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DIAGONAL SWAP PROCEDURES AND CHARACTERIZATIONS OF 2D-DELAUNAY TRIANGULATIONS (*)

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Abstract. — *A proof of a result currently used in automated 2D mesh adaptation is proposed. After recalling some basic principles on Voronoï and Delaunay meshes, we prove under some assumptions that any triangulation can be changed in a Delaunay triangulation by a sequence of local reconnections, and we give some characterizations of Delaunay triangulations.*

Résumé. — *Le but de cet article est de fournir une preuve d'un résultat couramment utilisé lors de la mise en œuvre de méthodes de volumes finis ou d'éléments finis en dimension 2 pour l'approximation de certaines équations aux dérivées partielles. Après avoir rappelé quelques résultats sur les maillages de Voronoï et Delaunay, on prouve, sous certaines hypothèses, que toute triangulation d'un polygone peut être modifiée par une suite de procédures de changement de diagonale afin d'obtenir une triangulation de Delaunay. On en déduit plusieurs caractérisations des triangulations de Delaunay.*

INTRODUCTION

Given two edge-adjacent triangles (abc) and (bdc), we call diagonal swap procedure (DS) the change of the triangles $\{(abc), (bdc)\}$ into the brace $\{(abd), (adc)\}$ (see *fig.* 1).

Several strategies can be imagined for using diagonal swap procedures. We call the diagonal swap according to the « empty bowl » test (DSEB) the rule of putting the brace $\{(abd), (adc)\}$ in place of $\{(abc), (bdc)\}$ if and only if d is located within the circumcircle of the triangle (abc) (that is to say if and only if a is located within the circumcircle of the triangle (bdc)). We call the diagonal swap according to the smallest diagonal test

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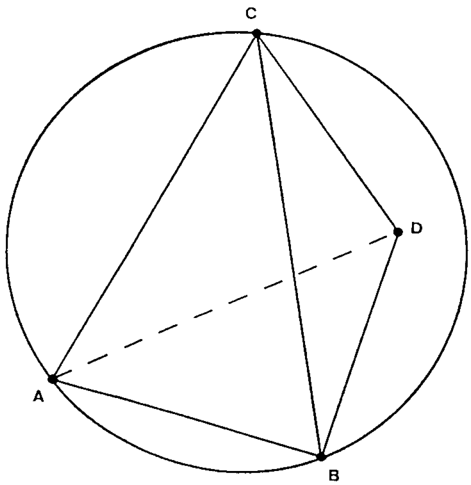


Figure 1.

(DSSD) the rule of putting $\{(abd), (adc)\}$ in place of $\{(abc), (bdc)\}$ if and only if $ad < bc$.

These procedures are often used to improve automatically finite element meshes ; in particular for fluid dynamics computations ([1], [2], [3]). The problem is to know if an algorithm based on these procedures always generates a « good » triangulation, in our particular case a Delaunay triangulation. This is not the case for the DSSD procedure, as it is immediately shown in figure 2 (triangles (abd) and (adc) form a Delaunay triangulation even though $bc < ad$). The DSEB procedure is currently

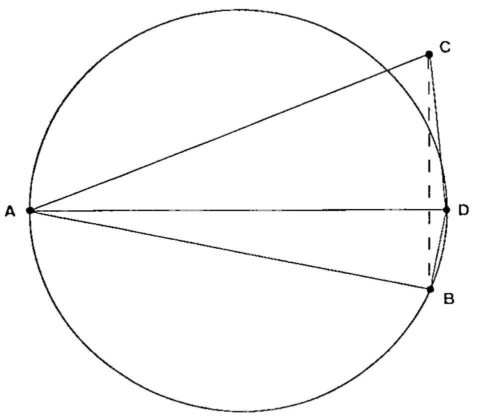


Figure 2.

considered as leading to a Delaunay triangulation. C. L. Lawson proved this result for convex polygons [4]. The purpose of this paper is to present a proof of this result in a more general case. Especially we prove that Delaunay triangulations minimize the sum of the circumbowl's radii — which gives a new characterization of these triangulations. As a consequence we also prove that an acute triangulation is necessary single.

1. VORONOÏ MESHES AND DELAUNAY TRIANGULATIONS

We begin by recalling some definitions and results on Voronoï and Delaunay meshes ([5], [6], [7], [8]). In the following a « bowl » will be an open bowl.

