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An upwind finite element method for singularly perturbed elliptic problems and local estimates in the L^∞-norm

AN UPWIND FINITE ELEMENT METHOD
FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS
AND LOCAL ESTIMATES IN THE L^∞-NORM (*)

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Communicated by P. G. CIARLET

Abstract. — We consider a finite element method for solving singularly perturbed second order elliptic problems in 2 D domains. We give a strategy for proving local L^∞-error estimates for a class of upwind-FEM and use this strategy to derive such local estimates for a special finite element method.

Résumé. — On étudie une méthode d'éléments finis adaptée à la résolution de problèmes elliptiques du second ordre avec perturbations singulières, posés sur des domaines bidimensionnels. On décrit une méthode permettant d'établir des estimations L^∞ locales de l'erreur pour une classe de méthodes « upwind », et on en déduit des estimations locales pour une méthode d'éléments finis particulière.

1. INTRODUCTION

We consider the problem

$$- \varepsilon \Delta u + b(x) \nabla u + c(x)u = f(x) \quad \text{in} \quad \Omega \subset R^2$$
$$u = 0 \quad \text{on} \quad \Gamma = \partial \Omega$$

with the small parameter $0 < \varepsilon \ll 1$. The qualitative behaviour of the solution is characterized by the existence of boundary layers (narrow regions where the norms $\|u\|_{k,p}$, $k \geq 1$, are not bounded independently of ε) which influence the properties of discretization methods. So the

application of standard finite element methods results in undesired oscil-
lations in the numerical solution u_h unless the discretization parameter is
very small. These oscillations can spread over a region much larger than the
boundary layer. A strong dependence on ε also is to be seen in the classical
error estimates which for piecewise linear discrete solutions can be written
in the form

$$\sqrt{\varepsilon}\,|u - u_h|_{1,2} + \|u - u_h\|_{0,2} \leqslant Ch\,\|u\|_{2,2}$$

where generally $\|u\|_{2,2} = O\left(\varepsilon^{-\frac{3}{2}}\right)$.

($\|\cdot\|_{k,p,G}$ is the norm in $W^{k,p}(G)$. If $G = \Omega$ we omit the index G.)

It is obvious to try to improve the estimates in subdomains of where
boundary layers are excluded (local estimates).

One objective of local estimates is with their help to ensure that the
numerical method « recognizes » boundary layers. Some literature on the
subject of local estimates can be found in [9] (using asymptotic means), [3],
[4], [9], [10] (using cut-off techniques, [10] for $\varepsilon = O(1)$).

In this article we derive local estimates for a special upwind FEM in the
L^∞- norm using both cut-off functions similar to [4] and a method proposed
already in 1973 by Ciarlet-Raviart [1] and generalized by Tabata [8] for
proving global L^∞-estimates. It is known that the method of Ciarlet-Raviart
even for global estimates often doesn't give the optimal rate of convergence
with respect to h. However, due to the difficulties of the dominating
convection term and especially due to the fact that in our special upwind
FEM the approximation error is only of order h it is not clear whether other
methods of proof can yield better estimates.

We call our special FEM « hybrid upwind-FEM » due to the fact
(contrary to some other usual applications of the word « hybrid ») that in
the discretization both FEM- and FDM-ideas are used. It is only one
representative of a class of FEM which all have the same main objective, i.e.
preserving inverse-monotony (and by that such meaningful physical proper-
ties as non-negativity of the solutions and often L^∞-a priori estimates and
L^∞-stability) for the discrete problem. A good survey of these methods and
an extensive bibliography can be found in [2]. Additionally a rather easy to
construct but nevertheless very interesting nonconforming variant is de-
scribed in [5].

In this article we describe our hybrid upwind FEM and note some
properties (Section 2), then we give a more general principle to get local
L^∞-error estimates (theorem 1, Section 3) and in section 4 we use theorem 1
to obtain L^∞-error estimates for the hybrid upwind FEM. Finally, in
Section 5, a numerical example is given.

2. THE HYBRID UPWIND FEM

The weak formulation of our problem is :

Find $u \in V_0 = H_0^1(\Omega)$ s.t.
$$\langle Lu, v \rangle \equiv l(u, v) \equiv \varepsilon(\nabla u, \nabla v) + (b \nabla u, v) + (cu, v) = \langle f, v \rangle \qquad (P)$$
$$\forall v \in V_0$$

Let b, c be sufficiently smooth (we will need b, $c \in C^{0+1}(\bar{\Omega})$ and sometimes, for (2.3), $b \in C^{1+1}(\bar{\Omega})$).

Let be a bounded polygonal domain divided into triangles the angles of which are less equal $\frac{\pi}{2}$. Let the triangulation $\{\mathfrak{T}_h\}$ be regular in the usual sense and h denote the maximal diameter of all triangles.

We consider also a dual decomposition of Ω which can be constructed in the following way.

To each node P_i corresponds a dual polygon D_i bounded by parts Γ_{ij} of the mid-perpendiculars of the adjoining triangles.
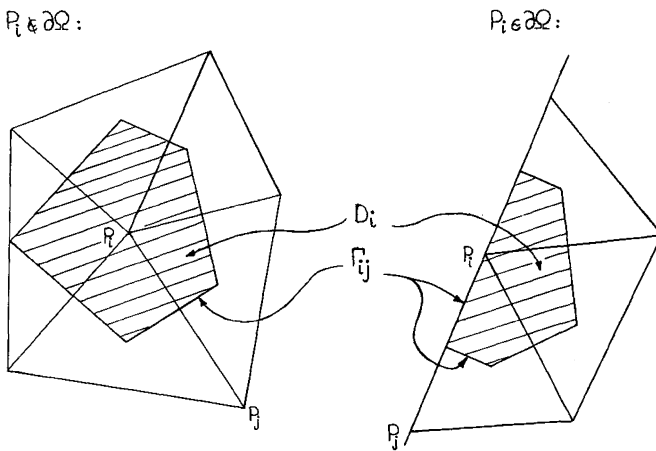

Figure 1.

Furthermore we use the notations :

Λ_i : set of indices of the nodes adjoining to P_i
n_i : outer normal vector to D_i.

Let $V_h = \left\{v_h \in C(\bar{\Omega}) \mid v_h|_T \in P_1(T)\right\}$, $V_{0h} = \left\{v_h \in V_h \mid v_h|_\Gamma = 0\right\}$ denote spaces of piecewise linear functions.

