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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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ON FINDING THE LARGEST ROOT OF A POLYNOMIAL (*)

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Abstract. — *The problem considered here is to find a good upper bound for the largest modulus of the roots of a given complex polynomial. We propose to first use a few iterations of Graeffe's method and then an upper bound given by Knuth.*

Résumé. — *Le problème considéré ici est la recherche d'un bon majorant pour le plus grand module des racines d'un polynôme donné. Nous proposons d'abord d'appliquer quelques itérations de la méthode de Graeffe puis une borne due à Knuth.*

We consider a polynomial with complex coefficients

$$f(X) = X^k + a_{k-1} X^{k-1} + \cdots + a_0 .$$

The question we wish to solve is « find an R such that all roots of f have absolute value at most R ».

This quantity appears in many bounds in computer algebra, and figures to a very high power in the bounds for factoring polynomials over algebraic number fields (see [6]).

1. CAUCHY'S METHOD AND ITS INHERENT WEAKNESS

Since Cauchy [1] (p. 122), it is known that R can be chosen as the unique positive real root $C(f)$ of the polynomial

$$f^*(X) = X^k - |a_{k-1}| X^{k-1} - \cdots - |a_0| .$$

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Let ρ be the absolute value of the largest root of the polynomial f. It is easy to see that, for any positive real x,

$$f^*(x) \geq 2\, x^k - (x+\rho)^k .$$

Hence $C(f)$ satisfies

$$(*) \qquad \rho \leq C(f) \leq \rho\,(2^{1/k} - 1)^{-1} ,$$

where both inequalities are sharp. The left inequality is an equality when $f = f^*$, whereas the right inequality is an equality when $f(x) = (x+\rho)^k$.

This shows in particular that $C(f)$ may be too big by a factor $\approxeq l/\mathrm{Log}\, 2$.

One would prefer a bound based on the $|a_i|$ which did not require the explicit computation of the root $C(f)$ of f^*. There are many such bounds Cauchy [1], Knuth [5] (ex. 4.6.2 : 20), Dieudonné [3] (p. 66) : all based on an analysis of f^*. Knuth's is

$$(**) \quad R \leq K(f) = 2 \max \left\{ |a_{k-1}|, |a_{k-2}|^{1/2}, |a_{k-3}|^{1/3}, \ldots, |a_0|^{1/k} \right\} ,$$

and can also be found in Henrici [4] (cor. 6.4k, p. 457).

Knuth shows that $K(f) \leq 2\, k\rho$. In our notation this follows from

$$\binom{k}{i}^{1/i} \leq k \quad \text{for} \quad 1 \leq i \leq k .$$

2. THE POWER OF GRAEFFE'S METHOD

In this note we use Graeffe's method to bound the roots of f as closely as we require. We remark that this method was also used in [2] to compute a good upper bound of the measure of a polynomial.

If we apply Graeffe's method to f, we obtain a polynomial f_1 whose roots are the squares of the roots of f. This process can be repeated, to obtain f_n, whose roots are the 2^n-th powers of the roots of the polynomial f. The computation is very easy : suppose that

$$f_i(X) = g_i(X^2) + X h_i(X^2) , \quad i \geq 0 ,$$

then

$$f_{i+1}(X) = g_i^2(X) - X h_i^2(X) .$$

If we apply $(*)$ to the polynomial f_n we get

$$\rho \leq C(f_n)^{2^{-n}} \leq (k/\mathrm{Log}\, 2)^{2^{-n}} \rho ,$$

and the last term tends rapidly to ρ as n increases ($n = \max\ \{3, [\mathrm{Log}\, k]\}$ gives a very accurate upper bound).

As previously remarked, one would prefer an explicit bound. Using (∗∗) instead of (∗), we get the same behaviour : rapid convergence for small n. More precisely, we have

$$\rho \leqslant K(f_n)^{2^{-n}} \leqslant (2\,k)^{2^{-n}}\,\rho\,.$$

Conclusion : A few iterations of Graeffe's root-squaring method followed by an application of Knuth's inequality will give a very tight bound for the absolute value of the roots of a polynomial, with comparatively little effort.

3. AN EXAMPLE

We consider the polynomial

$$P(X) = X^6 + X^5 + 6\,X^4 - 5\,X^3 + 3\,X^2 + 2\,.$$

When we apply Greaffe's method we get successively

$$P_1(X) = X^6 + 11\,X^5 + 52\,X^4 + 15\,X^3 + 33\,X^2 + 12\,X + 4\,,$$

$$P_2(X) = X^6 - 17\,X^5 + 2\,440\,X^4 + 2\,951\,X^3 + 1\,145\,X^2 + 120\,X + 16\,,$$

$$P_3(X) = X^6 + 4\,591\,X^5 + 6\,056\,224\,X^4 - 31\,116\,689\,X^3 + 680\,865\,X^2 + 22\,240\,X + 256\,,$$

$$P_4(X) = X^6 - 8\,964\,833\,X^5 + 36\,706\,467\,938\,304\,X^4 - 1\,467\,012\,622\,369\,X^3 + 602\,308\,261\,633\,X^2 - 146\,014\,720\,X + 65\,536\,.$$

The application of Cauchy's bound to P_4 gives $\rho < 2.771$. Indeed, it is proved in [2] that P has only two complex roots outside of the unit circle, so that $\rho = (M(P))^{1/2} < 2.654$.

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