Given n distinct points $x_1 \dots x_n$ of the plane, the set

$$V_i = \{x/\forall j, 1 \leq j \leq n, \quad d(x, x_i) \leq d(x, x_j)\},$$

is called the Voronoï polygon associated to x_i. Voronoï polygons are convex, contain interior points and form a mesh V, called Voronoï mesh, of the whole plane. If the points $x_1 \dots x_n$ are not on a same line, every Voronoï polygon has at least a vertex s; any vertex s can then be associated to the convex hull C_s of the points among $x_1 \dots x_n$ whose Voronoï polygon admits s as a vertex. It is assumed that C_s is a polytope (i.e. a compact and convex set without interior point) called Delaunay polytope, inscribed in a bowl centered on s that does not include any of the points $x_1 \dots x_n$. The set of those polytopes forms a mesh of the convex hull of the points $x_1 \dots x_n$ ([5], [6]). According to Delaunay's terminology, the mesh D will be called « special » if at least one element of D is not a triangle. Dividing these non simplicial elements into triangles gives then a triangulation of the convex hull of $x_1 \dots x_n$ from D. Generally speaking, we call Delaunay triangulation associated to the points $x_1 \dots x_n$ the triangulation D, or any triangulation deduced from D if D is « special ».

Figure 3*a* (extracted from [8]) illustrates a non special Delaunay mesh ; an example of special Delaunay mesh is shown in figure 3*b* ; the polytopes $x_2x_3x_7x_8$ and $x_3x_4x_5x_6x_7$ are indeed inscribed in circles.

Given a bounded polygon Ω having interior points and a set P of points of Ω containing at least the extremal points of Ω, a mesh M of Ω is said « adapted to P » if P is the set of the vertices of M. Ω is said « Delaunay-admissible » if there exists a mesh of Ω adapted to P and enclosed in the Delaunay mesh of the convex hull of Ω.

To give a characterization of a Delaunay-admissible polygon, the notion of « singular face » must be introduced ([7], [8]). Given a face $F = [x_i, x_j]$ of polygon Ω, line $[x_i, x_j]$ divides the space in two opened half planes E_1 and E_2. Since the elements of P are not on the same line, at least one of

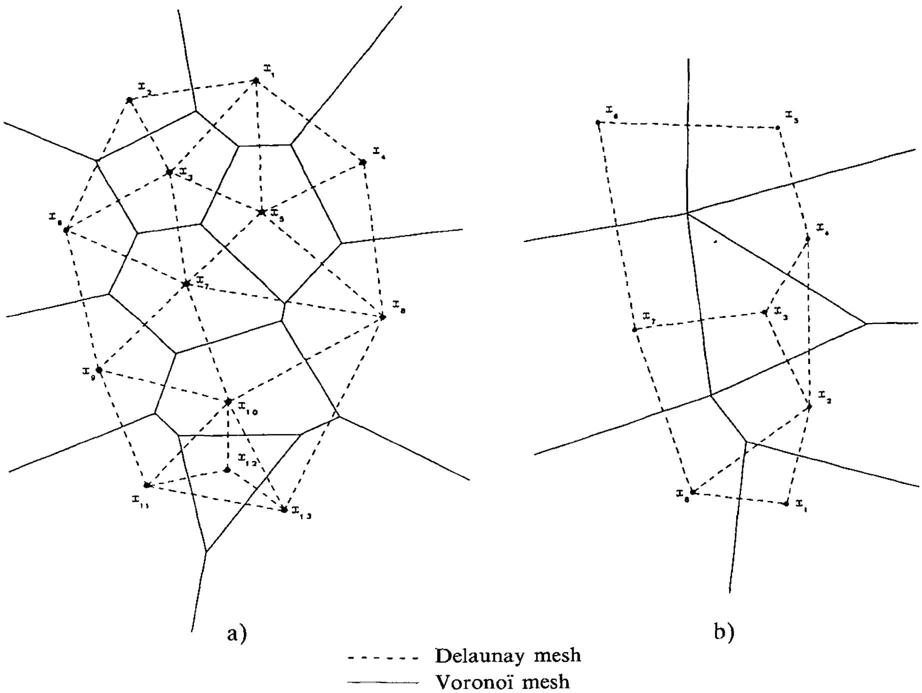


Figure 3.