In contrast to standard FEM we approximate $(b\,\nabla u_h, v_h)$, (cu_h, v_h) by

$$b_h(u_h, v_h) = \sum_i v_h(P_i) \sum_{j \in \Lambda_i} \beta_{ij}(\lambda_{ij} - 1)(u_h(P_i) - u_h(P_j))\,,$$

$$c_h(u_h, v_h) = \sum_i \text{meas}\,D_i\,u_h(P_i)\,v_h(P_i)$$

where β_{ij} is an approximation of $\int_{\Gamma_{ij}} bn_i\,d\Gamma_{ij}$ and $\lambda_{ij} = \dfrac{1}{2}(1 + \text{sgn}\,\beta_{ij})$.

The discretization of the convection term is motivated by the splitting

$$(b\,\nabla u_h, v_h) = (\text{div}\,(u_h\,b), v_h) - (u_h\,\text{div}\,b, v_h)$$

and the transformations (using integration by parts)

$$(\text{div}\,(u_h\,b), v_h) = \sum_i \int_{D_i} \text{div}\,(u_h\,b)\,v_h\,dx \approx \sum_i v_h(P_i) \int_{D_i} \text{div}\,(u_h\,b)\,dx$$

$$\approx \sum_i v_h(P_i) \sum_{j \in \Lambda_i} \int_{\Gamma_{ij}} bn_{ij}\,u_h\,d_{ij}$$

$$\approx \sum_i v_h(P_i) \sum_{j \in \Lambda_i} \beta_{ij}(\lambda_{ij}\,u_h(P_i) + (1 - \lambda_{ij})\,u_h(P_j))$$

and

$$(u_h\,\text{div}\,b, v_h) = \sum_i \int_{D_i} v_h\,u_h\,\text{div}\,b\,dx \approx$$

$$\approx \sum_i v_h(P_i)\,u_h(P_i) \int_{D_i} \text{div}\,b\,dx \approx \sum_i v_h(P_i) \sum_{j \in \Lambda_i} \beta_{ij}\,u_h(P_i)\,.$$

We will need β_{ij} such that

$$\beta_{ij} + \beta_{ji} = 0 \qquad \text{if} \qquad \Gamma_{ij} \cap \partial\Omega = \varnothing \tag{2.1}$$

and

$$\left| \beta_{ij} - \int_{\Gamma_{ij}} bn_i\,d\Gamma_{ij} \right| \leqslant Ch^2\,. \tag{2.2}$$

Sometimes instead of (2.2) we demand

$$\left| \beta_{ij} - \int_{\Gamma_{ij}} bn_i\,d\Gamma_{ij} \right| \leqslant Ch^3\,. \tag{2.3}$$

If $\Gamma_{ij} \cap \partial\Omega = \varnothing$ then (2.1) and (2.2) can be obtained (with $C = O(\|b\|_{C^{0+1}(\Omega)})$) e.g. for $\beta_{ij} = b(P_{ij})\,n_i\,\text{meas}\,\Gamma_{ij}$ where P_{ij} is the

midpoint of $\overline{P_i P_j}$; (2.3) can be obtained (with $C = O(\|b\|_{C^{1+1}(\bar{\Omega})})$) for $\beta_{ij} = b(\hat{P}_{ij}) n_i$ meas Γ_{ij} where $\hat{P}_{ij}$ is the midpoint of Γ_{ij}.

Now our discrete problem is

$$\left. \begin{array}{l} \text{Find } u_h \in V_{0h} \text{ s.t.} \\ l_h(u_h, v_h) \equiv \varepsilon(\nabla u_h, \nabla v_h) + b_h(u_h, v_h) + c_h(u_h, v_h) = \langle f, v_h \rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v_h \in V_{0h} \end{array} \right\} \quad (P_h) .$$

Taking into consideration that (P_h) corresponds to a linear system of equations

$$L_h z_h \equiv (\varepsilon A_h + B_h + C_h) z_h = f_h \qquad (2.4)$$

our discretization ensures that C_h is a diagonal matrix and that (due to the fact that we have no obtuse-angled triangles and due to the special choice of the λ_{ij}) the off-diagonal elements of A_h and B_h are non-positive.

Let us note the following essential facts about (P) and (P_h) (for proofs see [6]).

Let

(H1) $$c(x) \geqslant c_0 \geqslant 0$$

or

(H2) $$c(x) - \frac{1}{2} \operatorname{div} b(x) \geqslant \alpha_0 \geqslant 0$$

be fulfilled. Then the continuous problem (P) admits a unique solution and L is inverse-monotone, i.e., from $Lu \leqslant Lv$ and sup $(u - v, 0) \in H_0^1(\Omega)$ it follows $u \leqslant v$. Moreover, (H1) with $c_0 > 0$ yields L^∞-stability

$$\|u - v\|_{0,\infty} \leqslant C \|Lu - Lv\|_{0,\infty}$$

with a constant C independent on ε and from (H2) we obtain the V_0-ellipticity of the bilinearform $\langle L.,. \rangle$.

Are similar properties valid for the discrete problem ?

Under (H1) the hybrid upwind FEM preserves the properties of inverse-monotony, unique solvability and L^∞-stability for all h.

This is the main objective of the method and it is based on the fact that, contrary to standard FEM, the matrix L_h becomes an M-matrix. Furthermore, under (H2) with $\alpha_0 > 0$ the discrete bilinear-form is V_{0h}-elliptic for all $h \leqslant h_0$ independent of ε provided that (2.3) is fulfilled.

Under (H1), (H2) with $\alpha_0 > 0$ and (2.3) we get the estimate

$$\|I_h u - u_h\|_{0,\infty} \leqslant C(\kappa) h^\kappa \varepsilon^{-\frac{1}{2}} \|u\|_{2,p} \quad \text{with} \quad \kappa \to 1 \quad \text{for} \quad p \to \infty$$

where I_h denotes the interpolation operator.

For triangulations of parallelogramm type (see below, def. 1) we obtain

$$\| I_h u - u_h \|_{0,\infty} \leqslant C(\kappa) h^\kappa \| u \|_{2,p} \,.$$

Unfortunately, the norm $\| u \|_{2,p}$ is not bounded uniformly with respect to ε since boundary layers appear.

3. THE STRATEGY OF DERIVING THE LOCAL L^∞-ERROR ESTIMATES

The central result of this section is theorem 1 where the local estimate for the hybrid upwind FEM is given in some more general formulation to make clearer the main principle of deriving the estimates.

First some notations.

