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DIVERGENCE STABILITY IN CONNECTION WITH THE *P*-VERSION OF THE FINITE ELEMENT METHOD (*)

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Abstract. — *Many problems in continuum mechanics involve an incompressibility condition, usually in the form of a divergence constraint. The numerical discretization of such a constraint presents some interesting problems with regard to stability. In this paper we analyze certain stability properties, typical of high degree, conforming finite element approximations for problems with a divergence constraint. The results in this paper complement the results already published in [18] and [24].*

Résumé. — *De nombreux problèmes en mécanique des milieux continus font appel à une condition d'incompressibilité, le plus souvent sous forme d'une contrainte sur l'opérateur de divergence. La discrétisation numérique d'une contrainte présente quelques problèmes intéressants en ce qui concerne la stabilité. Dans cet article nous analysons certaines des propriétés les plus courantes qui découlent des approximations, par des méthodes d'éléments finis conformes de degré élevé, dans le cadre de problèmes soumis à cette contrainte sur l'opérateur de divergence. Les résultats de cet article complètent les résultats déjà publiés dans les références [18] et [24].*

0. INTRODUCTION

Many problems in continuum mechanics involve an incompressibility condition, usually in the form of a divergence constraint. The numerical discretization of such a constraint presents some interesting problems with regard to stability. As an important exemple we consider the two-dimensional Stokes equations

(1)
$$- \Delta \underline{U} + \nabla P = \underline{F} \quad \text{in } \Omega \subseteq \mathbb{R}^2,$$
$$\nabla \cdot \underline{U} = 0 \quad \text{in } \Omega,$$

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with appropriate boundary conditions on $\partial\Omega$. This has the standard weak formulation

(2)   Find $\underline{U} \in \mathcal{V} \subseteq [H^1(\Omega)]^2$ and $P \in \mathcal{W} \subseteq L^2(\Omega)$ such that

$$a(\underline{U}, \underline{v}) + b(\underline{v}, P) = (\underline{F}, \underline{v}) \quad \forall \underline{v} \in \mathcal{V}$$

$$b(\underline{U}, q) = 0 \quad \forall q \in \mathcal{W}.$$

The bilinear forms a and b are given by

$$a(\underline{U}, \underline{v}) = 2 \int_{\Omega} \sum_{i,j} \varepsilon_{ij}(\underline{U}) \, \varepsilon_{ij}(\underline{v}) \, d\underline{x}$$

$$b(\underline{v}, P) = - \int_{\Omega} \nabla \cdot \underline{v} P \, d\underline{x},$$

and $(\underline{F}, \underline{v})$ denotes the usual $[L^2(\Omega)]^2$ inner product. The tensor $\varepsilon_{ij}(\underline{v})$ is the symmetric derivative $\dfrac{1}{2} \left(\dfrac{\partial}{\partial x_i} v_j + \dfrac{\partial}{\partial x_j} v_i \right)$. The spaces $\mathcal{V}$ and $\mathcal{W}$ depend on the boundary conditions. For no-slip boundary conditions : $\mathcal{V} = [\mathring{H}^1(\Omega)]^2$ and $\mathcal{W} = L^2(\Omega) \cap \left\{ \int_{\Omega} q = 0 \right\}$; for stress-free boundary conditions : $\mathcal{V} =$ the orthogonal complement of $\{\varepsilon_{ij}(\underline{v}) = 0\}$ in $[H^1(\Omega)]^2$ and $\mathcal{W} = L^2(\Omega)$. A natural discretization of (2) consists in choosing finite dimensional spaces $\mathcal{V}_N \subseteq \mathcal{V}$, $\mathcal{W}_N \subseteq \mathcal{W}$ and determining

(3) $\underline{U}_N \in \mathcal{V}_N$ and $P_N \in \mathcal{W}_N$ such that

$$a(\underline{U}_N, \underline{v}) + b(\underline{v}, P_N) = (\underline{F}, \underline{v}) \quad \forall \underline{v} \in \mathcal{V}_N$$

$$b(\underline{U}_N, q) = 0 \quad \forall q \in \mathcal{W}_N.$$

The main obstacle in connection with (3) is to find spaces $\mathcal{V}_N$ and $\mathcal{W}_N$ so that the discretization is stable and at the same time has good approximation properties. A reasonable requirement concerning stability seems to be

(4) $\| \underline{U} - \underline{U}_N \|_{H^1} + \| P - P_N \|_{L^2} \leqslant$

$$\leqslant C \left(\min_{\underline{v} \in \mathcal{V}_N} \| \underline{U} - \underline{v} \|_{H^1} + \min_{q \in \mathcal{W}_N} \| P - q \|_{L^2} \right),$$

with C independent of the dimension variable N. It is well known that the Babuška-Brezzi condition

(5)
$$\min_{q \in \mathcal{W}_N \setminus \{0\}} \max_{\underline{v} \in \mathcal{V}_N} \frac{\displaystyle\int_\Omega \nabla \cdot \underline{v} q \, d\underline{x}}{\|\underline{v}\|_{H^1} \|q\|_{L^2}} \geq c > 0 ,$$

with c independent of N, is sufficient to guarantee (4) ($cf.$ [2], [7]). If the pressure spaces $\mathcal{W}_N$ are chosen equal to $\nabla \cdot \mathcal{V}_N$, then (5) is equivalent to the requirement that the divergence operator

$$\nabla \cdot : \mathcal{V}_N \to \mathcal{W}_N$$

has corresponding right inverses

$$(\nabla \cdot)_N^{-1} : \mathcal{W}_N \to \mathcal{V}_N ,$$

that are uniformly bounded in $\mathcal{B}(L^2 ; H^1)$. In this case (5) is both a necessary and sufficient condition that the quasi-optimality estimate

$$\|\underline{U} - \underline{U}_N\|_{H^1} \leq C \min_{\underline{v} \in \mathcal{V}_N} \|\underline{U} - \underline{v}\|_{H^1}$$

holds for arbitrary admissible force $\underline{F}$ ($cf.$ [19]).

The most natural low degree finite element spaces often fail to satisfy the Babuška-Brezzi condition, except on very special triangulations. A remedy is to appropriately enlarge the velocity space or to deplete the pressure space ; such approaches are analyzed for continuous piecewise linear (bilinear) velocities with piecewise constant pressures in [6] and [12] respectively. For continuous piecewise quadratic velocities one has the well known Taylor-Hood element, with continuous piecewise linear pressures (for the analysis leading to (5), see [4] and [22]). Enlarging the velocity space or depleting the pressure space is also in general necessary for cubic velocities and quadratic pressures ($cf.$ [19]).

For continuous piecewise polynomial velocities of total degree $\leq p$, $p \geq 4$, the situation is quite different. For an arbitrary triangulation the range of the divergence operator acting on the velocity space has a very simple characterization — it consists of all piecewise polynomials of total degree $\leq p - 1$, except for a certain constraint at so-called singular vertices ($cf.$ [18]). Furthermore, for fixed $p \geq 4$, the divergence operator possesses maximal right inverses, the norms of which are bounded independently of the mesh size h (provided non-singular vertices do not degenerate). To paraphrase : the condition (5) is satisfied for such velocities if the pressure space, $\mathcal{W}_N$, is chosen to be $\nabla \cdot \mathcal{V}_N$. Using the analysis in [24] we were able to prove that the same right inverses have $\mathcal{B}(L^2 ; H^1)$ norms, which are bounded by some power of p, for fixed h.

In this note we complement the results of [18] and [24]. We demonstrate

with a few examples, theoretical as well as computational, that it is *not* in general possible to find maximal right inverses for the divergence operator, acting on entire polynomials, the norms of which are bounded in $\mathscr{B}(L^2; H^1)$, uniformly in p. We discuss both spaces of total and separate degree $\leqslant p$, as well as spaces with and without boundary conditions.

The lack of uniformly bounded right inverses for the discrete case is somewhat surprising when compared to the continuous case : it is easy to see that there exists a right inverse $(\nabla \cdot)^{-1}$ which maps $H^s = \nabla \cdot (H^{s+1})^2$ boundedly into $(H^{s+1})^2$, $\forall s \geqslant 0$. A similar result holds with homogeneous Dirichlet boundary conditions, even for non-smooth (polygonal) domains Ω (*cf.* [1]).

Methods that use high degree polynomials to approximate the solution to the Stokes equations are quite common, whether they be variationally based spectral methods, or collocation based pseudo-spectral methods (*cf.* [10]). Another possibility is the so-called *p*-version of the finite element method (*cf.* [3]) : it uses a rather coarse mesh (triangulation or lattice) and achieves convergence by including, in a variational formulation, piecewise polynomials of high degree relative to this mesh. Even though the Babuska-Brezzi condition may only be satisfied with a constant approaching zero as some negative power of p, these methods often have optimal convergence rates as far as the velocity is concerned. We briefly return to an explanation of this (at least for variational methods) towards the very end of this paper. The lack of divergence stability may possibly reduce the convergence rates of the computed pressures. We conclude this paper with a numerical example that demonstrates such reduction in the case of a « driven-cavity » flow problem.

In case of periodic boundary conditions it is normal to consider spectral or pseudospectral methods based on trigonometric polynomials instead of polynomials. The resulting methods are much more likely to be uniformly divergence stable (see, e.g. [8]), however, they are restricted in their applicability due to the special boundary conditions.