the two sets $P \cap E_1$ and $P \cap E_2$ is not empty. Supposing for example $P \cap E_1$ is not empty, we can then find an element x_k of $P \cap E_1$ such that the bowl B circumscribed about triangle $(x_i x_j x_k)$ contains no point of $P \cap E_1$. F is said to be a singular face if :

$$\bar{B} \cap P \cap E_2 \neq \varnothing .$$

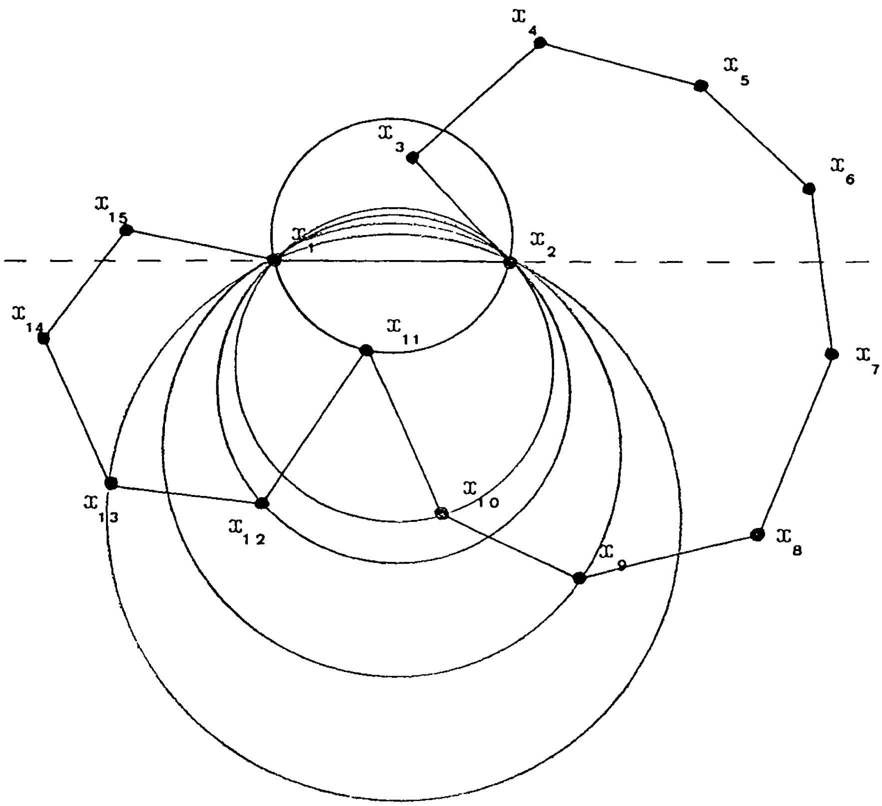
This is equivalent to say that F is singular if there exists a point x_k in $P \cap E_1$ such that the bowl P circumscribed about triangle $(x_i x_j x_k)$ satisfies the relation :

$$\bar{B} \cap P \cap E_2 \neq \varnothing .$$

An example of singular face is shown in figure 4 (extracted from [8]).

It is assumed that Ω is Delaunay admissible for P if and only if Ω has no singular face [7].

A polytope Ω is clearly Delaunay-admissible for every set of points including the extremal points of Ω. Assume Ω is not convex. By adding points on the singular faces of Ω, it is always possible to find a set of points P' including P for which Ω is Delaunay-admissible [7].



$P = \{x_1, ..., x_{15}\}$
$F = [x_1, x_2]$
$x_k = x_3$

Figure 4.

Given a triangulation T of Ω adapted to a set of points P, T is said acute if all the interior angles of the triangle of T are strictly inferior to $\pi/2$.

Given a polygon Ω and a set of points P including the extremal points of Ω, $\tau = \{T_1, ..., T_i, ..., T_m\}$ would denote the set of the triangulations of Ω adapted to P; if t_i^j is a triangle of T_i, $b(t_i^j)$ will denote the bowl circumscribed about t_i^j, $\bar{b}(t_i^j)$ its closure and $r(t_i^j)$ its radius.