For
$$z_h \in V_h \,, \quad \delta \in R$$

let $\quad z_{h,\delta} = \sum_i z_{i,\delta} \, \varphi_i \quad$ where $\quad z_{i,\delta} = \max\,(0, z_h(P_i) - \delta)$

and
$$\varphi_i \in V_h \,, \quad \varphi_i(P_j) = \delta_{ij} \,.$$

For $\Psi \in C(\bar\Omega)$, $\Psi \geqslant 0$, $p \geqslant 1$ we define the seminorms

$$|z|_{k,p,\Psi} = \left(\sum_{|\alpha| = k} \| \sqrt{\Psi}\, D^\alpha z \|^p_{0,p} \right)^{\frac{1}{p}} \,, \qquad p < \infty$$

$$\|z\|_{k,p,\Psi} = \left(\sum_{l=0}^{k} |z|^p_{l,p,\Psi} \right)^{\frac{1}{p}} \,, \qquad p < \infty$$

$$|z|_{k,\infty,\Psi} = \max_{|\alpha| = k} \left(\operatorname*{ess\,sup}_{\Omega} | \sqrt{\Psi}\, D^\alpha z | \right)$$

$$\|z\|_{k,\infty,\Psi} = \max_{l = 0(1)k} |z|_{l,\infty,\Psi}$$

$$\| | z | \|_\Psi = \varepsilon^{1/2} |z|_{1,2,\Psi} + |z|_{0,2,\Psi}$$

and the norm

$$\|z\|_\varepsilon = \varepsilon^{\frac{1}{2}} |z|_{1,2} + \|z\|_{0,2} \,.$$

In the following let I_h be an operator $C(\bar\Omega) \to V_h$, and for all $p \geqslant 1$ we define p' by $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Let us note some lemmata needed below.

LEMMA 1 : *For all* $z \in W^{k,p}(\Omega)$, $k \geq 0$, $2 \leq p \leq \infty$ *is*

$$|z|_{k,p',\Psi} \leq |z|_{k,2,\Psi} \, (\text{meas supp } (\Psi z))^{\frac{1}{2} - \frac{1}{p}}.$$

Proof: It is sufficient to derive the result for $k = 0$.
The proof is a simple consequence of Hölder's inequality.
We have (for $p > 2$)

$$|z|_{0,p',\Psi} = \left(\int_{\text{supp } (\Psi z)} |\sqrt{\Psi} z|^{p'} \cdot 1 \, dx \right)^{\frac{1}{p'}}$$

$$\leq \left(\int_{\Omega} |\sqrt{\Psi} z|^{p' \cdot \frac{2}{p'}} \, dx \right)^{\frac{p'}{2} \cdot \frac{1}{p'}} \left(\int_{\text{supp } (\Psi z)} 1^{\frac{2}{2-p'}} \, dx \right)^{\frac{2-p'}{2} \cdot \frac{1}{p'}}$$

$$\leq |z|_{0,2,\Psi} \, (\text{meas supp } (\Psi z))^{\frac{1}{2} - \frac{1}{p}}. \quad \square$$

LEMMA 2 : *For all* $1 \leq q < \infty$ *there is a constant* $C_1 < \infty$ *s.t. for all* β, $\delta \in R^1$ *with* $\beta > \delta$ *and for all* $z_h \in V_h$ *it holds :*

$$C_1 \|z_{h,\delta}\|_{0,q} \geq (\beta - \delta)(\text{meas supp } z_{h,\beta})^{\frac{1}{q}}.$$

Proof: See [1]. $\square$

LEMMA 3 : *For all* $2 < q < \infty$ *there is a constant* $C_2 < \infty$ *s.t. for all* $z_h \in V_{0h}$ *it holds :*

$$\|z_h\|_{0,q} \leq C_2 \min \left(h^{\frac{2}{q} - 1}, \varepsilon^{-\frac{1}{2}} \right) \|z_h\|_{\varepsilon}.$$

Proof: From Sobolev's imbedding theorem there follows the estimate

$$\|z_h\|_{0,q} \leq C \|z_h\|_{1,2} \leq C \varepsilon^{-\frac{1}{2}} \|z_h\|_{\varepsilon}.$$

The estimate $\|z_h\|_{0,q} \leq C h^{\frac{2}{q} - 1} \|z_h\|_{0,2}$ can be derived from

$$\|z_h\|_{0,q} \leq C \left(\sum_i |z_h(P_i)|^q \, \text{meas } D_i \right)^{\frac{1}{q}} \leq C h^{\frac{2}{q}} \left(\sum_i |z_h(P_i)|^q \right)^{\frac{1}{q}}$$

and

$$\|z_h\|_{0,2} \geq C \left(\sum_i |z_h(P_i)|^2 \, \text{meas } D_i \right)^{\frac{1}{2}} \geq C h \left(\sum_i |z_h(P_i)|^2 \right)^{\frac{1}{2}}. \quad \square$$

The following generalization of a lemma due to Stampacchia is the basis of our estimates.

LEMMA 4 : *Let ζ be a real-valued function that is defined on the measurable set $M \subset R^1$, monotone non-increasing and non-negative.*

Let $\quad \infty > \zeta_0 \geq \zeta(\delta) \quad \forall \delta \in M , \quad \delta_0 = \inf \{\delta \,|\, \delta \in M\} .$

Let there exist constants C_3, q, v with $0 < q$, $C_3 < \infty$, $1 < v < \infty$ s.t. for all $\beta, \delta \in M$ with $\beta > \delta$ the inequality

$$\zeta(\beta) \leq \frac{C_3}{(\beta - \delta)^q} (\zeta(\delta))^v \tag{3.1}$$

holds.

Then there exists a constant $C_4 < \infty$ (depending only on q, v, ζ_0) s.t. $\zeta(\gamma) = 0$ for all $\gamma \in M$ with meas $(M \cap (\delta_0, \gamma)) \geq C_4 C_3^{\frac{1}{q}}$.

Proof: Let $\gamma \in M$ arbitrary with $\zeta(\gamma) = \eta > 0$.
For all $l \geq 1$ we carry out the following construction.
For given $\delta_l \in M$ let

$$\alpha_l = 2^{\frac{1}{q}} C_3^{\frac{1}{q}} (\zeta(\delta_1))^{\frac{v-1}{q}} ,$$

$$\delta_{l+1}^* = \inf \{\delta \in M \,|\, \delta \geq \delta_l + \alpha_l\} ,$$

$$\delta_{l+1} \in M \quad \text{s.t.} \quad \delta_{l+1}^* \leq \delta_{l+1} \leq \delta_{l+1}^* + \vartheta \cdot 2^{-(l+1)}$$

with a certain positive constant ϑ.

$\delta_1 \in M$ is chosen s.t. $\delta_1 \leq \delta_0 + \dfrac{\vartheta}{2}$.