Methods that use high degree polynomials have also been proposed for mixed formulations of second order scalar elliptic problems, *cf.* [20]. In that connection the required stability estimate is very closely related to a bound on the $\mathscr{B}(L^2; L^2)$-norm of a right inverse for the divergence operator. The estimate is much weaker than the divergence stability estimate (5) that we are concerned with here, and it has been verified to hold (essentially) uniformly in p for the so called Raviart-Thomas element as well as the Brezzi-Douglas-Marini element, *cf.* [20].

To complete the introduction, let us give an interpretation of the constant

$$\min_{q \in \mathscr{W}_N \setminus \{0\}} \max_{\underline{v} \in \mathscr{V}_N} \frac{\displaystyle\int_\Omega \nabla \cdot \underline{v} q \, d\underline{x}}{\|\underline{v}\|_{H^1} \|q\|_{L^2}} = \mu_N ,$$

in terms of the associated matrices, when $\mathscr{W}_N = \nabla \cdot \mathscr{V}_N$. We first specify our choice of norm on $[H^1]^2$: of the many equivalent norms we take

$$\|\underline{v}\|_{H^1} = \left(\sum_{i=1}^{2} \int_\Omega \left| \frac{\partial}{\partial x_i} \underline{v} \right|^2 dx + \left| \int_\Omega \underline{v}\, d\underline{x} \right|^2 \right)^{1/2}.$$

Let $\{\varphi_\ell\}_{\ell=1}^{N}$ be a basis for $\mathscr{V}_N$ and let $\{\psi_k\}_{k=1}^{M}$ be a basis for $\mathscr{W}_N = \nabla \cdot \mathscr{V}_N$. The matrices $A = (a_{k\ell})_{k=1, \ell=1}^{M, N}$, $B = (b_{k\ell})_{k=1, \ell=1}^{N, N}$ and $C = (c_{k\ell})_{k=1, \ell=1}^{M, M}$ are defined by

(6a) $$\nabla \cdot \underline{\varphi}_\ell = \sum_{k=1}^{M} a_{k\ell}\, \psi_k, \quad 1 \leq \ell \leq N,$$

(6b) $$b_{k\ell} = \sum_{i=1}^{2} \int_\Omega \frac{\partial}{\partial x_i} \underline{\varphi}_k \cdot \frac{\partial}{\partial x_i} \underline{\varphi}_\ell\, d\underline{x} + \int_\Omega \underline{\varphi}_k\, d\underline{x} \cdot \int_\Omega \underline{\varphi}_\ell\, d\underline{x}$$
$$1 \leq k, \ell \leq N,$$

and

(6c) $$c_{k\ell} = \int_\Omega \psi_k\, \psi_\ell\, d\underline{x}, \quad 1 \leq k, \ell \leq M.$$

A is the discrete representation of the divergence operator and $(B., .)$ and $(C., .)$ represent the quadratic forms $\|\underline{v}\|_{H^1}^2$ and $\|q\|_{L^2}^2$ respectively.

With these definitions it is easy to see that μ_N is the smallest singular value of the $N \times M$ matrix $B^{-1/2} A^T C^{1/2}$, and this in turn is the square root of the smallest eigenvalue of the positive definite symmetric $M \times M$ matrix

(7) $$D = C^{1/2} A B^{-1} A^T C^{1/2}.$$

For any $q \in \mathscr{W}_N$ let $(\nabla \cdot)_N^{-1} q \in \mathscr{V}_N$ denote the element of minimal H^1-norm that has q for its divergence. By a « worst possible pressure » (as far as divergence stability is concerned) we mean a $q_0 \in \mathscr{W}_N$, $\|q_0\|_{L^2} = 1$, for which the « minimal norm » right inverse $(\nabla \cdot)_N^{-1}$ attains its operator norm. If $\underline{x} \in \mathbb{R}^M$ denotes a unit eigenvector for the matrix D, corresponding to the smallest eigenvalue, μ_N^2, then

(8) $$q_0 = \sum_{j=1}^{M} (C^{-1/2} \underline{x})_j\, \psi_j \in \mathscr{W}_N$$

is a worst possible pressure.

1. A RESULT FOR THE SPACES $\mathcal{Q}_p$

Let R denote the square $(-1, 1) \times (-1, 1)$. In this section we consider polynomials of separate degree $\leqslant p$, i.e., the velocity space is $(\mathcal{Q}_p)^2$ with

$$(9) \qquad \mathcal{Q}_p = \text{span} \left\{ x_1^m x_2^n : 0 \leqslant m, n \leqslant p \right\},$$

and the corresponding pressure space is

$$(10) \qquad \nabla \cdot (\mathcal{Q}_p)^2 = \text{span} \left\{ x_1^m x_2^n : 0 \leqslant m, n \leqslant p, \ m + n < 2p \right\}.$$

Note that we use p as a subscript instead of the dimension variable $N = (p+1)^2$. We use the notation $(\nabla \cdot)_p^{-1}$ for the right inverse with minimal $H^1(R)$ norm.

PROPOSITION 1 : *The operator* $(\nabla \cdot)_p^{-1} : \nabla \cdot (\mathcal{Q}_p)^2 \to (\mathcal{Q}_p)^2$, $p \geqslant 1$, *considered as an operator from a subspace of $L^2(R)$ to a subspace of $(H^1(R))^2$ satisfies*

$$cp \leqslant \left\| (\nabla \cdot)_p^{-1} \right\|_{\mathscr{B}(L^2 ; H^1)} \leqslant Cp,$$

with constants $0 < c$ and C independent of p.

Before giving a proof of Proposition 1 we make a few observations about orthogonal polynomials (*cf.* [14]). Let $\ell_n(x)$ denote the Legendre polynomial of degree n, with the standard normalization

$$(11) \qquad \int_{-1}^{1} \ell_n^2 \, dx = \frac{2}{2n+1} .$$

It is not difficult to see that

$$(12) \qquad \int_{-1}^{x} \ell_n = \frac{1}{2n+1} (\ell_{n+1}(x) - \ell_{n-1}(x)), \quad n \geqslant 1 .$$

The polynomial ℓ_n may be written as a telescoping series

$$\ell_n = \sum_{j=0}^{[n/2]-1} (\ell_{n-2j} - \ell_{n-2(j+1)}) + \begin{cases} \ell_1, & n \text{ odd} \\ \ell_0, & n \text{ even}, \end{cases}$$

and consequently

$$(13) \quad \ell_n(x) = \sum_{j=0}^{[n/2]-1} (2(n-2j-1)+1) \int_{-1}^{x} \ell_{n-2j-1} + \begin{cases} \ell_1, & n \text{ odd} \\ \ell_0, & n \text{ even}. \end{cases}$$

From this we conclude that $\int_{-1}^{1} \left(\frac{d}{dx} \ell_n \right)^2 dx$ is of the order $\sum_{j=0}^{[n/2]-1} j$, i.e.,

(14) $$\int_{-1}^{1} \left(\frac{d}{dx} \ell_n \right)^2 dx \text{ is of the order } n^2.$$

Let $q_n(x)$, $0 \leqslant n \leqslant p$ denote the polynomials

(15)
$$q_n(x) = \ell_n(x), \quad 0 \leqslant n < p$$

$$q_p(x) = \int_{-1}^{x} \ell_{p-1}$$

(the notation should properly be $q_n^{(p)}$, since the definition of q_n depends on p, but we drop the superscript for convenience). An elementary computation shows that the polynomial

$$r_p(x) = \alpha \ell_{p-2} + \beta \int_{-1}^{x} \ell_{p-1}, \quad p \geqslant 2,$$

satisfies

$$\frac{2}{3} \left(\frac{\alpha^2}{2p+1} + \frac{\beta^2}{(2p+1)(2p-1)^2} \right) \leqslant \int_{-1}^{1} (r_p(x))^2 \, dx \leqslant$$

$$\leqslant 6 \left(\frac{\alpha^2}{2p-3} + \frac{\beta^2}{(2p-3)(2p-1)^2} \right).$$

Since $\{\ell_n\}_{n=1}^{p-3} \cup \{\ell_{p-1}\}$ are mutually orthogonal in L^2, and since these are also orthogonal to ℓ_{p-2} and $\int_{-1}^{x} \ell_{p-1}$, it follows that

(16) $$\int_{-1}^{1} \left(\sum_{n=0}^{p} \alpha_n q_n(x) \right)^2 dx \text{ is equivalent to}$$

$$\sum_{n=0}^{p-1} \alpha_n^2 (n+1)^{-1} + \alpha_p^2 (p)^{-3}, \quad \text{with constants}$$

that are independent of p.

It is convenient to work with

$$\{ q_m(x_1) \, q_n(x_2) : 0 \leqslant m, n \leqslant p, \ m+n < 2p \}$$

as a basis for $\nabla \cdot (\mathcal{Q}_p)^2$. Based on (16) we get that if

$$q = \sum_{\substack{m, n = 0 \\ m + n < 2p}}^{p} \alpha_{mn} \, q_m(x_1) \, q_n(x_2) \, ,$$

then

(17) $\displaystyle\int_R (q)^2 \, d\underline{x}$ is equivalent to

$$\sum_{\substack{m < p \\ n < p}} (\alpha_{mn})^2 (m + 1)^{-1} (n + 1)^{-1} + \sum_{m < p} ((\alpha_{mp})^2 + (\alpha_{pm})^2) \, (m + 1)^{-1} p^{-3} \, ,$$

with constants that are independent of p.