2. TWO PRELIMINARY LEMMAS

LEMMA 1 : *Given two edge-adjacent triangles $t_1 = abc$ and $t_2 = bdc$ so that $d \in b(t_1)$, $t_3 = abd$ and $t_4 = adc$ are non degenerate triangles with positive* *areas. Let r_1, r_2, r_3, r_4 (resp. ω_1, ω_2, ω_3, ω_4) denote the radii (resp. centers) of the circumcircles of t_1, t_2, t_3, t_4 (cf. fig. 5), the following relations are satisfied :*

$$\sup(r_3, r_4) < \sup(r_1, r_2),$$

and :

$$r_3 + r_4 < r_1 + r_2 .$$

Proof : Show at first that :

$$(r_3 < r_1) \quad \text{or} \quad (r_3 < r_2) . \tag{1}$$

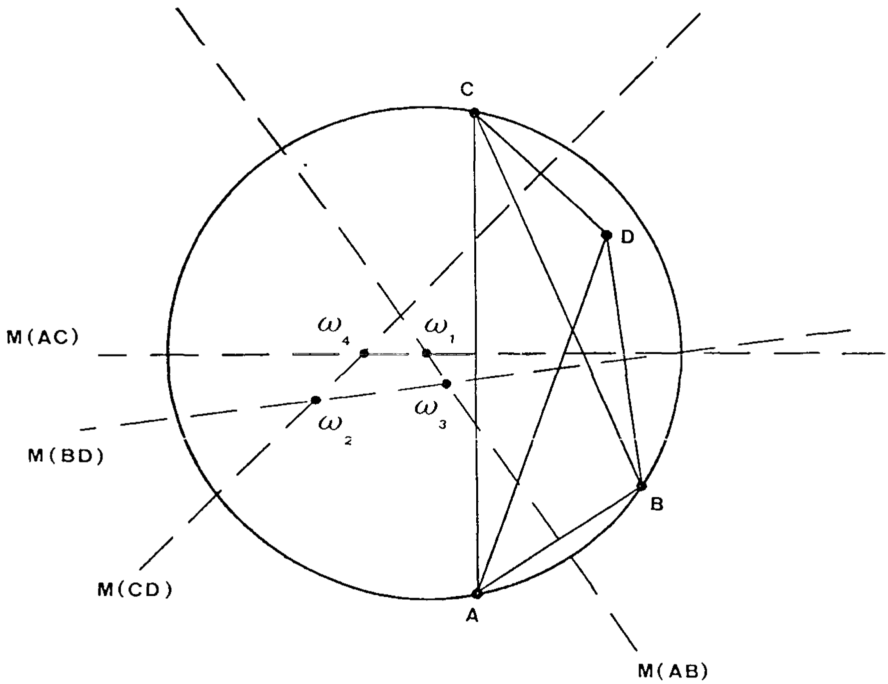
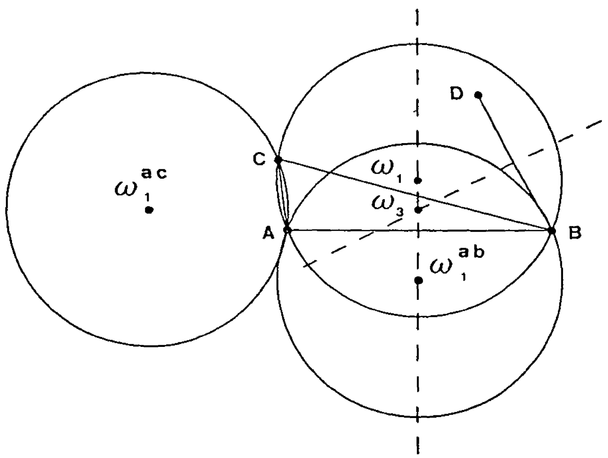


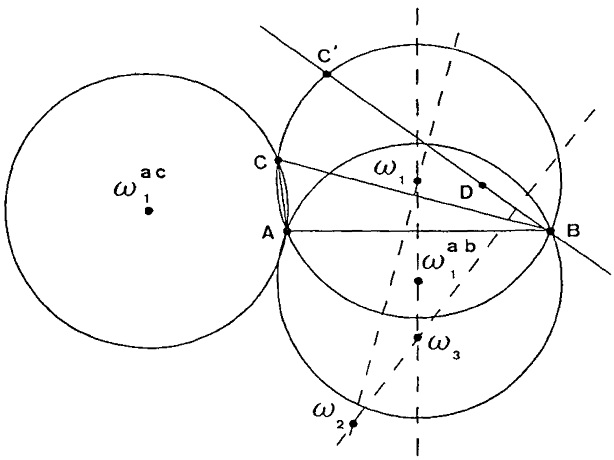
Figure 5.