Then it holds

$$\zeta(\delta_{l+1}) \leq \frac{C_3}{2 C_3 (\zeta(\delta_l))^{v-1}} (\zeta(\delta_l))^v = \frac{1}{2} \zeta(\delta_l) \leq \cdots \leq \frac{1}{2^l} \zeta(\delta_1) \leq \frac{1}{2^l} \zeta_0 .$$

Hence there is a $k = k(\eta) < \infty$ with $\zeta(\delta_k) < \eta$.
Furthermore, we have

$$\text{meas } (M \cap (\delta_0, \delta_k)) \leq \sum_{j=1}^{k-1} (\alpha_j + \vartheta \cdot 2^{-(j+1)}) + \frac{\vartheta}{2}$$

$$\leq \sum_{j=1}^{k-1} 2^{\frac{1}{q}} C_3^{\frac{1}{q}} \left(\frac{1}{2^{j-1}} \zeta_0\right)^{\frac{v-1}{q}} + \vartheta \sum_{j=1}^{k} \left(\frac{1}{2}\right)^j$$

$$< (2 C_3)^{\frac{1}{q}} (\zeta_0)^{\frac{v-1}{q}} \left(1 - \left(\frac{1}{2}\right)^{\frac{v-1}{q}(k-1)}\right) \left(1 - \left(\frac{1}{2}\right)^{\frac{v-1}{q}}\right)^{-1} + \vartheta .$$

Thus for sufficiently small ϑ it holds

$$\text{meas } (M \cap (\delta_0, \delta_k)) < C_3^{\frac{1}{q}} \, \zeta_0^{-\frac{\nu-1}{q}} \, 2^{\frac{\nu}{q}} \left(2^{\frac{\nu-1}{q}} - 1 \right)^{-1} .$$

Hence, choosing $C_4 = 2^{\frac{\nu}{q}} \left(2^{\frac{\nu-1}{q}} - 1 \right)^{-1} \zeta_0^{-\frac{\nu-1}{q}}$, we obtain

$$\text{meas } (M \cap (\delta_0, \delta_k)) < C_4 \, C_3^{\frac{1}{q}} .$$

By virtue of $\zeta(\delta_k) < \eta = \zeta(\gamma)$ and the monotony of ζ it follows $\gamma < \delta_k$ which proves the assertion. $\square$

The main idea in deriving the local estimates of the error w_h will be proving the assumptions of lemma 4 for $\zeta(\delta) = \text{meas } (\text{supp } w_{h,\delta} \cap \Omega')$ for a certain $\Omega' \subset \Omega$ with $\delta_0 = 0$ and small C_3 because then on Ω' the estimate $w_h \leqslant C_4 \, C_3^{\frac{1}{q}}$ follows. Now the general formulation of our local estimates.

THEOREM 1 : *Let* $\mu_{\max} = C_5 |\ln h|$ *with a sufficiently large* $C_5 < \infty$ *(for exact definition of* C_5 *see lemma 5).*

Assume the existence of a set off « cut-off functions » Ψ_μ, $\mu = 1(1)\mu_{\max}$, *and of domains* Ω_μ, $\mu = 0(1)\mu_{\max}$, *s.t.*

$$\Omega \supset \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_{\mu \max}$$

$$\Psi_\mu = 0 \quad on \quad \Omega \backslash \Omega_{\mu-1}$$

$$\Psi_\mu \geqslant 1 \quad on \quad \Omega_\mu$$

$$\Psi_\mu \geqslant 0 , \qquad \Psi_\mu \in C(\bar{\Omega}) .$$

For a certain bilinearform $l_h(.,.)$ *on* $V_h \times V_h$ *assume for all* $z_h \in V_{0h}$, $\mu = 1(1)\mu_{\max}$

(H3) $\qquad l_h(z_{h,\delta}, I_h(\Psi_\mu z_{h,\delta})) \leqslant l_h(z_h, I_h(\Psi_\mu z_{h,\delta}))$

(H4) (Ψ_μ-*weighted pseudo-ellipticity*).
There are constants $C_6 < \infty$ *and* $0 \leqslant K_1 = K_1(\varepsilon, h) < \infty$ *s.t.*

$$\| z_h \|^2_{\Psi_\mu} \leqslant C_6 \, l_h(z_h, I_h(\Psi_\mu z_h)) + K_1^2 \| z_h \|^2_{0,2,\Omega_{\mu-1}} .$$

(H5) (*error inequality*).
For a certain element $w_h \in V_{0h}$ (w_h *is the element we want to estimate, e.g.* $w_h = u_h - I_h u$) *and a certain* $2 < p \leqslant \infty$ *there exist* $K_i = K_i(w_h, \varepsilon, h) < \infty$, $i = 2, 3, 4$, *s.t.*

$$|l_h(w_h, I_h(\Psi_\mu z_h))| \leqslant K_2 |z_h|_{1,p',\Psi_\mu} + K_3 \| z_h \|_{0,p',\Psi_\mu} + K_4 \| z_h \|_{0,p',\Omega_{\mu-1}} .$$

Under these assumptions there holds

$$\|w_h\|_{0,\infty,\Omega_{\mu_{max}}} \le C \,|\ln h|\, \min\left(h^{\frac{2}{q}-1}, \varepsilon^{-\frac{1}{2}}\right) \times$$

$$\times \left(C_6\left(\varepsilon^{-\frac{1}{2}} K_2 + K_3 + \kappa K_4\right) + \left(\frac{1}{\kappa} + K_1\right) \|w_h\|_{0,\infty,\Omega_0}\right) \quad (3.2)$$

q *is a constant with* $q \ge \dfrac{p}{p-2}$, *C is proportional to* $C_1 C_2 C_4 C_5$. C_1 *and* C_2 *are derived from lemmata 2 and 3,* C_4 *is derived from lemma 4 for* $\nu = q\left(\dfrac{1}{2} - \dfrac{1}{p}\right)$ *and* ζ_0 = meas Ω_0, κ *is an arbitrary positive constant.*

For an application of (3.2) the assumptions of theorem 1 with very small K_1, K_2, K_3, K_4 and meas $(\Omega_{\mu-1}\backslash\Omega_\mu)$ are to be shown.

Proof of theorem 1 : Let $\delta \ge 0$ arbitrary, let $E(\delta)$ = supp $w_{h,\delta}$.

