 B}{1 - \gamma} \right) \|\nabla \Theta^0\|^2. \end{aligned}$$

Let us notice that, under the stability condition (2.13), we have :

$$\left( \frac{1 + \gamma}{B} - \frac{\tau \tilde{C}^2}{h^2} \right) \geq \frac{1 - \gamma}{B} > 0.$$

Hence, if we apply the discrete Gronwall inequality to the last relation, we obtain  $\max_{0 \leq n \leq N} \|\nabla \Theta^n\| \leq C$ , and finally inequality (2.14).

It still remains to prove (2.15). To this purpose, we first notice that, by virtue of the assumptions on  $\beta$  and  $b$ , relation (2.14) entails :

$$\max_{0 \leq n \leq N} \|\Theta^n\|_h + \max_{0 \leq n \leq N} \|U^n\|_h + \max_{0 \leq n \leq N} \|B^n\|_h \leq C. \tag{2.18}$$

Next, let us take  $\phi = \tau^2 \partial U^{n+1} \in V_h$  in equation (2.8). We get :

$$\tau^2 \|\partial U^{n+1}\|_h^2 + \tau^2 (\nabla \Theta^n + B^n, \nabla \partial U^{n+1}) = \text{I} + \text{II} = 0. \tag{2.19}$$

According to relations (2.14), (2.18), (2.7) and (2.13), we have :

$$|\text{II}| \leq C \tau^2 \|\nabla \partial U^{n+1}\| \leq C \tau^2 \frac{\tilde{C}}{h} \|\partial U^{n+1}\|_h \leq C \tau + \frac{\tau^2}{2} \|\partial U^{n+1}\|_h^2.$$

Hence, summing over  $n$  from 0 to  $N - 1$  equation (2.19), we obtain :

$$\sum_{n=0}^{N-1} \tau^2 \|\partial U^{n+1}\|_h^2 \leq C,$$

which completes the proof. ■

### 3. THE REGULARIZED PROBLEM

In order to prove error estimates for both variables  $U^n$  and  $\Theta^n$ , solutions of problem  $(P_{h\tau})$ , we will introduce some auxiliary problems. First of all, we will consider a family of nonlinear parabolic boundary value problems  $(P_\epsilon)$ , which are regularized approximations of problem  $(P)$ , obtained by



smoothing the constitutive function  $\beta$ . Such a procedure has already been used in theoretical works in order to establish existence, uniqueness and regularity results for the solution of the basic problem  $(P)$  (see e.g. Kamenomostskaya [13], Friedman [10] or Ladyzenskaya-Solonnikov-Ural’Ceva [14]), but also in numerical studies (see Jerome-Rose [12] or Nochetto [17] among others).

**3.1. The regularized problem**

Let us introduce the following strictly increasing function :

$$\begin{aligned} \beta_\varepsilon(\xi) &= \beta(u_1) + \varepsilon(\xi - u_1) \quad \forall \xi \in [u_1, \xi_\varepsilon], \\ \beta_\varepsilon(\xi) &= \beta(\xi) \quad \text{otherwise,} \end{aligned} \tag{3.1}$$

where  $\xi_\varepsilon$  is the solution of  $\beta(\xi) = \beta(u_1) + \varepsilon(\xi - u_1)$ , and consider the following regularized problem :

Problem  $(P_\varepsilon)$  : Find  $u_\varepsilon \in H^1(0, T; V')$  such that :

$$\left\{ \begin{aligned} \beta_\varepsilon(u_\varepsilon) &\in L^2(0, T; V) \end{aligned} \right. \tag{3.2}$$

$$\left\{ \begin{aligned} \left( \frac{\partial u_\varepsilon}{\partial t}, \phi \right) + (\nabla \beta_\varepsilon(u_\varepsilon) + b(\beta_\varepsilon(u_\varepsilon)), \nabla \phi) &= 0, \quad \forall \phi \in V, \end{aligned} \right. \tag{3.3}$$

$$\left\{ \begin{aligned} u_\varepsilon(\cdot, 0) &= \beta_\varepsilon^{-1}(\theta_0(\cdot)), \end{aligned} \right. \tag{3.4}$$

Existence, uniqueness and error estimates related to problem  $(P_\varepsilon)$  are known. Therefore, we will only summarize here the main results. A detailed description of their proofs can be found, e.g., in Ladyzenskaya-Solonnikov-Ural’Ceva [14], Jerome-Rose [12] and in Nochetto-Verdi [24] for error estimates results.

