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A. SZEPESSY

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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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CONVERGENCE OF A STREAMLINE DIFFUSION FINITE ELEMENT METHOD FOR SCALAR CONSERVATION LAWS WITH BOUNDARY CONDITIONS (*)

A. SZEPESSY [1]

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Abstract. — *A higher order accurate shock-capturing streamline diffusion finite element method for general scalar conservation laws is analysed ; convergence towards the unique solution is proved for several space dimensions with initial and boundary conditions, using a uniqueness theorem for measure valued solutions. Furthermore, some numerical results are given.*

Résumé. — *On démontre, pour des solutions approchées par la méthode d'éléments finis « streamline diffusion », la convergence vers la solution exacte dans le cas d'une loi de conservation scalaire générale dans un domaine borné Ω de R^d, $d \geq 1$, avec des conditions aux limites sur la frontière de Ω. On utilise un théorème d'unicité de la solution qui peut être une mesure. Finalement, quelques exemples numériques sont considérés.*

0. INTRODUCTION

In this note convergence of a higher-order accurate shock-capturing streamline diffusion finite element method (*SC*-method) is proved for general scalar conservation laws in several space dimensions with initial and boundary conditions, using the uniqueness theorem for measure valued solutions in [Sz III]. This theorem, which is a generalization of the corresponding result for the pure initial value problem by DiPerna [Di], yields convergence in L_p, $1 \leq p \leq \infty$, towards the unique solution, for approximate solutions of a scalar conservation law provided they are

(A) uniformly bounded in the L_∞-norm,
(B) weakly consistent with all entropy inequalities,
(C) strongly consistent with the initial condition.

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(1) INADA, Royal Institute of Technology, 10044 Stockholm, Suède.

In Section 3 the SC-method is proved to satisfy (A) and in Section 4 the conditions (B) and (C) are verified. We note that the convergence proof does not require estimates of the total variation, which are usually used together with classical compactness arguments to prove convergence of finite difference schemes. The only previous results for finite difference methods applied to scalar conservation laws with boundary conditions are given in [LR I] and [LR II], where convergence was obtained by classical total variation estimates. We also remark that in the case of scalar conservation laws in an unbounded domain, i.e. without boundary conditions, one can replace the assumption (A) by « uniformly bounded in L_q-norm, $1 \leq q \leq p$ » if the flux involved in the conservation law growths as a polynomial of degree not more than p in infinity, see [Sz II].

We now give some background material on the SC-method. The streamline diffusion method is a general finite element method for hyperbolic problems which may be viewed as a certain combination of the standard Galerkin method and a least squares method giving added stability through the weighted least squares control of the residual. In the shock-capturing variant artificial viscosity is added with the viscosity coefficient depending locally on the residual and the mesh parameter h. The effect of the shock-capturing term is to add substantial artificial viscosity locally where the solution is non smooth, which improves the quality of the approximations near shock fronts compared to the streamline method. The shock-capturing streamline diffusion method combines $\mathcal{O}(h^{k+1/2})$ accuracy for smooth solutions approximated by polynomials of degree k, with good stability obtained through the least squares control of the residual and the shock-capturing artificial viscosity.

The result presented here extends to higher order elements the analysis of SC-methods with piecewise linear elements initialized in [JSz I, JSz III] and continued in [Sz I]. In [JSz III] numerical results were presented for the Euler equations in two space dimensions and convergence was established for a Cauchy problem in one dimension for Burgers' equation using the theory of compensated compactness. Further, in [Sz I] convergence to the unique solution was proved for a SC-method applied to a Cauchy problem for a scalar conservation law in two space dimensions by using the uniqueness result for measure valued solutions of DiPerna [Di].

In Section 5 the result of some numerical experiments are presented.

1. MEASURE VALUED SOLUTIONS WITH BOUNDARY CONDITION

In this section we recall some results for measure valued solutions of scalar conservation laws with initial and boundary conditions given in [Sz III]. The proof of convergence of the finite element solutions will be based on Theorem 1.1 below.

Let Ω be a bounded open set of R^d with smooth boundary $\Gamma = \partial\Omega$ with outward unit normal n. We consider for $u : \Omega \times R_+ \to R$ the conservation law

$$u_t + \sum_{j=1}^{d} f_j(u)_{x_j} = 0 \quad \text{in} \quad \Omega \times R_+ , \tag{1.1}$$

with initial condition

$$u(.,0) = u_0 \quad \text{in} \quad \Omega , \tag{1.2}$$

and boundary condition : for all $k \in R$, $(\bar{x}, t) \in \Gamma \times R_+$

$$(\operatorname{sgn}(u(\bar{x},t)-k) - \operatorname{sgn}(a(\bar{x},t)-k))(f(u(\bar{x},t)) - f(k)) \,.\, n(\bar{x}) \geqslant 0 , \tag{1.3}$$

where $f = (f_1, \ldots, f_d) : R \to R^d$, $u_0 : \Omega \to R$ and $a : \Gamma \times R_+ \to R$ are given smooth functions and the function $\operatorname{sgn} : R \to R$ is defined by

$$\operatorname{sgn}(x) = \begin{cases} x/|x| , & \text{if} \quad x \neq 0 , \\ 0 & \text{if} \quad x = 0 . \end{cases}$$

We remark that if f is linear, then (1.3) requires u to be equal to the given boundary data a on the inflow boundary (where $f' \,.\, n \leqslant 0$), but does not impose any boundary condition on outflow boundaries (where $f' \,.\, n > 0$). More generally, in [BLN] it is proved that solutions u^ε of parabolic regularizations of (1.1) defined by

$$u_t^\varepsilon + \sum_{j=1}^{d} f_j(u^\varepsilon)_{x_j} = \varepsilon \, \Delta u^\varepsilon \quad \text{in} \quad \Omega \times R_+ ,$$
$$u^\varepsilon(.,0) = u_0 \quad \text{on} \quad \Omega ,$$
$$u^\varepsilon = a \quad \text{on} \quad \Gamma \times R_+ ,$$

converge a.e. to a function $u \in BV(\Omega \times R_+)$ as $\varepsilon \to 0+$, which is the unique weak entropy satisfying solution (the BV-solution) of (1.1-1.3). Moreover, the trace of u satisfies (1.3). Dubois and Le Floch [DL] have pointed out that (1.3), in the case of one space dimension, is equivalent to u being a solution of a certain Riemann problem on the boundary.

Let us now recall the concept of measure valued solution of (1.1-3) given in [Sz III] which will be used to prove convergence of the finite element method. Let then $\{u_j\}$ be a uniformly bounded sequence in $L_\infty(\Omega \times R_+)$, i.e.,

$$\|u_j\|_{L_\infty(\Omega \times R_+)} \leqslant K , \quad j = 1, 2, 3, \ldots , \tag{1.4}$$

(in the applications the u_j will be approximate solutions of (1.1-1.3)). Then according to Youngs theorem there exists, *cf.* [Di], [Ta], a subsequence, which we still label $\{u_j\}$, and an associated measurable measure valued mapping $\nu_{(.)} : \Omega \times R_+ \to \operatorname{Prob}(R)$, such that

$$\operatorname{supp} \nu_{(x,t)} \subset \{\lambda : |\lambda| \leqslant K\} , \quad \text{a.e.,} \quad (x,t) \in \Omega \times R_+ , \tag{1.5}$$

and $\forall g \in \mathscr{C}(R)$ the $L_\infty(\Omega \times R_+)$ weak star limit

(1.6a) $$g(u_j(.)) \quad \bar{g}(.) \text{ as } j \to \infty ,$$

exists, where

(1.6b)
$$\bar{g}(x, t) = \int_R g(\lambda)\, d\nu_{(x,t)}(\lambda) \equiv \langle \nu_{(x,t)}, g(\lambda) \rangle , \quad \text{a.e.,} \quad (x, t) \in \Omega \times R_+ .$$

Here Prob (R) is the space of all positive Borel measures on R of unit mass and $\nu_{(.)}$ is measurable means that $\langle \nu_y, g \rangle$ is measurable with respect to $y \in \Omega \times R_+$ for each continuous function g. The mapping ν is a Young measure associated with the sequence $\{u_j\}$.

To define measure valued solutions satisfying boundary conditions we shall associate with a given Young measure $\nu : \Omega \times R_+ \to$ Prob (R) satisfying (1.5), in general in a non unique way, a Young measure $\gamma\nu_{(.)} : \Gamma \times R_+ \to$ Prob (R), which we consider to be a « trace » of ν on $\Gamma \times R_+$. For this purpose we introduce the change of coordinates $x \to (\bar{x}, y)$ for x in a neighbourhood of Γ :

(1.7) $$x = \bar{x} - yn(\bar{x}) ,$$

where $(\bar{x}, y) \in \Gamma \times (0, \varepsilon)$, for some $\varepsilon > 0$.

LEMMA 1.1 [Sz III] : *Let $\nu : \Omega \times R_+ \to$ Prob (R) be a Young measure associated with a sequence $\{u_j\}$ satisfying* (1.4). *Then there is a sequence $\{y_j \in (0, \varepsilon)\}$, where $y_j \to 0$, and there is a measurable Young measure $\gamma\nu : \Gamma \times R_+ \to$ Prob (R) such that*

$$\operatorname{supp} \gamma\nu_{(\bar{x},t)} \subset \{\lambda : |\lambda| \leq K\} \quad \text{a.e.,} \quad (\bar{x}, t) \in \Gamma \times R_+ ,$$

and, for every $g \in \mathscr{C}(R)$, the $L_\infty(\Gamma \times R_+)$ weak star limit

$$\langle \nu_{(x(.,y_j),.)}, g(\lambda) \rangle \rightharpoonup \bar{g}(.,.) , \quad \text{as} \quad j \to \infty ,$$

exists, i.e.

(1.8a) $$\lim_{j \to \infty} \int_{\Gamma \times R_+} \langle \nu_{(x(\bar{x}, y_j), t)}, g(\lambda) \rangle \varphi \, ds\, dt = \int_{\Gamma \times R_+} \bar{g}(\bar{x}, t)\, \varphi \, ds\, dt ,$$

for all $\varphi \in L_1(\Gamma \times R_+)$, where ds is the Lebesgue measure on Γ, and

(1.8b) $$\bar{g}(\bar{x}, t) = \int_R g(\lambda)\, d\gamma\, \nu_{(\bar{x},t)} \equiv \langle \gamma\nu_{(\bar{x},t)}, g(\lambda) \rangle ,$$

for a.e. $(\bar{x}, t) \in \Gamma \times R_+$.

DEFINITION [Sz III] : *A Young measure ν, associated to a sequence $\{u_j\}$ which satisfies* (1.4), *is a measure valued solution (mv-solution) to* (1.1-1.3) *if for all $\phi \in \mathscr{C}_0^1(\bar{\Omega} \times R_+)$, $\phi \geq 0$, and for all $k \in R$, we have*

$$(1.9a) \quad \int_{\Omega \times R_+} \left(\langle \nu_{(x,t)}, |\lambda - k| \rangle \phi_t + \langle \nu_{(x,t)}, (\operatorname{sgn}(\lambda - k)) \times \right.$$
$$\left. \times (f(\lambda) - f(k)) \rangle \cdot \nabla \phi \right) dx\, dt - \int_{\Gamma \times R_+} \langle \gamma \nu_{(\bar{x},t)}, f(\lambda) - f(k) \rangle$$
$$\times n(\bar{x}) \phi \operatorname{sgn}(a - k)\, ds\, dt \geq 0 ,$$

and

$$(1.9b) \qquad \lim_{t \to 0} \int_{\Omega} \langle \nu_{(x,t)}, |\lambda - u_0| \rangle \, dx = 0 .$$

Remark 1.1 : In general Lemma 1.1 associates with ν a trace $\gamma\nu$ in a non-unique way. However, in the proof of Theorem 1.1 in [Sz III] it is seen that the expected value $\langle \gamma\nu_{(.)}, f(\lambda) \rangle$, which appears in the definition (1.9*a*), is in fact uniquely defined. □

The following uniqueness result for *mv*-solutions is proved in [Sz III].

THEOREM 1.1 [Sz III] : *Suppose that a Young measure ν associated with the sequence $\{u_j\}$ is a mv-solution to* (1.1-1.3) *and let w denote the unique BV-solution of* (1.1-1.3). *Then*

$$\nu_{(x,t)} = \delta_{w(x,t)} \qquad a.e. ,$$

i.e., $\nu_{(x,t)}$ reduces a.e. to the Dirac measure concentrated at $w(x,t)$, and the sequence $\{u_j\}$ converges strongly in $L_1^{loc}(\Omega \times R_+)$ to w.

