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THE p-VERSION OF THE FINITE ELEMENT METHOD FOR ELLIPTIC EQUATIONS OF ORDER $2l$ (*)

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Abstract. — *The approximation of solutions of elliptic problems of order* 2ℓ *over two-dimensional polygonal domains by the p-version of the finite element method is investigated. Optimal rates of convergence are established for the case when elements possessing* $C^{\ell-1}$ *continuity are used.*

Résumé. — *L'approximation des solutions de problèmes elliptiques d'ordre* 2ℓ *dans un domaine à deux dimensions est traitée par la version* $-p$ *de la méthode des éléments finis. On obtient une majoration optimale de l'erreur en utilisant les éléments qui sont* $C^{\ell-1}$ *continus.*

1. INTRODUCTION

The finite element method has three versions : the *h-version,* the *p-version* and the *h-p version*. In the h-version, increased accuracy is achieved by decreasing the mesh size h while keeping p, the degree of elements used fixed (usually $p = 1, 2, 3$). In the p-version, a fixed mesh is used while the degrees p of elements are either uniformly or selectively increased to achieve accuracy. The h-p version is a combination of both.

The standard h-version has been thoroughly investigated and many commercial and research programs are available. In particular, the solution of elliptic problems of order 2ℓ using elements that are $C^{\ell-1}$ continuous has been analyzed and optimal convergence rates established (see, for e.g. [8] and the references therein).

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The p- and h-p versions are recent developments. There is only one commercial code, the system PROBE (Noetic Tech, St.-Louis) and the first papers discussing theoretical aspects appeared only in 1981 ([6], [2]). For an account of today's state of the art, see [1]. See also [3], [4], [5], [9], [10], [12], [13].

In [6], a second-order model elliptic problem was considered over a bounded, two-dimensional polygonal domain. It was shown that if C^0 elements belonging to H_0^1 were used, then the rate of convergence was optimal up to an arbitrarily small $\varepsilon > 0$,

$$\|e\|_{H^1} \leq C(\varepsilon) p^{-(k-1)+\varepsilon} \|u\|_{H^k}. \tag{1.1}$$

The dependence upon ε was removed in [3] to yield an optimal convergence rate for C^0 elements, namely

$$\|e\|_{H^1} \leq C p^{-(k-1)} \|u\|_{H^k}. \tag{1.2}$$

Several higher order problems occur in engineering however, like problems of plates and shells ([19], [20]) for which elements possessing a greater amount of continuity than C^0 elements are required. To this end, the results of [6] were extended in [16] to C^1 elements to obtain the following rate of convergence :

$$\|e\|_{H^2} \leq C(\varepsilon) p^{-(k-2)+\varepsilon} \|u\|_{H^k}. \tag{1.3}$$

Unfortunately, for the case where the solution lies in $H^k \cap H_0^2$, the proof in [16] is predicated on an interpolation assumption ([16], eq. (2.37)) without which (1.3) does not hold. This assumption is stated by us in Section 4, Remark 4.2. Moreover, the proof of [16] indicates that the term $C(\varepsilon)$ can grow quickly with $\varepsilon \to 0$. Nevertheless, computational experience indicates that (1.3) holds without the term ε.

In [9], an alternate approach is used to show that for $C^{\ell-1}$ elements, $\ell \geq 1$,

$$\|e\|_{H^\ell} \leq C(\varepsilon) p^{-(k-\ell)+\varepsilon} \|u\|_{H^k}. \tag{1.4}$$

The proof of (1.4) does not specifically deal with the case of boundary conditions. In order to extend (1.4) to conforming $C^{\ell-1}$ elements with $\ell - 1$ vanishing normal derivatives on the boundary, an assumption similar to the one in [16] is used implicitly in the proof of [9], Theorem 3.4.

In this paper, we investigate the approximation of functions in $H^k \cap H_0^\ell$ by the p-version, using polynomials in $C^{\ell-1} \cap H_0^\ell$. For $k > 2\ell - \frac{1}{2}$, we obtain the optimal approximation result

$$\|e\|_{H^\ell} \leq C p^{-(k-\ell)} \|u\|_{H^k}. \tag{1.5}$$

In proving (1.5), we do not use the interpolation assumption used in [9], [16]. The use of this assumption, however, allows us to extend (1.5) to the case $\ell < k \leqslant 2\ell - \frac{1}{2}$.

In the case for which the solution exhibits singular behaviour of the type $u \approx r^{\alpha}$ ((r, θ) being polar coordinates) and the vertex of the elements is at the origin, we obtain the optimal estimate

$$\|e\|_{H^{\ell}} \leqslant C p^{-2(\alpha - \ell + 1)}. \tag{1.6}$$

This improves upon the rate of convergence found in [16] (and [9]-[10]) which is optimal up to an arbitrarily small $\varepsilon > 0$.

In Section 2, we describe the notation used and our model problem. Section 3 deals with approximation properties of polynomials on the square. In Section 4, we analyze the rate of approximation for functions in $H^k \cap H_0^{\ell}$ and prove (1.5). Section 5 deals with the case when the solution exhibits singular behaviour and proves (1.6). In Section 6, we summarize our results and briefly address some generalizations.

2. PRELIMINARIES

2.1. Notation

Let R^2 be the usual Euclidean space with $x = (x_1, x_2) \in R^2$. $\Omega \subset R^2$ will denote a bounded polygonal domain with vertices A_i, $i = 0, 1, \ldots, M$, $A_0 = A_M$ and with boundary $\Gamma = \sum_{i=1}^{M} \bar{\Gamma}_i$ where Γ_i are open straight lines with end points A_{i-1}, A_i. The internal angle between Γ_i and Γ_{i+1} will be denoted by ω_i, $i = 1, \ldots, M$, $0 < \omega_i \leqslant 2\pi$. The case $\omega_i = 2\pi$ results in a slit domain for which the boundary is two-sided (in an obvious sense).

By $L_2(\Omega) = H^0(\Omega)$ and $H^k(\Omega)$, $k > 0$ we denote the standard Sobolev spaces (with index 2). Also, $H_0^k(\Omega)$ denotes the subspace of functions with $k - 1$ vanishing normal derivatives on Γ. For $k > 0$ note an integer, we define $H^k(\Omega)$, $H_0^k(\Omega)$ as the usual interpolation spaces (by the K-method, see [7]) :

$$H^{\ell+\theta}(\Omega) = (H^{\ell}(\Omega), H^{\ell+1}(\Omega))_{\theta, q}$$

where $\ell + \theta = k$, $0 < \theta < 1$, $q = 2$. For $k > 1$, we define $H_0^{k+\theta}(\Omega) = H^{k+\theta}(\Omega) \cap H_0^k(\Omega)$. We will also deal with the Sobolev spaces $H^k(\Gamma_i)$, $H^k(I)$, $I = (a, b)$ which are defined for k integer in an analogous way. The spaces $H^k(\Omega)$, $H_0^k(\Omega)$, $H^k(\Gamma_i)$, etc. are Hilbert spaces and their inner products will be denoted by $(.,.)_{H^k(\Omega)}$, etc.

For $\kappa > 0$, we let

$$R(\kappa) = \{(x_1, x_2) \mid |x_1| < \kappa, |x_2| < \kappa\} . \tag{2.1}$$

$H_{PER}^k(R(\kappa)) \subset H^k(R(\kappa))$ will denote the space of all periodic functions with period 2κ.

By $\mathscr{P}_p^1(\Omega)$, respectively $\mathscr{T}_p^1(R(\kappa))$, we denote the space of all algebraic, respectively trigonometric (with period 2κ), polynomials of *total* degree at most p on Ω, respectively $R(\kappa)$. By $\mathscr{P}_p^2(\Omega)$, respectively $\mathscr{T}_p^2(R(\kappa))$, we denote the space of all algebraic, respectively trigonometric polynomials of degree at most p in each variable on Ω, respectively $R(\kappa)$. We also define $\mathscr{P}_p(\Gamma_i)$, $\mathscr{P}_p(I)$ $(I = (a, b))$ to be the space of all algebraic polynomials of degree at most p on Γ_i, respectively I.

Let $\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i$ where Ω_i are (open) triangles or parallelograms. We shall assume that $\Omega_i \cap \Omega_j = \phi$ for $i \neq j$ and $\bar{\Omega}_i \cap \bar{\Omega}_j$ is either the empty set or an entire side or a vertex common to Ω_i and Ω_j. We will assume that all vertices of Ω are the vertices of some Ω_j. Ω_j will be called elements.

Let $Q = (-1, 1) \times (-1, 1)$ and $T = \{(x_1, x_2) \mid |x_1| < 1, -1 < x_2 < x_1\}$ denote the standard square and standard triangle respectively. Let F_i be an affine mapping with Jacobian having positive determinant which maps Ω_i onto Q if Ω_i is a parallelogram and onto T if Ω_i is a triangle. Let now $S_p(\Omega) \subset L^2(\Omega)$ denote the set of functions u such that if u_{Ω_i} denotes the restriction of u to Ω_i then $u_{\Omega_i} \circ (F_i)^{-1} \in \mathscr{P}_p^2(Q)$ if Ω_i is a parallelogram and $u_{\Omega_i} \circ (F_i)^{-1} \in \mathscr{P}_p^1(T)$ if Ω_i is a triangle. We will then write $u_{\Omega_i} \in \mathscr{P}_p(\Omega_i)$ and $u \in S_p(\Omega)$. Furthermore, we denote for $\ell \geq 1$ integer,

$$S_p^\ell(\Omega) = H^\ell(\Omega) \cap S_p(\Omega)$$
$$_0S_p^\ell(\Omega) = S_p^\ell(\Omega) \cap H_0^\ell(\Omega) .$$

It is possible to show that $S_p^\ell \subset C^{(\ell-1)}(\Omega)$ and $_0S_p^\ell \subset C_0^{(\ell-1)}(\bar{\Omega})$, where $C^{(\ell)}(\bar{\Omega})$ is the space of all functions with ℓ continuous derivatives on $\bar{\Omega}$ and $C_0^{(\ell)}(\bar{\Omega}) \subset C^{(\ell)}(\bar{\Omega})$ the subspace of functions for which derivatives of order up to ℓ vanish on Γ.

For $r = (r_1, r_2)$, $0 \leq r_1, r_2 \leq |r| = r_1 + r_2$ and u a function defined on Ω, $D^{(r)} u$ will denote the derivative

$$D^{(r)} u = \frac{\partial^{|r|} u}{\partial x_1^{r_1} \partial x_2^{r_2}} .$$

2.2. The model problem and its properties

Let L be an operator defined on $H^{2\ell}(\Omega)$, $\ell \geq 1$ by

$$Lu = (-1)^\ell \Delta^\ell u + (-1)^{\ell-1} \Delta^{\ell-1} u \dots - \Delta u + u = \sum_{j=0}^{\ell} (-1)^j \Delta^j u . \quad (2.2)$$

Let $\frac{\partial}{\partial n}$ denote differentiation with respect to the outward normal to Ω on Γ. Then we will consider the following model problem

$$Lu = F \quad \text{on} \quad \Omega \quad (2.3)$$

$$\frac{\partial^r u}{\partial n^r} = 0 \quad \text{on} \quad \Gamma \quad r = 0, 1, \dots, \ell - 1 . \quad (2.4)$$

Note that (2.4) is equivalent to

$$D^{(r)} u = 0 \quad \text{on} \quad \Gamma \quad 0 \leq |r| \leq \ell - 1 .$$

The model problem (2.2)-(2.4) is a classical case of an elliptic equation over a nonsmooth domain. The properties of such problems have been well studied (see [17], [18] and the references therein).

We assume here that the solution of (2.2)-(2.4) can be written in the following form :

$$u = u_1 + \sum_{i=1}^{M} u_2^{[i]} \quad (2.5)$$

where

$$u_1 \in H^k(\Omega) \cap H_0^\ell(\Omega)$$

$$u_2^{[i]} = \operatorname{Re} \left\{ \sum_{j=1}^{n_i} C_j^{[i]} |\log r_i|^{\gamma_j^{[i]}} r_i^{\alpha_j^{[i]}} \phi_j^{[i]}(\theta_i) \chi^{[i]}(r_i) \right\} \in H_0^\ell(\Omega) \quad (2.6)$$

with $k > \ell$, $\operatorname{Re}(\alpha_j^{[i]}) > \ell - 1$, $\operatorname{Re}(\alpha_{j+1}^{[i]}) \geq \operatorname{Re}(\alpha_j^{[i]})$, $\gamma_j^{[i]} \geq 0$ real, $\phi_j^{[i]}(\theta_i)$ and $\chi^{[i]}(r_i)$ are real C^∞ (or sufficiently smooth) functions, $\chi^{[i]}(r_i) = 1$ for $0 < r_i < \rho^{[i]} < \frac{1}{4}$, $\chi^{[i]}(r_i) = 0$ for $r_i > 2\rho^{[i]}$. By (r_i, θ_i) we have denoted the polar coordinates with the origin at the vertex A_i of the polygon Ω. The partition (2.5) is typical for problem (2.2)-(2.4). The function u_1 represents the smooth component of the solution and is the only one present when the domain Ω is smooth. However, due to the presence of corners, the solution exhibits singularities of the form r^{α} near the corners, which leads to the components $u_2^{[i]}$. In general, it is these singularity components which limit the rate of approximation that can be obtained by the use of polynomial subspaces. For details and proofs of the partition, we refer to [17], [18].

It may be noted that we have only dealt here with essential homogeneous boundary conditions. Instead of (2.4), we may specify natural boundary conditions as well (which may be inhomogeneous), consisting of normal derivatives of order $\ell \leq r \leq 2\ell - 1$. Our results remain valid for the case when different types of conditions are specified on different portions of Γ. In Section 6 we shall comment on this and other generalizations of our results.