Denote by $b^{ab}(t_1)$ the bowl symmetrical of $b(t)$ about ab, and by ω_1^{ab} the center of this bowl. We have therefore to consider two cases (*cf. fig. 6a* and *fig. 6b*).

a) If $d \notin \bar{b}^{ab}(t_1)$, the locus of the center ω_3 of bowl $b(t_3)$ when d describes $b(t_1)/\bar{b}^{ab}(t_1)$ is segment $]\omega_1, \omega_a^{ab}[$; consequently $r_3 < r_1$.



a)



b)

Figure 6.

b) If $d \in \bar{b}^{ab}(t_1)$ let c' be the intersection of bd with the boundary of bowl $b(t_1)$. Since bc separates a and d, point c describes arc of circle $\widehat{ac}'$ which is on $b(t_1)$, and $c \neq a$, c'. When c describes $\widehat{ac}'$, the locus of center ω_2 of $b(t_2)$ is half line $]\infty, \omega_3[$ enclosed in mediator of $[bd]$ and therefore $r_3 < r_2$.

In the same way we have for edge ac :

$$(r_4 < r_1) \quad \text{or} \quad (r_4 < r_2) \,. \tag{2}$$

Relations (1) and (2) give us the first inequality of Lemma 1.

Show now that :

$$(r_1 \leq r_3 \Rightarrow r_4 < r_1) \quad \text{and} \quad (r_1 \leq r_4 \Rightarrow r_3 < r_1). \tag{3}$$

Denote by $b^{ac}(t_1)$ the bowl symmetrical of $b(t_1)$ about ac. If $d \in b(t_1) \cap \bar{b}^{ab}(t_1) \cap \bar{b}^{ac}(t_1)$ then angles $\widehat{adb}$ and $\widehat{adc}$ are greater than $\pi/2$, which is inconsistent with the hypothesis that quadrilatere $abdc$ is strictly convex.

It follows that if $d \in \bar{b}^{ab}(t_1)$ (resp. $d \in \bar{b}^{ac}(t_1)$) i.e. if $r_1 \leq r_3$ (resp. $r_1 \leq r_4$) then $d \notin \bar{b}^{ac}(t_1)$ (resp. $d \notin \bar{b}^{ab}(t_1)$), that is to say $r_4 < r_1$ (resp. $r_3 < r_1$).

On the same way, one can check by inverting points a and d that :

$$(r_2 \leq r_3 \Rightarrow r_4 < r_2) \quad \text{and} \quad (r_2 \leq r_4 \Rightarrow r_3 < r_2). \tag{4}$$

From (1), (2), (3), (4) we get the second inequality of Lemma 1. ■

LEMMA 2 : *Let Ω be a polygon, P a set of points including the extremal points of Ω and T a triangulation of Ω adapted to P so that the bowl circumscribed about at least one element of T contains a point of P. If $t_1 \dots t_n$ denote the elements of T satisfying this property, suppose t_1 is associated with the greatest circumcircle's radius :*

$$\forall j, \quad 1 \leq j \leq n, \quad r(t_j) \leq r(t_1).$$

Then, on the assumption that Ω is Delaunay admissible for P, $b(t_1)$ contains a vertex of one of the triangles adjacent to t_1.

Proof: Put $t_1 = abc$; let x be a point of P belonging to $b(t_1)$. One of the lines ab, ac or bc, bc for example, separates x and t_1. Given that Ω is Delaunay admissible for P, bc can't be a boundary face. It may be seen that we can find a point $d \in P$ like $t_j = bdc \in T$ (*fig.* 7).

If $d \notin b(t_1)$, it is very easy to check, by similar argument as in Lemma 1, that :

$$r(t_1) \leq r(t_j) \quad \text{and} \quad x \in b(t_j).$$

The case $r(t_1) < r(t_j)$ would be inconsistent with the hypothesis. Hence $r(t_1) = r(t_j)$ and we can resume the argument, fulfilling in addition the condition $d(t_j, x) < d(t_i, x)$. It may be seen that we get $x \in b(t_j)$ after a finite number of times. ■

Suppose Ω is Delaunay-admissible for P. A Delaunay triangulation T of Ω, adapted to P, is known to be characterized by the fact that the circumbowl of any element of T does not contain any point of P ([7]) :

$$\forall t \in T, \quad b(t) \cap P = \emptyset.$$

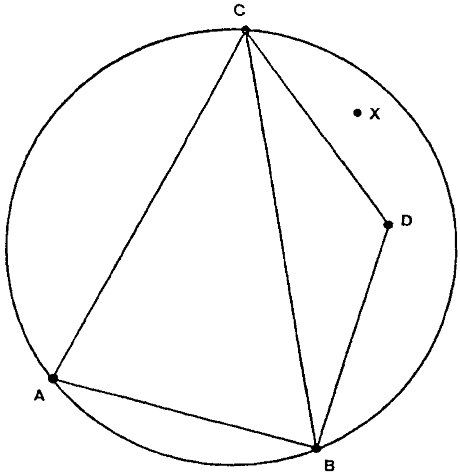


Figure 7.