**THEOREM 3.1 :** *Under the assumptions (1.1)-(1.4), problem  $(P_\varepsilon)$  admits a unique solution  $u_\varepsilon$ . Moreover, there exists a constant  $C$ , independent of  $\varepsilon$ , such that :*

$$\begin{aligned} \|u - u_\varepsilon\|_{L^\infty(0, T; V')}^2 &+ \|\beta(u) - \beta_\varepsilon(u_\varepsilon)\|_{L^2(Q)}^2 + \\ &+ \varepsilon \|u - u_\varepsilon\|_{L^2(Q)}^2 \leq C (\varepsilon \text{ meas } (A_\varepsilon(u)) + \varepsilon^2 |\log \varepsilon|), \end{aligned} \tag{3.5}$$

with :

$$A_\varepsilon(u) = \{ (x, t) \in Q, u_1 \leq u(x, t) \leq \xi_\varepsilon \},$$

and where  $u$  denotes the solution of problem  $(P)$ .

Let us now consider the following regularized discrete scheme.

**3.2. The regularized explicit scheme**

Problem  $(P_{\varepsilon h\tau})$ : For  $n = 0, 1, \dots, N - 1$ , find  $U_\varepsilon^{n+1} \in V_h$  such that :

$$(\partial U_\varepsilon^{n+1}, \phi)_h + (\nabla \Theta_\varepsilon^n + B_\varepsilon^n, \nabla \phi) = 0, \quad \forall \phi \in V_h, \tag{3.6}$$

where  $\Theta_\varepsilon^n$  and  $B_\varepsilon^n$  denote respectively the discrete functions :

$$\Theta_\varepsilon^n = r_h \beta_\varepsilon(U_\varepsilon^n), \quad B_\varepsilon^n = r_h b(\Theta_\varepsilon^n), \tag{3.7}$$

and where the initial datum  $U_\varepsilon^0$  is given by :

$$U_\varepsilon^0 = r_h \beta_\varepsilon^{-1}(\theta_0), \tag{3.8}$$

$r_h, V_h$  and  $(\cdot, \cdot)_h$  being defined in § 2.

*Remark 3.1* : The discrete problem  $(P_{\varepsilon h\tau})$  is a system of linear algebraic equations which clearly admits a unique solution for  $n, 0 \leq n \leq N - 1$ .

Moreover the following stability results hold.

**PROPOSITION 3.1** : *For any triangulation  $(\mathcal{T}_h)$  satisfying assumptions (2.1), (2.2) and under the stability condition (2.13), there exists a constant  $C$ , independent of  $\varepsilon, h$  and  $\tau$ , such that :*

$$\max_{0 \leq n \leq N} \|\nabla \Theta_\varepsilon^n\| + \frac{1}{\tau} \sum_{n=0}^{N-1} \|\Theta_\varepsilon^{n+1} - \Theta_\varepsilon^n\|_h^2 \leq C, \tag{3.9}$$

$$\max_{0 \leq n \leq N} \|U_\varepsilon^n\|_h + \sum_{n=0}^{N-1} \|U_\varepsilon^{n+1} - U_\varepsilon^n\|_h^2 \leq C, \tag{3.10}$$

$$\sum_{n=0}^{N-1} \|U_\varepsilon^{n+1} - U_\varepsilon^n\|_h^2 \leq C \tau \varepsilon^{-1}. \tag{3.11}$$

*Proof*: We first remark that the proof of Proposition 2.1 remains unchanged if we substitute  $\beta$  by  $\beta_\varepsilon$ ; consequently, assertions (3.9)-(3.10) are verified. In order to obtain relation (3.11), let us notice that, according to definition (3.1), we have :

$$\sum_{n=0}^{m-1} (\partial U_\varepsilon^{n+1}, \partial \Theta_\varepsilon^{n+1})_h \geq \varepsilon \sum_{n=0}^{m-1} \|\partial U_\varepsilon^{n+1}\|_h^2.$$

Hence, choosing  $\phi = \tau \partial \Theta_\varepsilon^{n+1}$  in (3.6) and using a technique similar to the one developed in the proof of Proposition 2.1 easily leads to the required result. ■

4. ERROR ESTIMATES RELATIVE TO THE REGULARIZED SCHEME

In order to prove some error estimates, we need some additional notations. For any triangulation  $(\mathcal{T}_h)$  and for any time step  $\tau$ , we set :

$$U_{h\tau}(\cdot, t) = U^n(\cdot) \quad \text{for } t \in [n\tau, (n+1)\tau[, \quad (4.1)$$

$$\Theta_{h\tau}(\cdot, t) = \Theta^n(\cdot) = r_h \beta(U^n)(\cdot) \quad \text{for } t \in [n\tau, (n+1)\tau[, \quad (4.2)$$

$U^n$  denoting the solution of  $(P_{h\tau})$  and analogously for  $U_{\varepsilon h\tau}$  and  $\Theta_{\varepsilon h\tau}$ .