2.3. The *p*-version of the finite element method

Problem (2.2)-(2.4) may be put into the following equivalent variational form $(F \in L_2(\Omega))$: find $u \in H_0^{\ell}(\Omega)$ satisfying

$$B(u, v) = \int_{\Omega} Fv \, dx \qquad \text{for all} \qquad v \in H_0^{\ell}(\Omega) \tag{2.7}$$

where $B(.,.)$, the bilinear form defined on $H^{\ell}(\Omega) \times H^{\ell}(\Omega)$ associated with the operator L is equal to $(.,.)_{H^{\ell}(\Omega)}$ so that

$$B(u, u) \geq \|u\|^2_{H^{\ell}(\Omega)} \tag{2.8}$$

holds for any $u \in H^{\ell}(\Omega)$.

The p-version of the finite element method consists now of finding $u_p \in {}_0S_p^{\ell}(\Omega)$ such that

$$B(u_p, v) = \int_{\Omega} Fv \, dx \qquad \text{for all} \qquad v \in {}_0S_p^{\ell}(\Omega)\,. \tag{2.9}$$

3. APPROXIMATION PROPERTIES OF $\mathcal{P}_p(\Omega)$

Let Q be the standard square with sides γ_i, $i = 1, 2, 3, 4$ and diagonal γ_5 corresponding to $x_1 = x_2$. Let T be the standard triangle enclosed by sides γ_1, γ_2 and γ_5.

LEMMA 3.1 : *Let $S = Q$ or T. Then there exists a family of operators $\{\hat{\Pi}_p\}$, $p = 1, 2, \ldots$, $\hat{\Pi}_p : H^k(S) \to \mathscr{P}_p(S)$, such that for any $u \in H^k(S)$, for $k \geqslant 0$, $0 \leqslant q \leqslant k$,*

$$\left\|u - \hat{\Pi}_p u\right\|_{H^q(S)} \leqslant C p^{-(k-q)} \|u\|_{H^k(S)} \tag{3.1}$$

for $r = (r_1, r_2)$, $k > |r| + \frac{1}{2}$, $0 \leqslant s < k - |r| - \frac{1}{2}$, $i = 1, 2, \ldots, 5$,

$$\left\|D^{(r)}(u - \hat{\Pi}_p u)\right\|_{H^s(\gamma_i)} \leqslant C p^{-\left(k - |r| - s - \frac{1}{2}\right)} \|u\|_{H^k(S)} \tag{3.2}$$

for $r = (r_1, r_2)$, $k > |r| + 1$, $x \in S$,

$$\left|D^{(r)}(u - \hat{\Pi}_p u)(x)\right| \leqslant C p^{-(k - |r| - 1)} \|u\|_{H^k(S)} \tag{3.3}$$

where the constant C is independent of u and p and where we denote $\mathscr{P}_p(S) = \mathscr{P}_p^2(Q)$ for $S = Q$ and $\mathscr{P}_p(S) = \mathscr{P}_p^1(T)$ for $S = T$.

Moreover, if $u \in \mathscr{P}_p(S)$, then $\hat{\Pi}_p u = u$.

Proof: Let $d > 1$. Then $\bar{S} \subset R(d)$ (see (2.1)). Since S is a Lipschitz domain there exists an extension operator $\mathscr{E}$ mapping $H^k(S)$ into $H^k(R(2d))$ such that

$$\mathscr{E}u = 0 \quad \text{on} \quad R(2d) - R\left(\frac{3}{2}d\right) \tag{3.4}$$

$$\|\mathscr{E}u\|_{H^k(R(2d))} \leqslant C \|u\|_{H^k(S)} \tag{3.5}$$

where C is independent of u. A concrete construction of $\mathscr{E}$ may be found in [21].

Let ϕ be the one-to-one mapping of $R\left(\frac{\pi}{2}\right)$ onto $R(2d)$:

$$R(2d) \ni x = (x_1, x_2) = \phi(\xi) = (2d \sin \xi_1, 2d \sin \xi_2)$$

with $(\xi_1, \xi_2) = \xi \in R\left(\frac{\pi}{2}\right)$. Further, we let

$$\tilde{R} = \phi^{-1}\left[R\left(\frac{3}{2}d\right)\right] \subset R\left(\frac{\pi}{2}\right)$$

where ϕ^{-1} denotes the inverse mapping of ϕ.

Let $v = \mathscr{E}u$ and

$$V(\xi) = v(\phi(\xi)) . \tag{3.6}$$

Because of (3.5), we easily see that

$$\operatorname{supp} V(\xi) \subset \tilde{R} . \tag{3.7}$$

In addition, it can be readily seen that

a) $V(\xi)$ is a periodic function with period 2π

b) V satisfies

$$\| V(\xi) \|_{H^k(R(\pi))} \leqslant C \| v \|_{H^k\left(R\left(\frac{3}{2}d\right)\right)} \leqslant C \| u \|_{H^k(S)} \tag{3.8}$$

and hence $V \in H^k_{PER}(R(\pi))$.

c) $V(\xi)$ is a symmetric function with respect to the lines $\xi_i = \pm \frac{\Pi}{2}$, $i = 1, 2$.

Let us expand the function V in terms of its Fourier Series

$$V(\xi_1, \xi_2) = \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_{j\ell}\, e^{i(j\xi_1 + \ell\xi_2)} . \tag{3.9}$$

For any $p \geqslant 1$, we define $V_p = \hat{\Pi}_p V$ by

$$V_p(\xi_1, \xi_2) = \sum_{j=-p}^{p} \sum_{\ell=-p}^{p} a_{j\ell}\, e^{i(j\xi_1 + \ell\xi_2)} \quad \text{for} \quad S = Q \tag{3.10a}$$

$$V_p(\xi_1, \xi_2) = \sum_{|j| + |\ell| \leqslant p} a_{j\ell}\, e^{i(j\xi_1 + \ell\xi_2)} \quad \text{for} \quad S = T . \tag{3.10b}$$

Obviously, $V_p \in \mathcal{C}_p(R(\pi))$.

We have

$$\| V \|^2_{H^k(R(\pi))} \approx \sum_{j,\ell} |a_{j\ell}|^2 \left((1 + j^2 + \ell^2)^{\frac{1}{2}} \right)^{2k} \tag{3.11}$$

where $\approx$ has the usual meaning of equivalency. (3.11) yields immediately for $0 \leqslant q \leqslant k$

$$\begin{aligned} \| V - V_p \|_{H^q(R(\pi))} &\leqslant C p^{-(k-q)} \| V \|_{H^k(R(\pi))} \\ &\leqslant C p^{-(k-q)} \| u \|_{H^k(S)} \end{aligned} \tag{3.12}$$

using (3.8b) with C independent of u.

In what follows, we assume $S = Q$. The case $S = T$ follows similarly.

Let $\hat{\gamma}_i$, $i = 1, \ldots, 4$ be the sides of $R(\pi)$ and let $\xi_2 = \hat{\xi}_2$ be one of the sides.

Then for $r = (r_1, r_2)$, $0 \leqslant |r| < k - \frac{1}{2}$,

$$D^{(r)}(V - V_p)(\xi_1, \hat{\xi}_2) =$$

$$= \left(\sum_{|j| > p} \sum_{|\ell| \leqslant p} + \sum_{|j| \leqslant p} \sum_{|\ell| > p} + \sum_{|j| > p} \sum_{|\ell| > p} \right) (ij)^{r_1} (i\ell)^{r_2} a_{j\ell} e^{i(j\xi_1 + \ell\hat{\xi}_2)}$$

$$= \sum_{|j| > p} (ij)^{r_1} b_j^{[1]} e^{ij\xi_1} + \sum_{|j| \leqslant p} (ij)^{r_1} b_j^{[2]} e^{ij\xi_1} + \sum_{|j| > p} (ij)^{r_1} b_j^{[3]} e^{ij\xi_1} \quad (3.13)$$

where, for $|j| > p$:

$$\left| (ij)^{r_1} b_j^{[1]} \right|^2 = j^{2r_1} \left(\sum_{|\ell| \leqslant p} \ell^{r_2} a_{j\ell} e^{i\ell\hat{\xi}_2} \right)^2$$

$$\leqslant j^{2r_1} p^{2r_2} \left(\sum_{|\ell| \leqslant p} |a_{j\ell}| \right)^2$$

$$\leqslant j^{2r_1} p^{2r_2} \left(\sum_{|\ell| \leqslant p} |a_{j\ell}|^2 (1 + j^2 + \ell^2)^k \right) \left(\sum_{|\ell| \leqslant p} (1 + j^2 + \ell^2)^{-k} \right)$$

$$\leqslant \frac{C j^{2r_1} p^{2r_2 + 1} A_j}{(1 + j^2)^k}$$

where we denote

$$A_j = \sum_{\ell = -\infty}^{\infty} |a_{j\ell}|^2 (1 + j^2 + \ell^2)^k . \quad (3.14)$$

Consider now the function

$$f(x) = \frac{x^{2\mu}}{(1 + x^2)^k}$$

for $x > 0$. We have for $\mu > 0$

$$f'(x) = \frac{2x^{2\mu - 1}}{(1 + x^2)^{k+1}} [\mu - (k - \mu) x^2] .$$

Hence, for $k > \mu$ and $x^2 > \frac{\mu}{k - \mu}$, $f(x)$ is a decreasing function of x. Moreover, when $\mu = 0$, $f(x)$ is decreasing for all $x > 0$. Hence, there exists a constant $C = C(k, \mu)$ such that for $|j| > p$, $k > \mu$,

$$\frac{j^{2\mu}}{(1 + j^2)^k} \leqslant \frac{C p^{2\mu}}{(1 + p^2)^k} . \quad (3.15)$$

so that taking $\mu = r_1$, for $k > r_1$,

$$\left|(ij)^{r_1} b_j^{[1]}\right|^2 \leqslant C A_j p^{-(2k-2|r|-1)}. \tag{3.16}$$

For $|j| \leqslant p$,

$$\begin{aligned}\left|(ij)^{r_1} b_j^{[2]}\right|^2 &\leqslant p^{2r_1}\Big(\sum_{|\ell|>p} \ell^{r_2}|a_{j\ell}|\Big)^2 \\ &\leqslant p^{2r_1} A_j\Big(\sum_{|\ell|>p} \ell^{2r_2}(1+j^2+\ell^2)^{-k}\Big) \\ &\leqslant C p^{2r_1} A_j \int_{p+1}^{\infty} \frac{dx}{x^{2(k-r_2)}} \\ &\leqslant C A_j p^{-(2k-2|r|-1)} \end{aligned} \tag{3.17}$$

provided $k > r_2 + \frac{1}{2}$.

Analogously, for $|j| > p$,

$$\begin{aligned}\left|(ij)^{r_1} b_j^{[3]}\right|^2 &\leqslant j^{2r_1}\Big(\sum_{|\ell|>p} \ell^{r_2}|a_{j\ell}|\Big)^2 \\ &\leqslant C A_j \int_{p+1}^{\infty} \frac{j^{2r_1} x^{2r_2}}{(1+j^2+x^2)^k}\,dx \\ &\leqslant C A_j \int_{p+1}^{\infty} \frac{dx}{(1+j^2+x^2)^{k-r_1-r_2}} \\ &\leqslant C A_j p^{-(2k-2|r|-1)} \end{aligned} \tag{3.18}$$

provided $k > |r| + \frac{1}{2}$. Hence, using (3.13)-(3.18), we see that for $i = 1, 2, 3, 4$

$$\begin{aligned}\|D^{(r)}(V-V_p)\|^2_{H^0(\hat{\gamma}_i)} &\leqslant \\ &\leqslant C\Big[\sum_{|j|>p} \left|j^{r_1} b_j^{[1]}\right|^2 + \sum_{|j|\leqslant p} \left|j^{r_1} b_j^{[2]}\right|^2 + \sum_{|j|>p} \left|j^{r_1} b_j^{[3]}\right|^2\Big] \\ &\leqslant C p^{-2\left(k-|r|-\frac{1}{2}\right)} \sum_{j=-\infty}^{\infty} A_j \\ &\leqslant C p^{-2\left(k-|r|-\frac{1}{2}\right)} \|u\|^2_{H^k(Q)}\end{aligned}$$

provided $k > |r| + \frac{1}{2}$. From this, it follows easily that for $s \geq 0$, s integer,

$$\|D^{(r)}(V - V_p)\|_{H^s(\tilde{\gamma}_i)} \leq Cp^{-\left(k - |r| - s - \frac{1}{2}\right)} \|u\|_{H^k(Q)} \tag{3.19}$$

provided $k > |r| + s + \frac{1}{2}$.