2. FORMULATION OF THE FINITE ELEMENT METHOD

In Sections 2-4 we shall use Theorem 1.1 to prove strong convergence in L_1^{loc} towards the unique *BV*-solution of (1.1-1.3) for the *SC*-method. First we consider a one-dimensional problem, i.e. we let $\Omega = (0, 1)$ and $\Gamma = \partial\Omega = \{0, 1\}$. The generalization to several dimensions is straight forward, see Remark 2.2. We shall assume that there is a constant C such that the smooth function f also satisfies

$$(2.1) \qquad \sup_{y \in R} |f''(y)| \leq C .$$

This is no essential restriction since the exact solution is bounded and thus $f(y)$ may be modified for large $|y|$ so as to satisfy (2.1). Below we denote by C a positive constant independent of h, not necessarily the same at each occurrence.

Let us now introduce the finite element spaces of the SC-method for (1.1-1.3). Let $0 = t_0 < t_1 < t_2 \dots < t_{\tilde{N}} = T$ be a sequence of time levels, set $I_n = (t_n, t_{n+1})$ and introduce the « slabs » $S_n = \Omega \times I_n$ and sets $\Omega_n = \Omega \times \{t_n\}$. For $h > 0$ and $n = 0, 1, 2, \dots$ let T_h^n be a quasi-uniform triangulation of S_n into triangles K of diameter $h_K \sim h$ with smallest angle uniformly bounded away from zero and let K have one right angle if $\bar{K} \cap \Gamma \neq \emptyset$ (in the case of several space dimensions this is generalized to mean that the set of space coordinates, of the vertices of $K \in T_h^n$ such that $K \cap \Gamma \neq \emptyset$, are the same for $t = t_{n+1}$ and $t = t_n$). Define for a given natural number $k \geq 1$

$$V_h^n = \left\{ v \in H^1(S_n) : v|_K \in P_k(K), K \in T_h^n, v|_{\Gamma \times R_+} = 0 \right\},$$

$$V_h = \prod_{n \geq 0} V_h^n,$$

where $P_k(K)$ denotes the set of polynomials of degree at most k on K. In other words, V_h^n consists of continuous piecewise polynomials on the slab S_n. Typically, $t_{n+1} - t_n \sim h$ with the slab S_n one element wide. To define the shock-capturing modification also for $k > 1$, we divide each $K \in T_h^n$ into similar triangles $\hat{K}_i$, $i = 1, 2, 3, \dots, k^2$ and introduce $\hat{T}_h^n = \{\hat{K}_i : i = 1, 2, 3, \dots, k^2, K \in T_h^n\}$,

$$\hat{V}_h^n = \left\{ v \in H^1(S_n) : v|_{\hat{K}_i} \in P_1(\hat{K}_i), i = 1, \dots, k^2, K \in T_h^n, v|_{\Gamma \times R_+} = 0 \right\},$$

$$\hat{V}_h = \prod_{n \geq 0} \hat{V}_h^n .$$

We also introduce the usual nodal interpolation operators

$$\pi : \prod_{n \geq 0} \mathscr{C}(S_n) \to V_h ,$$

$$\hat{\pi} : \prod_{n \geq 0} \mathscr{C}(S_n) \to \hat{V}_h ,$$

where the degrees of freedom of V_h and $\hat{V}_h$ are the values at the vertices of $\hat{K}_i \in \hat{T}_h^n$.

We will seek an approximate solution

$$u_h = U_h + \bar{a} ,$$

where $U_h \in V_h$ and $\bar{a} : \bar{\Omega} \times R_+$ is a smooth extension of a. Note that the functions in V_h and $\hat{V}_h$ are zero on $\Gamma \times R_+$ and continuous in x and possibly discontinuous in t at the discrete time levels t_n.

We shall need the following standard interpolation error estimate (2.2), « super-approximation » result (2.3) and inverse estimate (2.4) (a proof of (2.3) is given in the appendix).

LEMMA 2.1 : *There are constants C such that for* $w \in W^{s,p}(\omega) \cap \mathscr{C}(S_n)$, $v \in V_h$, $n = 0, 1, 2, \ldots$

(2.2a) $$\| w - \pi w \|_{W^{r,\infty}(\omega)} \leq Ch^{s-r} \| w \|_{\dot{W}^{s,\infty}(\omega)}, \quad s = 1, \ldots, k+1, \quad r=0, 1, \quad p=\infty,$$

(2.2b) $$\| w - \pi w \|_{H^r(\omega)} \leq Ch^{\bar{k}+1-r} \| w \|_{\dot{H}^{\bar{k}+1}(\omega)}, \quad r = 0, 1, \quad 1 \leq \bar{k} \leq k, \quad p = 2,$$

(2.3a) $$\| vw - \pi(vw) \|_{W^{r,\infty}(\omega)} \leq Ch^{1-r} \| v \|_{L_\infty(\omega)} \| w \|_{W^{1,\infty}(\omega)}, \quad r = 0, 1, \quad p = \infty,$$

(2.3b) $$\| vw - \pi(vw) \|_{H^r(\omega)} \leq C \| v \|_{L_\infty(\omega)} \sum_{i=1}^{k+1} h^{i-r} \| w \|_{\dot{H}^i(\omega)}, \quad r = 0, 1, \quad p = 2,$$

(2.3c) $$\| vw - \pi(vw) \|_{L_2(\Omega_n)} \leq C \| v \|_{L_\infty(S_n)} \sum_{i=1}^{k+1} h^{i-\frac{1}{2}} \| w \|_{\dot{H}^i(S_n)}, \quad p = 2,$$

(2.4a) $$\| v \|_{W^{r,p}(\omega)} \leq Ch^{-r} \| v \|_{L_p(\omega)}, \quad r = 0, \ldots, k, \quad 1 \leq p \leq \infty,$$

(2.4b) $$\| v \|_{L_p(S_n)} \leq Ch^{-2/p} \| v \|_{L_\infty(S_n)}, \quad 1 \leq p \leq \infty,$$

where $\omega = \Omega_n$, S_n, $K \cap \Omega_n$ or $K \cap S_n$ for $K \in T_h$ and $W^{s,p}$ is the usual Sobolev space (here dot denotes semi norm and $W^{s,2} = H^s$). The same estimates hold if we replace π by $\hat{\pi}$, V_h by $\hat{V}_h$ and k by 1.

The SC method for (1.1-1.3) can now be formulated as follows :

Find $u_h = U_h + \bar{a}$ where $U \equiv U_h \in V_h$ such that for $n = 0, 1, 2, \ldots$,

(2.5) $$\int_{S_n} L(U)(v + \delta(v_t + f'(U + \bar{a}) v_x)) \, dx \, dt + \int_{S_n} \varepsilon_1(U) \nabla \hat{U} \cdot \nabla \hat{v} \, dx \, dt + \int_{S_n} \varepsilon_2(U) \hat{U}_x \hat{v}_x \, dx \, dt + \int_{\Omega_n} (U_+ - U_-) v_+ \, dx = 0, \quad \forall v \in V_h^n,$$

where

$$\hat{w} \equiv \hat{\pi} w\,, \quad \forall w \in V_h\,,$$

$$L(U) \equiv U_t + f'(U+\bar{a})\,U_x + g(U,x,t)\,,\ g(U,x,t) \equiv f'(U+\bar{a})\,\bar{a}_x + \bar{a}_t\,,$$

$$\varepsilon_1(U)\big|_K \equiv \bar{\delta}\int_K |L(U)|\,(1+|f'(U+\bar{a})|)\,dx\,dt \Big/ \int_K dx\,dt\,,\ \forall K \in T_h^n\,,$$

$$\varepsilon_2(U)\big|_K \equiv \tilde{\psi}(x) + \bar{\bar{\delta}}\int_{K\cap\Omega_n} |U_+ - U_-|\,dx \Big/ \int_{K\cap\Omega_n} dx\,,\ \forall K \in T_h^n\,,$$

$$\tilde{\psi}(x)\big|_K = \begin{cases} \varepsilon & \text{if } \bar{K}\cap(\Gamma\times\mathbb{R}_+) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}\,,\ \forall K \in T_h\,,$$

$$v_\pm(x,t) = \lim_{s\to 0\pm} v(x,t+s)\,,\quad U_-(.,0) = u_0\,.$$

Further δ, $\bar{\delta}$, $\bar{\bar{\delta}}$, ε are positive parameters satisfying

$$\text{(2.6)}\quad \delta = Ch\,,\quad \bar{\delta} = Ch^{\alpha_1}\,,\quad \bar{\bar{\delta}} = Ch^{\alpha_2}\,,$$
$$(h + \delta + \bar{\delta}/h + \bar{\bar{\delta}})/\varepsilon \to 0 \quad \text{as} \quad h \to 0\,,$$

where the α_i are constants with $\frac{3}{2} < \alpha_1 < 2$, $\frac{1}{2} < \alpha_2 < 1$. From now on $U = U_h$ will denote a solution of (2.5).

Existence of a solution to (2.5) follows from a variant of Brouwer's fixed point theorem as in [Li], [JS].

Remark 2.1 : Comparing with the *SC*-method for the pure initial value problem considered in [JSz III] and [Sz I], we consider here an inhomogeneous problem with source term $g \neq 0$, and the shock-capturing term is accordingly modified to contain the total residual $L(U)$. Further, we add extra diffusion ($\tilde{\psi} u_{xx}$) on the elements next to the boundary $\Gamma = \partial\Omega$. In [Sz I] we proved that the method (2.5) with $k = 1$ is accurate of order $\mathcal{O}(h^{3/2})$ for a corresponding linear conservation law with smooth exact solution. This analysis easily extends to the method (2.5) in the nonlinear case with $k > 1$ if the exact solution is smooth, giving accuracy $\mathcal{O}(h^{k+1/2})$ away from the boundary (and $\mathcal{O}(h^{(1+d)/2})$ in a neighbourhood of the boundary) by using Lemma 3.4 below. □

The main result is the following.

THEOREM 2.1 : *The functions $u_h = U_h + \bar{a}$, where $U_h \in V_h$ satisfies* (2.5), *converge strongly in $L_1^{\mathrm{loc}}(\Omega \times R_+)$ to the unique BV-solution of* (1.1-1.3) *as h tends to zero.*

The proof is divided into Lemma 2.2 and Lemma 2.3, which are proved in Sections 3 and 4, respectively.

LEMMA 2.2 : *There is a constant C such that the functions u_h in Theorem* 2.1 *satisfy*

$$\|u_h\|_{L_\infty(\Omega \times R_+)} \leqslant C , \quad h > 0 . \tag{2.7}$$

LEMMA 2.3 : *There is a subsequence of the functions u_h given in Theorem* 2.1 *that generate an mv-solution v of* (1.1-1.3).

Remark 2.2 : The proof of Theorem 2.1 extends with the following to a general scalar conservation law in several dimensions yielding convergence of the *SC*-method applied to the conservation law (1.1-1.3) with $d \geqslant 1$.

The only major modifications in the case with $d > 1$ occur in the proofs of Lemma 3.2 and Lemma 3.3. In [Sz I] the corresponding results are Lemma 4.1 and Lemma 4.2, and they are proved for $d = 2$ for a special triangulation of $\Omega \times \{t_n\}$ into triangles K with one right angle and dividing each prism $K \times (t_n, t_{n+1})$ into three tetrahedrons. The proof of Lemma 4.2 in [Sz I] easily extends to an element given by the d-simplex $E_d = \left\{x \in R^d : 0 \leqslant x_i, \sum_{i=1}^{d} x_i \leqslant h\right\}$. In particular this proves Lemma 3.2 below for elements which at $t = t_n$ reduce to sets $E_{d-1} \times \{t_n\}$ (with the new constant chp^{-2} which does not change the analysis). For general triangulations with $d \geqslant 1$, Lemma 3.2 and 3.3 thus remain true for a *SC*-method if the shock-capturing terms are defined by

$$\sum_{\hat{K} \in \hat{T}_h^n} \int_K \varepsilon_1(U)\, \nabla' U \,.\, \nabla' v \; dx\, dt ,$$

$$\sum_{\hat{K} \in \hat{T}_h^n} \int_K \varepsilon_2(U)\, \nabla_x' U \,.\, \nabla_x' v \; dx\, dt ,$$

with ε_i as before and where ∇' denotes the gradient calculated in the orthonormal coordinates on E_d which are given by the linear transformation of $\hat{K}$ onto E_d and ∇_x' is the space-gradient in orthonormal coordinates on E_{d-1} which are given by the linear transformation of $\hat{K} \cap \Omega_n$ onto $E_{d-1} \times \{t_n\}$.