Moreover, notice that, from now, we will refer to the solution of problem  $(P_{h\tau})$  (resp.  $P_{\varepsilon h\tau}$ ), as being either  $U^n$  (resp.  $U_\varepsilon^n$ ) or  $(U_{h\tau}, \Theta_{h\tau})$  (resp.  $U_{\varepsilon h\tau}, \Theta_{\varepsilon h\tau}$ ).

For the sake of simplicity, we also introduce the following notations :

$$e = u_\varepsilon - U_{\varepsilon h\tau}; \quad f = \beta_\varepsilon(u_\varepsilon) - \Theta_{\varepsilon h\tau};$$

$$E^n = \tilde{u}_\varepsilon^n - U_\varepsilon^n; \quad F^n = \tilde{\beta}_\varepsilon(u_\varepsilon^n) - \Theta_\varepsilon^n;$$

Finally, let us consider the operators  $G : V' \rightarrow V$  and  $G_h : V' \rightarrow V_h$ , defined by :

$$(\nabla G \psi, \nabla \phi) = (\psi, \phi), \quad \forall \phi \in V, \quad \psi \in V', \quad (4.3)$$

$$(\nabla G_h \psi, \nabla \phi) = (\psi, \phi), \quad \forall \phi \in V_h, \quad \psi \in V'. \quad (4.4)$$

Since  $\Omega$  is a convex polygonal domain and since  $(\mathcal{T}_h)$  is regular, it is well-known that there exists a constant  $C$ , independent of  $h$ , such that :

$$\| (G - G_h) \psi \| \leq C h^2 \| \psi \|, \quad \forall \psi \in L^2(\Omega), \quad (4.5)$$

$$\| G_h \psi \|_H \leq C \| \psi \|_{V'}, \quad \forall \psi \in V'. \quad (4.6)$$

According to these notations, we are now ready to establish the following result.

**THEOREM 4.1 :** *Let assumptions (1.1)-(1.4), (2.1) and (2.2) hold. Then, under the stability condition (2.13), there exists a constant  $C$ , independent of  $\varepsilon$ ,  $h$  and  $\tau$ , such that :*

$$\| u_\varepsilon - U_{\varepsilon h\tau} \|_{L^\infty(0, T, V')}^2 + \| \beta_\varepsilon(u_\varepsilon) - \Theta_{\varepsilon h\tau} \|_{L^2(Q)}^2 + \varepsilon \| u_\varepsilon - U_{\varepsilon h\tau} \|_{L^2(Q)}^2 \leq$$

$$\leq C \left[ \tau + \frac{h^2}{\varepsilon} + \frac{\tau^2}{\varepsilon} + \frac{h^4}{\varepsilon^2} + (\varepsilon + h)^2 |\log(\varepsilon + h)| \right], \quad (4.7)$$

where  $u_\varepsilon$  and  $(U_{\varepsilon h\tau}, \Theta_{\varepsilon h\tau})$  denote the respective solutions of problems  $(P_\varepsilon)$  and  $(P_{\varepsilon h\tau})$ .

To prove this theorem, we will use a technique similar to the one developed by Nchetto-Verdi [22]. However, since this proof is lengthy,

some intermediate results, already established by these authors, will be recalled without details. Moreover, we point out that, hereafter,  $C$  will denote various constants, which are all independent of  $\varepsilon$ ,  $h$  and  $\tau$ . Besides, since no confusion is possible, we will omit the subscript  $\varepsilon$ .

*Proof:* Take  $\phi = GE^{n+1}$  in (3.3) and integrate on  $[n\tau, (n+1)\tau[$ , then choose  $\phi = \tau G_h E^{n+1}$  in (3.6), take their difference and finally sum over  $n$  for  $0 \leq n \leq m-1$ ,  $1 \leq m \leq N$ . We obtain, after reordering :

$$\begin{aligned} & \sum_{n=1}^m \tau [(\partial u^n - \partial U^n, GE^n) + (\nabla \tilde{\beta}(u^n), \nabla G E^n) - (\nabla \Theta^n, \nabla G_h E^n) + \\ & + (\tilde{b}^n, \nabla G E^n) - (B^{n-1}, \nabla G_h E^n)] \\ & + \sum_{n=1}^m \tau [(\partial U^n, GE^n) - (\partial U^n, G_h E^n)_h] \\ & + \sum_{n=1}^m \tau [(\nabla \Theta^n, \nabla G_h E^n) - (\nabla \Theta^{n-1}, \nabla G_h E^n)] = 0, \end{aligned} \tag{4.8}$$

i.e.  $I + II + III = 0$ .