Now we estimate $\|D^{(r)}(V - V_p)\|_{H^s(\tilde{\gamma}_5)}$. For $0 \leq |r| < k - \frac{1}{2}$,

$$D^{(r)}(V - V_p)(\xi, \xi) =$$

$$= \left(\sum_{|j| > p} \sum_{|\ell| \leq p} + \sum_{|j| \leq p} \sum_{|\ell| > p} + \sum_{|j| > p} \sum_{|\ell| > p} \right) (ij)^{r_1} (i\ell)^{r_2} a_{j\ell} e^{i(j+\ell)\xi}$$

$$= \sum_{q=-\infty}^{\infty} (C_q^{[1]} + C_q^{[2]} + C_q^{[3]}) e^{iq\xi} \tag{3.20}$$

where

$$C_q^{[1]} = \sum_{\substack{j+\ell=q \\ |j|>p,\ |\ell| \leq p}} (ij)^{r_1} (i\ell)^{r_2} a_{j\ell}$$

$$C_q^{[2]} = \sum_{\substack{j+\ell=q \\ |j|\leq p,\ |\ell| > p}} (ij)^{r_1} (i\ell)^{r_2} a_{j\ell}$$

$$C_q^{[3]} = \sum_{\substack{j+\ell=q \\ |j|>p,\ |\ell| > p}} (ij)^{r_1} (i\ell)^{r_2} a_{j\ell}\,.$$

By the Schwarz inequality,

$$|C_q^{[1]}|^2 \leq \left(\sum_{\substack{j+\ell=q \\ |j|>p,\ |\ell| \leq p}} |a_{j\ell}|^2 (1 + j^2 + \ell^2)^k \right) \times \left(\sum_{\substack{j+\ell=q \\ |j|>p,\ |\ell| \leq p}} j^{2r_1} \ell^{2r_2} (1 + j^2 + \ell^2)^{-k} \right) \leq A_q I_q^{[1]}$$

where

$$A_q = \sum_{j+\ell=q} |a_{j\ell}|^2 (1 + j^2 + \ell^2)^k$$

$$I_q^{[1]} = \sum_{\substack{j+\ell=q \\ |j|>p,\ |\ell| \leq p}} (q-\ell)^{2r_1} \ell^{2r_2} (1 + j^2 + \ell^2)^{-k} \leq 2^{2r_1 - 1} \sum_{\substack{j+\ell=q \\ |j|>p,\ |\ell| \leq p}} (q^{2r_1} + \ell^{2r_1}) \ell^{2r_2} (1 + j^2 + \ell^2)^{-k}\,.$$

For $q^2 \leqslant p^2$, this gives

$$I_q^{[1]} \leqslant \frac{Cp^{2|r|}}{(1+p^2)^k} N(p, q)$$

where $N(p, q)$, the number of terms in the sum, is obviously $\leqslant 2p$, so that

$$I_q^{[1]} \leqslant Cp^{-(2k-2|r|-1)}.$$

For $p^2 < q^2$, we have

$$q^2 = (j+\ell)^2 \leqslant 2(j^2+\ell^2)$$

so that

$$\begin{aligned} I_q^{[1]} &\leqslant \frac{C(q^{2r_1}+p^{2r_1})\,p^{2r_2}}{\left(1+\frac{q^2}{2}\right)^k} N(p, q) \\ &\leqslant C|q|^{-2(k-|r|)}p \\ &\leqslant Cp^{-(2k-2|r|-1)} \end{aligned}$$

provided $k > |r|$. Hence,

$$|C_q^{[1]}|^2 \leqslant CA_q\, p^{-(2k-2|r|-1)}. \tag{3.21}$$

Similarly,

$$|C_q^{[2]}|^2 \leqslant CA_q\, p^{-(2k-2|r|-1)}. \tag{3.22}$$

Finally,

$$|C_q^{[3]}|^2 \leqslant CA_q\, I_q^{[2]}$$

where

$$I_q^{[2]} = \sum_{\substack{j+\ell=q \\ |j|>p,\ |\ell|>p}} \frac{(q^{2r_1}+\ell^{2r_1})\,\ell^{2r_2}}{(1+j^2+\ell^2)^k}.$$

For $q^2 \leqslant p^2$

$$\begin{aligned} I_q^{[2]} &\leqslant \sum_{|\ell|>p} \frac{(p^{2r_1}+\ell^{2r_1})\,\ell^{2r_2}}{(1+\ell^2)^k} \\ &\leqslant Cp^{-(2k-2|r|-1)} \end{aligned}$$

provided $k > |r| + \frac{1}{2}$.

If $q^2 > p^2$, then

$$\begin{aligned} I_q^{[2]} &\leq \sum_{\substack{j+\ell=q \\ |j|>p,\ |\ell|>p}} \frac{q^{2r_1}\ell^{2r_2}}{(1+j^2+\ell^2)^k} + Cp^{-(2k-2|r|-1)} \\ &\leq \sum_{\substack{j+\ell=q \\ |j|>p, p<|\ell|<|q|}} \frac{q^{2|r|}}{\left(1+\frac{q^2}{2}\right)^k} + \sum_{|\ell|>|q|} \frac{\ell^{2|r|}}{(1+\ell^2)^k} + Cp^{-(2k-2|r|-1)} \\ &\leq C|q|^{-(2k-2|r|-1)} + Cp^{-(2k-2|r|-1)} \\ &\leq Cp^{-(2k-2|r|-1)} \end{aligned}$$

provided $k > |r| + \frac{1}{2}$. Hence

$$|C_q^{[3]}|^2 \leq CA_q\, p^{-(2k-2|r|-1)}. \tag{3.23}$$

Combining (3.20)-(3.23), we obtain

$$\|D^{(r)}(V - V_p)\|_{H^0(\hat{\gamma}_5)} \leq Cp^{-(2k-2|r|-1)}$$

which leads to (3.19) for $\hat{\gamma}_5$.

We now estimate $|D^{(r)}(V - V_p)(x)|$. Because $V - V_p \in H^k_{PER}(R(\pi))$, we can assume without loss of generality that $(\xi_1, \xi_2) \in \hat{\gamma}_i$ given by $\xi_2 = \hat{\xi}_2$. Then once again we have (3.13) and for $\varepsilon > 0$

$$\left(\sum_{|j|>p} |(ij)^{r_1} b_j^{[1]}|\right)^2 \leq \left(\sum_{|j|>p} \left|(ij)^{\left(r_1+\frac{1}{2}+\varepsilon\right)} b_j^{[1]}\right|^2\right)\left(\sum_{|j|>p} \frac{1}{j^{1+2\varepsilon}}\right). \tag{3.24a}$$

Taking $\mu = r_1 + \frac{1}{2} + \varepsilon$ in (3.15), we obtain analogously to (3.16)

$$\left|j^{\left(r_1+\frac{1}{2}+\varepsilon\right)} b_j^{[1]}\right|^2 \leq CA_j\, p^{-(2k-2|r|-2-2\varepsilon)}$$

provided $k > r_1 + \frac{1}{2} + \varepsilon$, so that (3.24$a$) yields

$$\left(\sum_{|j|>p} |(ij)^{r_1} b_j^{[1]}|\right)^2 \leq Cp^{-2(k-|r|-1)} \|u\|^2_{H^k(Q)}. \tag{3.24b}$$

Moreover, using (3.17), we see that

$$\begin{aligned} \left(\sum_{|j|\leq p} |(ij)^{r_1} b_j^{[2]}|\right)^2 &\leq \left(\sum_{|j|\leq p} |(ij)^{r_1} b_j^{[2]}|^2\right) p \\ &\leq Cp^{-2(k-|r|-1)} \|u\|^2_{H^k(Q)} \end{aligned} \tag{3.25}$$

provided $k > r_2 + \frac{1}{2}$. Also,

$$\left(\sum_{|j|>p} \left|(ij)^{r_1} b_j^{[3]}\right|\right)^2 \leqslant \left(\sum_{|j|>p} \left|(ij)^{\left(r_1+\frac{1}{2}+\varepsilon\right)} b_j^{[3]}\right|^2\right)\left(\sum_{|j|>p} \frac{1}{j^{1+2\varepsilon}}\right)$$
$$\leqslant C\left(\sum_{|j|>p} A_j p^{-(2k-2|r|-2-2\varepsilon)}\right) p^{-2\varepsilon}$$
$$\leqslant C p^{-2(k-|r|-1)} \|u\|^2_{H^k(Q)} \tag{3.26}$$

provided $k > |r| + 1 + \varepsilon$, where we have used (3.18) with $r_1 = r_1 + \frac{1}{2} + \varepsilon$. Combining (3.24b)-(3.26), we get for $k > |r| + 1$,

$$\left|D^{(r)}(V - V_p)(x)\right| \leqslant C p^{-(k-|r|-1)} \|u\|_{H^k(Q)} . \tag{3.27}$$

Because of (3.8c), $V_p(\phi^{-1}(x)) \in \mathcal{P}_p(Q)$. Further, ϕ is a regular mapping of $R(d)$ on Q, $\left(d < \frac{\pi}{2}\right)$ and $\phi(\hat{\gamma}_i) \supset \gamma_i$. Hence, for k, q, s integers, (3.1) follows immediately from (3.12), (3.2) from (3.19) and (3.23), and (3.3) from (3.27). Using a standard interpolation argument, (3.1)-(3.3) will hold for non-integral values of k, q and s as well.

The following one-dimensional result is from [5].

LEMMA 3.2 : *Let $I = (-1, 1)$ and $u \in H^k(I)$, $k \geqslant 1$. Then there exists a polynomial $z_p \in \mathcal{P}_p(I)$, $p \geqslant 2k - 1$, such that*

$$z_p^{(i)}(\pm 1) = u^{(i)}(\pm 1) \qquad i = 0, 1, \ldots, k-1 \tag{3.28}$$

and for $s = 0, 1, 2, \ldots, k$

$$\|u - z_p\|_{H^s(I)} \leqslant C p^{-(k-s)} \|u\|_{H^k(I)} \tag{3.29}$$

where C depends on k but not on u and p.

4. THE APPROXIMATION OF FUNCTIONS IN $H^k(\Omega)$

In this section, we will analyze the rate of approximation in the $H^\ell(\Omega)$ norm of u by a piecewise polynomial in ${}_0S_p^\ell(\Omega)$ for the case when $u \in H^k(\Omega) \cap H_0^\ell(\Omega)$ i.e. when $u_2^{[i]} = 0$ in (2.5). The main result we obtain is the following.

THEOREM 4.1 : *Let ℓ be an integer, $\ell \geqslant 1$. Let $u \in H^k(\Omega) \cap H_0^\ell(\Omega)$, $k > 2\ell - \frac{1}{2}$. Then there exists a sequence $\{u_p\}$, satisfying $u_p \in {}_0S_p^\ell(\Omega)$ and*

$$\frac{\partial^r u_p}{\partial n^r} = 0 \qquad r = 0, 1, \ldots, \ell - 1 \quad \text{on} \quad \Gamma \tag{4.1}$$

$$\| u - u_p \|_{H^\ell(\Omega)} \leqslant C p^{-(k-\ell)} \| u \|_{H^k(\Omega)} \tag{4.2}$$

where the constant C depends upon the partition of Ω, k and ℓ but is independent of u and p.

We first prove the following lemmas.

LEMMA 4.1 : *Let $I = (-1, 1)$. Given an integer $t \geqslant 1$, there exists a sequence of polynomials $\{\chi_p^r\} = \{\chi_p^{r,t}\}$, $p \geqslant 2t - 1$, $r = 0, 1, \ldots, t-1$, in $\mathcal{P}_p(I)$ satisfying*

$$\begin{aligned} \frac{d^j \chi_p^r}{dx^j}(-1) &= 1 \quad if \quad j = r \quad j = 0, 1, \ldots t-1\,; \quad r = 0, 1, \ldots, t-1\,, \\ &\ 0 \quad if \quad j \neq r \end{aligned} \tag{4.3a}$$

$$\frac{d^j \chi_p^r}{dx^j}(+1) = 0 \quad j = 0, 1, \ldots, t-1\,; \quad r = 0, 1, \ldots, t-1\,, \tag{4.3b}$$

$$\| \chi_p^r \|_{H^s(I)} \leqslant C p^{s - r - \frac{1}{2}} \quad s = 0, 1, \ldots, t-1\,; \quad r = 0, 1, \ldots, t-1\,, \tag{4.4}$$

where the constant C depends on t but is independent of p, r and s.

Proof: We first define, for $p \geqslant 2t - 1$, functions ϕ_p^r given by

$$\phi_p^r(x) = \left(\sum_{i=0}^{t-1} C_i^{r,p} (x+1)^i \right) e^{-p(x+1)} \tag{4.5}$$

where the constants $C_i^{r,p}$ are given by

$$\begin{aligned} C_i^{r,p} &= 0 && \text{for} \quad 0 \leqslant i \leqslant r - 1 \\ &= \frac{1}{r!} \frac{p^{i-r}}{(i-r)!} && \text{for} \quad r \leqslant i \leqslant t-1\,. \end{aligned} \tag{4.6}$$

Hence,

$$|C_i^{r,p}| \leqslant K p^{i-r}\,. \tag{4.7}$$

By (4.5), for any integer $0 \leqslant m \leqslant t-1$,

$$\frac{d^m \phi_p^r}{dx^m} = (-p)^m \sum_{i=0}^{t-1} C_i^{r,p} (x+1)^i e^{-p(x+1)}$$
$$+ m(-p)^{m-1} \sum_{i=1}^{t-1} C_i^{r,p} i (x+1)^{i-1} e^{-p(x+1)} + \cdots$$
$$\cdots + \sum_{i=m}^{t-1} C_i^{r,p} \frac{i!(x+1)^{i-m} e^{-p(x+1)}}{(i-m)!} \tag{4.8}$$

so that

$$\left\| \frac{d^m \phi_p^r}{dx^m} \right\|_{H^0(I)} \leqslant K \sum_{j=0}^{m} p^{m-j} \sum_{i=j}^{t-1} |C_i^{r,p}| \left(\int_{-1}^{+1} (x+1)^{2(i-j)} e^{-2p(x+1)} dx \right)^{\frac{1}{2}} .$$

Now for p sufficiently large,

$$\int_{-1}^{+1} (x+1)^s e^{-2p(x+1)} dx \leqslant K p^{-(1+s)} .$$

Hence, using (4.7),

$$\left\| \frac{d^m \phi_p^r}{dx^m} \right\|_{H^0(I)} \leqslant K \sum_{j=0}^{m} \sum_{i=j}^{t-1} p^{m-j} p^{i-r} p^{-\frac{1}{2}-i+j}$$
$$\leqslant K p^{m-r-\frac{1}{2}} .$$

This shows that for any $s = 0, 1, \ldots, t-1$,

$$\|\phi_p^r\|_{H^s(I)} \leqslant C p^{s-r-\frac{1}{2}}$$

i.e., $\{\phi_p^r\}$ satisfy (4.4) with χ_p^r replaced by ϕ_p^r. We now show that with our choice of $C_i^{r,p}$, $\{\phi_p^r\}$ also satisfy (4.3a). By (4.8), we have for $0 \leqslant m \leqslant t-1$

$$A_m^{r,p} = \frac{d^m \phi_p^r}{d\chi^m}(-1) = \sum_{j=0}^{m} \frac{m!}{j!} (-p)^j C_{m-j}^{r,p} .$$

Using (4.5), we get

$$A_m^{r,p} = 0 \quad \text{for} \quad m \leqslant r-1$$
$$= 1 \quad \text{for} \quad m = r .$$

Also, for $r < m \leqslant t-1$, we have

$$A_m^{r,p} = p^{m-r}\left(\sum_{j=0}^{m-r} \frac{(-1)^j (m-r)!}{(m-r-j)!\, j!}\right) = 0$$

since the term inside the brackets is the binomial expansion of $(1-1)^{m-r}$.