We are now ready for the proof of proposition 1

Given $q \in \nabla \cdot (\mathcal{Q}_p)^2$ we have

$$q = \sum_{\substack{m, n = 0 \\ m + n < 2p}}^{p} \alpha_{mn} \, q_m(x_1) \, q_n(x_2)$$

for some set of coefficients $\{\alpha_{mn}\}$. Define

$$\underline{u} = \begin{pmatrix} \displaystyle\sum_{n < m < p} \alpha_{mn} \left(\int_{-1}^{x_1} q_m \right) q_n(x_2) + \sum_{m < p} \alpha_{mp} \left(\int_{-1}^{x_1} q_m \right) q_p(x_2) \\[4mm] \displaystyle\sum_{m \leqslant n < p} \alpha_{mn} \, q_m(x_1) \int_{-1}^{x_2} q_n + \sum_{n < p} \alpha_{pn} \, q_p(x_1) \int_{-1}^{x_2} q_n \end{pmatrix} .$$

It is clear that $\underline{u} \in (\mathcal{Q}_p)^2$, with $\nabla \cdot \underline{u} = q$. It remains to estimate the H^1-norm of $\underline{u}$. Using (17) we immediately get that

$$\left\| \frac{\partial}{\partial x_1} u_1 \right\|_{L^2(R)} + \left\| \frac{\partial}{\partial x_2} u_2 \right\|_{L^2(R)} \leqslant C \, \|q\|_{L^2(R)} \, ;$$

also

$$\int_R \underline{u} = -\frac{4}{3} \begin{pmatrix} \alpha_{10} \\ \alpha_{01} - 3 \, \alpha_{00} \end{pmatrix} , \quad \text{for} \quad p \geqslant 2 \, ,$$

and therefore

$$\left| \int_R \underline{u} \right| \leqslant C \, \|q\|_{L^2(R)} \, .$$

Concerning $\dfrac{\partial}{\partial x_2} u_1$:

(18) $\quad \left\| \dfrac{\partial}{\partial x_2} u_1 \right\|^2_{L^2(R)} = \left\| \displaystyle\sum_{n < m < p} \alpha_{mn} \left(\int_{-1}^{x_1} \ell_m \right) \left(\dfrac{d}{dx} \ell_n \right) (x_2) \right.$

$$+ \sum_{m < p} \alpha_{mp} \left(\int_{-1}^{x_1} \ell_m \right) \ell_{p-1}(x_2) \Bigg\|^2_{L^2(R)}$$

$$\leqslant C \sum_{m < p} (m + 1)^{-3}$$

$$\times \left\| \sum_{n=0}^{m-1} \alpha_{mn} \left(\dfrac{d}{dx} \ell_n \right) + \alpha_{mp} \ell_{p-1} \right\|^2_{L^2(-1, 1)} .$$

The last estimate comes from the identities (11) and (12) and the L^2-orthogonality of the Legendre polynomials. From (18) we get by means of the triangle inequality, Schwarz inequality, and the estimate (14), that

$$\left\| \dfrac{\partial}{\partial x_2} u_1 \right\|^2_{L^2(R)} \leqslant C \sum_{m < p} (m + 1)^{-3} \left(\sum_{n=0}^{m-1} (\alpha_{mn})^2 \cdot \sum_{n=0}^{m-1} n^2 + (\alpha_{mp})^2 p^{-1} \right) ,$$

the right hand side of which is bounded by

$$C \left(\sum_{n < m < p} (\alpha_{mn})^2 + \sum_{m < p} (\alpha_{mp})^2 (m + 1)^{-3} p^{-1} \right) .$$

Using the above estimate in combination with (17) we get

$$\left\| \dfrac{\partial}{\partial x_2} u_1 \right\|^2_{L^2(R)} \leqslant C p^2 \| q \|^2_{L^2(R)} .$$

The same estimate holds for $\left\| \dfrac{\partial}{\partial x_1} u_2 \right\|^2_{L^2(R)}$. In summary we have thus established

$$\| \underline{u} \|_{H^1(R)} \leqslant C p \| q \|_{L^2(R)} ,$$

and since $(\nabla \cdot)_p^{-1} q$ is by definition the field of minimal H^1-norm it follows that

$$\left\| (\nabla \cdot)_p^{-1} \right\|_{\mathcal{B}(L^2 ; H^1)} \leqslant C p .$$

To verify the second inequality of this proposition, take

$$q^* = r(x_1) \ell_p(x_2)$$

for some fixed nonzero polynomial r, independent of p (we consider only $p \geqslant \text{degree}(r) + 1$, so that $q^* \in \nabla \cdot (\mathcal{Q}_p)^2$). As a basis for $\mathcal{Q}_p$ we choose

$$\{ q_m(x_1) \, q_n(x_2) : 0 \leqslant m, n \leqslant p \}.$$

For an arbitrary $\underline{u} \in (\mathcal{Q}_p)^2$ there exist coefficients $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ such that

$$u_1 = \sum_{m, n = 0}^{p} \alpha_{mn} \, q_m(x_1) \, q_n(x_2)$$

$$u_2 = \sum_{m, n = 0}^{p} \beta_{mn} \, q_m(x_1) \, q_n(x_2) \, .$$

If $\nabla \cdot \underline{u} = \dfrac{\partial}{\partial x_1} u_1 + \dfrac{\partial}{\partial x_2} u_2 = q^*$, then we must necessarily have

$$\sum_{m = 0}^{p} \alpha_{mp} \left(\frac{d}{dx} q_m \right) (x_1) \int_{-1}^{1} q_p \, \ell_p \, dx =$$

$$= \int_{-1}^{1} \nabla \cdot \underline{u}(x_1, x_2) \, \ell_p(x_2) \, dx_2 = r(x_1) \int_{-1}^{1} \ell_p^2 \, dx \, .$$

Due to (12) and (15) it follows that

$$\frac{d}{dx} \left(\sum_{m = 0}^{p} \alpha_{mp} \, q_m \right) = (2 p - 1) \, r \, ,$$

and therefore

$$(19) \quad u_1 = \sum_{n < p} q_n(x_2) \sum_{m \leqslant p} \alpha_{mn} \, q_m(x_1) + q_p(x_2)(2 p - 1) \left(\int_{-1}^{x_1} r + c \right) .$$

Differentiation with respect to x_2 yields

$$\frac{\partial}{\partial x_2} u_1 = \sum_{n < p} \left(\frac{d}{dx} q_n \right) (x_2) \sum_{m \leqslant p} \alpha_{mn} \, q_m(x_1) +$$

$$+ \, \ell_{p-1}(x_2)(2 p - 1) \left(\int_{-1}^{x_1} r + c \right) ,$$

from which it now follows (by orthogonality) that

$$\int_{R} \left(\frac{\partial}{\partial x_2} u_1 \right)^2 d\underline{x} \geqslant 2(2 p - 1) \int_{-1}^{1} \left(\int_{-1}^{x} r + c \right)^2 dx \geqslant$$

$$\geqslant 2(2 p - 1) \left\| \int_{-1}^{x} r - \frac{1}{2} \int_{-1}^{1} \int_{-1}^{x} r \, dx \right\|_{L^2(-1, 1)}^{2}$$

At the same time

$$\| q^* \|^2_{L^2(R)} = \frac{2}{2p+1} \, \| r \|^2_{L^2(-1,1)} \, ,$$

and so we have proven that for any $\underline{u} \in \mathcal{Q}_p$ with $\nabla \cdot \underline{u} = q^*$ one has

$$\| \underline{u} \|_{H^1(R)} \geqslant cp \, \| q^* \|_{L^2(R)} \, ,$$

for a fixed r (the constant c of course depends on our choice of r). This verifies the lower bound on the norm of $(\nabla \cdot)_p^{-1}$.   ■

Remark 1.1 :   If $\displaystyle\int_{-1}^{1} r = 0$ and $p \geqslant \text{degree } (r) + 1$ then $q^* = $

$r(x_1) \displaystyle\int_{-1}^{x_2} \ell_{p-1}$ is an element of $\nabla \cdot (\mathring{\mathcal{Q}}_p)^2$, where $\mathring{\mathcal{Q}}_p = \mathcal{Q}_p \cap \{ u = 0 \text{ on } \partial R \}$.

Indeed $\nabla \cdot \underline{u}^* = q^*$ for $\underline{u}^* = \left(\displaystyle\int_{-1}^{x_1} r \int_{-1}^{x_2} \ell_{p-1}, 0 \right)$. It follows now by a slight change of the argument in the last part of the previous proof that the minimal norm right inverse $(\nabla \cdot)_p^{-1} : \nabla \cdot (\mathring{\mathcal{Q}}_p)^2 \rightarrow (\mathring{\mathcal{Q}}_p)^2$ also must satisfy $cp \leqslant \| (\nabla \cdot)_p^{-1} \|_{\mathscr{B}(L^2, H^1)}$.

Remark 1.2 : The estimate in Theorem 3.1 of [17] is somewhat related to the upper bound in Proposition 1. The estimate in [17], however, concerns the gradient operator ; it is therefore much closer to an $\mathcal{O}(p)$-estimate of the $\mathscr{B}(H^{-1} ; L^2)$-norm of a right inverse for the « adjoint » divergence operator. We refer the reader to [26] for other estimates related to Proposition 1.