Using Nochetto-Verdi results ([22, p. 797-800]), we have :

$I \geq$

$$\begin{aligned} & (C - \mu) \|E^m\|_{V'}^2 + (C - \mu) \sum_{n=1}^m \|E^n - E^{n-1}\|_{V'}^2 + C \varepsilon \int_0^{m\tau} \|e(t)\|^2 dt + \\ & + C(1 - \mu) \int_0^{m\tau} \|f(t)\|^2 dt - \varepsilon \mu \sum_{n=1}^m \tau \|E^n\|^2 - \frac{C}{\mu} \sum_{n=1}^m \tau \|E^n\|_{V'}^2, \\ & - C(1 + \mu) \left( h^2 + \frac{h^2}{\varepsilon} + \tau \right) - \frac{C}{\mu} \tau - C \|u_{0\varepsilon} - U_\varepsilon^0\|_{V'}^2, \quad \forall \mu > 0. \end{aligned} \tag{4.9}$$

Besides, since assumption (1.4) holds, we get (see Nochetto-Verdi [24]) :

$$\|u_{0\varepsilon} - U_\varepsilon^0\|_{V'}^2 \leq C(\varepsilon + h)^2 |\log(\varepsilon + h)| = \sigma_0^2. \tag{4.10}$$

Thus, it still remains to estimate the terms II and III. To this purpose, we first observe that :

$$\begin{aligned} II = & \sum_{n=1}^m \tau (\partial U^n, (G - G_h) E^n) + \\ & + \sum_{n=1}^m \tau ((\partial U^n, G_h E^n) - (\partial U^n, G_h E^n)_h) = X + Y. \end{aligned}$$

The term  $X$  can be bounded by means of Cauchy-Schwarz and Young inequalities combined with relations (2.5), (4.5) and (3.11); namely

$$X \leq Ch^2 \sum_{n=1}^m \tau \|\partial U^n\|_h \|E^n\| \leq \varepsilon \mu \sum_{n=1}^m \tau \|E^n\|^2 + C \frac{h^4}{\varepsilon^2 \mu}, \quad \forall \mu > 0. \quad (4.11)$$

Besides, according to relations (2.5)-(2.7), (3.11) and (4.6), we have for any constant  $\mu > 0$ :

$$Y \leq Ch \sum_{n=0}^{m-1} \tau \|\partial U^n\|_h \|E^n\|_{V'} \leq C \mu \frac{h^2}{\varepsilon} + \frac{C}{\mu} \sum_{n=1}^m \tau \|E^n\|_{V'}^2. \quad (4.12)$$

It remains to analyse the term III. To this purpose, notice that relations (4.4) and (3.9) yield:

$$\text{III} = \sum_{n=1}^m \tau (\Theta^n - \Theta^{n-1}, E^n) \leq \varepsilon \mu \sum_{n=1}^m \tau \|E^n\|^2 + \frac{C}{\mu} \frac{\tau^2}{\varepsilon}, \quad \forall \mu > 0. \quad (4.13)$$

Collecting estimates (4.9) to (4.13), we obtain for any constant  $\mu > 0$ :

$$\begin{aligned} & \|E^m\|_{V'}^2 + \sum_{n=0}^{m-1} \|E^{n+1} - E^n\|_{V'}^2 + \varepsilon \int_0^{m\tau} \|e(t)\|^2 dt + \int_0^{m\tau} \|f(t)\|^2 dt \leq \\ & \leq C(1 + \mu) \left[ \frac{h^2}{\varepsilon} + \tau \right] + \frac{C}{\mu} \left[ \frac{\tau^2}{\varepsilon} + \tau + \frac{h^4}{\varepsilon^2} \right] + C \sigma_0^2 + \frac{C}{\mu} \sum_{n=1}^m \tau \|E^n\|_{V'}^2, \\ & + C \mu \left[ \|E^m\|_{V'}^2 + \sum_{n=0}^{m-1} \|E^{n+1} - E^n\|_{V'}^2, \right. \\ & \left. + \varepsilon \sum_{n=1}^m \tau \|E^n\|^2 + \int_0^{m\tau} \|f(t)\|^2 dt \right]. \end{aligned}$$