Using one after another (H4), (H3), (H5), lemma 1 and the inequality $ab \le \kappa a^2 + \dfrac{1}{4\kappa} b^2$ which holds for all $\kappa > 0$ we obtain

$$\|\|w_{h,\delta}\|\|^2_{\Psi_\mu} \le C_6 l_h(w_{h,\delta}, I_h(\Psi_\mu w_{h,\delta})) + K_1^2 \|w_{h,\delta}\|^2_{0,2,\Omega_{\mu-1}}$$

$$\le C_6 l_h(w_h, I_h(\Psi_\mu w_{h,\delta})) + K_1^2 \|w_{h,\delta}\|^2_{0,2,\Omega_{\mu-1}}$$

$$\le C_6(K_2|w_{h,\delta}|_{1,p',\Psi_\mu} + K_3\|w_{h,\delta}\|_{0,p',\Psi_\mu} + K_4\|w_{h,\delta}\|_{0,2,\Omega_{\mu-1}})$$
$$\quad + K_1^2\|w_{h,\delta}\|^2_{0,2,\Omega_{\mu-1}}$$

$$\le C_6(\text{meas}\,(E(\delta) \cap \Omega_{\mu-1}))^{\frac{1}{2}-\frac{1}{p}}\,(K_2|w_{h,\delta}|_{1,2,\Psi_\mu} + K_3\|w_{h,\delta}\|_{0,2,\Psi_\mu}$$
$$\quad + K_4\|w_{h,\delta}\|_{0,2,\Omega_{\mu-1}}) + K_1^2\|w_{h,\delta}\|^2_{0,2,\Omega_{\mu-1}}$$

$$\le C_6^2(\text{meas}\,(E(\delta) \cap \Omega_{\mu-1}))^{1-\frac{2}{p}}\left(\kappa_1 K_2^2 + \kappa_2 K_3^2 + \frac{1}{2}\kappa^2 K_4^2\right)$$
$$\quad + \frac{1}{4}\left(\frac{1}{\kappa_1}|w_{h,\delta}|^2_{1,2,\Psi_\mu} + \frac{1}{\kappa_2}\|w_{h,\delta}\|^2_{0,2,\Psi_\mu}\right)$$
$$\quad + \left(\frac{1}{2\kappa^2} + K_1^2\right)\|w_{h,\delta}\|^2_{0,2,\Omega_{\mu-1}}\,.$$

The choice $\kappa_1 = \dfrac{1}{2\varepsilon}$, $\kappa_2 = \dfrac{1}{2}$ yields

$$\|\|w_{h,\delta}\|\|_{\Psi_\mu} \le C_6(\text{meas}\,(E(\delta) \cap \Omega_\mu))^{\frac{1}{2}-\frac{1}{p}}\left(\varepsilon^{-\frac{1}{2}} K_2 + K_3 + \kappa K_4\right) +$$
$$\quad + \left(\frac{1}{\kappa} + K_1\sqrt{2}\right)\|w_{h,\delta}\|_{0,2,\Omega_{\mu-1}}\,.$$

From

$$\| w_{h, \delta} \|_{0, 2, \Omega_{\mu - 1}} \leq C \| w_{h, \delta} \|_{0, \infty, \Omega_{\mu - 1}} \left(\text{meas } (E(\delta) \cap \Omega_{\mu - 1}) \right)^{\frac{1}{2}}$$

$$\leq C_7 \| w_{h, \delta} \|_{0, \infty, \Omega_{\mu - 1}} \left(\text{meas } (E(\delta) \cap \Omega_{\mu - 1}) \right)^{\frac{1}{2} - \frac{1}{p}}$$

(C_7 depends on meas Ω) there follows

$$\| w_{h, \delta} \|_{\Psi_\mu} \leq C_8 \left(\text{meas } (E(\delta) \cap \Omega_{\mu - 1}) \right)^{\frac{1}{2} - \frac{1}{p}}$$

with $C_8 = C_6 \left(\varepsilon^{-\frac{1}{2}} K_2 + K_3 + \kappa K_4 \right) + C_7 \left(\frac{1}{\kappa} + K_1 \sqrt{2} \right) \| w_{h, \delta} \|_{0, \infty, \Omega_{\mu - 1}}.$

By virtue of lemma 2 and lemma 3 we get for all $\beta > \delta$ and $q > 2$

$$\left(\text{meas } (E(\delta) \cap \Omega_\mu) \right)^{\frac{1}{q}} \leq \frac{C_1}{\beta - \delta} \| w_{h, \delta} \|_{0, q, \Omega_\mu}$$

$$\leq \frac{C_1 C_2}{\beta - \delta} \min \left(h^{\frac{2}{q} - 1}, \varepsilon^{-\frac{1}{2}} \right) \| w_{h, \delta} \|_{\varepsilon, \Omega_\mu}$$

$$\leq \frac{C_1 C_2}{\beta - \delta} \min \left(h^{\frac{2}{q} - 1}, \varepsilon^{-\frac{1}{2}} \right) \| w_{h, \delta} \|_{\Psi_\mu}$$

$$\leq \frac{C_9}{\beta - \delta} \left(\text{meas } (E(\delta) \cap \Omega_{\mu - 1}) \right)^{\frac{1}{2} - \frac{1}{p}} \qquad (3.3)$$

with $$C_9 = C_1 C_2 C_8 \min \left(h^{\frac{2}{q} - 1}, \varepsilon^{-\frac{1}{2}} \right).$$

Now the only problem for an application of lemma 4 with $= \text{meas } (E(\cdot) \cap \Omega_\mu)$ lies in the fact that in (3.3) on the right hand side it stands $\Omega_{\mu - 1}$ instead of Ω_μ. To overcome this difficulty we use the following lemma :

LEMMA 5 : *Let* $z_h \in V_h$, $E(\delta) = \text{supp } z_{h, \delta}$.
Let $C_5 (= | \ln h |^{-1} \mu_{max})$ *be chosen such that*

$$\left(\text{meas } \Omega_0 \right) h^{C_5 \ln 2} \leq \min \{ \text{meas } T \,|\, T \in \mathcal{C}_h \}.$$

(By virtue of the assumed regularity of the family of triangulations we can find such a $C_5 < \infty$ independently of h.)
Then we have for all $\delta \geq 0$:
If not
(i) *there is a* $\mu \in \{ 1, ..., \mu_{max} \}$ *s.t.*

$$\text{meas } (E(\delta) \cap \Omega_\mu) \geq \frac{1}{2} \text{meas } (E(\delta) \cap \Omega_{\mu - 1})$$

then
   (ii) $z_{h,\delta} = 0$ *on* $\Omega_{\mu_{max}}$.

Proof of lemma 5 : If not (i) then

$$\text{meas } \Omega_0 \geqslant \text{meas } (E(\delta) \cap \Omega_0) > 2 \text{ meas } (E(\delta) \cap \Omega_1) > \cdots >$$
$$> 2^{\mu_{max}} \text{ meas } (E(\delta) \cap \Omega_{\mu_{max}}) ,$$

hence

$$\text{meas } (E(\delta) \cap \Omega_{\mu_{max}}) < 2^{-C_5 |\ln h|} \text{ meas } \Omega_0 = h^{C_5 \ln 2} \text{ meas } \Omega_0 ,$$

hence

$$\text{meas } (E(\delta) \cap \Omega_{\mu_{max}}) = 0 . \quad \square$$

Now let us turn back to the proof of theorem 1 :

Let $v = q \left(\dfrac{1}{2} - \dfrac{1}{p} \right)$ (by virtue of $q > \dfrac{p}{p-2}$ is $v > 1$), $\zeta_0 = \text{meas } \Omega_0$ and let C_4 be the constant from lemma 4 for these q, v and ζ_0.