The localization result in Lemma 4.1 for $d > 1$ follows if we let the normal coordinate y, which is defined in (1.7), play the role of « x ». Further, in the super approximation result (4.12) requiring $k + 1 > d/2$, we can instead use L_1 and L_∞ bounds as in (4.10) combined with (2.3a) in which case no restrictions on d enter. □

3. PROOF OF LEMMA 2.2

First we state and prove the following basic L_2 stability estimate.

LEMMA 3.1 : *For* $N = 0, 1, 2, \ldots$, *we have*

$$\frac{1}{2}\left(\int_{\Omega_{N+1}} (U_-)^2\,dx + \sum_{n=0}^{N} \int_{\Omega_n} (U_+ - U_-)^2\,dx\right) +$$

$$+ \delta\left(\int_{S^N} (U_t + f'\,U_x)^2\,dx\,dt\right) + \int_{S^N} \varepsilon_1(U)\left|\nabla \hat{U}\right|^2 dx\,dt$$

$$+ \int_{S^N} \varepsilon_2(U)\,\hat{U}_x^2\,dx\,dt \leqslant C\left(\int_{\Omega} u_0^2\,dx + \int_{S^N} g(0, x, t)^2\,dx\,dt\right),$$

where $S^N = \bigcup_{n=0}^{N} S_n$ *and integrals over* S^N *are interpreted as a sum of integrals over the* S_n.

Proof: Taking $v = U$ in (2.5) we obtain

$$(3.1)\quad 0 = \int_{S_n} L(U)(U + \delta(U_t + f'\,U_x))\,dx\,dt + \int_{\Omega_n} (U_+ - U_-)\,U_+\,dx +$$

$$+ \int_{S_n} \varepsilon_1(U)\left|\nabla \hat{U}\right|^2 dx\,dt + \int_{S_n} \varepsilon_2(U)\,\hat{U}_x^2\,dx\,dt$$

$$= \frac{1}{2}\left(\int_{\Omega_{n+1}} (U_-)^2\,dx - \int_{\Omega_n} (U_-)^2\,dx + \int_{\Omega_n} (U_+ - U_-)^2\,dx\right)$$

$$+ \delta \int_{S_n} (U_t + f'\,U_x)^2\,dx\,dt + \int_{S_n} \varepsilon_1(U)\left|\nabla \hat{U}\right|^2 dx\,dt$$

$$+ \int_{S_n} \varepsilon_2(U)\,\hat{U}_x^2\,dx\,dt + \int_{S_n} (f'\,U_x\,U + gU)\,dx\,dt$$

$$+ \delta \int_{S_n} g(f'\,U_x + U_t)\,dx\,dt\,.$$

Define now the function F_m for $m \in \mathbb{N}$ by

$$(3.2)\qquad F_m(s, x, t) = \int_0^s f'(w + \bar{a})\,w^m\,dw\,.$$

We have by (2.1)

(3.3)

$$\left|\int_\Omega f' U_x U\,dx\right| = \left|\int_\Omega \frac{d}{dx}F_1(U,x,t)\,dx - \int_\Omega \int_0^U (f'(w+\bar a))_x w\,dw\,dx\right| =$$
$$= \left|-\int_\Omega \int_0^U (f'(w+\bar a))_x w\,dw\,dx\right| \leqslant C \int_\Omega U^2\,dx\,.$$

Further by (2.1)

$$|g(U,x,t)| \leqslant C|U| + |g(0,x,t)|\,, \tag{3.4}$$

so that by summing over $n = 0, 1, 2, \ldots, N$ in (3.1), we get

$$\frac{1}{2}\left(\int_{\Omega_{N+1}} (U_-)^2\,dx + \sum_{n=0}^{N} \int_{\Omega_n} (U_+ - U_-)^2\,dx + \right.$$
$$\left. + \delta \int_{S^N} (U_t + f' U_x)^2\,dx\,dt\right)$$
$$+ \int_{S^N} \varepsilon_1(U)\left|\nabla \hat U\right|^2 dx\,dt + \int_{S^N} \varepsilon_2(U)\,\hat U_x^2\,dx\,dt$$
$$\leqslant \frac{1}{2}\int_\Omega u_0^2\,dx + C\left(\int_{S^N} U^2\,dx\,dt + \int_{S^N} (g(0,x,t))^2\,dx\,dt\right). \tag{3.5}$$

Also, for $t_N < t' \leqslant t_{N+1}$

$$\int_\Omega U^2(x,t')\,dx = \int_{\Omega_{N+1}} U_-^2\,dx - \int_{t'}^{t_{N+1}} 2\left(U_t U + \frac{d}{dx}F_1\right) dx\,dt =$$
$$= \int_{\Omega_{N+1}} U_-^2\,dx - \int_{t'}^{t_{N+1}} 2(U_t + f' U_x)\,U\,dx\,dt$$
$$- \int_t^{t_{N+1}} 2\int_0^U (f'(w+\bar a))_x w\,dw\,dx\,dt$$
$$\leqslant \int_{\Omega_{N+1}} U_-^2\,dx + \delta\int_{S_N} (U_t + f' U_x)^2\,dx\,dt + \left(C + \frac{1}{\delta}\right)\int_{t'}^{t_{N+1}} U^2\,dx\,dt\,.$$

Hence, by a Gronwall inequality

$$\int_\Omega U^2(x,t')\,dx \leqslant C\left(\int_{\Omega_{N+1}} U_-^2\,dx + \delta\int_{S_N} (U_t + f' U_x)^2\,dx\,dt\right). \tag{3.6}$$

Letting now

$$W_N \equiv \sum_{n=0}^{N} \int_{\Omega_n} U_-^2 \, dx ,$$

we then have by (3.5)

$$W_{N+1} - W_N \leqslant C \left(\int_{\Omega} u_0^2 \, dx + \int_{S^N} (g(0, x, t))^2 \, dx \, dt \right) + ChW_N .$$

Using a discrete Gronwall argument we get

$$hW_{N+1} \leqslant C \left(\int_{\Omega} u_0^2 \, dx + \int_{S^N} (g(0, x, t))^2 \, dx \, dt \right) , \tag{3.7}$$

which together with (3.6) inserted into (3.5) proves the lemma. □

Now we turn to the proof of the L_∞-estimate (2.6). This result will follow by letting p tend to infinity in L_p-estimates obtained by multiplication with $\pi(U^{p-1})$. First we give three preliminary results in Lemma 3.2-3.4.

LEMMA 3.2 : *There is a positive constant c, independent of p such that for* $p = 2m$, $m = 1, 2, 3, \ldots$, *and* $n = 0, 1, 2, \ldots$

$$\text{ch} \sum_{\hat{K} \in \hat{T}_h^n} \int_{\hat{K} \cap \Omega_n} \varepsilon_2(U)((\hat{U}_+)_x)^2 \| \hat{U}_+ \|_{L_\infty(\hat{K} \cap \Omega_n)}^{p-2} dx \leqslant \tag{3.8}$$

$$\leqslant \int_{S_n} \varepsilon_2(U) \, \hat{U}_x (\hat{\pi}(\hat{U}^{p-1}))_x \, dx \, dt .$$

The proof of this result is analogous to the proof of Lemma 4.2 in [JSz III]. Further we have.

LEMMA 3.3 : *There is a constant* $c > 0$ *independent of p such that for* $p = 2m$, $m = 1, 2, 3, \ldots$, $n = 0, 1, 2, \ldots$

$$\int_{S_n} \varepsilon_1(U) \, \nabla \hat{U} \, . \, \nabla \hat{\pi}(\hat{U}^{p-1}) \, dx \, dt \geqslant \tag{3.9}$$

$$\geqslant \frac{c}{p^2} \sum_{\hat{K} \in \hat{T}_h^n} \int_{\hat{K}} \varepsilon_1(U) |\nabla \hat{U}|^2 \, \| \hat{U} \|_{L_\infty(K)}^{p-2} \, dx \, dt .$$

Proof: Considering a triangle $\hat{K} \in \hat{T}_h$ and the three associated heights, there is always one that stands on a side of the triangle. Choose this side and the orthogonal height as coordinate directions (*cf. fig.* 3.1).

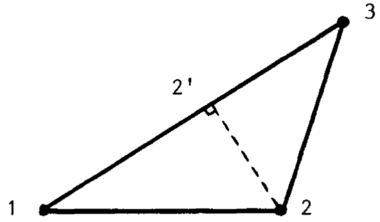

Figure 3.1.

Since ε_1, $\nabla\hat{U} . \nabla\hat{\pi}(\hat{U}^{p-1})$ and $|\nabla\hat{U}|^2 \, \|\hat{U}\|^{p-2}_{L_\infty(\hat{K})}$ are constant on $\hat{K}$, it is enough to prove that $\nabla\hat{U} . \nabla\hat{\pi}(\hat{U}^{p-1}) \geqslant \frac{c}{p^2} |\nabla\hat{U}|^2 \, \|\hat{U}\|^{p-2}_{L_\infty(\hat{K})}$. Define the function $f_p : S^2 \setminus \{\pm 3^{-1/2}(1, 1, 1)\} \to R$ by

$$f_p(y_1, y_2, y_3) = \frac{(y_3 - y_1)(y_3^{p-1} - y_1^{p-1}) + (y_2 - y_{2'})(y_2^{p-1} - (y^{p-1})_{2'})}{((y_3 - y_1)^2 + (y_2 - y_{2'})^2) \max (y_1^{p-2}, y_2^{p-2}, y_3^{p-2})},$$

where y_1, y_2, y_3 and y_2' are the values associated with the vertices and to the point $2'$, respectively, i.e. there is a $\beta \in [0, 1]$ such that

$$y_2' = \beta y_1 + (1 - \beta) y_3 ,$$

and

$$(y^{p-1})_{2'} = \beta (y_1)^{p-1} + (1 - \beta)(y_3)^{p-1} .$$

We note that f_p is continuous on S^2 so that by the definition of f_p we have $\nabla\hat{U} . \nabla\hat{\pi}(\hat{U}^{p-1}) \geqslant \min_{y \in S^2} f_p(y) |\nabla\hat{U}|^2 \, \|\hat{U}\|^{p-2}_{L_\infty(\hat{K})}$. We shall now estimate $\min_{S^2} f_p$. Let us first assume that $|y_2| = \max (|y_1|, |y_2|, |y_3|)$ (we may then assume that $|y_2 - y_3| \leqslant |y_2 - y_1|$). In this case we have

$$\begin{aligned} f_p(y_1, y_2, y_3) &\geqslant \frac{1}{4} \frac{(y_3 - y_1)^2 (y_1^{p-2} + y_3^{p-2}) + (y_2 - y_3)^2 (y_2^{p-2} + y_3^{p-2})}{((y_3 - y_1)^2 + (y_2 - y_{2'})^2) \, y_2^{p-2}} \geqslant \\ &\geqslant \frac{1}{4} \left(\frac{(y_2 - y_3)^2}{(y_3 - y_1)^2 + (y_2 - y_2')} + \left(\frac{y_3}{y_2} \right)^{p-2} \frac{(y_2 - y_3)^2 + (y_3 - y_1)^2}{(y_3 - y_1)^2 + (y_2 - y_2')^2} \right) \\ &\geqslant \frac{1}{32} (y_2 - y_3)^2 + \frac{1}{12} \left(1 - \frac{y_2 - y_3}{y_2} \right)^{p-2} \geqslant \frac{c}{p^2}. \end{aligned}$$

Further, in the case when $|y_3| = \max (|y_1|, |y_2|, |y_3|)$,

$$(y_2 - y_2')(y_2^{p-1} - (y^{p-1})_{2'}) \geqslant - \beta (1 - \beta)(y_3 - y_1)(y_3^{p-1} - y_1^{p-1}) ,$$

so that if also $|y_2 - y_1| \leqslant |y_1 - y_3|$, then

$$f_p(y_1, y_2, y_3) \geqslant \frac{1}{5}\frac{(y_3 - y_1)(y_3^{p-1} - y_1^{p-1})(1 - \beta(1-\beta))}{(y_3 - y_1)^2\, y_3^{p-2}}$$
$$\geqslant \frac{1}{20}(1 - \beta(1-\beta)) \geqslant \frac{3}{80},$$

and if $|y_2 - y_1| > |y_1 - y_3|$ (we still have $|y_3| = \max_i (|y_i|)$, then

$$f_p(y_1, y_2, y_3) \geqslant \frac{1}{4}\frac{(y_3 - y_1)^2 (y_3^{p-2} + y_1^{p-2}) + (y_1 - y_2)^2 (y_1^{p-2} + y_2^{p-2})}{((y_1 - y_3)^2 + (y_2 - y_2')^2)\, y_3^{p-2}} \geqslant$$
$$\geqslant \frac{1}{4}\left(\frac{(y_3 - y_1)^2}{(y_3 - y_2)^2 + (y_2 - y_2')^2} + \left(\frac{y_1}{y_3}\right)^{p-2}\frac{(y_1 - y_2)^2 + (y_3 - y_1)^2}{(y_1 - y_3)^2 + (y_2 - y_2')^2}\right)$$
$$\geqslant \frac{1}{32}(y_3 - y_1)^2 + \left(1 - \frac{y_3 - y_1}{y_3}\right)^{p-2}\frac{1}{12} \geqslant \frac{c}{p^2}.$$