Moreover, let us notice that:

$$\sum_{n=1}^m \tau \|E^n\|^2 \leq \int_0^{m\tau} \|e(t)\|^2 dt.$$

Therefore, we can choose  $\mu$  small enough in order to absorb the terms

$$\sum_{n=1}^m \|E^n\|^2, \quad \sum_{n=0}^{m-1} \|E^{n+1} - E^n\|_{V'}^2, \quad \|E^m\|_{V'}^2, \quad \text{and} \quad \int_0^{m\tau} \|f(t)\|^2 dt$$

in the left hand side of the previous inequality. We then obtain :

$$\begin{aligned} \|E^m\|_{V'}^2 + \int_0^{m\tau} (\varepsilon \|e(t)\|^2 + \|f(t)\|^2) dt &\leq \\ &\leq C \left( \tau + \frac{\tau^2}{\varepsilon} + \frac{h^2}{\varepsilon} + \frac{h^4}{\varepsilon^2} + \sigma_0^2 \right) + C \sum_{n=1}^m \tau \|E^n\|_{V'}^2 . \end{aligned}$$

Hence, if we apply the discrete Gronwall inequality, it follows that :

$$\begin{aligned} \|E^m\|_{V'}^2 + \int_0^{m\tau} (\varepsilon \|e(t)\|^2 + \|f(t)\|^2) dt &\leq \\ &\leq C \left( \tau + \frac{\tau^2}{\varepsilon} + \frac{h^2}{\varepsilon} + \frac{h^4}{\varepsilon^2} + \sigma_0^2 \right) . \end{aligned} \tag{4.14}$$

Finally, let us notice that, since  $u_\varepsilon \in H^1(0, T; V')$ , we have :

$$\sup_{t \in [0, T]} \|e(t)\|_{V'} \leq \max_{0 \leq m \leq N} \|E^m\|_{V'} + C \tau^{1/2} .$$

Applying this last result to inequality (4.14) completes the proof. ■

### 5. GLOBAL ERROR ESTIMATES

In order to prove global error estimates, it still remains to analyse the regularization effects on discrete problems.

#### 5.1. Error estimates relative to regularization in discrete problems

Let us first notice that, using a technique similar to the one developed in Nochetto-Verdi (see [24], lemma 3, p. 1184) one can prove that :

LEMMA 5.1 : *Let assumptions (1.1)-(1.4), (2.1) and (2.2) hold. Then, there exists a constant C independent of  $\varepsilon$ , h and  $\tau$ , such that :*

$$\|U^0 - U_\varepsilon^0\|_{V'}^2 \leq C (h + \varepsilon)^2 |\log(\varepsilon + h)| . \tag{5.1}$$

THEOREM 5.1 : *Let assumptions (1.1)-(1.4), (2.1) and (2.2) hold. Then, under the stability condition (2.13), there exists a constant C independent of  $\varepsilon$ , h and  $\tau$ , such that :*

$$\begin{aligned} \|U_{h\tau} - U_{\varepsilon h\tau}\|_{L^\infty(0, T, V')}^2 + \|\Theta_{h\tau} - \Theta_{\varepsilon h\tau}\|_{L^2(Q)}^2 + \varepsilon \|U_{h\tau} - U_{\varepsilon h\tau}\|_{L^2(Q)}^2 &\leq \\ &\leq C (h^2 + \tau + (h + \varepsilon)^2 |\log(\varepsilon + h)| + \varepsilon \text{meas}(A_\varepsilon(U_{h\tau}))) , \end{aligned}$$

where  $A_\varepsilon(U_{h\tau}) = \{(x, t) \in Q, u_1 \leq U_{h\tau} \leq \xi_\varepsilon\}$  , (5.2)

$(U_{h\tau}, \Theta_{h\tau})$  and  $(U_{\varepsilon h\tau}, \Theta_{\varepsilon h\tau})$  being the solutions of  $(P_{h\tau})$  and  $(P_{\varepsilon h\tau})$ .

*Proof* : If we subtract equation (3.6) from (2.8) and sum their difference over  $n$ , from 0 to  $m - 1$ , ( $1 \leq m \leq N$ ), we obtain for any test functions  $\phi^0, \phi^1, \dots, \phi^m$  belonging to  $V_h$  :

$$\sum_{n=0}^{m-1} (\partial \mathcal{V}^{n+1}, \phi^n)_h + \sum_{n=0}^{m-1} (\mathcal{B}^n, \nabla \phi^n) + \sum_{n=0}^{m-1} (\nabla \mathcal{O}^n, \nabla \phi^n) = \text{I} + \text{II} + \text{III} = 0, \quad (5.3)$$

where, for the sake of simplicity, we have set :

$$\mathcal{V}^n = U^n - U_\varepsilon^n;$$

$$\mathcal{B}^n = B^n - B_\varepsilon^n \quad \text{and} \quad \mathcal{O}^n = \mathcal{O}^n - \mathcal{O}_\varepsilon^n, \quad \text{for any } n, \quad 0 \leq n \leq N.$$