In order to obtain a function satisfying (4.3b) as well, we let $U_p^r = \phi_p^r \psi$ where $\psi \in C^\infty(I)$ is a smooth cut-off function satisfying

$$\begin{aligned} \psi(x) &= 1 \quad \text{for} \quad -1 \leqslant x \leqslant -\frac{1}{2} \\ &= 0 \quad \text{for} \quad \frac{1}{2} \leqslant x \leqslant 1 . \end{aligned}$$

Then it may be easily verified that U_p^r satisfies (4.3)-(4.4). We now use lemma 3.2 to approximate U_p^r by a polynomial z_p^r in $\mathscr{P}_p(I)$, $p \geqslant 2t-1$, satisfying (3.28)-(3.29). (3.28) implies then that z_p^r satisfies (4.3). Also, using (3.29) and the fact that U_p^r satisfies (4.4), we have

$$\begin{aligned} \| U_p^r - z_p^r \|_{H^s(I)} &\leqslant C p^{-(k-s)} \| U_p^r \|_{H^k(I)} \\ &\leqslant C p^{-(k-s)} p^{k-r-\frac{1}{2}} \\ &\leqslant C p^{s-r-\frac{1}{2}} \end{aligned}$$

so that by the triangle inequality, z_p^r satisfies (4.4). Taking $\chi_p^r = z_p^r$ yields the lemma. □

LEMMA 4.2 : *Let $v(x)$ be a function defined on I satisfying*

$$\frac{d^s v}{dt^s}(1) = 0 \qquad s = 0, 1, \ldots, r-1 . \tag{4.9}$$

Then

$$\left\| \frac{v(x)}{(x-1)^r} \right\|_{H^0(I)} \leqslant C \| v \|_{H^r(I)} . \tag{4.10}$$

Proof: Let $f(\xi)$ be a function defined on $[0, \infty)$ such that $f = 0$ for $x > 2$. Define

$$f_r(\xi) = \frac{1}{\Gamma(r)} \int_0^\xi (\xi - t)^{r-1} f(t)\, dt . \tag{4.11}$$

Then by [15], page 245, No. (9.9.5), we have

$$\int_0^2 \left(\frac{f_r(\xi)}{\xi^r} \right)^2 d\xi \leq C \int_0^2 (f(\xi))^2 \, d\xi \,. \tag{4.12}$$

Now, take

$$f(\xi) = v^{(r)}(1-\xi) \,.$$

Then, substituting in (4.11), integrating by parts r times and using (4.9), we see that

$$f_r(\xi) = Cv(1-\xi)$$

so that (4.12) becomes

$$\int_0^2 \left(\frac{v(1-\xi)}{\xi^r} \right)^2 d\xi \leq C \int_0^2 (v^{(r)}(1-\xi))^2 \, d\xi \,.$$

Substituting $1 - \xi = x$ gives (4.10). □

Proof of Theorem 4.1 : Let Ω_i, $i = 1, 2, \ldots, N$ be the elements of the partition of Ω. We first construct the functions $z_{p,1}^{[i]} = \hat{\Pi}_p \, u^{[i]}$ as in Lemma 3.1. The lemma is applicable because a linear transformation maps the parallelogram or triangular element onto the standard square or triangle, preserving the polynomials. Hence for $\ell \geq 1$ integer, $k \geq \ell$,

$$\| u - z_{p,1}^{[i]} \|_{H^\ell(\Omega_i)} \leq C p^{-(k-\ell)} \| u \|_{H^k(\Omega_i)} \,. \tag{4.13}$$

Our first step is to add a function $y_p^{[i]} \in \mathcal{P}_p(\Omega_i)$, $p \geq 4\ell - 3$ to $z_{p,1}^{[i]}$ so that the function $z_p^{[i]} = z_{p,1}^{[i]} + y_p^{[i]}$ satisfies (3.1), (3.2) and

$$D^{(r)} z_p^{[i]} = D^{(r)} u \quad \text{for} \quad 0 \leq |r| \leq 2\ell - 2 \tag{4.14}$$

at the vertices of Ω_i. Let us first assume that Ω_i is a parallelogram, which we may take to be the standard square Q without loss of generality. Let

$$\alpha^{(r)} = D^{(r)}(u - z_p^{[i]})(-1, -1) \tag{4.15}$$

where $r = (r_1, r_2)$. Define the function

$$w_1 = \sum_{0 \leq |r| \leq 2\ell - 2} \alpha^{(r)} \chi_p^{r_1}(x_1) \, \chi_p^{r_2}(x_2) \,. \tag{4.16}$$

Here, $\chi_p^{r_i} = \chi_p^{r_i, t}$, $i = 1, 2$ are as in Lemma 4.1, with $t = 2\ell - 1$, so that $\chi_p^{r_i} \in \mathcal{P}_p(I)$, $p \geq 4\ell - 3$. This implies that we may construct $w_1 \in \mathcal{P}_p(Q)$

provided $p \geqslant 4\ell - 3$. We see then that with $k > 2\ell - 1$,

$$\|w_1\|_{H^\ell(\Omega_i)} \leqslant \sum_{0 \leqslant |r| \leqslant 2\ell-2} \sum_{\ell_1+\ell_2=\ell} |\alpha^{(r)}| \, \|\chi_p^{r_1}\|_{H^{\ell_1}(I)} \|\chi_p^{r_2}\|_{H^{\ell_2}(I)}$$

$$\leqslant C \sum_{0 \leqslant |r| \leqslant 2\ell-2} \sum_{\ell_1+\ell_2 \leqslant \ell} p^{-k+|r|+1} p^{\ell_1 - r_1 - \frac{1}{2}} p^{\ell_2 - r_2 - \frac{1}{2}} \|u\|_{H^k(\Omega_i)}$$

$$\leqslant C p^{-(k-\ell)} \|u\|_{H^k(\Omega_i)} \tag{4.17}$$

where we have used (3.3) with $k > |r| + 1$ i.e. $k > 2\ell - 1$ and (4.4). Moreover, on the side $\Gamma_1 = \{(x, -1) \mid -1 < x < 1\}$, we have by (3.3) and Lemma 4.1, for $j = (j_1, j_2)$, $s \geqslant 0$ integer, $0 \leqslant |j| + s \leqslant 2\ell - 2$, $k > 2\ell - 1$,

$$\|D^{(j)} w_1\|_{H^s(\Gamma_1)} = \sum_{0 \leqslant |r| \leqslant 2\ell-2} |\alpha^{(r)}| \left\| \frac{d^{j_1}\chi_p^{r_1}}{dx_1^{j_1}} \frac{d^{j_2}\chi_p^{r_2}}{dx_2^{j_2}}(-1) \right\|_{H^s(\Gamma_1)}$$

$$= \sum_{0 \leqslant r_1 + j_2 \leqslant 2\ell - 2} |\alpha^{(r_1, j_2)}| \left\| \frac{d^{j_1}\chi_p^{r_1}}{dx_1^{j_1}} \right\|_{H^s(\Gamma_1)}$$

$$\leqslant C p^{-(k - r_1 - j_2 - 1)} p^{s + j_1 - r_1 - \frac{1}{2}} \|u\|_{H^k(\Omega_i)}$$

$$= C p^{-(k - |j| - s - \frac{1}{2})} \|u\|_{H^k(\Omega_i)} \tag{4.18}$$

(4.18) will also be true for the side $\Gamma_2 = \{(-1, y) \mid -1 < y < 1\}$ and will hold trivially on the other two sides of Q, where $w_1 = 0$. We can repeat this construction for each of the four nodes of Q to obtain w_j, $j = 1, 2, 3, 4$. Then defining $y_p^{[i]} = \sum_{j=1}^{4} w_j$, we see that $z_p^{[i]} = z_{p,1}^{[i]} + y_p^{[i]}$ satisfies (3.1), (3.2) and (4.14).

For Ω_i a triangle, we assume that Ω_i is the rotated standard triangle T with vertices $P_1(-1, -1)$, $P_2(1, -1)$ and $P_3(-1, 1)$. Define $\alpha^{(r)}$ as in (4.15). Let $p \geqslant 10\ell - 6$ so that $\tilde{p} = [(p - 2\ell + 1)/2] \geqslant 4\ell - 3$ and define $w_1^{(0,0)} \in \mathscr{P}_p(T)$ by

$$w_1^{(0,0)} = \alpha^{(0,0)} \chi_{\tilde{p}}^0(x_1)\, \chi_{\tilde{p}}^0(x_2)(x_1 + x_2)^{2\ell - 1}(-2)^{-2\ell+1} \tag{4.19}$$

with $\chi_{\tilde{p}}^0 = \chi_{\tilde{p}}^{0, 2\ell - 1}$. Then we see that for $0 \leqslant |r| \leqslant 2\ell - 2$, $k > 1$,

$$D^{(r)} w_1^{(0,0)} = 0 \quad \text{on} \quad P_2 P_3 ,$$

$$\|D^{(r)} w_1^{(0,0)}\|_{H^s(\Gamma)} \leqslant C p^{-\left(k - |r| - s - \frac{1}{2}\right)} \|u\|_{H^k(\Omega_i)}$$

for $\Gamma = P_1 P_2$ or $P_1 P_3$, $s \geq 0$ integer, $|r| + s \leq 2\ell - 2$ and

$$|D^{(r)} w_1^{(0,0)}(-1,-1)| \leq C |\alpha^{(0,0)}| \leq C p^{-(k-1)} \|u\|_{H^k(\Omega_i)}$$

$$\|w_1^{(0,0)}\|_{H^\ell(\Omega_i)} \leq C p^{-(k-\ell)} \|u\|_{H^k(\Omega_i)} .$$

We now define $w_1^{(r)}$ recursively for $|r| > 0$. Let

$$\beta_{(m)}^{(r)} = D^{(r)} w_1^{(m)}(-1,-1) . \tag{4.20}$$

$\beta_{(m)}^{(r)}$ will be non-zero only when $r_i \geq m_i$, $i = 1, 2$. Now defining for $0 < |j| \leq 2\ell - 2$

$$w_1^{(j)} = \left(\alpha^{(j)} - \sum_{\substack{|m| < |j| \\ 0 \leq m_i \leq j_i \ (i=1,2)}} \beta_m^{(j)} \right) (-2)^{-2\ell+1} \chi_{\bar{p}}^{j_1}(x_1)\, \chi_{\bar{p}}^{j_2}(x_2)(x_1 + x_2)^{2\ell-1} \tag{4.21}$$

(with $\chi_{\bar{p}}^{j_i} = \chi_{\bar{p}}^{j_i, 2\ell-1}$), we see that $D^{(r)} w_1^{(j)}\ (-1,-1) = 0$ whenever $r_i < j_i$ $(i = 1, 2)$ and

$$D^{(r)} w_1^{(j)}(-1,-1) = \left(\alpha^{(j)} - \sum_{\substack{|m| < |j| \\ 0 \leq m_i \leq j_i \ (i=1,2)}} \beta_m^{(j)} \right) \quad \text{for} \quad r = j .$$

Hence, for $p \geq 10\ell - 6$,

$$w_1 = \sum_{|j| \leq 2\ell - 1} w_1^{(j)} \in \mathscr{P}_p(\Omega) \tag{4.22}$$

satisfies, for $0 \leq |r| + s \leq 2\ell - 2$, $k > 2\ell - 1$,

$$D^{(r)} w_1(-1,-1) = \alpha^{(r)} \tag{4.23a}$$

$$D^{(r)} w_1 = 0 \quad \text{on} \quad P_2 P_3 \tag{4.23b}$$

$$\|w_1\|_{H^\ell(\Omega_i)} \leq C p^{-(k-\ell)} \|u\|_{H^k(\Omega_i)} \tag{4.23c}$$

$$\|D^{(r)} w_1\|_{H^s(\Gamma)} \leq C p^{-\left(k - |r| - s - \frac{1}{2}\right)} \|u\|_{H^k(\Omega_i)} \tag{4.23d}$$

where $\Gamma = P_1 P_2$ or $P_2 P_3$.

A similar construction as that of w_1 may be used to obtain functions w_2 and w_3 associated with the nodes P_2 and P_3 respectively, after first mapping Ω_i in a suitable way onto the standard triangle T. Then we take $y_p^{[i]} = \sum_{j=1}^{3} w_j$.

Let now $\gamma = \bar{\Omega}_j \cap \bar{\Omega}_m$ and A_1, A_2 be the end points of γ. Now $D^{(r)} z_p^{[j]} \neq D^{(r)} z_p^{[m]}$ on γ. Denote the « jumps » of z_p on γ by

$$w_{jm}^{(r)}(x, y) = D^{(r)}(z_p^{[j]} - z_p^{[m]})|_\gamma \quad \text{for} \quad (x, y) \in \gamma . \tag{4.24}$$

Then, because $D^{(r)} z_p^{[j]}(A_i) = D^{(r)} u(A_i) = D^{(r)} z_p^{[m]}(A_i)$, we have $w_{jm}^{(r)}(A_i) = 0$ for $i = 1, 2$; $0 \leq |r| \leq 2\ell - 2$ and also, by (3.2), (4.18) and (4.23d),

$$\| w_{jm}^{(r)} \|_{H^t(\gamma)} \leq C p^{-\left(k - |r| - t - \frac{1}{2}\right)} (\| u \|_{H^k(\Omega_j)} + \| u \|_{H^k(\Omega_m)}) \tag{4.25}$$

for $0 \leq |r| + t \leq 2\ell - 1$, $k > 2\ell - \frac{1}{2}$.