This proves Lemma 3.3, since $f_p(y_1, y_2, y_3)$ is symmetric in y_1 and y_3. □

LEMMA 3.4 : *There are constants c, $C > 0$ independent of p and h such that for $w \in V_h$ and $K \in T_h$*

$$c\|\nabla\hat{w}\|_{L_\infty(\hat{K}_i)} \leqslant \|\nabla w\|_{L_\infty(\hat{K}_i)} \leqslant C \max_i \|\nabla\hat{w}\|_{L_\infty(\hat{K}_i)}, \tag{3.10}$$

$$\|\hat{w}\|_{L_\infty(\hat{K}_i)} \leqslant \|w\|_{L_\infty(\hat{K}_i)} \leqslant C \max_i \|\hat{w}\|_{L_\infty(\hat{K}_i)}, \tag{3.11}$$

$$\|\nabla w\|^2_{L_\infty(K)}\|w\|^p_{L_\infty(K)} \leqslant C^p p^2 \sum_{i=1}^{k^2} \| |\nabla\hat{w}|^2 |\hat{w}|^p \|_{L_\infty(\hat{K}_i)}, \tag{3.12}$$

$$p = 1, 2, 3, \ldots, .$$

The same inequalities holds with ∇ replaced by $\frac{\partial}{\partial x}$, K by $K \cap \Omega_n$ and $\hat{K}_i$ by $\hat{K}_i \cap \Omega_n$.

Proof: First we note that $\hat{w} = \hat{\pi} w$, so by standard interpolation estimates we obtain the left inequality in (3.10). Further, the polynomial $w|_K$ has coefficients which depend linearly on the nodal values of $\hat{w}|_{\hat{K}_i}$, $i = 1, \ldots, k^2$. Therefore we can define $w|_K$ using one nodal value together with $k' \equiv (k+1)(k+2)/2 - 1$ linearly independent differences $(\beta_1, \ldots, \beta_{k'}) \equiv \beta$ of nodal values of $\hat{w}|_{\hat{K}_i}$. These variables also determine

$\hat{w}|_{\hat{K}_i}$. In order to prove the right inequality in (3.10) we define the function $f : R^{k'} \setminus \{0\} \to R$ by

$$f(\beta) = \frac{\| \nabla w \|_{L_\infty(K)}}{\max_i \| \nabla \hat{w} \|_{L_\infty(\hat{K}_i)}} .$$

In the case when $\beta = 0$ we have $\max_i \| \nabla \hat{w} \|_{L_\infty(\hat{K}_i)} = 0$ and then (3.10) is trivially true. Next, when $\max_i \| \nabla \hat{w} \|_{L_\infty(\hat{K}_i)} \neq 0$ we can define the new variables $(\beta'_1, \ldots, \beta'_{k'}) \equiv \beta'$ by $\beta'_j = \beta_j / |\beta|$. Since the coefficients of the polynomials $\nabla w|_K$ and $\nabla \hat{w}|_{\hat{K}_i}$ depend linearly on β_j and $\beta = 0$ implies $\nabla w|_K \equiv \nabla \hat{w}|_{\hat{K}_i} \equiv 0$ the function $f(\,.\,)$ is homogeneous of degree zero, i.e. $f(\beta) = f(\beta')$. Further we easily see that ∇w and $1/\nabla \hat{w}$ are continuous on the compact set $S^{k'-1}$. This implies that there is a constant C such that $\| \nabla w \|_{L_\infty(\hat{K}_i)} \leq C \max_i \| \nabla \hat{w} \|_{L_\infty(\hat{K}_i)}$, which proves (3.10).

The inequality (3.11) follows as above taking now β equal to the nodal values of $\hat{w}$ and defining $f(\beta) = \| w \|_{L_\infty(K)} / \max_i \| \hat{w} \|_{L_\infty(\hat{K}_i)}$.

In order to prove (3.12) we first note that if $\max_i \| \hat{w} \|_{L_\infty(\hat{K}_i)} = 0$ then (3.12) holds for all $C > 0$. Next, we consider the case when $\max_i \| \hat{w} \|_{L_\infty(\hat{K}_i)} \neq 0$. Dividing then the inequality (3.12) by $h^{-2} \max_i (\| \hat{w} \|^p_{L_\infty(\hat{K}_i)})$ we may in the following assume that $\max_i \| \hat{w} \|_{L_\infty(\hat{K}_i)} = 1$ and $c < \operatorname{diam}(\hat{K}_i) < 1$ for all $\hat{K}_i \subset K$. Suppose now that

$$\sum_{i=1}^{k^2} \| \, |\nabla \hat{w}|^2 \, |\hat{w}|^p \|_{L_\infty(\hat{K}_i)} = \varepsilon^2 . \tag{3.13}$$

We shall prove that

$$\| \nabla w \|_{L_\infty(K)} \leq C p \varepsilon . \tag{3.14}$$

Together with (3.11), (3.14) proves (3.12). To prove (3.14), we first note that by (3.13)

$$\varepsilon \geq \| \nabla \hat{w} \|_{L_\infty(\hat{K}_i)} \| \hat{w} \|^{p/2}_{L_\infty(\hat{K}_i)} \geq C^{p/2} (\| \nabla \hat{w} \|^{p/2+1}_{L_\infty(\hat{K}_i)}) \quad \forall i ,$$

which, in the case when $\varepsilon \geqslant \dfrac{\ln 2}{4\,kp}$, by (3.10) implies

$$\|\nabla w\|_{L_\infty(K)} \leqslant C \max_i \left(\|\nabla \hat{w}\|_{L_\infty(\hat{K}_i)}\right) \leqslant C\varepsilon^{1/(p/2+1)} \leqslant Cp\varepsilon .$$

Next, for $\varepsilon < \dfrac{\ln 2}{4\,kp}$, using the notation $\varepsilon_i \equiv \|\nabla \hat{w}\|_{L_\infty(\hat{K}_i)}$ and a relevant numbering of $\hat{K}_i$, we have since $\nabla \hat{w}$ is constant on $\hat{K}_i$

$$\|\hat{w}\|_{L_\infty(\hat{K}_1)} = 1 ,$$

$$\|\hat{w}\|_{L_\infty(\hat{K}_i)} \geqslant 1 - \sum_{j=1}^{i-1} \varepsilon_j , \quad i = 2, \dots, k^{\,2} . \tag{3.15}$$

Hence by combining (3.13) and (3.15), we get $\varepsilon_i \left(1 - \sum_{j=1}^{i-1} \varepsilon_j\right)^{p/2} \leqslant \varepsilon$, $i = 1, \dots, k^{\,2}$ which by (3.15) implies $\varepsilon_1 \leqslant \varepsilon$ and $\varepsilon_2 \leqslant \varepsilon(1 + 2\,\varepsilon_1)^{p/2} \leqslant \varepsilon \exp(\varepsilon p) \leqslant 2\,\varepsilon$. Continuing in this way we get $\varepsilon_j \leqslant 2\,\varepsilon$, $j = 1, \dots, k^{\,2}$, which by (3.10) proves (3.14) also for this case.

The inequalities (3.10-3.12) with ∇ replaced by $\dfrac{\partial}{\partial x}$, K by $K \cap \Omega_n$ and $\hat{K}_i$ by $\hat{K}_i \cap \Omega_n$ follows as above by noting that $\hat{\pi}$ is also an interpolant on $K \cap \Omega_n$. □

Taking now $v = \pi(U^{p-1})$ in (2.5) where p is an even integer greater than 2, we get

$$\begin{aligned}
0 = &\int_{S_n} L(U)\, U^{p-1}\, dx\, dt + \int_{\Omega_n} (U_+ - U_-)(U_+)^{p-1}\, dx - \\
&- \int_{S_n} L(U)(U^{p-1} - \pi(U^{p-1}))\, dx\, dt \\
&- \int_{\Omega_n} (U_+ - U_-)(U_+^{p-1} - \pi(U_+^{p-1}))\, dx \\
&+ \delta \int_{S_n} L(U)((U^{p-1})_t + f'(U^{p-1})_x)\, dx\, dt \\
&- \delta \int_{S_n} L(U)((U^{p-1})_t - \pi(U^{p-1})_t + f'((U^{p-1}))_x - (\pi(U^{p-1}))_x)\, dx\, dt \\
&+ \int_{S_n} \varepsilon_1(U)\, \nabla \hat{U} \,.\, \nabla \hat{\pi}(\hat{U}^{p-1})\, dx\, dt \\
&+ \int_{S_n} \varepsilon_2(U)\, \hat{U}_x(\hat{\pi}(\hat{U}^{p-1}))_x\, dx\, dt \equiv \sum_{i=1}^{8} E_n^i .
\end{aligned}$$

Using (2.2), the fact that $U \in V_h$ implies $\| U \|_{\dot{W}^{k+1,\infty}(K)} = 0$ and (2.4a), we have

$$\| (I - \pi) U^{p-1} \|_{W^{r,\infty}(K)} \leqslant C h^{k+1-r} \| U^{p-1} \|_{\dot{W}^{k+1,\infty}(K)} \leqslant$$
$$\leqslant C p^{k+1} h^{2-r} \| \nabla U \|^2_{L_\infty(K)} \| U \|^{p-3}_{L_\infty(K)} , \quad r = 0, 1 ,$$

which combined with Lemma 3.4 yields

$$|E_n^3| + |E_n^6| \leqslant C p^{k+1} h(h + \delta) \sum_{K \in T_h^n} \times$$
$$\times \int_K |L(U)| (1 + |f'|) \, dx \, dt \, \| \nabla U \|^2_{L_\infty(K)} \| U \|^{p-3}_{L_\infty(K)}$$
$$\leqslant C^p p^{k+3} \frac{h\delta}{\bar{\delta}} \left[\sum_{\hat{K} \in \hat{T}_h^n} \int_{\hat{K}_i \cap \{ |U| > 1\}} \varepsilon_1(U) |\nabla \hat{U}|^2 \, \| \hat{U} \|^{p-2}_{L_\infty(\hat{K}_i)} \, dx \, dt \right.$$
$$\left. + \int_{S_n} \varepsilon_1(U) |\nabla U|^2 \, dx \, dt \right] \equiv I_n + II_n ,$$

where by Lemma 3.3

$$|I_n| \leqslant C^p p^{k+5} \frac{h\delta}{\bar{\delta}} \int_{S_n} \varepsilon_1(U) \nabla \hat{U} \,.\, \nabla \hat{\pi}(\hat{U}^{p-1}) \, dx \, dt ,$$

and by Lemma 3.1

$$\sum_{n=0}^{N} |II_n| \leqslant C^p p^{k+3} h\delta / \bar{\delta} .$$

Further, using Lemma 3.1-3.4 we get as above

$$|E_n^4| \leqslant C p^{k+1} h^2 \sum_{K \in T_h^n} \int_{K \cap \Omega_n} |U_+ - U_-| \, dx \, \| U_x \|^2_{L_\infty(K \cap \Omega_n)} \, \| U \|^{p-3}_{L_\infty(K \cap \Omega_n)} \leqslant$$
$$\leqslant C^p p^{k+3} \frac{h^2}{\bar{\bar{\delta}}} \sum_{\hat{K} \in \hat{T}_h^n} \int_{\hat{K} \cap \Omega_n} \varepsilon_2(U) |\hat{U}_x|^2 \, \| \hat{U} \|^{p-2}_{L_\infty(K \cap \Omega_n)} \, dx$$
$$+ C^p p^{k+3} \frac{h^2}{\bar{\bar{\delta}}} \sum_{\hat{K} \in \hat{T}_h^n} \int_{\hat{K} \cap \Omega_n} \varepsilon_2(U) |\hat{U}_x|^2 \, dx$$
$$\leqslant C^p p^{k+3} \frac{h}{\bar{\bar{\delta}}} \left(\int_{S_n} \varepsilon_2(U) \, \hat{U}_x (\hat{\pi}(\hat{U}^{p-1}))_x \, dx \, dt + \int_{S_n} \varepsilon_2(U) |\hat{U}_x|^2 \, dx \, dt \right) .$$