Let us now choose the test functions  $\phi^n \in V_h$  defined as follows :

$$\begin{aligned} \phi^n &= \tau^2 \sum_{k=n}^{m-1} (\mathcal{O}^k - \mathcal{O}_\varepsilon^k) = \tau^2 \sum_{k=n}^{m-1} \mathcal{O}^k, \quad 0 \leq n \leq m-1; \\ \phi^n &= 0, \quad m-1 < n \leq N. \end{aligned}$$

If we analyse separately terms I to III in (5.3), we get :

$$\begin{aligned} \text{I} &= - \sum_{n=0}^{m-1} (\mathcal{V}^{n+1}, \partial \phi^n)_h - \frac{1}{\tau} (\mathcal{V}^0, \phi^0)_h \\ &= \sum_{n=0}^{m-1} \tau (\mathcal{V}^n, \mathcal{O}^n)_h + \sum_{n=0}^{m-1} \tau^2 (\partial \mathcal{V}^{n+1}, \mathcal{O}^n)_h - \frac{1}{\tau} (\mathcal{V}^0, \phi^0)_h \\ &= X + Y + Z. \end{aligned} \quad (5.4)$$

The term  $X$  can easily be bounded by means of Nchetto-Verdi results (see [24], p. 1187) combined with  $(\cdot, \cdot)_h$  definition ; namely

$$X \geq C \varepsilon \sum_{n=0}^{m-1} \tau \|\mathcal{V}^n\|_h^2 + \frac{1}{2B} \sum_{n=0}^{m-1} \tau \|\mathcal{O}^n\|_h^2 - C \varepsilon \text{meas} (A_\varepsilon(U_{h\tau})). \quad (5.5)$$

Moreover, using relations (2.15) and (3.10), we get for any  $\mu > 0$  :

$$\begin{aligned} Y &\leq \sum_{n=0}^{m-1} \tau^2 \|\partial \mathcal{V}^{n+1}\|_h \|\mathcal{O}^n\|_h \\ &\leq \frac{C}{\mu} \sum_{n=0}^{m-1} \tau^3 \|\partial \mathcal{V}^{n+1}\|_h^2 + \mu \sum_{n=0}^{m-1} \tau \|\mathcal{O}^n\|_h^2 \\ &\leq \frac{C}{\mu} \tau + \mu \sum_{n=0}^{m-1} \tau \|\mathcal{O}^n\|_h^2. \end{aligned} \quad (5.6)$$

The term  $Z$  requires a different analysis. Using relations (2.5)-(2.6) and (5.1), we have for any function  $\phi \in V_h$  :

$$\begin{aligned} (\mathcal{V}^0, \phi)_h &= (\mathcal{V}^0, \phi) + (\mathcal{V}^0, \phi)_h - (\mathcal{V}^0, \phi) \\ &\leq (\|\mathcal{V}^0\|_{V'} + Ch\|\mathcal{V}^0\|)\|\phi\|_H \\ &\leq C((h + \varepsilon)|\log(\varepsilon + h)|^{1/2} + h)\|\phi\|_H. \end{aligned} \tag{5.7}$$

Hence, taking  $\phi = \phi^0$  and applying Young inequality, we obtain :

$$Z \leq \frac{C}{\mu} ((h + \varepsilon)^2 |\log(\varepsilon + h)| + h^2) + C\mu \left\| \tau \sum_{k=0}^{m-1} \nabla \mathcal{O}^k \right\|^2. \tag{5.8}$$

Besides, notice that,  $b$  Lipschitz-continuity property combined with relations (2.5), (2.14) and (3.9) yields :

$$\begin{aligned} \|\mathcal{B}^n\| &\leq \|B^n - b(\mathcal{O}^n)\| + \|b(\mathcal{O}^n) - b(\mathcal{O}_\varepsilon^n)\| + \|b(\mathcal{O}_\varepsilon^n) - B_\varepsilon^n\| \\ &\leq Ch\|\nabla \mathcal{O}^n\| + C\|\mathcal{O}^n\|_h + Ch\|\nabla \mathcal{O}_\varepsilon^n\| \\ &\leq Ch + C\|\mathcal{O}^n\|_h. \end{aligned} \tag{5.9}$$