Lemma 2 gives then the following corollary which yields a more stringent characterization of Delaunay triangulations. A proof of this result is proposed in [9] for n-dimensional polytopes.

COROLLARY 1 : *Given a polygon Ω Delaunay admissible for P, a triangulation T of Ω adapted to P is a Delaunay triangulation if and only if for any $t \in T$ the bowl circumscribed about t does not include any vertex of the triangles adjacent to t.*

Proof: If the triangulation T is not a Delaunay triangulation the bowl circumscribed to at least one element t of T includes a point belonging to P. It follows from Lemma 2 that the circumbowl of t contains at least one vertex of the triangles adjacent to t, which is inconsistent with the hypothesis. ■

Lemma 2 gives also the following characterization of acute triangulations.

COROLLARY 2 : *Let Ω be a Delaunay admissible polygon for P. If there exists an acute triangulation T of Ω adapted to P, then P is not a special family of points, and T is the only Delaunay triangulation of Ω adapted to P.*

Proof: If T is not a Delaunay triangulation, we can find an element t of T whose circumbowl contains a vertex of a triangle t' adjacent to t. By elementary geometric criteria, we find that at least one of triangles t and t' has an interior angle greater than $\pi/2$, which is inconsistent with the hypothesis that T is acute.

Suppose T is a special triangulation. There is a Delaunay polytope d

which is not a simplex. Then d is inscribed in a circle, so any triangulation of d has an angle greater than $\pi/2$. Therefore we can assume that P is not special and T is consequently the only Delaunay triangulation of Ω adapted to P. ■

3. THEOREM

Theorem below proofs particularly that DSEB procedure applied a suffisant number of times to the triangles gives a Delaunay triangulation. We will deduct then a third characterization of Delaunay triangulations.

THEOREM : *Let P be a set of points and let Ω be a Delaunay admissible polygon for P. Then any triangulation T_i adapted to P can be changed to a Delaunay triangulation T_j adapted to P by a sequence of* DSEB *procedures.*

Proof: Let $t_1, \dots, t_n$ be the T_i elements whose circumbowl contains a point of P and among them $t_{r_i} \dots t_{n_i}$ whose circumbowl's radius r_i is the greatest. The couple (r_i, n_i) may be associated to T_i. By agreement, if there is no bowl circumscribed about an element of T_i that encloses a point of P (that is to say if T_i is a Delaunay triangulation), we set $r_i = 0$ and $n_i = 0$. We can define a pre-order relation (i.e. a reflexive and transitive relation) on the set τ of the triangulations of Ω adapted to P by :

$$T_i < T_j \Leftrightarrow [(r_i < r_j) \text{ or } (r_i = r_j \text{ and } n_i \leqslant n_j)] .$$

We get an equivalence relation R on τ by :

$$T_i \, R \, T_j \Leftrightarrow [T_i < T_j \text{ and } T_j < T_i] .$$

Let $\dot{T}$ denote the R-equivalence set for a triangulation T of Ω adapted to P. We obtain therefore a total order relation $\leqslant$ on τ/R by :

$$\dot{T}_i \leqslant \dot{T}_j \Leftrightarrow T_i < T_j .$$

The lower bound of τ/R for this order relation is the set of the Delaunay triangulations of Ω (which is a singleton if P is not special).

Let $T_i \in \tau$ be a triangulation of which an element $t = abc$ admits a circumbowl including at least a point x of P. From Lemma 2 the circumbowl of t contains a vertex of one of the triangles adjacent to t that belongs not to t (*cf. fig.* 7).