Choosing now p such that

$$C^p p^{k+6} \leqslant \min\left(\frac{\bar{\delta}}{h^2}, \frac{\bar{\bar{\delta}}}{h}\right), \tag{3.16}$$

and combining the above estimates we get by summation over $n = 0, 1, 2, \ldots, N$,

$$\sum_{n=0}^{N}\left(\int_{\Omega_{n+1}}(U_-)^p\,dx - \int_{\Omega_n}(U_+)^p\,dx - \int_{\Omega_n}(U_- - U_+)\,p(U_+)^{p-1}\,dx\right) +$$

$$+ \delta p(p-1)\int_{S^N}(U_t + f'\,U_x)^2\,U^{p-2}\,dx\,dt$$

$$\leqslant C^p p^{k+6}(h^2/\bar{\delta} + h/\bar{\bar{\delta}}) + \delta p(p-1)\int_{S^N}|g|\,|U_t + f'\,U_x|\,|U|^{p-2}\,dx\,dt$$

$$+ \left|p\int_{S^N} f'\,U_x\,U^{p-1}\,dx\,dt + p\int_{S^N} g U^{p-1}\,dx\,dt\right|.$$

Using the convexity of the function $U \to U^p$, (3.4), (2.1) and the same argument as in (3.3) with F_1 replaced by F_{p-1}, we have

$$\| U_- \|^p_{L_p(\Omega_{N+1})} + \frac{\delta p(p-1)}{2}\int_{S^N}(U_t + f'\,U_x)^2\,U^{p-2}\,dx\,dt \leqslant \tag{3.17}$$

$$\leqslant \|u_0\|^p_{L_p(\Omega)} + C^p p^{k+6}(h^2/\bar{\delta} + h/\bar{\bar{\delta}})$$

$$+ Cp(1 + \delta(p-1))\left(\int_{S^N} U^p\,dx\,dt + \int |g(0, x, t)|^p\,dx\,dt\right).$$

The next step is to obtain L_p estimates for all $t \in (0, T)$. For $t_n < t' \leqslant t_{n+1}$ we have

$$\| U(\,.\,, t')\|^p_{L_p(\Omega)} = \| U_- \|^p_{L_p(\Omega_{n+1})} - p\int_{t'}^{t_{n+1}}\int_\Omega (U_t\,U^{p-1} + (F_{p-1})_x)\,dx\,dt =$$

$$= \| U_- \|^p_{L_p(\Omega_{n+1})} - p\int_{t'}^{t_{n+1}}\int_\Omega (U_t + f'\,U_x)\,U^{p-1}\,dx\,dt$$

$$- p\int_{t'}^{t_{n+1}}\int_\Omega\int_0^U (f'(w + \bar{a}))_x\,w^{p-1}\,dw\,dx\,dt \leqslant \| U_- \|^p_{L_p(\Omega_{n+1})}$$

$$+ p\left(\int_{S_n}(U_t + f'\,U_x)^2\,U^{p-2}\,dx\,dt\int_{t'}^{t_{n+1}}\int_\Omega U^p\,dx\,dt\right)^{1/2}$$

$$+ C \int_{t'}^{t_{n+1}} \int_\Omega U^p \, dx \, dt \leq \| U_- \|^p_{L_p(\Omega_{n+1})} + \delta p (p-1)$$

$$\times \int_{S_n} (U_t + f' U_x)^2 U^{p-2} \, dx \, dt$$

$$+ \left(C + \frac{p}{4(p-1)\delta} \right) \int_t^{t_{n+1}} \int_\Omega U^p \, dx \, dt ,$$

so that by using Gronwalls inequality we obtain for $t_n < t' \leq t_{n+1}$

$$\text{(3.18)} \quad \| U(\cdot, t) \|^p_{L_p(\Omega)} \leq C \Big(\| U_- \|^p_{L_p(\Omega_{n+1})} + \delta p (p-1) \times \times \int_{S_n} (U_t + f' U_x)^2 U^{p-2} \, dx \, dt \Big) .$$

Further, using a discrete Gronwall inequality we have by (3.17)

$$\text{(3.19)} \quad \| U_- \|^p_{L_p(\Omega_{n+1})} + \delta p (p-1) \int_{S^N} (U_t + f' U_x)^2 U^{p-2} \, dx \, dt \leq \leq \exp(C(p + p^2 h))(\| u_0 \|^p_{L_p(\Omega)} + p \| g(0, .) \|^p_{L_p(S^N)} + 1) .$$

This proves by (2.6), (3.18), and (3.16) the existence of positive constants c, C independent of p and h such that

$$\sup_{t \leq T} \| U(\cdot, t) \|_{L_p(\Omega)} \leq C , \quad \text{for} \quad 4 \leq p \leq c \ln \ (1/h) .$$

Finally, using an inverse estimate we have

$$\| U_h \|_{L_\infty(\Omega \times (0,T))} \leq C (ph^{-1})^{2/p} \| U_h \|_{L_p(\Omega \times (0,T))} \leq \leq C \exp \left(2 \left(\frac{\ln p}{p} + \frac{\ln (1/h)}{c \ln (1/h)} \right) \right) \leq C ,$$

which proves the lemma. □

4. PROOF OF LEMMA 2.3

To prove Lemma 2.3 we first note that by Lemma 2.2 the solutions U_h of (2.5) are uniformly bounded in the L_∞-norm, so that the sequence $\{u_h\}$ defined by $u_h = U_h + \bar{a}$ satisfies (1.4). Then there exists according to Youngs theorem a Young measure $\nu_{(.)} : \Omega \times R_+ \to \text{Prob}\ (R)$ associated to a subsequence $\{u_{h_j}\}$ $h_j \to 0$ such that ν satisfies (1.5-1.6), and by Lemma 1.1 there exists an associated Young measure $\gamma\nu : \Gamma \times R_+ \to \text{Prob}\ (R)$ satisfying (1.8). We shall prove the following propositions.

PROPOSITION 4.1 : *The Young measure* ν *associated to* $\{u_{h_j}\}$ *is a mv-solution in the interior domain, i.e.* $\forall k \in R$

$$\frac{\partial}{\partial t} \langle \nu_{(x,t)}, |\lambda - k| \rangle + \frac{\partial}{\partial x} \langle \nu_{(x,t)}, (\operatorname{sgn}(\lambda - k))(f(\lambda) - f(k)) \rangle \leq 0 ,$$

in $\mathcal{D}'(\Omega \times R_+)$.

PROPOSITION 4.2 : *The Young measure* $\gamma\nu$ *associated to* ν *given in Proposition* 4.1 *satisfies* $\forall k \in R$

$$\langle \gamma\nu_{(\bar{x},t)}, (\operatorname{sgn}(\lambda - k) - \operatorname{sgn}(a - k))(f(\lambda) - f(k)) \rangle \,.\, n \geq 0, \quad \text{in} \quad \mathcal{D}'(\Gamma \times R_+) .$$

We postpone the proofs of Proposition 4.1 and 4.2 to the end of this section. First we prove that they imply that ν and $\gamma\nu$ satisfy (1.9a). To this end we let $\phi \in \mathcal{C}^1(\bar{\Omega} \times R_+)$, $\phi \geq 0$ and

$$\chi_\delta(x(\bar{x}, y)) \equiv \begin{cases} 0 & 0 \leq y < \delta , \\ \frac{1}{2} + \frac{3}{4}(y - 2\delta)/\delta - \frac{1}{4}\left(\frac{y - 2\delta}{\delta}\right)^3 & \delta \leq y < 3\delta , \\ 0 & y \geq 3\delta , \end{cases}$$

where $\bar{x}$, and y are defined in (1.7), and write

$$\begin{aligned} &\int_{\Omega \times R_+} (\langle \nu_{(x,t)}, |\lambda - k| \rangle \phi_t + \\ &+ \langle \nu_{(x,t)}, (\operatorname{sgn}(\lambda - k))(f(\lambda) - f(k)) \rangle \phi_x)\, dx\, dt \\ &- \int_{\Gamma \times R_+} \langle \gamma\nu_{(\bar{x},t)}, f(\lambda) - f(k) \rangle \,.\, n\phi \operatorname{sgn}(a - k)\, ds\, dt \\ &= \int_{\Omega \times R_+} (\langle \nu_{(x,t)}, |\lambda - k| \rangle (\chi_\delta \phi) + \\ &+ \langle \nu_{(x,t)}, (\operatorname{sgn}(\lambda - k))(f(\lambda) - f(k)) \rangle (\chi_\delta \phi)_x)\, dx\, dt \\ &+ \int_{\Omega \times R_+} (\langle \nu_{(x,t)}, |\lambda - k| \rangle ((1 - \chi_\delta)\phi)_t \\ &+ \langle \nu_{(x,t)}, (\operatorname{sgn}(\lambda - k))(f(\lambda) - f(k)) \rangle ((1 - \chi_\delta)\phi)_x)\, dx\, dt \\ &- \int_{\Gamma \times R_+} \langle \gamma\nu_{(\bar{x},t)}, f(\lambda) - f(k) \rangle \,.\, n\phi \operatorname{sgn}(a - k)\, ds\, dt \\ &\equiv I_\delta + II_\delta + III_\delta . \end{aligned}$$

By Proposition 4.1 we get $I_\delta \geq 0$ since $\chi_\delta \phi \in \mathscr{C}_0^1(\Omega \times R_+)$. Further by letting δ tend to zero we obtain as in [(1.16), Sz III] and by Proposition 4.1 and 4.2

$$\lim_{\delta \to 0+} (II_\delta + III_\delta) = \int_{\Gamma \times R_+} \langle \gamma \nu_{(\bar{x}, t)}, (\operatorname{sgn}(\lambda - k) - \operatorname{sgn}(a - k)) \\ (f(\lambda) - f(k)) \rangle \,.\, n\phi \, ds \, dt \geq 0 \,.$$

This proves that ν and $\gamma\nu$ satisfy (1.9*a*). The fact that ν also satisfies the initial condition (1.9*b*) follows as in [Lemma 3.3, Sz I]. Hence, ν is a *mv*-solution and by Theorem 1.1 this implies that u_h converges strongly in L^1_{loc} to the unique BV-solution of (1.1-1.3) as $h \to 0$.

It remains to prove Propositions 4.1 and 4.2. Let us introduce the notation

$$\operatorname{Int} S_n = \bigcup_{\substack{K \in T_h^n \\ \bar{K} \cap (\Gamma \times R_+) = \emptyset}} K \,, \quad \operatorname{Bdr} S_n = S_n \backslash \operatorname{Int} S_n \,,$$

$$\operatorname{Int} S^N = \bigcup_{n=0}^{N} \operatorname{Int} S_n \,, \quad \operatorname{Bdr} S^N = \bigcup_{n=0}^{N} \operatorname{Bdr} S_n \,,$$

$$\operatorname{Int} \Omega_n = \bigcup_{\substack{K \in T_h^n \\ \bar{K} \cap (\Gamma \times R_+) = \emptyset}} (K \cap \Omega_n) \,, \quad \operatorname{Bdr} \Omega_n = \Omega_n \backslash \operatorname{Int} \Omega_n \,.$$

We start by taking in (2.5) $v = \pi w$ with $w = j_\eta(U + \bar{a} - k)\, \phi\chi$ where $k \in R$ and for $\eta > 0$, $j_\eta = \operatorname{sgn} * \omega_\eta \in \mathscr{C}^\infty(R)$ is a standard mollification of sgn, where $\omega \in \mathscr{C}_0^\infty((-1, 1))$, $\omega \geq 0$, $\int_R \omega \, dy = 1$, $\omega_\eta(y) = \eta^{-1} \omega(y/\eta)$. We note that

$$j_\eta(s) = \begin{cases} 1 \,, & \text{if } \; s \geq \eta \,, \\ -1 \,, & \text{if } \; s \leq -\eta \,, \end{cases} \qquad (4.1)$$
$$j'_\eta(s) = 2\, \omega_\eta(s) \geq 0 \,.$$