Thus, applying Young inequality to the term  $\text{II}$ , we get after reordering :

$$\text{II} \leq C\mu h^2 + C\mu \sum_{n=0}^{m-1} \tau \|\mathcal{O}^n\|_h^2 + \frac{C}{\mu} \sum_{n=0}^{m-1} \tau \left\| \tau \sum_{k=0}^n \nabla \mathcal{O}^k \right\|^2 \quad \forall \mu > 0. \tag{5.10}$$

Finally, using the following well-known property :

$$\sum_{n=0}^{m-1} \alpha_n \sum_{k=n}^{m-1} \alpha_k \geq \frac{1}{2} \left( \sum_{n=0}^{m-1} \alpha_n \right)^2, \quad \forall \alpha_n \in \mathbb{R}, \quad 0 \leq n \leq m-1,$$

we have :

$$\text{III} = \sum_{n=0}^{m-1} \tau \left( \nabla \mathcal{O}^n, \tau \sum_{k=n}^{m-1} \nabla \mathcal{O}^k \right) \geq \frac{1}{2} \left\| \tau \sum_{n=0}^{m-1} \nabla \mathcal{O}^n \right\|^2. \tag{5.11}$$

Hence, collecting estimates (5.4)-(5.11) and choosing  $\mu$  small enough, relation (5.3) becomes :

$$\begin{aligned} \sum_{n=0}^{m-1} \tau \|\mathcal{O}^n\|_h^2 + \left\| \tau \sum_{n=0}^{m-1} \nabla \mathcal{O}^n \right\|^2 + \varepsilon \sum_{n=0}^{m-1} \tau \|\mathcal{V}^n\|_h^2 &\leq \\ &\leq C\sigma + C \sum_{n=0}^{m-1} \tau \left\| \tau \sum_{k=0}^n \nabla \mathcal{O}^k \right\|^2, \end{aligned}$$

where

$$\sigma = \sigma(\varepsilon, h, \tau) = h^2 + \tau + \varepsilon \text{meas}(A_\varepsilon(U_{h\tau})) + (h + \varepsilon)^2 |\log(\varepsilon + h)|.$$



Applying the discrete Gronwall inequality to this expression yields :

$$\sum_{n=0}^{m-1} \tau \|\mathcal{O}^n\|_h^2 + \left\| \tau \sum_{n=0}^{m-1} \nabla \mathcal{O}^n \right\|^2 + \varepsilon \sum_{n=0}^{m-1} \tau \|\mathcal{V}^n\|_h^2 \leq C \sigma . \quad (5.12)$$

It still remains to prove that :  $\max_{0 \leq n \leq N} \|\mathcal{V}^n\|_{V'}^2 \leq C \sigma$ .

To this purpose, let us choose  $\phi^n = \tau \phi \in V_h$  in (5.3). We get then :

$$\begin{aligned} \sum_{n=0}^{m-1} \tau (\partial \mathcal{V}^{n+1}, \phi)_h &= (\mathcal{V}^m - \mathcal{V}^0, \phi)_h \\ &= - \sum_{n=0}^{m-1} \tau (\nabla \mathcal{O}^n, \nabla \phi) - \sum_{n=0}^{m-1} \tau (\mathcal{B}^n, \nabla \phi) . \end{aligned}$$

Taking into account relations (5.7), (5.9) and (5.12), it follows that :

$$(\mathcal{V}^m, \phi)_h \leq C \left( \sigma^{1/2} + \sum_{n=0}^{m-1} \tau \|\mathcal{B}^n\| \right) \|\nabla \phi\| \leq C \sigma^{1/2} \|\nabla \phi\| .$$

Hence, using (2.15) and (3.10), we get for any function  $\phi \in \mathcal{D}(\Omega)$  :

$$\begin{aligned} (\mathcal{V}^m, \phi) &= \\ &= (\mathcal{V}^m, \phi - r_h \phi) + ((\mathcal{V}^m, r_h \phi) - (\mathcal{V}^m, r_h \phi)_h) + (\mathcal{V}^m, r_h \phi)_h \\ &\leq C \|\phi - r_h \phi\| + Ch \|r_h \phi\|_H + (\mathcal{V}^m, r_h \phi)_h \\ &\leq C \sigma^{1/2} \|\nabla \phi\| . \end{aligned}$$

Duality arguments complete the proof. ■

### 5.2. Global error estimates

The first immediate consequence of all our previous results is the following global estimate.