2. RESULTS FOR THE SPACES $\mathscr{P}_p$

We now consider polynomials of total degree $\leqslant p$ on the square R. Without boundary conditions the velocity space is $(\mathscr{P}_p)^2$, where

$$\mathscr{P}_p = \text{span} \{ x_1^m x_2^n : m + n \leqslant p \} \, ,$$

and the corresponding pressure space is $\nabla \cdot (\mathscr{P}_p)^2 = \mathscr{P}_{p-1}$. With Dirichlet boundary conditions on the entire boundary the velocity space is $(\mathring{\mathscr{P}}_p)^2$, where

$$\mathring{\mathscr{P}}_p = \mathscr{P}_p \cap \{ u|_{\partial R} = 0 \},$$

and the pressure space $\nabla \cdot (\mathring{\mathscr{P}}_p)^2$ is of codimension 9 in $\mathscr{P}_{p-1}$ $(p \geqslant 5)$.

PROPOSITION 2   *Let* $(\nabla)_p^{-1} : \mathscr{P}_{p-1} \to (\mathscr{P}_p)^2$, $p \geqslant 1$, *denote the right inverse with minimal H^1-norm Considered as an operator from a subspace of $L^2(R)$ to a subspace of $(H^1(R))^2$ this satisfies*

$$\left\| (\nabla)_p^{-1} \right\|_{\mathscr{B}(L^2 \ H^1)} \leqslant Cp ,$$

with C independent of p

Proof   Given $q \in \mathscr{P}_{p-1}$ we have

$$q = \sum_{\substack{m+n<p \\ m\ n \geqslant 0}} \alpha_{mn} \ell_m(x_1) \ell_n(x_2)$$

for some set of coefficients $\{\alpha_{mn}\}$ Define

$$\underline{u} = \begin{pmatrix} \sum\limits_{\substack{m+n<p \\ n<m}} \alpha_{mn} \left(\int_{-1}^{x_1} \ell_m \right) \ell_n(x_2) \\ \sum\limits_{\substack{m+n<p \\ m \leqslant n}} \alpha_{mn} \ell_m(x_1) \left(\int_{-1}^{x_2} \ell_n \right) \end{pmatrix}$$

It is clear that $\underline{u} \in (\mathscr{P}_p)^2$ with $\nabla \underline{u} = q$ Using the first part of the proof of Proposition 1 we get the estimate

$$\| \underline{u} \|_{H^1(R)} \leqslant Cp \| q \|_{L^2(R)}$$

(in this case we just have $\alpha_{mn} = 0$ for m, n that simultaneously satisfy $0 \leqslant m, n \leqslant p$ and $p \leqslant m + n < 2p$)   This verifies the desired estimate on the operator norm   ■

For the case of Dirichlet boundary conditions on the entire boundary we have

PROPOSITION 3   *Let* $(\nabla)_p^{-1} : \nabla (\mathring{\mathscr{P}}_p)^2 \to (\mathring{\mathscr{P}}_p)^2$, $p \geqslant 4$, *denote the right inverse with minimal H^1-norm Considered as an operator from a subspace of $L^2(R)$ to a subspace of $(H^1(R))^2$, this satisfies*

$$cp \leqslant \left\| (\nabla)_p^{-1} \right\|_{\mathscr{B}(L^2 \ H^1)} ,$$

with $0 < c$ independent of p

Proof   Consider

$$q^* = \ell_1(x_1) \int_{-1}^{x_2} \ell_{p-3}$$

It is clear that $q^* \in \nabla \cdot (\overset{\circ}{\mathscr{P}}_p)^2$, indeed,

$$\nabla \cdot \underline{v} = q^* \quad \text{for} \quad \underline{v} = \left(\int_{-1}^{x_1} \ell_1 \int_{-1}^{x_2} \ell_{p-3}, 0 \right).$$

On the other hand, if $\underline{u} \in (\overset{\circ}{\mathscr{P}}_p)^2$ is an arbitrary velocity field with $\nabla \cdot \underline{u} = q^*$, then

$$\underline{u} = \begin{pmatrix} \displaystyle\sum_{m,n=1}^{m+n \le p-2} \alpha_{mn} \int_{-1}^{x_1} \ell_m \int_{-1}^{x_2} \ell_n \\[2em] \displaystyle\sum_{m,n=1}^{m+n \le p-2} \beta_{mn} \int_{-1}^{x_1} \ell_m \int_{-1}^{x_2} \ell_n \end{pmatrix}$$

and

$$\sum_{m,n=1}^{m+n \le p-2} \left[\alpha_{mn} \ell_m(x_1) \left(\int_{-1}^{x_2} \ell_n \right) + \beta_{mn} \left(\int_{-1}^{x_1} \ell_m \right) \ell_n(x_2) \right] =$$

$$= \ell_1(x_1) \int_{-1}^{x_2} \ell_{p-3} .$$

Using (12) and the linear independence of the Legendre polynomials we get

$$(20) \qquad \sum_{n=1}^{p-3} \alpha_{1n} \int_{-1}^{x_2} \ell_n - \frac{1}{5} \sum_{n=1}^{p-4} \beta_{2n} \ell_n(x_2) = \int_{-1}^{x_2} \ell_{p-3} ,$$

since these are the respective coefficients of $\ell_1(x_1)$. The identity (20) implies

$$\alpha_{1\,p-3} = 1 ,$$

and since

$$\frac{\partial}{\partial x_2} u_1 = \sum_{m,n=1}^{m+n \le p-2} \alpha_{mn} \left(\int_{-1}^{x_1} \ell_m \right) \ell_n(x_2) =$$

$$= \sum_{m,n=1}^{m+n \le p-2} \frac{1}{2m+1} \alpha_{mn} (\ell_{m+1}(x_1) - \ell_{m-1}(x_1)) \ell_n(x_2) ,$$

it follows that

$$(21) \qquad \left\| \frac{\partial}{\partial x_2} u_1 \right\|_{L^2(R)}^2 \ge \frac{1}{9} \alpha_{1\,p-3}^2 \int_{-1}^{1} \ell_0^2 \int_{-1}^{1} \ell_{p-3}^2 = \frac{4}{9(2p-5)} .$$

A simple computation gives

$$\| q^* \|_{L^2(R)}^2 = \frac{8}{3} \frac{1}{(2p-3)(2p-5)(2p-7)} ,$$

and therefore, in light of (21),

$$\|u\|_{H^1(R)}^2 \geq \frac{1}{6}(2p-3)(2p-7)\|q^*\|_{L^2(R)}^2 .$$

Since $\underline{u}$ is an arbitrary field in $(\overset{\circ}{\mathscr{P}}_p)^2$, with $\nabla \cdot \underline{u} = q^*$, this gives the desired lower bound on the operator norm. ∎

We do not know the exact order of $\|(\nabla\cdot)_p^{-1}\|_{\mathscr{B}(L^2, H^1)}$ for any of the two cases covered by Propositions 2 and 3. It is easy to see that one always has $\frac{1}{\sqrt{2}} \leq \|(\nabla\cdot)_p^{-1}\|_{\mathscr{B}(L^2, H^1)}$. With Dirichlet boundary conditions on all of ∂R we can through a direct construction verify that $\|(\nabla\cdot)_p^{-1}\|_{\mathscr{B}(L^2, H^1)} \leq Cp^2$. At the end of this section we provide some numerical results concerning the cases covered by Propositions 2 and 3.

We have a somewhat tighter estimate with Dirichlet boundary conditions on one side only. Let $\tilde{\mathscr{P}}_p^{\,\circ}$ denote the space

$$\tilde{\mathscr{P}}_p^{\,\circ} = \mathscr{P}_p \cap \{u = 0 \text{ at } x_1 = -1\}.$$

For the velocity space $(\tilde{\mathscr{P}}_p^{\,\circ})^2$ and its corresponding pressure space $\nabla \cdot (\tilde{\mathscr{P}}_p^{\,\circ})^2 = \mathscr{P}_{p-1}$ we have

PROPOSITION 4 : *Let* $(\nabla\cdot)_p^{-1} : \mathscr{P}_{p-1} \to (\tilde{\mathscr{P}}_p^{\,\circ})^2$, $p \geq 1$, *denote the right inverse with minimal H^1-norm. Considered as an operator from a subspace of $L^2(R)$ to a subspace of $(H^1(R))^2$ this satisfies*

$$cp \leq \|(\nabla\cdot)_p^{-1}\|_{\mathscr{B}(L^2, H^1)} \leq Cp^{3/2},$$

with $0 < c$ and C independent of p.