Let now F_j be the affine transformation satisfying $F_j(\Omega_j) = S$, where $S = Q = \{(\xi, \eta) \mid |\xi| < 1, |\eta| < 1\}$, the standard square, if Ω_j is a parallelogram and $S = T = \{(\xi, \eta) \mid |\xi| < 1, -1 < \eta < -\xi\}$ the rotated standard triangle, if Ω_j is a triangle. Let γ be mapped by F_j onto the side $\Gamma_1 = \{(\xi, -1) \mid |\xi| < 1\}$ of S. Let $\boldsymbol{n} = \boldsymbol{n}(\xi, \eta)$ denote the outward unit normal to S along ∂S and let $\tilde{\boldsymbol{n}}$ be its image on $\partial\Omega_j$ under F_j^{-1}, so that $\frac{\partial \tilde{f}}{\partial \tilde{\boldsymbol{n}}} |_{\partial\Omega_j} = \frac{\partial f}{\partial \boldsymbol{n}} |_{\partial S}$ whenever $\tilde{f}(x, y) = f(\xi, \eta)$. Define, for $(x, y) \in \gamma$, $s \geq 0$ integer,

$$\tilde{\beta}_{jm}^s(x, y) = \frac{\partial^s}{\partial \tilde{\boldsymbol{n}}^s} (z_p^{[j]} - z_p^{[m]})|_\gamma . \tag{4.26}$$

Then, if $\beta_{jm}^s = \tilde{\beta}_{jm}^s \circ F_j^{-1}$ we see that for $(x, y) \in \gamma$, $(\xi, -1) = F_j(x, y)$,

$$\beta_{jm}^s(\xi) = \beta_{jm}^s(\xi, -1) = \tilde{\beta}_{jm}^s(x, y) . \tag{4.27}$$

Now $\tilde{\beta}_{jm}^s$ are obviously linear combinations of those $w_{jm}^{(r)}$ defined by (4.24) which satisfy $|r| = s$. Hence, it can be deduced from (4.25) and (4.27) that for $0 \leq s + t \leq 2\ell - 1$, $k > 2\ell - \frac{1}{2}$,

$$\| \beta_{jm}^s \|_{H^t(\Gamma_1)} \leq C p^{-\left(k - s - t - \frac{1}{2}\right)} (\| u \|_{H^k(\Omega_j)} + \| u \|_{H^k(\Omega_m)}) . \tag{4.28}$$

Moreover, since for $|r| \leq 2\ell - 2$, $D^{(r)} z_p^{[j]} = D^{(r)} z_p^{[m]}$ at the end points of

γ, we have for $t = 0, 1, \ldots, 2\ell - 2 - s$,

$$\frac{d^t \beta^s_{jm}}{d\xi^t}(\pm 1) = 0 . \tag{4.29}$$

For Ω_j a parallelogram, define

$$\zeta_{jm}(\xi, \eta) = \sum_{s=0}^{\ell-1} \beta^s_{jm}(\xi)\, \chi^s_p(\eta)$$

with $\chi^s_p = \chi^{s,\ell-1}_p$. Then using (4.3), (4.29) and (4.28), we see that

$$\frac{\partial^s \zeta_{jm}}{\partial \boldsymbol{n}^s} = \beta^s_{jm} \quad \text{on} \quad \Gamma_1 \quad \text{for} \quad 0 \leqslant s \leqslant \ell - 1 \tag{4.30a}$$

$$D^{(r)} \zeta_{jm} = 0 \quad \text{on} \quad \partial S - \Gamma_1 \quad \text{for} \quad 0 \leqslant |r| \leqslant \ell - 1 \tag{4.30b}$$

$$\begin{aligned} \|\zeta_{jm}\|_{H^\ell(S)} &\leqslant C \sum_{\ell_1 + \ell_2 \leqslant \ell} \sum_{s=0}^{\ell-1} \|\beta^s_{jm}\|_{H^{\ell_1}(\Gamma_1)} \|\chi^s_p\|_{H^{\ell_2}(I)} \\ &\leqslant C \sum_{\ell_1 + \ell_2 \leqslant \ell} \sum_{s=0}^{\ell-1} p^{-\left(k - s - \ell_1 - \frac{1}{2}\right)} p^{\ell_2 - s - \frac{1}{2}} \|u\|_{H^k(\Omega)} \\ &\leqslant C p^{-(k-\ell)} \|u\|_{H^k(\Omega)} \end{aligned} \tag{4.30c}$$

where we have taken $k > 2\ell - \dfrac{1}{2}$.

We now show the existence of a ζ_{jm} satisfying (4.30) in the case Ω_j is a triangle. By (4.29), we see that

$$\beta^s_{jm}(\xi) = (\xi - 1)^{2\ell - s - 1}\, \psi_s(\xi) \tag{4.31}$$

where ψ_s is a polynomial in ξ. We define, for $0 \leqslant s \leqslant \ell - 1$, a polynomial ζ^s_{jm} by

$$\zeta^s_{jm}(\xi, \eta) = \frac{1}{(\xi - 1)^\ell} \left(\beta^s_{jm}(\xi) - \tau^s_{jm}(\xi)\right) \chi^s_p(\eta) (\xi + \eta)^\ell \tag{4.32a}$$

with

$$\tau^s_{jm}(\xi) = \sum_{i=0}^{s-1} \frac{\partial^s \zeta^i_{jm}}{\partial \eta^s}(\xi, -1) . \tag{4.32b}$$

where $\chi^s_p = \chi^{s,\ell-1}_p$ and for $s = 0$, $\tau^s_{jm}(\xi) = 0$. We now define

$$\zeta_{jm} = \sum_{s=0}^{\ell-1} \zeta^s_{jm} . \tag{4.33}$$

Then, using, (4.29) and (4.32), it may be verified that ζ_{jm} satisfies (4.30a) and (4.30b). We now show that (4.30c) is also satisfied. To this end, we first show that τ^s_{jm} has a similar decomposition as does β^s_{jm} in (4.31). Using (4.32) with $s = 0$,

$$\zeta^0_{jm} = \frac{\beta^0_{jm}(\xi)\,\chi^0_p(\eta)(\xi+\eta)^\ell}{(\xi-1)^\ell}$$

so that for $0 < s \leqslant \ell - 1$,

$$\frac{\partial^s \zeta^0_{jm}}{\partial \eta^s}(\xi,-1) = \frac{\beta^0_{jm}(\xi)\,\ell(\ell-1)\dots(\ell-s+1)}{(\xi-1)^s}. \tag{4.34}$$

Using (4.31) with $s = 0$, this yields

$$\frac{\partial^s \zeta^0_{jm}}{\partial \eta^s}(\xi,-1) = (\xi-1)^{2\ell-s-1}\,\phi^0_s(\xi)\,, \quad 0 < s \leqslant \ell - 1\,. \tag{4.35}$$

Assume now that for $t \leqslant i - 1$,

$$\frac{\partial^s \zeta^t_{jm}}{\partial \eta^s}(\xi,-1) = (\xi-1)^{2\ell-s-1}\,\phi^t_s(\xi) \quad t < s \leqslant \ell - 1 \tag{4.36}$$

for some polynomial ϕ^t_s. Then, by (4.32), (4.31), (4.36),

$$\zeta^i_{jm}(\xi,\eta) = (\xi-1)^{\ell-1-i}\,\phi(\xi)\,\chi^i_p(\eta)(\xi+\eta)^\ell$$

for some polynomial $\phi(\xi)$. Hence, for $i < s \leqslant \ell - 1$,

$$\begin{aligned}\frac{\partial^s \zeta^i_{jm}}{\partial \eta^s}(\xi,-1) &= (\xi-1)^{\ell-1-i}\,\phi(\xi)\,\frac{d^i\chi^i_p}{d\eta^i}\,\frac{\partial^{s-i}(\xi+\eta)^\ell}{\partial\eta^{s-i}}\Bigg|_{(\xi,-1)}\\ &= (\xi-1)^{2\ell-s-1}\,\phi^i_s(\xi)\end{aligned} \tag{4.37}$$

for some polynomial ϕ^i_s. (4.35)-(4.37) imply, by induction, that (4.36) holds for all $t = 0, 1, \dots, \ell - 1$. From this, we see

$$\tau^s_{jm}(\xi) = (\xi-1)^{2\ell-s-1}\,\phi_s(\xi) \tag{4.38}$$

for some polynomial $\phi_s(\xi)$.

Now by the definition (4.32a) of ζ^s_{jm} we have

$$\|\zeta^s_{jm}\|_{H^\ell(S)} \leqslant \sum_{\ell_1+\dots+\ell_4 \leqslant l} \left\| \frac{\partial^{\ell_1+\ell_2}}{\partial\xi^{\ell_1}\,\partial\eta^{\ell_2}}\left(\frac{\xi+\eta}{\xi-1}\right)^\ell \frac{d^{\ell_3}}{d\xi^{\ell_3}}(\beta^s_{jm}(\xi) - \right.$$

$$\left. - \tau^s_{jm}(\xi))\,\frac{d^{\ell_4}}{d\eta^{\ell_4}}\chi^s_p(\eta)\right\|_{H^0(S)}.$$

Since on $S = T$

$$\left| \frac{\partial^{\ell_1+\ell_2}}{\partial \xi^{\ell_1} \partial \eta^{\ell_2}} \left(\frac{\xi+\eta}{\xi-1} \right)^l \right| \leq \frac{C}{|\xi-1|^{\ell_1+\ell_2}}$$

we see

$$\| \zeta_{jm}^s \|_{H^\ell(S)} \leq C \sum_{\ell_1+\cdots+\ell_4 \leq \ell} \left\| \frac{1}{(\xi-1)^{\ell_1+\ell_2}} \frac{d^{\ell_3}}{d\xi^{\ell_3}} (\beta_{jm}^s(\xi) - \tau_{jm}^s(\xi)) \right\|_{H^0(\Gamma_1)} \times \left\| \frac{d^{\ell_4}}{d\eta^{\ell_4}} \chi_p^s \right\|_{H^0(I)} \tag{4.39}$$

Using (4.31) and (4.38), we see that

$$\frac{d^t}{d\xi^t} \left(\frac{d^{\ell_3}}{d\xi^{\ell_3}} (\beta_{jm}^s - \tau_{jm}^s) \right) (1) = 0 \quad \text{for} \quad t = 0, 1, \ldots, 2\ell - 1 - s - \ell_3$$

Hence, using Lemma 4.2, for $l_5 = l_1 + l_2 + l_3$,

$$\left\| \frac{1}{(\xi-1)^{\ell_1+\ell_2}} \frac{d^{\ell_3}}{d\xi^{\ell_3}} (\beta_{jm}^s - \tau_{jm}^s) \right\|_{H^0(\Gamma_1)} \leq C (\| \beta_{jm}^s \|_{H^{\ell_5}(\Gamma_1)} + \| \tau_{jm}^s \|_{H^{\ell_5}(\Gamma_1)}) \tag{4.40}$$

We now show that for $i = 0, \ldots, \ell - 1, s \leq \ell - 1, r \leq \ell$,

$$\left\| \frac{\partial^s \zeta_{jm}^i}{\partial \eta^s} \right\|_{H^r(\Gamma_1)} \leq C p^{-\left(k-s-r-\frac{1}{2}\right)} \| u \|_{H^k(\Omega)} . \tag{4.41}$$

First, we see that using (4.34), (4.31), Lemma 4.2, and (4.28), (4.41) is satisfied for $i = 0$. Next, assume that (4.41) holds for all $i \leq n - 1$. Then, we have, using (4.32)

$$\begin{aligned} \frac{\partial^s \zeta_{jm}^n}{\partial \eta^s} (\xi, -1) &= \frac{C}{(\xi-1)^\ell} (\beta_{jm}^n(\xi) - \tau_{jm}^n(\xi))(\xi-1)^{\ell-s+n} \\ &= \frac{C}{(\xi-1)^{s-n}} \left(\beta_{jm}^n(\xi) - \sum_{i=0}^{n-1} \frac{\partial^n \zeta_{jm}^i}{\partial \eta^n} (\xi, -1) \right) \end{aligned}$$

so that

$$\begin{aligned} \left\| \frac{\partial^s \zeta_{jm}^n}{\partial \eta^s} \right\|_{H^r(\Gamma_1)} &\leq C \left[\left\| \frac{\beta_{jm}^n}{(\xi-1)^{s-n}} \right\|_{H^r(\Gamma_1)} + \sum_{i=0}^{n-1} \left\| \frac{1}{(\xi-1)^{s-n}} \frac{\partial^n \zeta_{jm}^i}{\partial \eta^n} \right\|_{H^r(\Gamma_1)} \right] \\ &\leq C \left[\| \beta_{jm}^n \|_{H^{r+s-n}(\Gamma_1)} + \sum_{i=0}^{n-1} \left\| \frac{\partial^n \zeta_{jm}^i}{\partial \eta^n} \right\|_{H^{r+s-n}(\Gamma_1)} \right] \end{aligned}$$

(where we have used (4.31), (4.37) and Lemma 4.2)

$$\leqslant Cp^{-\left(k-s-r-\frac{1}{2}\right)} \|u\|_{H^k(\Omega)}$$

by (4.28) and our hypothesis. Hence, by induction, (4.41) holds for all $i = 0, 1, \ldots, \ell - 1$. This shows that

$$\|\tau_{jm}^s\|_{H^{\ell_5}(\Gamma_1)} \leqslant Cp^{-\left(k-s-\ell_5-\frac{1}{2}\right)} \|u\|_{H^k(\Omega)} \tag{4.42}$$

so that, using (4.39), (4.40), (4.28) and (4.42) with Lemma 4.1, we see

$$\begin{aligned}\|\zeta_{jm}^s\|_{H^\ell(S)} &\leqslant C \sum_{\ell_1+\cdots+\ell_4 \leqslant \ell} (p^{-(k-s-\ell_1-\ell_2-\ell_3-\frac{1}{2})}\|u\|_{H^k(\Omega)})\left(p^{\ell_4-s-\frac{1}{2}}\right) \\ &\leqslant Cp^{-(k-\ell)}\|u\|_{H^k(\Omega)} .\end{aligned}$$

Using (4.33), the same estimate holds for ζ_{jm}, so that (4.30c) is proven.