Further, $\phi \in \mathscr{C}_0^\infty(\bar{\Omega} \times (0, T))$, $\phi \geq 0$ and $\chi \in V_h$ with χ linear on $K \in T_h$ and $\chi|_{\operatorname{Int} S^N} = 1$, so that in particular $\chi_t|_{S_n} = 0$. We have

$$(4.2) \quad 0 = \int_{S_n} L(U)\, w \, dx \, dt + \int_{\Omega_n} (U_+ - U_-)\, w_+ \, dx -$$
$$- \int_{\operatorname{Int} S_n} L(U)(w - \pi w)\, dx \, dt - \int_{\operatorname{Int} \Omega_n} (U_+ - U_-)(w_+ - \pi w_+)\, dx$$

$$+ \delta \int_{\text{Int } S_n} L(U)(w_t + f' w_x) \, dx \, dt$$

$$- \delta \int_{\text{Int } S_n} L(U)((w - \pi w)_t + f'(w - \pi w)_x) \, dx \, dt$$

$$+ \int_{\text{Int } S_n} \varepsilon_1(U) \nabla \hat{U} . \nabla \hat{\pi} w \, dx \, dt + \int_{\text{Int } S_n} \varepsilon_2(U) \hat{U}_x (\hat{\pi} w)_x \, dx \, dt$$

$$- \int_{\text{Bdr } S_n} L(U)(w - \pi w) \, dx \, dt - \int_{\text{Bdr } \Omega_n} (U_+ - U_-)(w - \pi w) \, dx$$

$$+ \delta \int_{\text{Bdr } S_n} L(U)((\pi w)_t + f'(\pi w)_x) \, dx \, dt$$

$$+ \int_{\text{Bdr } S_n} \varepsilon_1(U) \nabla \hat{U} . \nabla \hat{\pi} w \, dx \, dt + \int_{\text{Bdr } S_n} \varepsilon_2(U) \hat{U}_x (\hat{\pi} w)_x \, dx \, dt \equiv \sum_{i=1}^{13} E_n^i .$$

Let now

$$J_\eta(y, k) = \int_k^y j_\eta(s - k) \, ds ,$$

$$Q_\eta(y, k) = \int_k^y f'(s) j_\eta(s - k) \, ds .$$

Note that by (4.1) $J_\eta(., k)$ is convex. Integrating by parts in E_1 and summing over n, we get from (4.2)

$$(4.3) \quad - \int_{S^N} (J_\eta(u_h, k)(\phi\chi)_t + Q_\eta(u_h, k)(\phi\chi)_x) \, dx \, dt$$
$$+ \sum_{n=0}^{N} \left[\int_{\Omega_{n+1}} J_\eta(u_{h-}, k) \, \phi\chi \, dx - \right.$$
$$\left. - \int_{\Omega_n} (J_\eta(u_{h+}, k) - (u_{h+} - u_{h-}) j_\eta(u_{h+} - k)) \right]$$
$$= - \sum_{n=0}^{N} \sum_{i=3}^{13} E_n^i \equiv - \sum_{i=3}^{13} R^i .$$

By the convexity of $J_\eta(., k)$, we see that the sum over n on the left hand side of (4.3) is nonnegative. By arguments similar to those used in the proof of Proposition 5.1 in [Sz I] using now also Lemma (3.4), we further have

PROPOSITION 4.3 :

$$\liminf_{h \to 0} \sum_{i=3}^{8} R^i \geq 0 .$$

Taking $\phi \in \mathscr{C}_0^\infty(\Omega \times R_+)$, $\phi \geq 0$ in (4.3) we note that $\sum_{i=9}^{13} R^i = 0$ and letting then h_j tend to zero using Proposition 4.3 and (1.6), we get

$$\int_{\Omega \times R_+} (\langle \nu_{(x,t)}, J_\eta(\lambda, k) \rangle \phi_t + \langle \nu_{(x,t)}, Q_\eta(\lambda, k) \rangle \phi_x) \, dx \, dt \geq 0 ,$$

which proves Proposition 4.1 by dominated convergence when $\eta \to 0+$.

In order to prove Proposition 4.2 we shall first give two preliminary lemmas in which we estimate the solution $U = U_h$ near the boundary. Let C_M be a constant such that $\|U_h\|_{L_\infty} \leq C_M$ and define β by

$$\beta = 1 \Bigg/ \left(\sup_{\substack{|w| \leq C_M \\ (x,t) \in \Omega \times (0,T)}} \frac{4}{w^2} |F_1(w, x, t)| \right) , \tag{4.4}$$

where F_1 is defined in (3.2). Further let us introduce the direction

$$\bar{\beta} = (1 + \beta^2)^{-\frac{1}{2}} (\beta, 1) \in \mathbb{R}^2 ,$$

and for $n = 0, 1, 2, \ldots$, the point $(\hat{x}, \hat{t})$:

$$(\hat{x}, \hat{t}) \cdot \bar{\beta} = \max_{\substack{(x,t) \in \bar{K} \in T_h^n \\ \{0\} \times R_+ \cap \bar{K} \neq \emptyset}} (x, t) \cdot \bar{\beta} .$$

For $n = 0, 1, 2, \ldots$ we also define the cut-off function $\psi : S_n \to \mathbb{R}_+$.

$$\psi(x,t)|_{S_n} = \begin{cases} e^{-\frac{1}{\tau} \bar{\beta} \cdot (x - \hat{x}, t - \hat{t})} , & \bar{\beta} \cdot (x - \hat{x}, t - \hat{t}) > 0 , \\ 1 , & \bar{\beta} \cdot (x - \hat{x}, t - \hat{t}) \leq 0 , \end{cases}$$

where $\tau = (h + \hat{\delta}/h + \bar{\bar{\delta}}) \alpha$, and α is a sufficiently large constant. Note that ψ is equal to one on the part of Bdr S_n at $x = 0$ and decays exponentially in Int S_n.

We then have the following local stability result. In the case of linear convection problems with $\varepsilon_1 = \varepsilon_2 = 0$ (no shock-capturing) such results were proved in [JNP].

LEMMA 4.1 : *Under the above assumptions we have for all h sufficiently small*

$$\frac{1}{2}\int_{\Omega_{n+1}} (U_-)^2\,\psi\,dx + \frac{1}{2}\int_{\Omega_n} (U_+)^2\,\psi\,dx + \delta\int_{S_n} (U_t + f'\,U_x)^2\,\psi\,dx + \\ + \int_{S_n} \varepsilon_1(U)\left|\nabla\hat{U}\right|^2 \psi\,dx\,dt + \int_{S_n} \varepsilon_2(U)(\hat{U}_x)^2\,\psi\,dx\,dt \leqslant \\ \leqslant C\left(\int_{\Omega_n} \psi\,dx + \int_{S_n} \psi\,dx\,dt\right).$$

An analogous result holds for localization near $x = 1$.

Proof : First we note that

$$\max_{|y| < \tau} \frac{\psi(x+y)}{\psi(x)} \leqslant e\,, \tag{4.5a}$$

$$\nabla\psi = -\frac{\bar{\beta}}{\tau}\,\psi\,. \tag{4.5b}$$

Taking $v = \pi(U\psi)$ in (2.5) and using that $\hat{\pi}(U\psi) = \hat{\pi}(\hat{U}\psi)$, we get

$$\begin{aligned}
(4.6)\quad & \int_{S_n} L(U)\,U\psi\,dx\,dt + \delta\int_{S_n} L(U)(U_t + f'\,U_x)\,\psi\,dx\,dt + \\
& + \int_{\Omega_n} (U_+ - U_-)\,U_+\,\psi\,dx + \int_{S_n} \varepsilon_1(U)\left|\nabla\hat{U}\right|^2 \psi\,dx\,dt \\
& + \int_{S_n} \varepsilon_2(U)(\hat{U}_x)^2\,\psi\,dx\,dt = \int_{S_n} L(U)(U\psi - \pi(U\psi))\,dx\,dt \\
& + \delta\int_{S_n} L(U)((U\psi - \pi(U\psi)))_t + f'\,(U\psi - \pi(U\psi))_x\,dx\,dt \\
& - \delta\int_{S_n} L(U)(\psi_t + f'\,\psi_x)\,U\,dx\,dt + \int_{\Omega_n} (U_+ - U_-)(U\psi - \pi(U\psi))_+\,dx \\
& + \int_{S_n} \varepsilon_1(U)\,\nabla\hat{U}\,.\,\nabla(\hat{U}\psi - \hat{\pi}(\hat{U}\psi))\,dx\,dt - \int_{S_n} \varepsilon_1(U)\,\nabla\hat{U}\,.\,\nabla\psi\,\hat{U}\,dx\,dt \\
& + \int_{S_n} \varepsilon_2(U)\,\hat{U}_x(\hat{U}\psi - \hat{\pi}(\hat{U}\psi))_x\,dx\,dt - \int_{S_n} \varepsilon_2(U)\,\hat{U}_x\,\psi_x\,\hat{U}\,dx\,dt \equiv \sum_{i=1}^{8} E^i\,.
\end{aligned}$$

Combining now the super approximation (2.3a), with $\omega = K \in T_h$, $s = 0, 1$, together with the L_∞ estimate (2.6), the inverse estimate (2.4), Lemma 3.4 and (4.5a), we find the following bounds for the terms on the right hand side of (4.6)

$$|E^1| + |E^2| + |E^3| \leqslant C(h+\delta) \sum_{K \in T_h^n} \int_K |L(U)| \, \|U\|_{L_\infty(K)} |\nabla\psi| \, dx \, dt$$

$$\leqslant c \int_{S_n} U^2 |\nabla\psi| \, dx \, dt + C(h+\delta)^2 \int_{S_n} |L(U)|^2 |\nabla\psi| \, dx \, dt$$

$$\leqslant c \int_{S_n} U^2 |\nabla\psi| \, dx \, dt + C \frac{h^2}{\tau} \int_{S_n} (U_t + f' \, U_x)^2 \psi \, dx \, dt + \frac{\mathrm{Ch}^2}{\tau} \int_{S_n} \psi \, dx \, dt \, .$$

Here and below, c is a constant to be chosen sufficiently small. Further,

$$|E^4| \leqslant \mathrm{Ch} \sum_{K \in T_h^n} \int_{K \cap \Omega_n} |U_+ - U_-| \, \|U\|_{L_\infty(K \cap \Omega_n)} |\nabla\psi| \, dx$$

$$\leqslant C \frac{h}{\tau} \int_{\Omega_n} U_+^2 \psi \, dx + C \frac{h}{\tau} \int_{\Omega_n} \psi \, dx \, ,$$

$$|E^5| + |E^6| \leqslant C \sum_{K \in \hat{T}_h^n} \int_K \varepsilon_1(U) |\nabla \hat{U}| \, \|\hat{U}\|_{L_\infty(K)} |\nabla\psi| \, dx \, dt$$

$$\leqslant c \int_{S_n} U^2 |\nabla\psi| \, dx \, dt + C \frac{\bar{\delta}}{\tau h} \int_{S_n} \varepsilon_1(U) |\nabla \hat{U}|^2 \psi \, dx \, dt \, ,$$

$$|E^7| + |E^8| \leqslant C \sum_{K \in \hat{T}_h^n} \int_K (\varepsilon_2(U) - \tilde{\psi})(\hat{U}_x) |\nabla\psi| \, \|\hat{U}\|_{L_\infty(K)} \, dx \, dt \, +$$

$$+ \int_{S_n} \left| \tilde{\psi} \hat{U}_x (\hat{U}\psi - \pi(\hat{U}\psi))_x \right| dx \, dt + \int_{S_n} \left| \tilde{\psi} \hat{U}_x \, \hat{U} \psi_x \right| dx \, dt$$

$$\leqslant c \int_{S_n} U^2 |\nabla\psi| \, dx \, dt + C \frac{\bar{\bar{\delta}}}{\tau} \int_{S_n} \varepsilon_2(U) (\hat{U}_x)^2 \psi \, dx \, dt \, ,$$

where in the last inequality we used that $\psi(.)|_K = 1$ if $K \in T_h^n$ and $K \cap \mathrm{Supp}\, \tilde{\psi} \neq \varnothing$. Integrating by parts in the first term on the left hand side of (4.6) and using the above estimates, we obtain

(4.7)

$$\frac{1}{2} \int_{\Omega_{n-1}} (U_-)^2 \psi \, dx + \frac{1}{2} \int_{\Omega_n} (U_+)^2 \psi \, dx + \delta \int_{S_n} (U_t + f' \, U_x)^2 \psi \, dx \, dt \, +$$

$$+\int_{S_n} \varepsilon_1(U)|\nabla\hat{U}|^2 \psi\, dx + \int_{S_n} \varepsilon_2(U)(\hat{U}_x)^2 \psi\, dx\, dt$$

$$-\int_{S_n}\left(\frac{U^2}{2}\psi_t + F_1\psi_x\right) dx\, dt \leq C\left(\int_{\Omega_n}\psi\, dx + \int_{S_n}\psi\, dx\, dt\right)$$

$$+ c\int_{S_n} U^2|\nabla\psi|\, dx\, dt .$$

Finally by (4.4), (4.5*b*)

$$-\int_{S_n}\frac{U^2}{2}\left(\psi_t + \frac{2}{U^2}F_1(U)\psi_x\right) dx\, dt = -\int_{S_n}\frac{U^2}{2}\left(\frac{2}{U^2}F_1, 1\right)\cdot\nabla\psi\, dx\, dt =$$

$$= \frac{1}{\tau}\int_{S_n}\frac{U^2}{2}\left(\frac{2}{U^2}F_1, 1\right)\cdot\bar{\beta}\psi\, dx\, dt \geq c\int_{S_n} U^2\frac{\psi}{\tau}\, dx\, dt = c\int_{S_n} U^2|\nabla\psi|\, dx\, dt$$

which together with (4.7) proves the lemma. □

We have the following estimate of U near the boundary.