Proof : Given $q \in \mathscr{P}_{p-1}$ we have

$$q = \sum_{\substack{m, n \geq 0}}^{m+n<p} \alpha_{mn} \, \ell_m(x_1) \, \ell_n(x_2)$$

for some set of coefficients $\{\alpha_{mn}\}$. Define

$$\underline{u}^{(1)} = \begin{bmatrix} \displaystyle\sum_{\substack{n<m}}^{m+n<p} \alpha_{mn} \left(\int_{-1}^{x_1} \ell_m \right) \ell_n(x_2) \\[2em] \displaystyle\sum_{\substack{m \leq n}}^{m+n<p} \alpha_{mn} \, \ell_m(x_1) \left(\int_{-1}^{x_2} \ell_n \right) \end{bmatrix} .$$

It is clear that $\underline{u}^{(1)} \in (\mathscr{P}_p)^2$ with $\nabla \cdot \underline{u}^{(1)} = q$ in R and $u_1^{(1)} = 0$ at $x_1 = -1$. From the proof of Proposition 2 we get

(22) $$\| \underline{u}^{(1)} \|_{H^1(R)} \leqslant C p \| q \|_{L^2(R)} .$$

Let φ denote the polynomial

$$\varphi = \sum_{n=0}^{p-1} \left[\sum_{m=0}^{M(n,p)} \alpha_{mn} (-1)^{m + M(n,p)} \right] \left(\int_{-1}^{x_1} \ell_{M(n,p)} \right) \left(\int_{-1}^{x_2} \ell_n \right) ,$$

with $M(n, p) = \min(n, p - 1 - n)$, and set

$$\underline{u}^{(2)} = \nabla \times \varphi = \left(\frac{\partial}{\partial x_2} \varphi, - \frac{\partial}{\partial x_1} \varphi \right) \in (\mathscr{P}_p)^2 .$$

Since $\ell_m(-1) = (-1)^m$ (see equation (13)), it follows that

(23) $$u_2^{(2)} \big|_{x_1 = -1} = - \frac{\partial}{\partial x_1} \varphi \big|_{x_1 = -1}$$

$$= - \sum_{n=0}^{p-1} \left[\sum_{m=0}^{M(n,p)} \alpha_{mn} (-1)^m \right] \int_{-1}^{x_2} \ell_n$$

$$= - u_2^{(1)} \big|_{x_1 = -1} .$$

We also have $\qquad u_1^{(2)} \big|_{x_1 = -1} = \frac{\partial}{\partial x_2} \varphi \big|_{x_1 = -1} = 0 ,$

and as a consequence of this and (23), the field

$$\underline{u} = \underline{u}^{(1)} + \underline{u}^{(2)}$$

satisfies $\underline{u} = 0$ at $x_1 = -1$, i.e.,

$$\underline{u} \in (\overset{\circ}{\mathscr{P}}_p)^2 .$$

The field $\underline{u}$ also satisfies

$$\nabla \cdot \underline{u} = q \quad \text{in } R .$$

In order to show that

(24) $$\| \underline{u} \|_{H^1(R)} \leqslant C p^{3/2} \| q \|_{L^2(R)}$$

it suffices, in light of (22), to show that

$$\left\| \frac{\partial}{\partial x_1} \underline{u}^{(2)} \right\|_{L^2(R)} \quad \text{and} \quad \left\| \frac{\partial}{\partial x_2} \underline{u}^{(2)} \right\|_{L^2(R)}$$

are bounded by $C p^{3/2} \| q \|_{L^2(R)}$.

We consider the term

$$\left\| \frac{\partial}{\partial x_2} u^{(2)} \right\|_{L^2(R)}^2 = \left\| \left(\frac{\partial}{\partial x_2} \right)^2 \varphi \right\|_{L^2(R)}^2 + \left\| \frac{\partial^2}{\partial x_1 \partial x_2} \varphi \right\|_{L^2(R)}^2 .$$

Let β_n denote the coefficient

$$\beta_n = \sum_{m=0}^{M(n,p)} \alpha_{mn} (-1)^{m+M(n,p)} .$$

It is easy to see that

$$(25) \qquad \left\| \frac{\partial^2}{\partial x_1 \partial x_2} \varphi \right\|_{L^2(R)}^2 \leq C \sum_{n=0}^{p-1} \beta_n^2 (M(n,p)+1)^{-1}(n+1)^{-1} .$$

Using the formula (12) and the estimate (14) we get

$$(26) \qquad \left\| \left(\frac{\partial}{\partial x_2} \right)^2 \varphi \right\|_{L^2(R)}^2 = \left\| \sum_{n=0}^{p-1} \beta_n \left(\int_{-1}^{x_1} \ell_{M(n,p)} \right) \frac{d}{dx} \ell_n(x_2) \right\|_{L^2(R)}^2$$

$$\leq C \sum_{n=0}^{p-1} \beta_n^2 (M(n,p)+1)^{-3}(n+1)^2 ,$$

since the indices $M(n,p)$, $0 \leq n \leq p-1$, have at most two occurrences of the same fixed value. A combination of (25) and (26) gives

$$(27) \qquad \left\| \frac{\partial}{\partial x_2} u^{(2)} \right\|_{L^2(R)}^2 \leq C \sum_{n=0}^{p-1} \beta_n^2 (M(n,p)+1)^{-3}(n+1)^2 ,$$

since $M(n,p) \leq n$. Schwarz inequality implies that

$$(28) \qquad \beta_n^2 \leq \sum_{m=0}^{M(n,p)} \alpha_{mn}^2 (M(n,p)+1) .$$

Insertion of (28) into (27) now yields

$$\left\| \frac{\partial}{\partial x_2} u^{(2)} \right\|_{L^2(R)}^2 \leq C \sum_{n=0}^{p-1} \sum_{m=0}^{M(n,p)} \alpha_{mn}^2 (M(n,p)+1)^{-2}(n+1)^2$$

$$\leq Cp^3 \sum_{n=0}^{p-1} \sum_{m=0}^{M(n,p)} \alpha_{mn}^2 (m+1)^{-1}(n+1)^{-1}$$

$$= Cp^3 \sum_{\substack{m+n<p \\ m \leq n}} \alpha_{mn}^2 (m+1)^{-1}(n+1)^{-1} ,$$

which again is bounded by $Cp^3 \| q \|_{L^2(R)}^2$ (due to (11) and the orthogonality

of the Legendre polynomials). The term $\left\| \dfrac{\partial}{\partial x_1} \underline{u}^{(2)} \right\|_{L^2(R)}$ may be estimated in the exact same manner, and consequently we have established (24), i.e., we have verified the upper bound on the operator norm.

To verify the lower bound, take

$$q^* = \int_{-1}^{x_2} \ell_{p-2} \, .$$

Using the same argument as in the proof of Proposition 3, but considering the coefficients of $\ell_0(x_1)$, we get that any solution of $\nabla \cdot \underline{u} = q^*$ with $\underline{u} \in (\mathring{\tilde{\mathscr{P}}}_p)^2$ must satisfy

$$\|\underline{u}\|_{H^1(R)} \geq cp \, \|q^*\|_{L^2(R)} \, , \quad c > 0 \, .$$

This establishes the lower bound on the operator norm. ∎

We conclude this section with some computational results concerning the velocity spaces $(\mathscr{P}_p)^2$ and $(\mathring{\tilde{\mathscr{P}}}_p)^2$. As a basis for $\mathscr{P}_p$ in our computations we pick products of integrals of Legendre polynomials (supplemented by the constant function) :

$$(29) \qquad 1, \quad \int_{-1}^{x_1} \ell_m, \int_{-1}^{x_2} \ell_n \quad 0 \leq m, \, n \leq p - 1 \, ,$$

$$\int_{-1}^{x_1} \ell_m \int_{-1}^{x_2} \ell_n \quad 0 \leq m, n \quad \text{and} \quad m + n \leq p - 2 \, .$$

As a basis for $\nabla \cdot (\mathscr{P}_p)^2 = \mathscr{P}_{p-1}$ we pick

$$(30) \qquad \ell_m(x_1) \, \ell_n(x_2) \quad 0 \leq m, n \quad \text{and} \quad m + n \leq p - 1 \, .$$

The top plot in figure 1 shows the smallest eigenvalue of the matrix D (as defined in (7)) for p varying between 1 and 18 when no boundary conditions are imposed on the velocity fields. The eigenvalues were computed using two EISPACK subroutines : first the matrix was transformed (by orthogonal similarity transformations) into a tridiagonal matrix using subroutine TRED2, then the eigenvalues were computed by the QL method (an obvious variant of the QR method) using subroutine TQL2. The $\mathscr{B}(L^2; H^1)$ norm of the « minimal norm » right inverse is the reciprocal square root of the smallest eigenvalue. The numbers do not clearly indicate whether these right inverses are bounded independently of p — if anything they seem to indicate that the norms grow as $p \to \infty$, but only as a very small power or possibly a logarithm of p (the corresponding solid line was computed by linear regression on the last four points, it is proportional to

p $^{0\,3975}$) The linear bound on the $\mathscr{B}(L^2, H^1)$ norm given by Proposition 2 is clearly too conservative to represent the asymptotic rate