Hence for any $\gamma = \bar{\Omega}_j \cap \bar{\Omega}_m$, we have constructed a polynomial ζ_{jm} satisfying (4.30). Defining $\tilde{\zeta}_{jm} = \zeta_{jm} \circ F_j$, we see that $\left\|\tilde{\zeta}_{jm}\right\|_{H^\ell(\Omega_j)}$ also satisfies the bound in (4.30c). Moreover, by (4.30a, b), replacing $z_p^{[j]}$ by $z_p^{[j]} - \tilde{\zeta}_{jm}$ on Ω_j achieves the required $C^{\ell-1}$ continuity across γ without altering the jumps in z_p on the other sides of $\partial\Omega_j$. Repeating this process for each γ in the triangulation, we obtain a $u_p = z_p \in C^{(\ell-1)}(\Omega)$ satisfying (4.2). The essential boundary conditions (4.1) on Γ may be imposed on u_p by the same method. This completes the proof of the theorem. □

Remark 4.1 : The function u_p constructed by us belongs to ${}_0S_{p+n}^\ell$ for some n depending ℓ. By suitably changing the constant in (4.2), we obtain $u_p \in {}_0S_p^\ell$ such that (4.1)-(4.2) still hold.

Remark 4.2 : In Theorem 4.2, we have not specified explicitly what the minimum value of p can be. To observe any approximation, in general, we must have $p \geqslant p_0$ (some). This is because for p too small, ${}_0S_p^\ell$ may just contain the function 0, so that the corresponding u_p constructed in Theorem 4.1 may be zero. In general, there will be a p_0 such that $u_p \neq 0$ for $p \geqslant p_0$. This p_0 depends upon the mesh chosen. For a general mesh of triangles and parallelograms, using polynomials of *total* degree $p \geqslant p_0 = 4\ell - 3$ is sufficient (Theorem 6.1 of [8]). However, for triangular meshes, a result of de Boor and Höllig (Theorem 4.7 in [8]) gives $p_0 = 3\ell - 1$ as a less stringent sufficient condition. Even this is not necessary, though, if one considers global spline bases or looks at special meshes. See [8] for details.

In any case, taking the constant C in (4.2) to be large enough (for example $C_1 p_0^{k-\ell}$) technically allows us to assert Theorem 4.1 for all $p \geq 1$.

Remark 4.3 : For $\ell < k \leq 2\ell - \frac{1}{2}$, Theorem 4.1 still holds provided we assume that $u \in \Phi$ instead of $H^k(\Omega) \cap H_0^\ell(\Omega)$, where Φ is defined by interpolation using the K-method,

$$\Phi = (H^r(\Omega) \cap H_0^\ell(\Omega), H_0^\ell(\Omega))_{\theta, \infty} . \tag{4.43}$$

Here, $r > 2\ell - \frac{1}{2}$ and $(H^r(\Omega), H^\ell(\Omega))_{\theta,2} = H^k(\Omega)$. The proof is similar to that of Theorem 4.2 in [3] and is omitted here. Generally, however, the restriction $k > 2\ell - \frac{1}{2}$ is not a severe one, particularly in the light of results in the next section where corner singularities are treated.

As mentioned in the introduction, the results corresponding to Theorem 4.1 in [9], [16] are based on the assumption that $u \in \Phi$, which is not the usual result predicted by elliptic regularity theory.

Theorem 4.1 and Remark 4.2 lead to the following estimate for the rate of convergence of the p-version of the finite element method.

THEOREM 4.2 : *Let $u \in H^k(\Omega) \cap H_0^\ell(\Omega)$, $k > \ell$, be the solution of* (2.2)-(2.4). *Assume further that for $\ell < k \leq 2\ell - \frac{1}{2}$, $u \in \Phi$ defined by* (4.43). *Let u_p be the finite element solution based on the p-version satisfying* (2.9). *Then*

$$\|u - u_p\|_{H^\ell(\Omega)} \leq Cp^{-(k-\ell)} \|u\|_{H^k(\Omega)} \tag{4.44}$$

where C is a constant independent of p, u but depending on the partition of Ω.

Proof: The proof follows from Theorem 4.1 and the fact that

$$\|u - u_p\|_{H^\ell(\Omega)} \leq C \|u - z_p\|_{H^\ell(\Omega)}$$

for any $z_p \in {}_0S_p^\ell$. □

5. THE APPROXIMATION OF SINGULAR FUNCTIONS

In the previous section, we analyzed the approximation of functions which were known to be in $H^k(\Omega) \cap H_0^\ell(\Omega)$, $k > 2\ell - \frac{1}{2}$. In this section, we analyze functions of the type (2.6), which have a singularity at a corner of the domain.

5.1. An approximation result

Let $Q = (-1, 1) \times (-1, 1)$ as before. Let $\tilde{x}_i = x_i + 1$, $i = 1, 2$ and let (r, θ) be the polar coordinates with origin $(-1, 1)$; $r^2 = \tilde{x}_1^2 + \tilde{x}_2^2$, $\theta = \arctan(\tilde{x}_2/\tilde{x}_1)$. For $\kappa > 1$, $0 < \rho < 1$, define

$$\begin{aligned} S_\kappa &= \left\{ x \in Q \,\middle|\, \frac{1}{\kappa}\tilde{x}_1 < \tilde{x}_2 < \kappa\tilde{x}_1 \right\} \\ S_\kappa^\rho &= S_\kappa \cap \{x \mid \tilde{x}_1^2 + \tilde{x}_2^2 < \rho^2\} \\ Q_a &= \{x \mid 0 < \tilde{x}_1 < a, 0 < \tilde{x}_2 < a\}, \quad 0 < a < 1 \\ Q^b &= \{x \mid \tilde{x}_1 > b, \tilde{x}_2 > b\} \cap Q, \quad 0 < b < 1 \\ R_\kappa &= S_\kappa \cap Q_1 \quad \tilde{R}_\kappa = S_\kappa \cap Q_{1/2}\,. \end{aligned} \tag{5.1}$$

Let $\kappa_0 > \kappa > 1$. Figure 5.1 shows the domains under consideration

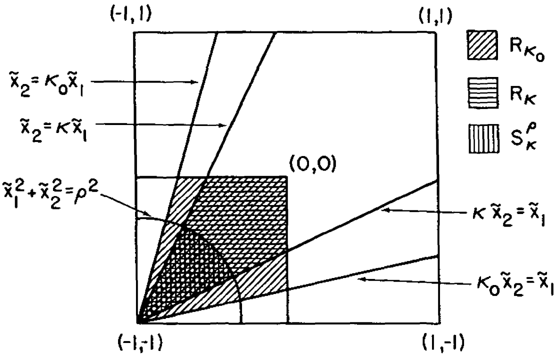

Figure 5.1.

Let

$$\xi(\tilde{x}_1, \tilde{x}_2) = (\tilde{x}_1 - \kappa\tilde{x}_2)^\ell (\kappa\tilde{x}_1 - \tilde{x}_2)^\ell = r^{2\ell}\, \phi_1(\theta)\,. \tag{5.2a}$$

Obviously, $\phi_1(\theta)$ is an analytic function in θ and ξ is a polynomial which satisfies

$$D^{(k)}\xi|_{L_i} = 0 \quad 0 \leqslant |k| \leqslant \ell - 1\,, \quad i = 1, 2 \tag{5.2b}$$

where L_1 and L_2 are the lines $\tilde{x}_1 = \kappa\tilde{x}_2$ and $\tilde{x}_1 = \tilde{x}_2/\kappa$ respectively.

Let, for $\operatorname{Re}\alpha > \ell_{-1}$, $\gamma \geqslant 0$ real,

$$u(\tilde{x}_1, \tilde{x}_2) = \operatorname{Re}\{r^\alpha |\log r|^\gamma \chi(r)\, \phi(\theta)\} \tag{5.3}$$

where $\Phi(\theta)$, $\chi(r)$ are sufficiently smooth real (e.g., C^∞) functions and

$$\begin{aligned}\chi(r) &= 1 \quad \text{for} \quad 0 \leqslant r \leqslant \frac{\rho}{3} \\ &= 0 \quad \text{for} \quad \frac{2\rho}{3} \leqslant r\,, \quad 0 < \rho < \frac{1}{2}\end{aligned}$$

is a function defined on Q. We shall assume that u satisfies (5.2b) on L_1, L_2 and has support in R_{κ_0}. Then we see that

$$\tilde{u}(\tilde{x}_1, \tilde{x}_2) = \xi(\tilde{x}_1, \tilde{x}_2)\, u_0(\tilde{x}_1, \tilde{x}_2) \tag{5.4a}$$

where

$$u_0(\tilde{x}_1, \tilde{x}_2) = \operatorname{Re}\{r^{\alpha - 2\ell} |\log r|^\gamma \chi(r)\, \Psi(\theta)\} \tag{5.4b}$$

with $\Psi(\theta)$ a smooth (e.g., C^∞) function.

The main approximation result we prove is

THEOREM 5.1 : *Let u be given by* (5.3) *and satisfy* (5.2b) *on L_1, L_2. Then there exists $z_p \in \mathcal{P}_p(Q)$ such that for $0 \leqslant |k| \leqslant \ell - 1$, $D^{(k)} z_p = 0$ on the lines L_i, $i = 1, 2$ and for $\kappa_0 > \kappa$,*

$$\|u - z_p\|_{H^\ell(\tilde{R}_{\kappa_0})} \leqslant C\, |\log p|^\gamma\, p^{-2(\operatorname{Re}\alpha - \ell + 1)} \tag{5.5}$$

where C is a constant independent of p.

We will require a series of lemmas to prove Theorem 5.1.

Let $\omega(r)$, $0 \leqslant r < \infty$ be a C^∞ function satisfying

$$\begin{aligned}\omega(r) &= 0 \quad \text{for} \quad 0 \leqslant r \leqslant 1 \\ &= 1 \quad \text{for} \quad 2 \leqslant r < \infty\,.\end{aligned}$$

For any $\Delta > 0$, define

$$\omega^\Delta(r) = \omega\left(\frac{r}{\Delta}\right). \tag{5.6}$$

Then we decompose u_0 by

$$u_0 = v + w \tag{5.7}$$

where

$$v = \omega^\Delta u_0 \tag{5.8a}$$

$$w = (1 - \omega^\Delta)\, u_0\,. \tag{5.8b}$$

It can be readily see that

$$\begin{aligned}v &= 0 \quad \text{for} \quad 0 \leqslant r \leqslant \Delta \\ w &= 0 \quad \text{for} \quad r \geqslant 2\Delta\,.\end{aligned}$$

LEMMA 5.1 : *Given* $k = (k_1, k_2)$, *there exists a constant* $C(k)$ *such that for* $x = (x_1, x_2) \in R_{\kappa_0}$

$$\begin{aligned} |D^{(k)} v| &\leq C(k) |\log \Delta|^{\gamma} (1 + x_i)^{\tilde{\alpha} - 2\ell - |k|} && \text{on } R_\kappa \\ &= 0 && \text{on } S_\kappa^\Delta \end{aligned} \tag{5.9}$$

where $\tilde{\alpha} = \mathrm{Re}\,(\alpha)$.

Proof: For α real the lemma follows by taking $\alpha = \alpha - 2\ell + 2$ in Lemma 5.1 of [3]. The result for α complex follows easily.

In what follows, we will assume that v satisfies (5.9) and not the explicit form (5.4b), (5.8a).

Let

$$v(x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} P_i(x_1) P_j(x_2) \tag{5.10}$$

where $P_i(x_\ell) = P_i(x_\ell, \beta, \beta)$, $\beta > -\frac{1}{2}$ are Jacobi polynomials of index β which will be determined later. Then

$$a_{ij} = C_i C_j (i+1)(j+1) \int_{-1}^{+1} \int_{-1}^{+1} v(x_1, x_2) P_i(x_1) P_j(x_2) (1 - x_1^2)^\beta (1 - x_2^2)^\beta \, dx_1 \, dx_2 \tag{5.11}$$

where C_i, C_j are bounded from above and below independently of i, j but depending on β (see [11], p. 841, formula 7.391.1). Define

$$v_p(x_1, x_2) = \sum_{i=0}^{p} \sum_{j=0}^{p} a_{ij} P_i(x_1) P_j(x_2) \tag{5.12}$$

$$b_i(x_2) = \sum_{j=0}^{\infty} a_{ij} P_j(x_2), \tag{5.13a}$$

$$b_i^{[p]}(x_2) = \sum_{j=0}^{p} a_{ij} P_j(x_2) \tag{5.13b}$$

with

$$b_i(x_2) = C_i (i+1) \int_{-1}^{+1} v(x_1, x_2)(1 - x_1^2)^\beta P_i(x_1) \, dx_1 . \tag{5.14}$$

It can be readily seen that

$$v = \sum_{i=0}^{\infty} b_i(x_2)\, P_i(x_1) \tag{5.15a}$$

$$v_p = \sum_{i=0}^{p} b_i^{[p]}(x_2)\, P_i(x_1)\,. \tag{5.15b}$$

Let

$$\psi_p(x_1, x_2) = \sum_{i=0}^{p} b_i(x_2)\, P_i(x_1) \tag{5.16}$$

then

$$v - v_p = (v - \psi_p) + (\psi_p - v_p) = \sigma_p + \rho_p \tag{5.17}$$

The following lemma follows immediately by taking $\alpha = \tilde{\alpha} - 2\ell + 2$ in Lemma 5.3 and 5.4 of [3].