LEMMA 4.2 : *There is a constant C such that for h sufficiently small*

$$\|U\|_{L_\infty(\mathrm{Bdr}\, S^N)} \leq C\sqrt{\frac{\tau}{\varepsilon}}, \tag{4.8}$$

which in particular implies that

$$\lim_{h\to 0}\|U\|_{L_\infty(\mathrm{Bdr}\, S^N)} = 0 .$$

Proof : Since $\hat{U}(., t)|_\Gamma = 0$, we have

$$\hat{U}(x, t) = \int_0^x \hat{U}_x(s, t)\, ds ,$$

so that by Lemma 3.4

$$\|U(., .)\|^2_{L_\infty(\mathrm{Bdr}\, S_n)} \leq C\|\hat{U}(., .)\|^2_{L_\infty(\mathrm{Bdr}\, S_n)} \leq C\int_{\mathrm{Bdr}\, S_n}\hat{U}_x^2\, dx\, dt .$$

Further, by Lemma 4.1

$$\int_{\mathrm{Bdr}\, S_n}\hat{U}_x^2\, dx\, dt \leq \frac{1}{\varepsilon}\int_{S_n}\tilde{\psi}\hat{U}_x^2\psi\, dx\, dt$$

$$\leq \frac{C}{\varepsilon}\left(\int_{\Omega_n}\psi\, dx + \int_{S_n}\psi\, dx\, dt\right) \leq \frac{C}{\varepsilon}(h+\tau),$$

which proves the lemma. □

Let $\phi = \tilde{\phi}\tilde{\chi}_\zeta$ where $0 \leq \tilde{\phi} \in \mathscr{C}_0^\infty(\bar{\Omega} \times R_+)$, and for $\zeta > 0$

$$H_\zeta = 1 + \frac{1}{2}(\operatorname{sgn}(. - 3\zeta/4) - \operatorname{sgn}(. + 3\zeta/4)) * \omega_{\zeta/4},$$

$$\tilde{\chi}_\zeta(x, t) = H_\zeta(\bar{a}(x, t) - k).$$

Then we have $\tilde{\chi}_\zeta \in \mathscr{C}^\infty(\bar{\Omega} \times R_+)$, Range $\tilde{\chi}_\zeta \subseteq [0, 1]$ and

$$\tilde{\chi}_\zeta(x, t) = \begin{cases} 0 & \text{if} \quad |\bar{a}(x, t) - k| < \zeta/2 \\ 1 & \text{if} \quad |\bar{a}(x, t) - k| > \zeta. \end{cases}$$

We thus obtain by Lemma 4.2

$$\pi\,(j_\eta(U + \bar{a} - k)\,\tilde{\phi}\tilde{\chi}_\zeta\,\chi)\big|_{\text{Bdr } S_n} = (\operatorname{sgn}(\bar{a} - k)\,\pi(\tilde{\phi}\tilde{\chi}_\zeta\,\chi))\big|_{\text{Bdr } S_n}, \tag{4.9}$$

for h sufficiently small (i.e. such that $C\sqrt{\tau/\varepsilon} < \zeta/3 - \eta$). With this choice of ϕ in (4.3), we have

PROPOSITION 4.4 :

$$\lim_{h \to 0} \sum_{i=9}^{13} R^i = -\int_{\Gamma \times R_+} \operatorname{sgn}(a - k)\langle \gamma\nu_{(\bar{x}, t)}, f(a) - f(\lambda)\rangle \,.\, n\phi\, ds\, dt\,.$$

Before proving this proposition we shall show that it implies Proposition 4.2. Letting $h \to 0$ in (4.3) we have as above $\liminf_{h \to 0} \sum_{i=3}^{8} R^i \geq 0$, and using the fact that supp $\chi_x \subset$ Bdr S^N, the continuity of Q_η and Lemma 4.2, we get

(4.9')

$$\int_{\Omega \times R_+} Q_\eta(U + \bar{a}, k)\,\chi_x(., t)\,\phi\, dx\, dt \to -\int_{\Gamma \times R_+} Q_\eta(a, k)\,.\, n\phi\, ds\, dt\,.$$

Hence, by letting $h_j \to 0$ in (4.3), using Proposition 4.4, (1.6) and Lemma (2.2) we get

$$\int_{\Omega \times R_+} (\langle \nu_{(x,t)}, J_\eta(\lambda, k)\rangle\,\phi_t + \langle \nu_{(x,t)}, Q_\eta(\lambda, k)\rangle\,\phi_x)\, dx\, dt -$$

$$- \int_{\Gamma \times R_+} Q_\eta(a, k)\,\phi\,.\, n\, ds\, dt + \int_{\Gamma \times R_+} \operatorname{sgn}(a - k) \times$$

$$\times \langle \gamma\nu_{(\bar{x}, t)}, f(a) - f(\lambda)\rangle\,.\, n\phi\, ds\, dt \geq 0\,.$$

Using dominated convergence when η tend to zero, we obtain (1.9a) with $\phi = \tilde{\phi}\tilde{\chi}_\zeta$. As in [(1.21$a$), Sz III] this implies

$$\int_{\Gamma \times R_+} \langle \gamma\nu_{(\bar{x}, t)}, (\operatorname{sgn}(\lambda - k) - \operatorname{sgn}(a - k))(f(\lambda) - f(k))\rangle \,.\; . n \lim_{\zeta \to 0+} \chi_\zeta \tilde{\phi}\, ds\, dt \geq 0 \,.$$

where we have used dominated convergence once more, now when $\zeta \to 0+$. Since $(\operatorname{sgn}(\lambda - .) - \operatorname{sgn}(a(\bar{x}, t) - .))(f(\lambda) - f(.))$ is locally Lipschitz continuous on $R \setminus \{a(\bar{x}, t)\}$, this yields as in [Sz III, (1.21a)].

$$\langle \gamma\nu_{(\bar{x}, t)}, (\operatorname{sgn}(\lambda - k) - \operatorname{sgn}(a(\bar{x}, t) - k))(f(\lambda) - f(k))\rangle \,.\; . n \geq 0 \quad \forall k \neq a(\bar{x}, t)$$

a.e. on $\Gamma \times R_+$. Letting then $k \to a(\bar{x}, t) \pm$, we finally obtain Proposition 4.2.

It remains to prove Proposition 4.4 by the super approximation (2.3) and (4.9) we get

$$(4.10) \quad |R^9| \leq \sum_{K \in T_h} \int_{K \cap \operatorname{Bdr} S^N} |L(U)| \times \|\operatorname{sgn}(\bar{a} - k)\,\phi\chi - \pi(\operatorname{sgn}(\bar{a} - k)\,\phi\chi)\|_{L_\infty(K)}\, dx\, dt$$

$$\leq \mathrm{Ch} \int_{S^N} |L(U)|\, dx\, dt \leq \mathrm{Ch}\, \|L(U)\|_{L_2(S^N)} \leq C\sqrt{h}\,,$$

$$|R^{10}| \leq \sum_{n=0}^{N} \sum_{K \in T_h^n} \int_{\operatorname{Bdr} \Omega_n \cap K} |U_+ - U_-| \times \|\operatorname{sgn}(\bar{a} - k)\,\phi\chi - \pi(\operatorname{sgn}(\bar{a} - k)\,\phi\chi)\|_{L_\infty(K \cap \Omega_n)}\, dx$$

$$\leq \mathrm{Ch} \sum_{n=0}^{N} \int_{\Omega_n} |U_+ - U_-|\, dx \leq C\sqrt{h}\,.$$

We shall now estimate the terms R^{11}, R^{12} and R^{13} by using the equation (2.5) with $v = \pi(\operatorname{sgn}(\bar{a} - k)\;\phi(\chi - \chi_\rho))$, where $\chi_\rho \in C_0^\infty(\Omega)$, $0 \leq \chi_\rho \leq 1$ and $\chi_\rho(x) = 1$ for all $x \in \Omega$, such that $\operatorname{dist}(x, \Gamma) > \rho$. To this end we add to $R^{11} + R^{12} + R^{13}$ the following sum

$$\sum_{n=0}^{N} \Bigg(\delta \int_{\operatorname{Int} S_n} L(U)\, ((\pi(\operatorname{sgn}(\bar{a} - k)\,\phi\chi))_t + f'(\pi(\operatorname{sgn}(\bar{a} - k)\,\phi\chi))_x)\, dx\, dt$$

$$+ \int_{\operatorname{Int} S_n} \varepsilon_1(U)\, \nabla\hat{U} \,.\, \nabla\hat{\pi}(\operatorname{sgn}(\bar{a} - k)\,\phi\chi)\, dx\, dt$$

$$+ \int_{\mathrm{Int}\, S_n} \varepsilon_2(U)\, \hat{U}_x(\hat{\pi}(\operatorname{sgn}(\bar{a}-k)\,\phi\chi))_x \, dx\, dt$$

$$- \delta \int_{S_n} L(U)((\pi(\operatorname{sgn}(\bar{a}-k)\,\phi\chi_\rho))_t + f'(\pi(\operatorname{sgn}(\bar{a}-k)\,\phi\chi_\rho))_x)\, dx\, dt$$

$$- \int_{S_n} \varepsilon_1(U)\, \nabla\hat{U} \,.\, \nabla\hat{\pi}(\operatorname{sgn}(\bar{a}-k)\,\phi\chi_\rho)\, dx\, dt$$

$$- \left. \int_{S_n} \varepsilon_2(U)\, \hat{U}_x(\hat{\pi}(\operatorname{sgn}(\bar{a}-k)\,\phi\chi_\rho))_x\, dx\, dt \right) \equiv G^1$$

and by the stability Lemma 3.1 we then have

$$G^1 \leqslant C(\rho)(\sqrt{\delta} + \sqrt{\bar{\delta}/\mathrm{h}} + \sqrt{\bar{\bar{\delta}} + \varepsilon})\,. \tag{4.11}$$

By using (2.5) with $v = \pi(\operatorname{sgn}(\bar{a}-k)\,\phi(\chi-\chi_\rho))$ we get

$$R^{11} + R^{12} + R^{13} + G^1 =$$

$$= \sum_{n=0}^{N} \left[\int_{S_n} \varepsilon_1(U)\, \nabla\hat{U} \,.\, \nabla\hat{\pi}(\operatorname{sgn}(\bar{a}-k)\,\phi(\chi-\chi_\rho))\, dx\, dt \, + \right.$$

$$+ \int_{S_n} \varepsilon_2(U)\, \hat{U}_x(\hat{\pi}(\operatorname{sgn}(\bar{a}-k)\,\phi(\chi-\chi_\rho)))_x\, dx\, dt$$

$$+ \delta \int_{S_n} L(U)((\pi(\operatorname{sgn}(\bar{a}-k)\,\phi(\chi-\chi_\rho)))_t$$

$$+ f'(\pi(\operatorname{sgn}(\bar{a}-k)\,\phi(\chi-\chi_\rho)))_x)\, dx\, dt]$$

$$= - \sum_{n=0}^{N} \left[\int_{S_n} L(U)\, \pi(\operatorname{sgn}(\bar{a}-k)\,\phi(\chi-\chi_\rho))\, dx\, dt \, + \right.$$

$$+ \int_{\Omega_n} (U_+ - U_-)\, \pi(\operatorname{sgn}(\bar{a}-k)\,\phi(\chi-\chi_\rho))\, dx]\,.$$