Let $\qquad\qquad C_9' = C_9 2^{\frac{1}{2} - \frac{1}{p}} , \qquad C_{10} = C_4 C_9' .$

If for any $\delta \leqslant C_{10}(\mu_{max} + 1)$ the case (ii) of lemma 5 occurs then there follows

$$w_h \leqslant C_{10}(\mu_{max} + 1) \qquad \text{on} \qquad \Omega_{\mu_{max}} . \qquad (3.4)$$

Now we assume that we have case (i) of lemma 5 for all

$$\delta \in [0, C_{10}(\mu_{max} + 1)] .$$

For all δ we denote by μ_δ one of the μ which fulfil

$$\text{meas } (E(\delta) \cap \Omega_\mu) \geqslant \frac{1}{2} \text{ meas } (E(\delta) \cap \Omega_{\mu - 1}) .$$

Then owing to (3.3) for all $\beta > \delta$ there holds

$$(\text{meas } (E(\beta) \cap \Omega_{\mu_\delta}))^{\frac{1}{q}} \leqslant \frac{C_9'}{\beta - \delta} (\text{meas } (E(\delta) \cap \Omega_{\mu_\delta}))^{\frac{1}{2} - \frac{1}{p}} . \qquad (3.5)$$

The values of δ vary in an interval of length $C_{10}(\mu_{max} + 1)$, and we have at most μ_{max} different values of μ_δ.
Hence there exists a $\bar{\mu} \in \{1, ..., \mu_{max}\}$ s.t.

$$\text{meas } M(\bar{\mu}) > C_{10} \qquad (3.6)$$

where the notation $M(\mu) = \left\{ \delta \in [0, C_{10}(\mu_{max} + 1)] \mid \mu_\delta = \mu \right\}$ is used.

By virtue of (3.5) the function $\zeta(\cdot) = \text{meas } (E(\cdot) \cap \Omega_{\bar\mu})$, restricted to $M(\bar\mu)$, fulfils all assumptions of lemma 4 (with $\delta_0 = \inf \{\delta \mid \delta \in M(\bar\mu)\}$, $\zeta_0 = \text{meas } \Omega_0$, $\nu = q \left(\dfrac{1}{2} - \dfrac{1}{p} \right)$). Hence we have $\zeta(\gamma) = 0$ for all $\gamma \in M(\bar\mu)$ with

$$\text{meas } (M(\bar\mu) \cap (\delta_0, \gamma)) \geq C_4 \, C_9'(= C_{10})$$

and owing to (3.6) the set of these γ is not empty.

Thus there follows on $\Omega_{\bar\mu}$

$$w_h \leq \gamma \leq C_{10}(\mu_{max} + 1) \qquad (3.7)$$

(3.4) and (3.7) imply in both cases of lemma 5 the estimate

$$w_h \leq C_{10}(\mu_{max} + 1) \qquad \text{on} \qquad \Omega_{\mu_{max}} \, .$$

Because the assumption (H3) is also fulfilled for $- w_h$ instead of w_h we get the same upper bound for $- w_h$. Hence we have proven

$$\|w_h\|_{0, \infty, \Omega_{\mu_{max}}} \leq C_{10}(\mu_{max} + 1) \, .$$

Recalling the construction of C_{10} and μ_{max} the assertion (3.1) follows. $\square$

4. LOCAL L^∞-ERROR ESTIMATES FOR THE HYBRID UPWIND FEM

Let the interpolation operator $I_h : C(\bar\Omega) \to V_h$ be defined by $(I_h z)(P_i) = z(P_i)$ for all nodes P_i.

l_h is again the bilinearform constructed in Section 2.

Let u, u_h be solutions of (P), (P_h).

Applying theorem 1 we want to derive local estimates for $w_h = u_h - I_h u$ of the form $\|w_h\|_{0, \infty, \Omega'} \leq C(h^\alpha + \varepsilon^\beta)$, $\Omega' \subset \Omega$. To this end we have to check the assumptions of theorem 1 with small K_1, K_2, K_3, K_4. Since K_2, K_3 and K_4 will prove themselves to be dependent on $W^{2, p}(\Omega_{\mu - 1})$-norms of u we have to choose Ω_0 such that boundary layers are excluded.

Of course $\Omega_{\mu_{max}}$ should be so large as possible. At least, to ensure that the numerical method « recognizes » boundary layers, we should fulfil

$$\sup \left\{ \text{dist } (x, \Omega_{\mu_{max}}) \mid x \in \Omega_0 \right\} \to 0 \qquad \text{for} \qquad h, \varepsilon \to 0 \, .$$

For this property very special construction of the cut-off functions Ψ_μ is needed (the main problem is ensuring the Ψ_μ-weighted pseudo-ellipticity (H4) with small K_1).

The Ψ_μ we use here are very similar to the cut-off function constructed by Nävert [4]. In [4] the cut-off technique is applied to the streamline diffusion method. There local estimates in energy norms are derived. The method we here propose for deriving L^∞-estimates is, however, not applicable to the streamline diffusion method because the discrete bilinearform does not become of non-negative type, i.e. (H1) cannot be reached. For more special information about the streamline diffusion method see [4].

The hybrid upwind FEM gives the possibility to ensure (H1) and (H2). However, for (H2) special type of triangulation in the neighbourhood of the boundary layer is needed.

DEFINITION 1 : *The triangulation is called of parallelogram type if it has arisen from the cut of three groups of parallels where the distance between two adjoining parallels is within each group constant.*
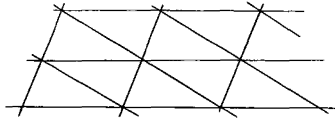

Figure 2. — Triangulation of parallelogram type.

We use the notations

$n(x)$ - outer normal vector to $\partial\Omega$ (there where it exists)

$$\Gamma^- = \{x \in \partial\Omega \,|\, b(x)\,n(x) < 0\} \,.$$

With a fixed constant $d > 0$ (d can be chosen small) we set

$$\Gamma^+ = \{x \in \partial\Omega \,|\, b(x)\,n(x) \geqslant d\}$$
$$\Gamma^0 = \{x \in \partial\Omega \,|\, 0 \leqslant b(x)\,n(x) < d\} \,.$$

Analogously we define e.g. Γ_μ^-, Γ_μ^+, Γ_μ^0.