LEMMA 5.2 : *Let* $\tilde{\alpha} - m + \frac{\beta}{2} - 2\ell + \frac{3}{4} < 0$. *Then*

$$\left| \frac{d^m b_i(x_2)}{dx_2^m} \right| \leq C (i+1)^{\frac{1}{2}} |\log \Delta|^{\gamma} (1 + x_2)^{\tilde{\alpha} - m + \frac{\beta}{2} - 2\ell + \frac{3}{4}} \tag{5.18a}$$

$$\left| \frac{d^m b_i(x_2)}{dx_2^m} \right| \leq \frac{C |\log \Delta|^{\gamma}}{(i+1)^{\frac{1}{2}}} (1 + x_2)^{\tilde{\alpha} - m + \frac{\beta}{2} - 2\ell + \frac{1}{4}}\,. \tag{5.18b}$$

Let us now analyze $\rho_p = \psi_p - v_p$ given in (5.17). We have

$$\rho_p(x_1, x_2) = \sum_{i=0}^{p} [b_i(x_2) - b_i^{[p]}(x_2)]\, P_i(x_1)$$

so that for $k = (k_1, k_2)$,

$$D^{(k)} \rho_p = \sum_{i=k_1}^{p} \frac{d^{k_2}}{dx_2^{k_2}} (b_i - b_i^{[p]})(x_2)\, P_i^{(k_1)}(x_1)\,.$$

Now, for $t > 0$ integer, we have by [5], formula A.2.15

$$P_i^{(t)}(x, \beta, \beta) = \frac{1}{2^t} (2\beta + i + 1) \dots (2\beta + i + t)\, P_{i-t}(x, \beta + t, \beta + t)\,.$$

Hence, for $0 \leqslant m + k_2 \leqslant p + 1$,

$$\begin{aligned}
A_1 &= \int_{-1}^{+1} \left(\int_{-1}^{+1} (D^{(k)} \rho_p(x_1, x_2))^2 (1 - x_1^2)^{\beta + k_1} dx_1 (1 - x_2^2)^{\beta + k_2} dx_2 \right) \\
&\leqslant C \int_{-1}^{+1} \sum_{i = k_1}^{p} i^{2k_1 - 1} \left(\frac{d^{k_2}}{dx_2^{k_2}} (b_i - b_i^{[p]})(x_2) \right)^2 (1 - x_2^2)^{\beta + k_2} dx_2 \\
&\leqslant C \sum_{i=1}^{p} i^{2k_1 - 1} \int_{-1}^{+1} \left(\sum_{p+1}^{\infty} a_{ij} P_j^{(k_2)}(x_2) \right)^2 (1 - x_2^2)^{\beta + k_2} dx_2 \\
&\leqslant C \sum_{i=1}^{p} i^{2k_1 - 1} \sum_{j=p+1}^{\infty} a_{ij}^2 j^{2k_2 - 1} \\
&\leqslant \frac{C}{p^{2m}} \sum_{i=1}^{p} i^{2k_1 - 1} \sum_{j=p+1}^{\infty} \frac{a_{ij}^2 (j + m + k_2)!}{(j - m - k_2)!\, j} \\
&\leqslant \frac{C}{p^{2m}} \sum_{i=1}^{p} i^{2k_1 - 1} \int_{-1}^{+1} \left(\frac{d^{m+k_2} b_i(x_2)}{dx_2^{m+k_2}} \right)^2 (1 - x_2^2)^{\beta + m + k_2} dx_2
\end{aligned} \tag{5.19}$$

Using (5.14), we see that the support of $b_i(x_2)$ lies in $I_1 = [-1 + \Delta \sin \theta_0, 0]$ where $\tan \theta_0 = \frac{1}{\kappa_0}$. Hence, from (5.18b) and (5.19), for $\tilde{\alpha} - m + \frac{\beta}{2} - 2\ell + \frac{3}{4} < 0$,

$$\begin{aligned}
A_1 &\leqslant \frac{C}{p^{2m}} \sum_{i=1}^{p} i^{2k_1 - 1} \int_{-1 + \Delta \sin \theta_0}^{0} \left(\frac{d^{m+k_2} b_i}{dx_2^{m+k_2}} \right)^2 (1 - x_2^2)^{\beta + m + k_2} dx_2 \\
&\leqslant \frac{C |\log \Delta|^{2\gamma}}{p^{2m}} \sum_{i=1}^{p} i^{2k_1 - 1} \int_{-1 + \Delta \sin \theta_0}^{0} \frac{1}{i} (1 + x_2)^{2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + \frac{1}{2}} dx_2 \\
&\leqslant \frac{C |\log \Delta|^{2\gamma}}{p^{2m - 2k_1 + 1}} \Delta^{2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + \frac{3}{2}}
\end{aligned} \tag{5.20a}$$

for the case $k_1 \geqslant 1$, provided $2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + \frac{3}{2} < 0$. For the case $k_1 = 0$, we use (5.18a) instead of (5.18b) to bound $\frac{d^{m+k_2} b_i}{dx_2^{m+k_2}}$ and obtain

$$A_1 \leqslant \frac{C |\log \Delta|^{2\gamma}}{p^{2m-1}} \Delta^{2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + \frac{5}{2}} \tag{5.20b}$$

provided $2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + \frac{5}{2} < 0$.

Similarly, we estimate the term $D^{(k)}\sigma_p$ with $0 \leqslant m + k_2 \leqslant p + 1$, $m \geqslant k_1$. We have

$$\sigma_p(x_1, x_2) = \sum_{i=p+1}^{\infty} b_i(x_2)\, P_i(x_1)$$

$$B_1 = \int_{-1}^{+1} \left(\int_{-1}^{+1} (D^{(k)}\sigma_p)^2 (1 - x_1^2)^{\beta + k_1}\, dx_1 \right) (1 - x_2^2)^{\beta + k_2}\, dx_2$$

$$\leqslant C \int_{-1}^{+1} \left(\sum_{i=p+1}^{\infty} i^{2k_1 - 1} \left(\frac{d^{k_2} b_i}{dx_2^{k_2}}(x_2) \right)^2 \right) (1 - x_2^2)^{\beta + k_2}\, dx_2$$

$$\leqslant \frac{C}{p^{2(m-k_1)}} \int_{-1}^{+1} \sum_{i=p+1}^{\infty} \left(\frac{d^{k_2} b_i(x_2)}{dx_2^{k_2}} \right)^2 \frac{(i+m)!}{i\,(i-m)!} (1 - x_2^2)^{\beta + k_2}\, dx_2$$

$$\leqslant \frac{C}{p^{2(m-k_1)}} \int_{-1}^{+1} \int_{-1}^{+1} \left(\frac{\partial^{m+k_2} v}{\partial x_1^m\, \partial x_2^{k_2}} \right)^2 (1 - x_1^2)^{\beta + m}\, dx_1 (1 - x_2^2)^{\beta + k_2}\, dx_2 \quad (5.21)$$

Since the support of v lies in $R_{\kappa_0} - S_{\kappa_0}^{\Delta}$, we can use Lemma 5.1 and obtain with

$$I_2 = \left[-1 + \frac{1}{\kappa_0}(1 + x_2),\ -1 + \kappa_0(1 - x_2) \right]$$

$$B_1 \leqslant \frac{C\,|\log \Delta|^{2\gamma}}{p^{2(m-k_1)}} \int_{-1+\Delta \sin\theta_0}^{0} \int_{I_2} \Big((1 + x_1)^{2(\tilde{\alpha} - 2\ell - m - k_2)} (1 - x_1^2)^{\beta + m} (1 - x_2^2)^{\beta + k_2} \Big)\, dx_1\, dx_2$$

$$\leqslant \frac{C\,|\log \Delta|^{2\gamma}}{p^{2(m-k_1)}} \int_{-1+\Delta \sin\theta_0}^{0} (1 + x_2)^{2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + 1}\, dx_2$$

$$\leqslant \frac{C\,|\log \Delta|^{2\gamma}}{p^{2(m-k_1)}} \Delta^{2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + 2} \quad (5.22)$$

provided that $2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + 2 < 0$. Hence, we obtain the following lemma.

LEMMA 5.3 : *Let ρ_p and σ_p be as defined in* (5.17). *Then for $0 \leqslant m + \ell \leqslant p + 1$ and $0 \leqslant |k| \leqslant \ell$*

$$\int_{-1}^{+1} \int_{-1}^{+1} (D^{(k)}\rho_p)^2 (1 - x_1^2)^{\beta + k_1} (1 - x_2^2)^{\beta + k_2}\, dx_1\, dx_2 \leqslant$$
$$\leqslant \frac{C\,|\log \Delta|^{2\gamma}}{p^{2m - 2\tilde{k}_1 + 1}} \Delta^{2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + \frac{3}{2} + t} \quad (5.23)$$

where $\tilde{k}_1 = \max\{1, k_1\}$ *and* $t = 1$ *if* $k_1 = 0$, $t = 0$ *otherwise, provided that* $\tilde{\alpha} - m + \frac{\beta}{2} - 2\ell + \frac{3}{4} < 0$ *and* $2\tilde{\alpha} - m + 2\beta - 4\ell + \frac{5}{2} < 0$

$$\int_{-1}^{+1}\int_{-1}^{+1} (D^{(k)}\sigma_p)^2 (1 - x_1^2)^{\beta + k_1} (1 - x_2^2)^{\beta + k_2}\, dx_1\, dx_2 \leqslant$$

$$\leqslant \frac{C\,|\log \Delta|^{2\gamma}}{p^{2(m - k_1)}}\,\Delta^{2\tilde{\alpha} - m + 2\beta - 4\ell - k_2 + 2} \qquad (5.24)$$

provided that $2\tilde{\alpha} - m + 2\beta - 4\ell + 2 < 0$. *The constant C is independent of* k, Δ, p *but depends on* $\tilde{\alpha}$, β, γ, m, ℓ.

Let $Q^{2\Delta}$ be as in (5.1) and define

$$R_\kappa^\Delta = R_\kappa \cap Q^{2\Delta}.$$

Also, for $f(x_1, x_2)$, $(x_1, x_2) \in Q$ and $\Delta < \frac{1}{4}$, define

$$\begin{aligned} f_\Delta(x_1, x_2) &= f(x_1 - 2\Delta, x_2 - 2\Delta), \quad &(x_1, x_2) \in Q^{2\Delta} \\ &= 0 &(x_1, x_2) \in Q - Q^{2\Delta}. \end{aligned} \qquad (5.25)$$

LEMMA 5.4 : *Let* $\xi(x_1, x_2)$ *be given by* (5.2a) *and let* $0 < \Delta < \frac{1}{4}$. *Then, on* $R_{\kappa_0}^\Delta$,

$$|D^{(k)}\xi_\Delta(x_1, x_2)| \leqslant C(1 - x_1^2)^{t_1}(1 - x_2^2)^{t_2} \qquad (5.26)$$

for any t_1, $t_2 \geqslant 0$ *satisfying* $t_1 + t_2 = 2\ell - |k|$.

Proof: The proof is essentially the same as that of Lemma 5.6 in [3] and is omitted here. □

LEMMA 5.5 : *Let* v *satisfy* (5.9) *and* v_p *be given by* (5.12). *Then for* $\Delta = p^{-2}$

$$\|\xi_\Delta(v - v_p)\|_{H^\ell(R_{\kappa_0}^\Delta)} \leqslant C\,|\log p|^\gamma\, p^{-2(\tilde{\alpha} - \ell + 1)} \qquad (5.27)$$

where C is a constant independent of p.

Proof: We first estimate $\|\xi_\Delta\,\rho_p\|_{H^\ell(R_{\kappa_0}^\Delta)}$ where ρ_p (and σ_p) are as in (5.17). To this end, let us estimate $D_1 = \|(D^{(r)}\xi_\Delta)(D^{(s)}\rho_p)\|_{L_2(R_{\kappa_0}^\Delta)}$ with $|r| + |s| \leqslant \ell$. Using (5.26), we have for any t_1, $t_2 \geqslant 0$ with $t_1 + t_2 = 2\ell - |r|$

$$D_1^2 \leqslant \iint_{R_{\kappa_0}^\Delta} (1 - x_1^2)^{2t_1}(1 - x_2^2)^{2t_2}(D^{(s)}\rho_p)^2\, dx_1\, dx_2.$$

Let us choose t_1, t_2 so that $-2t_i + s_i \geqslant -2\ell$, $i = 1, 2$. Assume $\beta > 2\ell$, so that $\beta - 2t_i + s_i > 0$. Then, because on $R^{\Delta}_{\kappa_0}$,

$$0 < C \leqslant \frac{1 - x_i^2}{2\Delta},$$

we get using Lemma 5.3

$$\begin{aligned} D_1^2 &\leqslant C \int_{-1}^{+1} \int_{-1}^{+1} \frac{(1 - x_1^2)^{\beta + s_1}}{\Delta^{\beta - 2t_1 + s_1}} \frac{(1 - x_2^2)^{\beta + s_2}}{\Delta^{\beta - 2t_2 + s_2}} (D^{(s)} \rho_p)^2 \, dx_1 \, dx_2 \\ &\leqslant \frac{C |\log \Delta|^{2\gamma}}{p^{2m - 2\tilde{s}_1 + 1}} \Delta^{2\tilde{\alpha} - m - s_2 + \frac{3}{2} - 2|r| - |s| + t} \end{aligned}$$

where $\tilde{s}_1 = \max\{1, s_1\}$ and $t = 1$ if $s_1 = 0$, $t = 0$ otherwise, provided that $\tilde{\alpha} - m + \frac{\beta}{2} - 2\ell + \frac{3}{4} < 0$ and $2\tilde{\alpha} - m + 2\beta - 4\ell + \frac{5}{2} < 0$. Choosing m large enough and $\Delta = p^{-2}$, we get

$$\begin{aligned} D_1^2 &\leqslant C |\log p|^{2\gamma} \frac{p^{-4\tilde{\alpha} + 2m + 2s_2 - 3 + 4|r| + 2|s| - 2t}}{p^{2m - 2\tilde{s}_1 + 1}} \\ &= C |\log p|^{2\gamma} p^{-4\tilde{\alpha} + 4(|r| + |s|) - 4} \\ &\leqslant C |\log p|^{2\gamma} p^{-4(\tilde{\alpha} - \ell + 1)}. \end{aligned}$$

We may similarly show that for $|r| + |s| \leqslant \ell$,

$$D_2^2 = \| (D^{(r)} \xi_\Delta)(D^{(s)} \sigma_p) \|^2_{L_2(R^{\Delta}_{\kappa_0})} \leqslant C |\log p|^{2\gamma} p^{-4(\tilde{\alpha} - \ell + 1)}.$$

This proves the lemma. □

For the case that $\tilde{\alpha} > 1$, $\tilde{\alpha}$ not an integer, let $k = [\tilde{\alpha}]$ be the largest integer less than $\tilde{\alpha}$. For q an integer, $0 \leqslant q \leqslant k$, we denote by $v^{[q]}$ the q-th derivative of v along the direction $\hat{n}$, where $\hat{n}$ is the unit vector along the line $x_1 = x_2$. Then $v^{[q]}$ will satisfy (5.9) in Lemma 5.1 with $\tilde{\alpha}$ replaced by $\tilde{\alpha} - q > 0$. Hence, using Lemma 5.5, we get

$$\| \xi_\Delta (v^{[q]} - v_p^{[q]}) \|_{H^{\ell}(R^{\Delta}_{\kappa_0})} \leqslant C |\log p|^{\gamma} p^{-2(\tilde{\alpha} - q - \ell + 1)}. \tag{5.28}$$

Let ω^{Δ} be defined by (5.6) and ω^{Δ}_{Δ} be its translation given by (5.25). Then (see (5.7))

$$\begin{aligned} u_{0\Delta} &= u_{0\Delta} \, \omega^{\Delta}_{\Delta} + u_{0\Delta} (1 - \omega^{\Delta}_{\Delta}) \\ &= v_\Delta + w_\Delta . \end{aligned} \tag{5.29}$$

Since $u \in H^{\ell}(R_{\kappa_0})$, then $u_\Delta \in H^{\ell}(R_{\kappa_0})$ and hence

$$\xi_\Delta w_\Delta = u_\Delta(1-\omega_\Delta^\Delta) \in H^\ell(R_{\kappa_0}^\Delta) .$$

LEMMA 5.6 : *Let* $\Delta = p^{-2}$, $\tilde{\Delta} = 2\sqrt{2}\,\Delta$. *Then for* $k = [\tilde{\alpha}]$, $\tilde{\alpha} > \ell - 1$ *non integer*

$$\left\| \xi_\Delta \left(v_\Delta - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}^i}{i!} v^{[i]} \right) \right\|_{H^\ell(R_{\kappa_0}^\Delta)} \leqslant C\,|\log p|^\gamma p^{-2(\tilde{\alpha}-\ell+1)} \quad (5.30)$$

$$\|\xi_\Delta w_\Delta\|_{H^\ell(R_{\kappa_0}^\Delta)} \leqslant C\,|\log p|^\gamma p^{-2(\tilde{\alpha}-\ell+1)} \quad (5.31)$$

where C *is independent of* p, Δ.