Lemma 1 yields therefore that applying procedure DSEB gives a triangulation $T_j \in \tau$ such that :

$$\dot{T}_j < \dot{T}_i .$$

We can thereby build a strictly decreasing sequence of τ/R, and consequently reach the lower bound of τ/R, that is the set of the Delaunay triangulations. ■

COROLLARY 1 : *Let Ω be a Delaunay admissible polygon and T a triangulation adapted to P. $S(T)$ will denote the radii sum of the bowls Circumscribed about elements of T. The Delaunay triangulations of Ω adapted to P are those which minimize $S(T)$ when T describes the set τ of the triangulations of Ω adapted to P.*

Proof: Given $T \in \tau$ which is not a Delaunay triangulation, we can find an element $t \in T$ whose circumbowl contains a point of P. According to the previous theorem a Delaunay triangulation $T_D \in \tau$ can be built from T by a sequence of procedures DSEB. Lemma 1 says that a procedure DSEB yields $S(T)$ strictly lower ; consequently $S(T_D) \leq S(T)$. ■

It may be seen that we can define another pre-order relation by :

$$T_i < T_j \Leftrightarrow S(T_i) \leq S(T_j) .$$

It follows from Corollary 1 that the associated order relation admits as lower bound the set of the Delaunay triangulations (see [10]).

COROLLARY 2 : *Let Ω be a Delaunay admissible polygon for a set of points P and τ the set of the triangulations of Ω adapted to P. A triangulation $T \in \tau$ may be changed to any triangulation $T' \in \tau$ by a sequence of* DS *procedures.*

Proof: From T we can get a Delaunay triangulation T_D by a sequence of DSEB procedures. In the same way T' may be modified in order to get a Delaunay triangulation T'_D.

It is obvious that triangulation T_D can be changed to T'_D by a sequence of DS procedures. This can be stated as follows :

$$T \xrightarrow{\text{DSEB}} T_D \overset{\text{DS}}{\leftrightarrow} T'_D \xrightarrow{\text{DSEB}^{-1}} T' ,$$

and proves the expected result. ■

A demonstration of this corollary in the case of a general polygon is proposed in [11].

4. EXTENSION TO 3D

Generalization of theorem of chapter 3 in the 3 dimensional case is an interesting problem although not an obvious one. We must indeed put in place of diagonal swap procedure according to the empty bowl a procedure adapted to 3D-case. Given five points *a, b, c, d, e* so that line *de* cuts triangle *abc* (*cf. fig.* 8), the following procedure may be defined :

a) if $e \in b(abcd)$ (or $d \in b(abec)$) tetrahedra *abcd, bcde* and *acde* are put in place of tetrahedra *abcd* and *aecb* ;

b) if $c \in b(abde)$ (or $a \in b(bcde)$, or $b \in b(acde)$), tetrahedra *abcd* and *abec* are put in place of tetrahedra *abde, bcde* and *acde* ;

c) if line *de* cuts *ab* (or *bc*, or *ac*), tetrahedron *abde* (or *bcde*, or *acde*) disappears and two solutions are possible : either change *abcd* and *aecb* in *bcde* and *acde*, either change *bcde* and *acde* in *abcd* and *aecb*.

The capability of changing any triangulation of Ω adapted to P in a Delaunay triangulation is an opened problem.

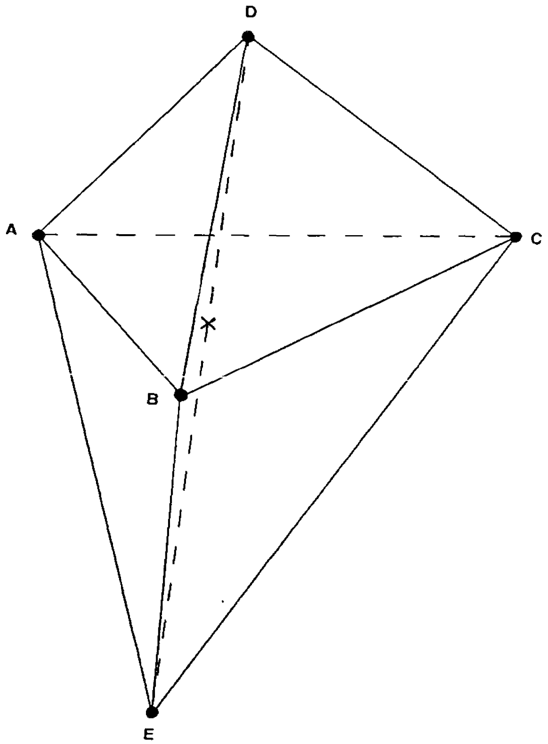


Figure 8.

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