Let $\sigma = \max(\varepsilon, h)$.

Now let Ω_μ be a domain with boundary Γ_μ that fulfils the following assumption :

(H6)

  (i) $\Omega_\mu \subset \Omega$, $\Gamma_\mu \in \mathscr{C}^\infty$ piecewise, Γ_μ local Lipschitz-continuous.

  (ii) $\Gamma_\mu^- \subset \Gamma^-$.

  (iii) $|b| \geqslant C > 0$ in Ω_μ.

  (iv) $|bn| \geqslant C > 0$ on Γ_μ^- .

(v) No characteristics of the reduced equation which start from Γ^0_μ are contained in Ω_μ.

(vi) The angles at edges of $\Gamma^0_\mu \cup \Gamma^+_\mu$ are less than π.

We construct our cut-off function in the following way. We begin with a one-dimensional cut-off function Φ which is characterized by $\Phi(t) = 0$ for $t \leqslant 0$, $\Phi(t) = 1$ for $t > C\sigma |\ln \sigma|$, and by an exponential behaviour for small positive t.
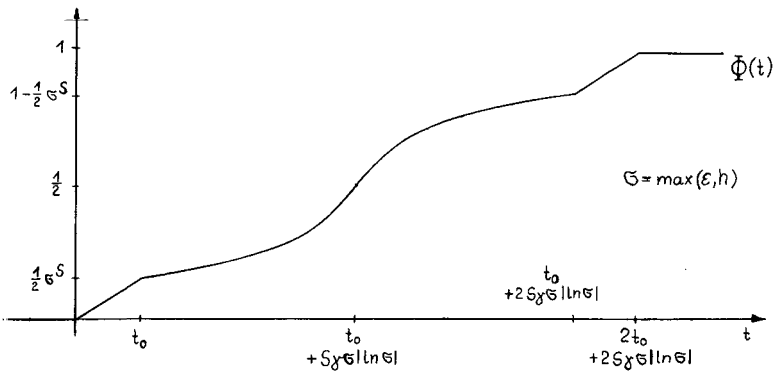

Figure 3.

In figure 3 S, γ are certain positive constants ; the value of t_0 is not essential (we can assume it as small in comparison with σ) ; essential is the behaviour of $\Phi(t)$ in $(t_0, t_0 + S\gamma\sigma |\ln \sigma|)$, there is

$$\Phi(t) = \frac{1}{2} \exp \left(\frac{t - S\gamma\sigma |\ln \sigma| - t_0}{\gamma\sigma} \right) .$$

For $t \geqslant t_0 + S\gamma\sigma |\ln \sigma|$ we can choose

$$\Phi(t) = 1 - \Phi(2 t_0 + 2 S\gamma\sigma |\ln \sigma| - t) .$$

Now we divide $\Gamma^+_\mu \cup \Gamma^0_\mu$ into their smooth parts.

For such a smooth part Γ^k of Γ^+_μ we define Ψ^k by $\Psi^k(x) = \Phi(\text{dist}(x, \Gamma^k))$ for $x \in \Omega_\mu$ and $\Psi^k(x) = 0$ for $x \in \Omega \backslash \Omega_\mu$.

For a smooth part Γ^k of Γ^0_μ the construction of Ψ^k is similar ; but we replace σ by $\sqrt{\sigma}$ and dist (x, Γ^k) by $\frac{1}{K} \hat{t}(x)$ where $\hat{t}(x) = \text{dist}(\chi^-(x), \Gamma^k)$, $\chi(x)$ is the characteristic through x, $\chi^-(x)$ is the part of $\chi(x)$ from Γ^- to x and K is defined such that in a neighbourhood of Γ^k we have $|\hat{t}(x) - \hat{t}(y)| \leqslant K |x - y|$, $\hat{t}(x) \geqslant K$ dist (x, Γ^k). Finally we define our cut-

off function $\Psi_{\mu+1}$ by $\hat{\Psi}_{\mu+1} = \prod_k \Psi^k$ and $\Psi_{\mu+1} = I_h \hat{\Psi}_{\mu+1}^2$ (the use of

$\Psi_{\mu+1}$ instead of $\hat{\Psi}_{\mu+1}$ is not essential and has only technical reasons).

For this $\Psi_{\mu+1}$ the following assumptions are valid :

LEMMA 6 : *There is a constant $C < \infty$ s.t. :*

For all Ω'_μ with

(i) $\Omega'_\mu \subset \Omega_\mu$.
(ii) dist $(\Omega'_\mu, \Gamma^0_\mu) \geq C \sqrt{\sigma} |\ln \sigma|$.
(iii) dist $(\Omega'_\mu, \Gamma^+_\mu) \geq C\sigma |\ln \sigma|$

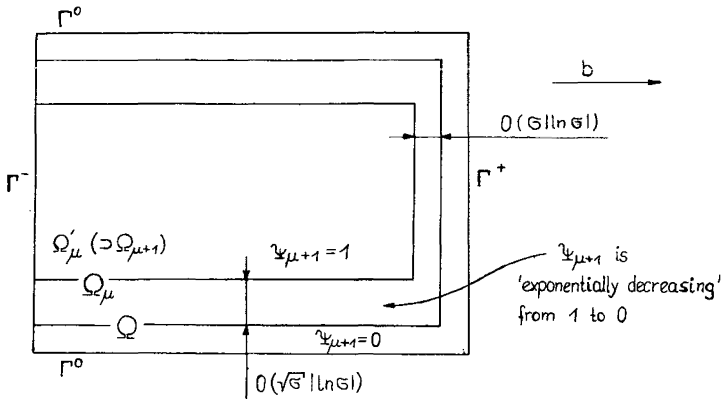*there holds $\Psi_{\mu+1} \equiv 1$ on Ω'_μ.*

Figure 4.