Proof: By Taylor's theorem and Lemma 5.1, for any $(x_1, x_2) \in R_{\kappa_0}^\Delta$, and $s = (s_1, s_2)$

$$\left| D^{(s)}\left(v_\Delta - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}^i}{i!} v^{[i]} \right)(x_1, x_2) \right| \leqslant$$

$$\leqslant C\,\Delta^{k+1} \left| \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right)^{k+1} D^{(s)} v(x_1 - \theta, x_2 - \theta) \right|$$

$$\leqslant C\,\Delta^{k+1}(1+x_1)^{\tilde{\alpha}-2\ell-|s|-k-1}|\log\Delta|^\gamma$$

where $0 < |\theta| < 2\Delta$. Hence, using Lemma 5.4, we get for $\Delta = p^{-2}$, $|r| + |s| = \ell$ and $t_1 + t_2 = 2\ell - |r|$

$$\left\| D^{(r)}(\xi_\Delta)\, D^{(s)}\left(v_\Delta - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}^i}{i!} v^{[i]} \right) \right\|^2_{L_2(R_{\kappa_0}^\Delta)}$$

$$\leqslant \quad C \iint_{R_\kappa^\Delta} (1-x_1^2)^{2t_1}(1-x_2^2)^{2t_2}\,\Delta^{2(k+1)}|\log\Delta|^{2\gamma} \times$$

$$\times\ (1+x_1)^{2\tilde{\alpha}-4\ell-2|s|-2k-2}\,dx_1\,dx_2$$

$$\leqslant C\,\Delta^{2(k+1)}|\log\Delta|^{2\gamma} \int_{2\Delta}^{1} (1+x_1)^{2(\tilde{\alpha}-k-|r|-|s|)-1}\,dx_1$$

$$\leqslant C\,\Delta^{2(k+1)}|\log\Delta|^{2\gamma}\,\Delta^{2(\tilde{\alpha}-k-|r|-|s|)}$$

$$\leqslant C\,|\log p|^{2\gamma} p^{-4(\tilde{\alpha}-\ell+1)} .$$

In the above inequality, we used the obvious fact that $\tilde{\alpha} - k - \ell < 0$. We may bound the other terms in (5.30) analogously.

Let us now prove (5.31). Let $K = \left\{ (r,\theta) \middle| 0 < r < 2\Delta, 0 < \theta < \frac{\pi}{2} \right\}$.

Then it may be seen that

$$\| \xi_\Delta w_\Delta \|_{H^\ell(R^\Delta_{\kappa_0})} = \| \xi w \|_{H^\ell(K)} .$$

Using (5.2a) we have

$$\left| \frac{\partial^t \xi(r, \theta)}{\partial r^t} \right| \leqslant C r^{2\ell - t} .$$

Also, by (5.8b), (5.3),

$$\left| \frac{\partial^s w(r, \theta)}{\partial r^s} \right| \leqslant C |\log r|^\gamma r^{\tilde{\alpha} - 2\ell - s} \quad \text{for} \quad r < 2\Delta \\ = 0 \quad \text{for} \quad r \geqslant 2\Delta . \tag{5.32}$$

For $\Delta = p^{-2}$ and $t + s = \ell$, we therefore get

$$\begin{aligned} \left\| \frac{\partial^t \xi}{\partial r^t} \frac{\partial^s w}{\partial r^s} \right\|^2_{L^2(K)} &= \int_0^{\frac{\pi}{2}} \int_0^{2\Delta} \left(\frac{\partial^t \xi}{\partial r^t} \right)^2 \left(\frac{\partial^s w}{\partial r^s} \right)^2 r \, dr \, d\theta \\ &\leqslant C \int_0^{2\Delta} |\log r|^{2\gamma} r^{4\ell - 2t + 2\tilde{\alpha} - 4\ell - 2s + 1} dr \\ &\leqslant C |\log \Delta|^{2\gamma} \Delta^{2(\tilde{\alpha} - \ell + 1)} \\ &\leqslant C |\log p|^{2\gamma} p^{-4(\tilde{\alpha} - \ell + 1)} \end{aligned}$$

where we have used the fact that $\tilde{\alpha} > \ell - 1$. The other terms in (5.31) can be treated similarly. This completes the proof of Lemma 5.6. □

We now prove our main result.

Proof of Theorem 5.1 : Let $\hat{x}_i = x_i - 2\Delta$, $i = 1, 2$ and let $\hat{S}_\kappa$ be the translation of S_κ obtained by this transformation. Let

$$z_{p\Delta} = \xi_\Delta \left(\sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}^i}{i!} v_p^{[i]} \right)$$

where $k = [\tilde{\alpha}]$. Then $z_{p\Delta} \in \mathcal{P}_{p+2\ell}(Q)$ and for $|r| \leqslant \ell - 1$, $D^{(r)} z^{p\Delta} = 0$ on the sides of $\hat{S}_\kappa$. Moreover

$$\begin{aligned} \| u_\Delta - z_{p\Delta} \|_{H^\ell(R^\Delta_{\kappa_0})} &= \left\| \xi_\Delta \left(u_{0\Delta} - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}^i}{i!} v_p^{[i]} \right) \right\|_{H^\ell(R^\Delta_{\kappa_0})} \\ &\leqslant \| \xi_\Delta w_\Delta \|_{H^\ell(R^\Delta_{\kappa_0})} + \left\| \xi_\Delta \left(v_\Delta - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}^i}{i!} v^{[i]} \right) \right\|_{H^\ell(R^\Delta_{\kappa_0})} \\ &\quad + \sum_{i=0}^{k} \frac{\tilde{\Delta}^i}{i!} \| \xi_\Delta (v^{[i]} - v_p^{[i]}) \|_{H^\ell(R^\Delta_{\kappa_0})} \\ &\leqslant C |\log p|^\gamma p^{-2(\tilde{\alpha} - \ell + 1)} \end{aligned}$$

where we have used (5.28), (5.30) and (5.31). We now translate back to S_κ and suitably adjust the constant C in (5.5) to get the theorem. □

Remark 5.1 : We have proven a slightly stronger result than Theorem 5.1. It is sufficient to assume that v and w defined by (5.8a), (5.8b) satisfy (5.9) and (5.32) respectively.

Remark 5.2 : For the case that Re α is an integer and $\gamma = 0$, u will be arbitrarily smooth. Hence, the above result is too pessimistic and the results from Section 4 will apply.

Remark 5.3 : From the proof, it may be seen that the internal angle ω_i between γ_i and γ_{i+1} could equal 2π, i.e., we may also consider the slit domain.

5.2. Approximation over the domain Ω

We now return to the problem of approximation of the functions $u_2^{[i]}$ given by (2.6).

Let the vertex A_ι of Ω be at the origin 0. Let the part of Ω containing the elements with vertices at 0 be as shown as in figure 5.2. We assume that we have only triangular elements. The case when elements are parallelograms does not change the argument.

Let $\bar{\Omega} = \bigcup_{\iota=1}^{m} \bar{T}_i$, $\tilde{\Gamma} = \bigcup_{\iota=1}^{m} \overline{B_iB}_{i+1}$. Let the line $\overline{OB}_j$ have the coordinate θ_j, $j = 1, \dots, m+1$. Denote $D_\rho = \{x \mid x_1^2 + x_2^2 < \rho\}$ and assume that $D_{\rho_0} \subset \Omega$, $0 < \rho_0 \leqslant 1$. We then obtain the following theorem.

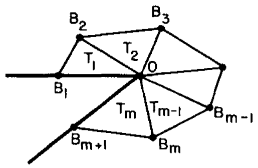

Figure 5.2.

THEOREM 5.2 : *Let u be the function given by (5.3) with $\rho \leqslant \rho_0\,\mu$, μ sufficiently small. Then there exists $z_p \in H^\ell(\Omega)$ satisfying $z_p \in \mathscr{P}_p(T_i)$, $i = 1, \dots, m$; $D^{(r)} z_p = 0$ on OB_1, OB_{m+1} and $\tilde{\Gamma}$ for $0 \leqslant |r| \leqslant \ell - 1$ and*

$$\| u - z_p \|_{H^\ell(\Omega)} \leqslant C\, |\log p\,|^{\gamma}\, p^{-2(\operatorname{Re}\alpha - \ell + 1)} \tag{5.33}$$

where C is independent of p.

The proof of the above theorem is very similar to that of [3], Theorem 5.2 and only a brief outline is given here. Essentially, we first consider the case for which $D^{(r)} \Phi(\theta_j) = 0$ for $j = 1, \ldots, m+1$, $0 \leq |r| \leq \ell - 1$. We may then map $S = \{(r, \theta) \mid \theta_j < \theta < \theta_{j+1}\}$ onto R_κ by a linear mapping $\mathcal{T}$ and consider the image $\tilde{u}$ of u on $\tilde{T}_j = \mathcal{T}(T_j)$. Let η_j be a polynomial function of degree $\leq \ell$ satisfying $D^{(r)} \eta_j = 0$ for $0 \leq r \leq \ell - 1$ on $\mathcal{T}(B_j \bar{B}_{j+1})$. Then, after suitably extending $\tilde{u}$ outside R_κ, the function $\tilde{u}/\eta_j$ satisfies the conditions mentioned in Remark 5.1 to Theorem 5.1. Hence, we may approximate $\tilde{u}_j/\eta_j$ by a function z_p^* satisfying (5.33) on $\tilde{T}_j$ and hence $z_p^* \eta_j = z_{p+\ell}^*$ satisfies (5.33) too, proving the result for this case.

For the case when $D^{(r)} \eta_j = 0$ for $j \neq j_0$, $0 \leq r \leq \ell - 1$, the triangles $T_{j_0 - 1}$, T_{j_0} are mapped together into R_κ and the argument repeated. The details may be found in [3].

Remark 5.4 : The function we constructed was in $\mathcal{P}_{p+\ell}(T_i)$. By suitably changing the constant in (5.33), we may obtain a function in $\mathcal{P}_p(T_i)$.

Remark 5.5 : (5.33) obviously yields the estimate

$$\|u - z_p\|_{Hs(\Omega)} \leq C \, |\log p|^\gamma \, p^{-2(\operatorname{Re} \alpha - s + 1)}$$

for $0 \leq s \leq \ell$.

6. THE RATE OF CONVERGENCE OF THE *p*-VERSION OF THE FINITE ELEMENT METHOD

We now summarize our results from Sections 4 and 5 and briefly remark on some generalizations.

The following theorem follows immediately from Theorem 4.2 and Theorem 5.2.

THEOREM 6.1 : *Let u be the solution of problem* (2.2)-(2.4). *Assume that u can be written in the form* (2.5), (2.6) *and in addition that for* $\ell < k \leq 2\ell - \frac{1}{2}$, $u \in \Phi$ *defined by* (4.43). *Let* u_p *be the finite element solution as described in Section* 2.3 *with triangular or parallelogram elements. Then*

$$\|u - u_p\|_{H^1(\Omega)} \leq C p^{-\mu} |\log p|^\nu R \tag{6.1}$$

where, letting $\alpha^* = \min_i \operatorname{Re} \alpha_1^{[i]}$,

$$\mu = \min_i (k - \ell, 2(\operatorname{Re} \alpha_1^{[i]} - \ell + 1)) = \min (k - \ell, 2(\alpha^* - \ell + 1)) \quad (6.2)$$

$$\nu = \max \{\gamma_1^{[j]} : \operatorname{Re} \alpha_1^{[j]} = \alpha^*\} \quad \text{if} \quad \mu = 2(\alpha^* - \ell + 1)$$
$$= 0 \quad \text{otherwise} \quad (6.3)$$

$$R = \|u_1\|_{H^k(\Omega)} + \sum_{L,I} |C_j^{[i]}| \quad (6.4)$$

Remark 6.1 : Theorem 6.1 has been stated only for the model problem (2.2)-(2.4). It is obvious, however, that the theorem holds for any elliptic problem or order 2 ℓ if the solution has the form (2.5), (2.6) or when (2.6) is different but has the same character concerning the growth of its derivatives. Moreover, as mentioned in Section 2.1, more general boundary conditions may be also be treated.

Remark 6.2 : We assumed that polynomials of the same degree are used over each element. Our results and proofs may be modified in an obvious way when different degrees are used over different elements.

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