Now, by (2.3) and (2.2) we have

$$\|(\chi-\chi_\rho)\operatorname{sgn}(\bar{a}-k)\,\phi - \pi((\chi-\chi_\rho)\operatorname{sgn}(\bar{a}-k)\,\phi)\|_{L_2(S^N)} \leqslant \tag{4.12a}$$

$$\leqslant \mathrm{Ch}\, \|\chi\|_{L_\infty(S^N)}\, \|\phi\|_{H^{k+1}(S^N)} + \mathrm{Ch}^2\, \|\chi_\rho\, \phi\|_{H^{k+1}(S^N)}\,,$$

$$\|(\chi-\chi_\rho)\operatorname{sgn}(\bar{a}-k)\,\phi - \pi((\chi-\chi_\rho)\operatorname{sgn}(\bar{a}-k)\,\phi)\|_{L_2(\Omega_n)} \leqslant \tag{4.12b}$$

$$\leqslant C\left(h^{\frac{1}{2}}\|\chi\|_{L_\infty(S_n)}\, \|\phi\|_{H^{k+1}(S_n)} + h^{3/2}\|\chi_\rho\, \phi\|_{H^{k+1}(S_n)}\right)\,.$$

Let now

$$G^2 \equiv \sum_{n=0}^{N} \left[\int_{S_n} L(U)(\chi - \chi_\rho) \operatorname{sgn}(\bar{a} - k) \phi \, dx \, dt + \right.$$

$$\left. + \int_{\Omega_n} (U_+ - U_-)(\chi - \chi_\rho) \operatorname{sgn}(\bar{a} - k) \phi \, dx \right].$$

Then it follows from (4.12) and Lemma 3.1 that

$$|R^{11} + R^{12} + R^{13} + G^1 + G^2| \leq C(\rho) \left(\frac{h}{\sqrt{\delta}} + \sqrt{h} \right). \tag{4.13}$$

Integrating by parts and using the fact that $\phi \nabla \operatorname{sgn}(\bar{a} - k) = 0$, we have

$$\begin{aligned} G^2 = \sum_{n=0}^{N} \Big[\int_{\Omega_{n+1}} & u_{h-} (\chi - \chi_\rho) \operatorname{sgn}(\bar{a} - k) \phi \, dx - \\ & - \int_{\Omega_n} (u_{h+} - u_{h+} + u_{h-})(\chi - \chi_\rho) \operatorname{sgn}(\bar{a} - k) \phi \, dx \\ & - \int_{S_n} u_h((\chi - \chi_\rho) \operatorname{sgn}(\bar{a} - k) \phi)_t \, dx \, dt \\ & - \int_{S_n} f(u_h)((\chi - \chi_\rho) \operatorname{sgn}(\bar{a} - k) \phi)_x \, dx \, dt \Big] \\ & = - \int_{S^N} (u_h \phi_t + f(u_h) \phi_x)(\chi - \chi_\rho) \operatorname{sgn}(\bar{a} - k) \, dx \, dt - \\ & - \int_{S^N} f(u_h)(\chi - \chi_\rho)_x \phi \operatorname{sgn}(\bar{a} - k) \, dx \, dt . \end{aligned} \tag{4.14}$$

Letting now $h \to 0$ using Lemma 2.2 and 4.2, (4.10) and (4.11-4.14), we find as in (4.9′)

$$\begin{aligned} \sum_{i=9}^{13} R^i \to \int_{S^N} & (\langle \nu_{(x,t)}, \lambda \rangle \phi_t + \langle \nu_{(x,t)}, f(\lambda) \rangle \phi_x) \times \\ & \times \operatorname{sgn}(\bar{a} - k)(1 - \chi_\rho) \, dx \, dt \\ & - \int_{S^N} \langle \nu_{(x,t)}, f(\lambda) \rangle (\chi_\rho)_x \operatorname{sgn}(\bar{a} - k) \phi \, dx \, dt - \\ & - \int_{\Gamma \times R_+} f(a) \, n\phi \operatorname{sgn}(a - k) \, ds \, dt , \end{aligned}$$

and by letting $\rho \to 0$ we then obtain as in [Sz III, (1.13)]

$$\sum_{i=9}^{13} R^i \to - \int_{\Gamma \times R_+} \langle \gamma \nu_{(x,t)}, f(a) - f(\lambda) \rangle \, n\phi \, \mathrm{sgn}\,(a-k)\, ds\, dt\,,$$

which finally proves Proposition 4.4 and the lemma. □

5. NUMERICAL RESULTS

In figure 5.1 we give numerical results for the SC-method on a uniform mesh with $k = 1$ applied to Burgers' equation in $(0,1) \times R_+$. More precisely, figure 5.1 shows the approximate solutions for the problem (1.1-1.3) with $d = 1$, $f(u) = f_1(u) = u^2/2$, initialdata $u_0 = \begin{cases} 1, & 0 \leq x \leq 0.5 \\ -0.5, & 0.5 < x \leq 1 \end{cases}$, and boundary values $a(0) = 1$, $a(1) = 0$.

Let v_j, $j = 1, \dots, N$ be a finite element basis for V_h. We base our numerical results for Burgers equation on the following slightly modified version of (2.5). Find $u \equiv U + 1 - x$, $U \in V_h$ such that for $n = 0, 1, 2, \dots$

$$F_n(u, v_j) = 0\,, \quad j = 1, \dots, N\,, \tag{5.1}$$

where

$$F_n(u, v) = \int_{S_n} (u_t v - f(u)\, v_x)\, dx\, dt + \int_{\Omega_n} (u_+ - u_-)\, v_+ \, dx + $$
$$+ \int_{S_n} \varepsilon_1(u)\, \nabla \hat{u} \,.\, \nabla \hat{v}\, dx\, dt + \int_{S_n} \varepsilon_2(u)\, \hat{u}_x\, \hat{v}_x\, dx\, dt$$
$$+ \delta \int_{S_n} (u_t + f(u)_x)(v_t + f'(\bar{u})\, v_x)\, dx\, dt\,,$$

where

$$\bar{u}|_K = \int_K u\, dx\, dt \quad K \in T_h\,.$$

The equation (5.1) was then solved iteratively on each time interval $(t_{n,}\, t_{n+1})$ by

$$u^{m+1} = u^m - \beta \sum_j v_j\, \gamma F_n(u^m, v_j)/(F_n(u^m + \gamma v_j, v_j) - F_n(u^m, v_j))\,, \tag{5.2}$$
$$m = 1, 2, \dots$$

with the relaxation parameter $\beta = 0.4$, $\gamma = 0.01$, $k = 1$, $h = 0.01$, $\delta = h$, $\bar{\delta} = 0.3\, h^{7/4}$ and $\bar{\bar{\delta}} = \bar{\delta}/h$.

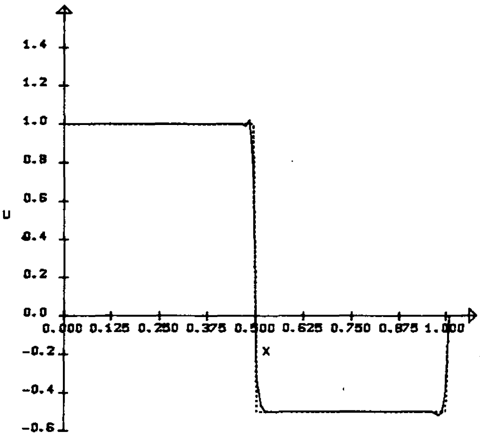

Figure 5.1*a*. — Method (5.1-5.2), one time step.

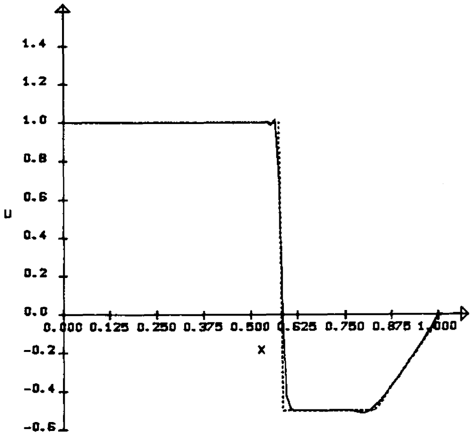

Figure 5.1*b*. — Method (5.1-5.2), 41 time steps.

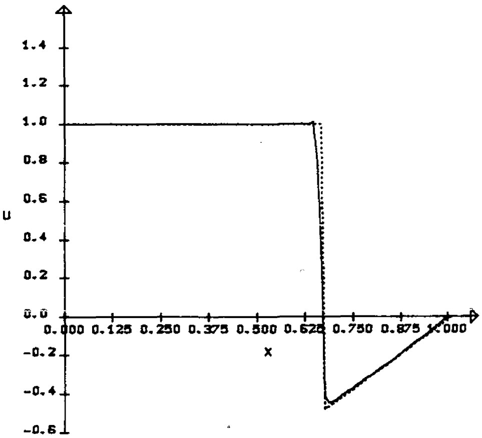

Figure 5.1*c*. — Method (5.1-5.2), 81 time steps.

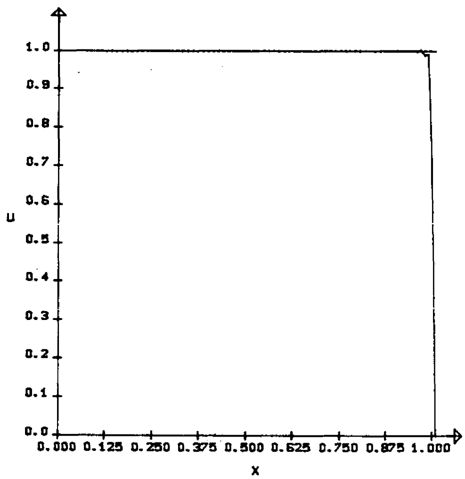

Figure 5.1*d*. — Method (5.1-5.2), 201 time steps.

APPENDIX

We now give a proof of the super approximation results (2.3) : Let us consider the case (2.3a) with $\omega = S_n$. It is then sufficient to consider one triangle $K \in T_h^n$, i.e. $\omega = K$.

Defining $\mathscr{P}\varphi = \int_K \varphi \, dx \, dt / \int_K dx \, dt$, $\varphi \,|_K \in L_1(K)$, we have

(A.1) $$\| w - \mathscr{P}w \|_{L_\infty(K)} \leq Ch \| \nabla w \|_{L_\infty(K)} .$$

Further

$$\|vw - \pi(vw)\|_{L_\infty(K)} \leq \|v\mathscr{P}w - \pi(v\mathscr{P}w)\|_{L_\infty(K)} + \\ + \|(I - \pi)(v(w - \mathscr{P}w))\|_{L_\infty(K)} \equiv T_1 + T_2 .$$

Now, since $v \in V_h^{\mu}$ and $\mathscr{P}w$ is constant on K, $\pi(v\mathscr{P}w) = v\mathscr{P}w$ on K and thus $T_1 = 0$. Further we have by (A.1) and (2.2b)

$$\begin{aligned} T_2 &\leq Ch \|\nabla(v(w - \mathscr{P}w))\|_{L_\infty(K)} \\ &\leq Ch(\|v\|_{L_\infty(K)} \|w\|_{\dot{W}^{1,\infty}(K)} + h\|\nabla w\|_{L_\infty(K)} \|v\|_{\dot{W}^{1,\infty}(K)}) \\ &\leq Ch\|v\|_{L_\infty(K)} \|w\|_{W^{1,\infty}(K)} . \end{aligned}$$

where we in the last step used the inverse estimate (2.4a). This proves (2.3a) with $\omega = S_n$, $r = 0$. The case $r = 1$ and $\omega = \Omega_n$ are similar.

The inequality (2.3b) follows as above by (2.2b) and (2.4a) using that $\|v\|_{\dot{H}^{k+1}(K)} = 0$:

$$h^r \|(I - \pi) vw\|_{H^r(S_n)} \leq Ch^{k+1} \|vw\|_{\dot{H}^{k+1}(S_n)} \leq C \|v\|_{L_\infty(S_n)} \sum_1^{k+1} h^i \|w\|_{\dot{H}^i(S_n)} .$$

Finally to prove (2.3c) we first note that for $f \in \mathscr{C}^\infty(S_n)$ we have

$$|f(x, t_n)| \leq \frac{1}{t} \int_{t_n}^{t_n + t} |f(x, s)| \, ds + \int_{t_n}^{t_n + t} |f_t(x, s)| \, ds ,$$

which gives

$$\frac{1}{2} |f(x, t_n)|^2 \leq \frac{1}{h} \int_{t_n}^{t_n + h} |f(x, s)|^2 \, ds + h \int_{t_n}^{t_n + h} |f_t(x, s)|^2 \, ds .$$

So that by a density argument we obtain for $f \in H^1(S_n)$

$$\|f\|_{L_2(\Omega_n)} \leq \sqrt{2}(h^{-1/2}\|f\|_{L_2(S_n)} + h^{1/2}\|f_t\|_{L_2(S_n)}) .$$

Combining now this estimate for $f = (I - \pi)(wv)$ with (2.3b) we get (2.3c). □

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