When Dirichlet conditions are imposed on all of ∂R, i e when the velocity space is $(\overset{\circ}{\mathscr{P}}_p)^2$, then the corresponding pressure space $\nabla\,(\overset{\circ}{\mathscr{P}}_p)^2 \subseteq \mathscr{P}_{p-1}$ is no longer the entire space $\mathscr{P}_{p-1}$ A simple count of dimensions gives that $\nabla\,(\overset{\circ}{\mathscr{P}}_p)^2$ has co-dimension 9 in $\mathscr{P}_{p-1}$, $p \geqslant 5$ $\nabla\,(\overset{\circ}{\mathscr{P}}_p)^2$ is the common nullspace for the following nine linearly independent functionals   the integral over R, point evaluation at each of the corners of ∂R and $\left(\dfrac{\partial}{\partial n}\right)^{p-2}$ evaluated at the center of each side of ∂R As a basis for $\overset{\circ}{\mathscr{P}}_p$ we take the same elements as in (29) except for those in the first line, and those corresponding to $m = 0$ or $n = 0$ in the second line Instead of computing the matrices A and C by using a basis for $\nabla\,(\overset{\circ}{\mathscr{P}}_p)^2$ we use a basis for $\mathscr{P}_{p-1} \cap \left\{\displaystyle\int_\Omega q = 0\right\}$, the same as in (30) except for the constant function The only effect of this in terms of eigenvalues and eigenvectors is

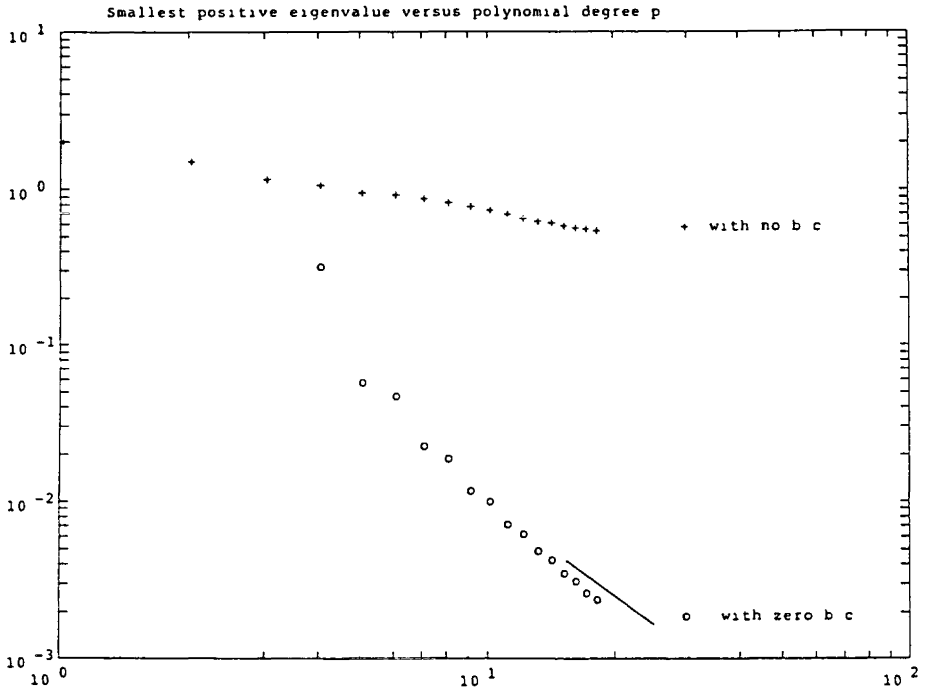Figure 1

to add 0 as an eigenvalue of D with multiplicity 8, $p \geqslant 5$ (in the case $p = 4$, $\nabla \cdot (\mathring{\mathscr{P}}_p)^2$ only has codimension 8 in $\mathscr{P}_{p-1}$, and the corresponding multiplicity of 0 becomes 7). The lower plot in figure 1 shows the smallest positive eigenvalue of D for p varying between 4 and 18 ; the eigenvalues were computed as before, using EISPACK. The $\mathscr{B}(L^2 ; H^1)$ norm of the « minimal norm » right inverse $(\nabla \cdot)_p^{-1} : \nabla \cdot (\mathring{\mathscr{P}}_p)^2 \to (\mathring{\mathscr{P}}_p)^2$ is the reciprocal square root of the smallest positive eigenvalue. The numbers clearly indicate that the norms grow at least linearly p (as they should according to Proposition 3). Based on these numbers it is safe to conjecture that the norms grow no faster than $Cp^{3/2}$ (the solid line near the plot of the eigenvalues is proportional to p^{-2}, corresponding to linear growth of the norms).

Figures 2 and 3 show contour plots of worst possible pressures, in the sense defined at the end of the introduction, for $p = 7$, 8, 14 and 15. These pressures are elements of $\nabla \cdot (\mathring{\mathscr{P}}_p)^2$, and they have the property that the right inverses $(\nabla \cdot)_p^{-1} : \nabla \cdot (\mathring{\mathscr{P}}_p)^2 \to (\mathring{\mathscr{P}}_p)^2$ attain their $\mathscr{B}(L^2 ; H^1)$ norms there. Solid lines in the plots correspond to positive values, dashed lines correspond to negative values. On each plot the interval between contour lines is indicated at the bottom and so is the entire value range of the pressure.

We note several features :

1) There is a marked difference between worst pressures for even and odd p. For even p the smallest positive eigenvalue has multiplicity > 1. The pressure obtained by interchanging the role of x_1 and x_2 in the figures shown for $p = 8$ and 14 are also worst possible. This difference correlates well with the lowest plot in figure 1, which really seems to consist of two slightly different plots, one for even p and one for odd p.

2) The value range for a worst pressure grows as p increases, and the extreme values are clearly attained on ∂R.

3) For p odd and sufficiently large there is a local checkerboard pattern developing, similar to that found in connection with some unstable low order elements (see, e.g., [5] for bilinear-constant velocity-pressure approximation).

3. CONCLUDING REMARKS

We have shown with a few examples that one cannot in general construct maximal right inverses for the divergence operator, whose $\mathscr{B}(L^2 ; H^1)$ norms are uniformly bounded as the polynomial degree increases. We do not think that the square domain is extremely special in this regard, and we
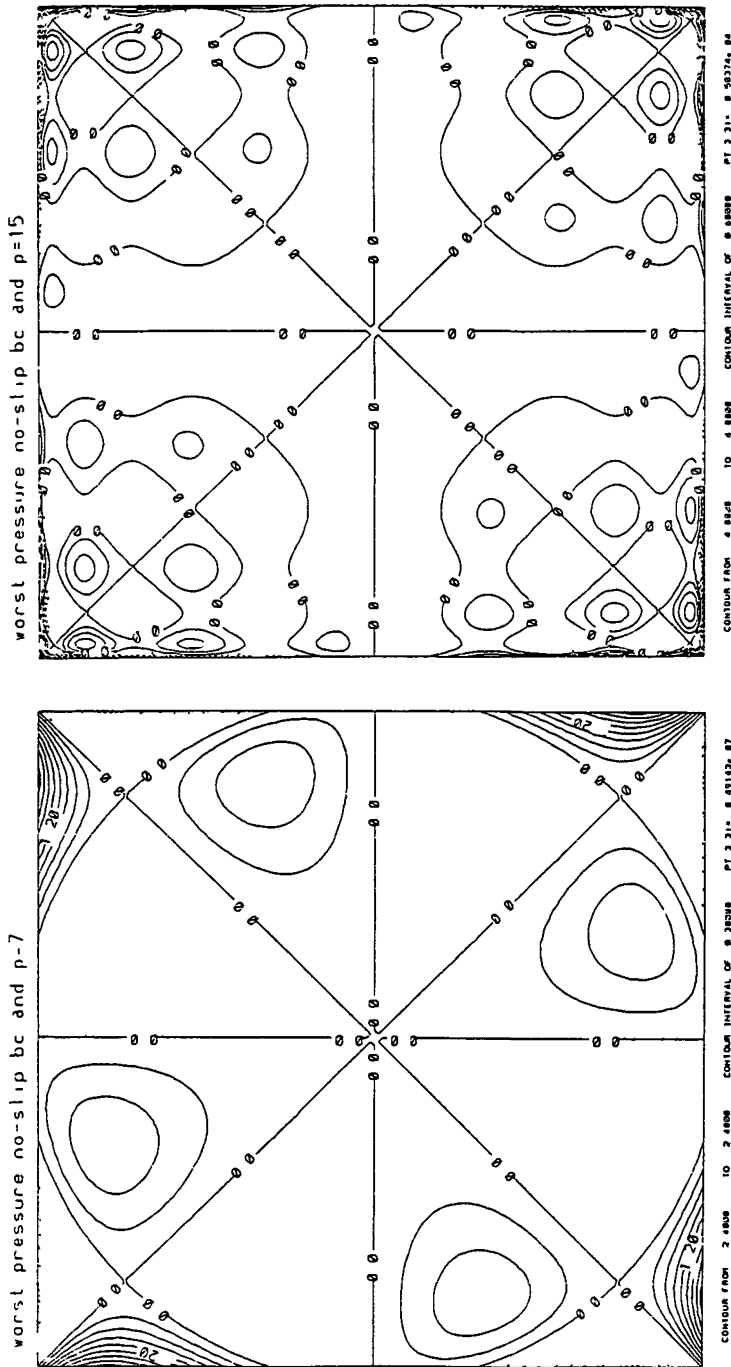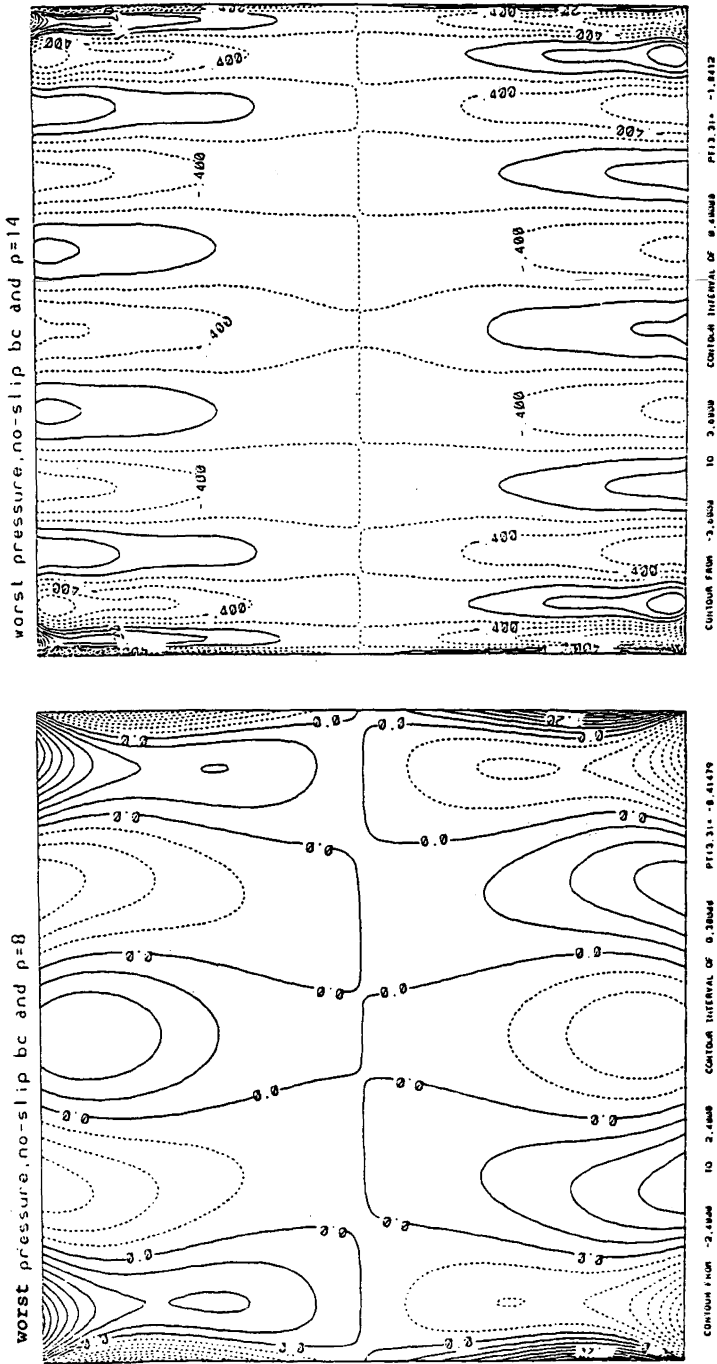
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Figure 2.