Let  $(u, \theta = \beta(u))$  and  $(U_{h\tau}, \Theta_{h\tau})$  be the respective solutions of problems  $(P)$  and  $(P_{h\tau})$ . If assumptions (1.1)-(1.4), (2.1) and (2.2) hold, then, under the stability condition (2.13), and for any  $\varepsilon > 0$  small enough, there exists a constant  $C$ , independent of  $\varepsilon, h$  and  $\tau$  such that :

$$\begin{aligned} \|u - U_{h\tau}\|_{L^\infty(0, T; V')}^2 + \|\theta - \Theta_{h\tau}\|_{L^2(Q)}^2 + \varepsilon \|u - U_{h\tau}\|_{L^2(Q)}^2 &\leq \\ &\leq C \left( \tau + \frac{h^2}{\varepsilon} + \frac{h^4}{\varepsilon^2} + \frac{\tau^2}{\varepsilon} + (h + \varepsilon)^2 |\log(\varepsilon + h)| \right. \\ &\quad \left. + \varepsilon^2 |\log \varepsilon| + \varepsilon \text{meas}(A_\varepsilon(U_{h\tau})) + \varepsilon \text{meas}(A_\varepsilon(u)) \right) , \quad (5.13) \end{aligned}$$

where  $A_\varepsilon(U_{h\tau})$  and  $A_\varepsilon(u)$  are defined in (5.2) and (3.5).

Thus, choosing  $\varepsilon$  in a good way, we obtain the following error estimates :

**PROPOSITION 5.1 :** *Let  $(u, \theta = \beta(u))$  and  $(U_{h\tau}, \Theta_{h\tau})$  be the respective solutions of problems  $(P)$  and  $(P_{h\tau})$ . If assumptions (1.1)-(1.4), (2.1) and (2.2) hold, then, under the stability condition (2.13), there exists a constant  $C$ , independent of  $h$  and  $\tau$  such that :*

$$\|u - U_{h\tau}\|_{L^\infty(0, T; V')} + \|\theta - \Theta_{h\tau}\|_{L^2(Q)} \leq Ch^{1/2}. \tag{5.14}$$

*Proof :* Let  $\varepsilon$  be chosen so that :  $\varepsilon = \alpha_1 h$ , where  $\alpha_1 > 0$  is an arbitrary constant. Then, condition (2.13) combined with (5.13) implies (5.14). ■

**PROPOSITION 5.2 :** *« The Non-Degenerate case. »*

*Let the assumptions of Proposition 5.1 hold. Then, under the stability condition (2.13) and under the additional properties :*

$$\text{meas}(A_\varepsilon(u)) \leq C\varepsilon, \quad \text{i.e. } (P) \text{ non-degenerate}, \tag{5.15}$$

$$\text{meas}(A_\varepsilon(U_{h\tau})) \leq C\varepsilon, \quad (\text{discrete non-degeneracy property}), \tag{5.16}$$

*there exists a constant  $C$ , independent of  $\varepsilon$ ,  $h$  and  $\tau$ , such that :*

$$\|u - U_{h\tau}\|_{L^\infty(0, T; V')} + \|\theta - \Theta_{h\tau}\|_{L^2(Q)} \leq Ch^{2/3}, \tag{5.17}$$

$$\|u - U_{h\tau}\|_{L^2(Q)} \leq Ch^{1/3}. \tag{5.18}$$

*Proof :* Choosing in (5.13)  $\varepsilon = \alpha_1 h^{2/3}$ ,  $\alpha_1 > 0$  being an arbitrary constant, and using condition (2.13) easily lead to the required results. ■

*Remark 5.1 :* A characterization of « non degenerate » problems (i.e. problems satisfying condition (5.15)) can be found in Nochetto [20]. In particular, he shows that, under some qualitative assumptions upon the data, property (5.15) holds. Against that, there exists, to our knowledge, no characterization of discrete non-degenerate problems.

*Remark 5.3 :* Sharp error estimates for both temperature and enthalpy have been proven, in « non-degenerate cases » by Nochetto [17] (i.e.  $\mathcal{O}(h)$ ) for a Euler backward scheme without numerical integration. On the other hand, Nochetto-Verdi [22] have obtained, for an Euler backward scheme combined with integration quadrature rules, results equivalent to ours.

*Remark 5.4 :* Let us notice that numerical experiments show better error estimates than those predicted theoretically (viz.  $\mathcal{O}(h)$ ). We refer for that to Amiez-Gremaud [1].

*Remark 5.5 :* Finally, we would like to emphasize that the stability condition (2.13), which is a priori quite restrictive, can be relaxed by the use

of a « super-step » method (see Gentzsch [11] or Droux [8] for numerical experiments).

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