Figure 3.

believe that similar examples can be found on many other polygonal domains. The square, however, is extremely convenient since it has a very simple L^2-orthogonal basis for the polynomials. The existence of this basis was most important for establishing the bounds in Propositions 1 through 4.

We do *not* claim that it is impossible to find domains for which uniformly bounded maximal right inverses exist. Indeed one such class of domains (for polynomials of total degree $\leqslant p$, with no boundary conditions) are the ellipses :

Example 3.1 : Let $E = \left\{ (x_1, x_2) : ax_1^2 + bx_2^2 < 1 \right\}$, $0 < a, b$. The Laplace operator $\Delta : \left(\dfrac{\partial}{\partial x_1} \right)^2 + \left(\dfrac{\partial}{\partial x_2} \right)^2$ maps $\mathring{\mathscr{P}}_{p+1}$ into $\mathscr{P}_{p-1}$. The space $\mathring{\mathscr{P}}_{p+1}$, on the domain E, is the same as $(ax_1^2 + bx_2^2 - 1)\,\mathscr{P}_{p-1}$, and since Δ has no nontrivial null vectors with homogeneous Dirichlet boundary conditions, it follows that Δ is an isomorphism from $\mathring{\mathscr{P}}_{p+1}$ onto $\mathscr{P}_{p-1}$. By elliptic estimates

$$\| \nabla \Delta^{-1} q \|_{H^1(E)} \leqslant \| \Delta^{-1} q \|_{H^2(E)} \leqslant C \| q \|_{L^2(E)} \, ,$$

and therefore we conclude that the operator $(\nabla \cdot)_p^{-1} = \nabla \Delta^{-1} : \mathscr{P}_{p-1} \to (\mathscr{P}_p)^2$ is a uniformly bounded maximal right inverse for the divergence operator on the domain E.   ∎

Without boundary conditions, unboundedness of the « minimal norm » right inverse in spaces of entire polynomials (on a square, say) will immediately lead to unboundedness in spaces of piecewise polynomials relative to a fixed partition (a lattice). We expect that the « minimal norm » right inverses for truly piecewise polynomials will inherit some of the (likely) extra unboundedness associated with homogeneous boundary conditions. For example, on a lattice (with more than one rectangle) it would not be totally surprising, if the right inverses corresponding to spaces of piecewise polynomials of separate degree $\leqslant p$ have $\mathscr{B}(L^2 ; H^1)$ norms, that grow faster than p.

Lack of divergence stability as evidenced by the fact that the best lower bound in (5) might behave like $p^{-\alpha}$, $\alpha > 0$, does not lead to suboptimal order of convergence for the velocities, as $p \to \infty$ (provided we use the divergence of the velocity space as the pressure space). The explanation for this is quite standard with stress-free boundary conditions on a simply connected domain (see e.g. [9], [25]). Since $\underline{U}_p$ is a projection of $\underline{U}$ onto $Z_p = (\mathscr{Q}_p)^2 \cap \{ \mathrm{div}\, \underline{v} = 0 \}$ we get

$$\| \underline{U} - \underline{U}_p \|_{H^1(\Omega)} \leqslant C \min_{\underline{v} \in Z_p} \| \underline{U} - \underline{v} \|_{H^1(\Omega)} \, .$$

Since Ω is simply connected

$$(31) \qquad \underline{U} = \nabla \times \Phi , \qquad \int_\Omega \Phi = 0 .$$

We also have

$$\{ \nabla \times \psi : \psi \in \mathcal{Q}_p \} \subseteq Z_p ,$$

and it follows from standard approximation results that

$$\min_{\underline{v} \in Z_p} \| \underline{U} - \underline{v} \|_{H^1(\Omega)} \leq \min_{\psi \in \mathcal{Q}_p} \| \nabla \times \Phi - \nabla \times \psi \|_{H^1(\Omega)}$$

$$\leq \min_{\psi \in \mathcal{Q}_p} \| \Phi - \psi \|_{H^2(\Omega)}$$

$$\leq C p^{-M} \| \Phi \|_{H^{M+2}(\Omega)}$$

$$\leq C p^{-M} \| \underline{U} \|_{H^{M+1}(\Omega)} .$$

Consequently

$$\| \underline{U} - \underline{U}_p \|_{H^1(\Omega)} \leq C p^{-M} \| \underline{U} \|_{H^{M+1}(\Omega)} ,$$

which represents the optimal rate of convergence for general $\underline{U} \in H^{M+1}(\Omega)$. The same argument works with $\mathcal{P}_p$ in place of $\mathcal{Q}_p$. This argument will also work for Dirichlet boundary conditions provided one shows that

$$(32) \qquad \min_{\psi \in \mathcal{Q}_p \cap \mathring{H}^2(\Omega)} \| \Phi - \psi \|_{\mathring{H}^2(\Omega)} \leq C p^{-M} \| \Phi \|_{H^{M+2}(\Omega)}$$

for any $\Phi \in \mathring{H}^2(\Omega) \cap H^{M+2}(\Omega)$. G. Sacchi Landriani states that the estimate (32) is proven for $\Omega = R = (-1, 1)^2$ in [15]. We also refer the reader to [27].

Remark 3.1 : The approximation rate suggested by the estimate

$$(32) \qquad \min_{\underline{v} \in (\mathcal{P}_p)^2} \| \underline{U} - \underline{v} \|_{H^1(\Omega)} \leq C p^{-M} \| \underline{U} \|_{H^{M+1}(\Omega)}$$

is not optimal for typical corner singularities that appear in the solution of elliptic boundary value problems on polygonal domains Ω. It follows from the analysis in [3], that if $\underline{U}$ is of the form

$$(33) \qquad \underline{U} = r^\gamma g (\ln r) \, \underline{\varphi}(\theta) ,$$

where (r, θ) are polar coordinates around a corner on $\partial\Omega$ and the functions g and φ are smooth with g and all its derivatives growing at most polynomially at $-\infty$, then one has

$$\min_{\underline{v} \in (\mathscr{P}_p)^2} \| \underline{U} - \underline{v} \|_{H^1(\Omega)} \leq C_\varepsilon \, p^{-2\gamma + \varepsilon}$$

as $p \uparrow \infty$ (for any $\varepsilon > 0$). This generally represents twice the approximation rate suggested by (32), since r^γ is in $H^{\gamma + 1 - \varepsilon}(\Omega)$, but not in $H^{\gamma + 1}(\Omega)$.

If the right hand side $\underline{F}$ is smooth, then the solution to the Stokes problem (1) is smooth except at the corners of the polygonal domain Ω. In general the corner singularities for the velocity and the pressure are of the form (33) for different γ (cf. [11]). Consequently we get that with a smooth right hand side on a polygonal domain the optimal approximation rates for the velocity and the pressure, as $p \uparrow \infty$, are generally twice those suggested by the regularity of $\underline{U}$ and P in standard Sobolev spaces †.

The function Φ, defined by (31), has a corner singularity of the form

$$\Phi = r^{\gamma + 1} \, \tilde{g}(\ln r) \, \tilde{\varphi}(\theta) \, .$$

In analogy with the $H^1(\Omega)$-estimate we expect that

$$\min_{\psi \in \mathscr{P}_p} \| \Phi - \psi \|_{H^2(\Omega)} \leq C_\varepsilon \, p^{-2\gamma + \varepsilon} \, ,$$

(twice the approximation rate suggested by the regularity of Φ in standard Sobolev spaces). From the analysis preceeding this remark it now follows that we should also generally expect the finite element error in velocity, $\| \underline{U} - \underline{U}_p \|_{H^1(\Omega)}$, to converge at twice the rate suggested by the regularity of $\underline{U}$ in standard Sobolev spaces. ∎

For discretizations of the equations of elasticity in displacement form, divergence stability is intimately connected with uniform convergence rates as Poisson's ratio ν approaches $1/2$. Using an interpolation argument it is possible to prove that the energies computed by the p-version of the finite element method converge with (essentially) optimal rates uniformly in $\nu \in [0, 1/2[$, cf. [23].

Both for the Stokes problem and the equations of elasticity a lack of divergence stability may lead to certain computational difficulties. In [16] it is reported that an increasing number of conjugate gradient steps are required to solve for the pressure for increasing p. In [21] it is found that a certain component of the elastic stress (corresponding to the hydrostatic pressure) is not well approximated using high degree elements and a standard displacement formulation of the equations of elasticity.

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† In the case of Dirichlet boundary conditions it will often normally be necessary to use piecewise polynomials on a fixed subdivision to achieve these rates.

We conclude this paper with a computational example that demonstrates how the lack of divergence stability may lead to a reduced approximation rate for the pressure in the Stokes problem. Consider the « driven cavity » flow problem

(34)
$$- \Delta \underline{U} + \nabla P = 0 \quad \text{in} \quad R = (-1, 1)^2$$
$$\nabla \cdot \underline{U} = 0 \quad \text{in} \quad R$$
$$\underline{U}(x_1, 1) = (1 - x_1^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for} \quad -1 < x_1 < 1 \,,$$
$$\underline{U} = 0 \quad \text{elsewhere on } \partial R \,.$$

We decompose $\underline{U}$ as $\underline{U} = \underline{U}^{(1)} + \underline{U}^{(0)}$ with $\underline{U}^{(1)} \in (\mathring{H}^1(R))^2$ and $\underline{U}^{(0)} = \frac{1}{2} (1 - x_1^2)(1 + x_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We approximate $\underline{U}^{(1)}$ and P through the use of spaces $(\mathring{\mathscr{P}}_p)^2$ and $\mathscr{P}_{p-1}$ respectively :

(35)
$$a(\underline{U}_p^{(1)}, \underline{v}) + b(\underline{v}, P_p) = - a(\underline{U}^{(0)}, \underline{v}) \quad \forall \underline{v} \in (\mathring{\mathscr{P}}_p)^2$$
$$b(\underline{U}_p^{(1)}, q) = - b(\underline{U}^{(0)}, q) \quad \forall q \in \mathscr{P}_{p-1}$$
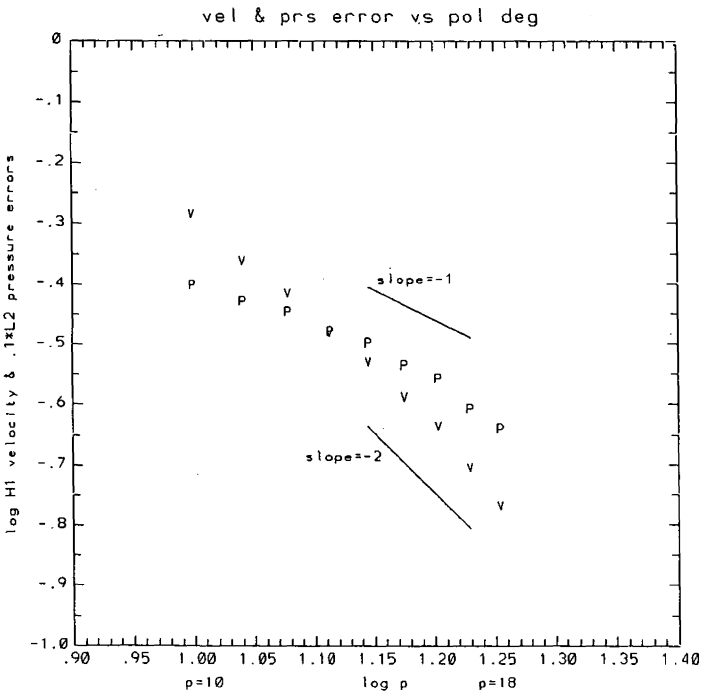

Figure 4. — $\underline{u}(x, 1) = (1 - x^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\underline{U}_p^{(1)} \in (\overset{\circ}{\mathscr{P}}_p)^2$, $P_p \in \mathscr{P}_{p-1}$. The linear system (35) does not have a unique pressure component P_p, nor does it have an exact solution. We get around this difficulty by solving (35) in the least squares sense. In reality this should be very close to using $\nabla \cdot (\overset{\circ}{\mathscr{P}}_p)^2$ for a pressure space. Figure 4 shows the H^1-« error » in velocity and 0.1 times the L^2-« error » in pressure as a function of p, p between 10 and 18 (V signifies velocity, P signifies pressure. The « errors » were calculated using the solution for $p = 20$ in place of the exact solution). The errors are essentially constant for p between 4 and 9, we are not displaying these.

It follows directly from Theorem 6.2 of [1] that the problem

$$\nabla \cdot \underline{V} = 0 \quad \text{in} \quad R$$

$$\underline{V}(x_1, 1) = (1 - x_1^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for} \quad -1 < x_1 < 1$$

$$\underline{V} = 0 \quad \text{elsewhere on } \partial R$$

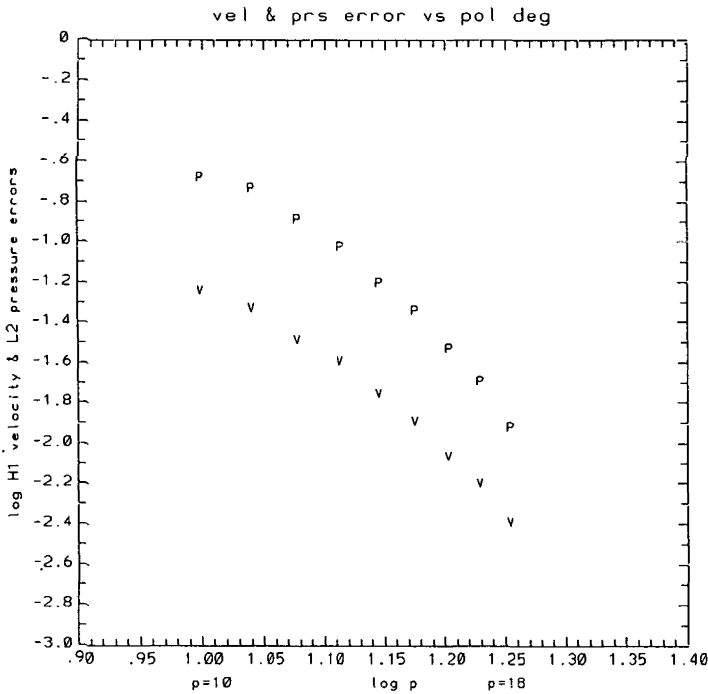has a solution in $(H^{2-\varepsilon}(R))^2$ and that no solution is in $(H^2(R))^2$. From

Figure 5. — $\underline{u}(x, 1) = (1 - x^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

known regularity results for the Stokes problem with homogeneous Dirichlet conditions in convex polygonal domains, *cf.* [13], it now follows that the solution to (34) will have $\underline{U} \in (H^{2-\varepsilon}(R))^2$, $P \in H^{1-\varepsilon}(R)$ for any $\varepsilon > 0$ (but not $\underline{U}$ in $(H^2(R))^2$ or P in $H^1(\Omega)$). The lack of further regularity is due to singularities at the top corners. Extrapolating from the discussion in Remark 3.1 we therefore expect optimal approximation rates for the velocity and the pressure to be $p^{-2+\varepsilon}$, in the H^1-norm and the L^2-norm respectively. As is evidenced by figure 4 the computed velocities converge at the expected optimal rate, whereas there is clearly a reduction in the order of convergence of the computed pressures. The convergence rate appears to have been reduced by at least a factor of *p*. This reduction in convergence rate can in our mind only be attributed to the lack of divergence stability demonstrated in Proposition 3.

We do want to emphasize that a reduction in the convergence rate for the pressures is not always observed. For a more regular solution, such as that obtained for instance by replacing $1 - x_1^2$ in (34) with $(1 - x_1^2)^2$, the convergence rate for the computed pressures shows no evidence of reduction (see figure 5). At this point we have not investigated exactly when a reduction will occur or the important question, what to do about it.

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