LEMMA 7 (Ensuring (H3)) : *Under (H1) there holds for all $z_h \in V_{0h}$, $\delta \in R$*

$$l_h(z_{h,\delta}, I_h(\Psi_\mu z_{h,\delta})) \leq l_h(z_h, I_h(\Psi_\mu z_{h,\delta})) .$$

LEMMA 8 (Ensuring (H4)) : *Let $\Omega_{\mu-1}$ fulfil (H6).*
Under (H2) with $\alpha_0 > 0$ and (2.3) there is a constant $C < \infty$ s.t. for sufficiently small σ, sufficiently large γ, S and for a triangulation of $\Omega \cap \{x \mid \Psi_\mu(x) < 1\}$ which is of parallelogram type there holds for all $z_h \in V_{0h}$, for all ε, h

$$\| z_h \|^2_{\Psi_\mu} \leq C \left(l_h(z_h, I_h(\Psi_\mu z_h)) + \sigma^{\frac{S}{2}+1} h^{-2} \| z_h \|^2_{0,2,\Omega_{\mu-1}} \right) .$$

LEMMA 9 (Ensuring (H5)) : *Let $1 < p \leq \infty$; let $u \in W^{2,p}(\Omega)$, $u_h \in V_{0h}$ be solutions of (P), (P_h). We denote again $w_h = u_h - I_h u$.*

Then there is a constant $C < \infty$ s.t. for sufficiently small σ and sufficiently large γ, S there holds for all $z_h \in V_{0h}$, for all ε, $h \leqslant \sigma$

$$l_h(w_h, I_h(\Psi_\mu z_h)) \leqslant Ch \|u\|_{2,p,\Omega_{\mu-1}} \left(\|z_h\|_{1,p',\Psi_\mu} + \sigma^{\frac{s}{2}} h^{-1} \|z_h\|_{0,p',\Omega_{\mu-1}} \right).$$

If, beyond this, Ω_μ is contained in a part of Ω the triangulation of which is of parallelogram type then we have

$$l_h(w_h, I_h(\Psi_\mu z_h)) \leqslant Ch \|u\|_{2,p,\Omega_{\mu-1}} \times$$
$$\times \left(\varepsilon \|z_h\|_{1,p',\Psi_\mu} + \|z_h\|_{0,p',\Psi_\mu} + \sigma^{\frac{s}{2}} h^{-1} \|z_h\|_{0,p',\Omega_{\mu-1}} \right).$$

Some remarks on the proofs of lemmata 6-9 (for detailed proofs see [6]) ; Lemma 6 follows from the construction of $\Psi_{\mu+1}$. Lemma 7 even holds for all $\Psi_\mu \in C(\bar{\Omega})$, $\Psi_\mu \geqslant 0$. The straightforward proof is based on the fact that the discrete matrix L_h is off-diagonal non-positive and that owing to $c \geqslant 0$ for all $i \sum_j (L_h)_{ij} \geqslant 0$. More extensive and rather technical are the proofs of lemma 9 and especially of lemma 8, the main steps of these proofs can be found in section 5. The constants in lemma 8, 9 depend on the $C^{0+1}(\bar{\Omega}_{\mu-1})$-norms of b and c.

Now we are able to formulate our local L^∞ error estimate for the hybrid upwind FEM :

THEOREM 2 : *If $2 < p \leqslant \infty$, $\Omega'' \subset \Omega$ fulfils (H6) (i)-(v), $u \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, $u_h \in V_{0h}$ solutions of (P), (P_h), (H1), (H2) with $\alpha_0 > 0$, $\sigma = \max(\varepsilon, h)$ sufficiently small, $q > \dfrac{2p}{p-2}$ then there are constants C_{11}, $C_{12} < \infty$ independent of h, ε s.t. for all $\Omega' \subset \Omega''$ with*

$$\text{dist}(\Omega', (\Gamma'')^0) \geqslant C_{11} \sqrt{\sigma} |\ln \sigma \ln h|,$$
$$\text{dist}(\Omega', (\Gamma'')^+) \geqslant C_{11} \sigma |\ln \sigma \ln h|$$

there holds :

(i) if $\Omega'' \backslash \Omega'$ is contained in a domain the triangulation of which is of parallelogram type then

$$\|u_h - I_h u\|_{0,\infty,\Omega'} \leqslant C_{12} |\ln h| \left\{ \min\left(h^{\frac{2}{q}} \varepsilon^{-\frac{1}{2}}, h\varepsilon^{-1} \right) \|u\|_{2,p,\Omega''} + \right.$$
$$\left. + \min\left(h^{\frac{2}{q}-1}, \varepsilon^{-\frac{1}{2}} \right) \sigma^{\frac{s}{4}+\frac{1}{2}} h^{-1} \|u_h - I_h u\|_{0,\infty,\Omega''} \right\}.$$

(ii) *If Ω'' is contained in a domain the triangulation of which is of parallelogram type then*

$$\|u_h - I_h u\|_{0,\infty,\Omega'} \leqslant C_{12} |\ln h| \left\{ \min \left(h^{\frac{2}{q}}, h\varepsilon^{-\frac{1}{2}} \right) \|u\|_{2,p,\Omega''} + \right.$$

$$\left. + \min \left(h^{\frac{2}{q}-1}, \varepsilon^{-\frac{1}{2}} \right) \sigma^{\frac{S}{4}+\frac{1}{2}} \|u_h - I_h u\|_{0,\infty,\Omega''} \right\} .$$

In both cases C_{12} depends on the $C^{0+1}(\bar{\Omega}'')$-norms of b and c. The positive constant S can be made arbitrary large if we make C_{11} large.

Proof : We can construct domains Ω_μ with the following properties :

$$\Omega'' = \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_{\mu_{\max}} \supset \Omega' .$$

all Ω_μ, $\mu = 0, 1, ..., \mu_{\max} - 1$ fulfil (H6) (i)-(v) ; for the above constructed $\Psi_{\mu+1}$ it holds $\Psi_{\mu+1} = 0$ on $\Omega \backslash \Omega_\mu$, $\Psi_{\mu+1} = 1$ on $\Omega_{\mu+1}$; the constants in lemma 6, 8, 9 are independent of μ ; $\mu_{\max}$ is the same as in theorem 1.

Further, covering Ω' by a finite number of suitable domains which fulfil (H6) (i)-(vi) we may assume that all Ω_μ fulfil (H6) (vi) too.

By virtue of the lemmata 6-9 then all assumptions of theorem 1 are fulfilled, where C_6 is the constant C from lemma 8,

$$K_1 = O\left(\sigma^{\frac{S}{4}+\frac{1}{2}} h^{-1} \right) , \quad K_3 = O(h\|u\|_{2,p,\Omega''}) , \quad K_4 = O\left(\sigma^{\frac{S}{2}} \|u\|_{2,p,\Omega''} \right) ,$$

$K_2 = O(h\|u\|_{2,p,\Omega''})$ for case (i), $K_2 = O(h\varepsilon\|u\|_{2,p,\Omega''})$ for case (ii), C_4 depends on p, q, meas Ω''.

So we can apply theorem 1. Choosing in (3.2) $\kappa = \sigma^{-\frac{S}{4}-\frac{1}{2}} h$ we obtain the assertion.

Owing to the construction of the Ψ_μ there follows that S is proportional to C_{11} and so we can make S large together with C_{11}. $\square$

Remark 1 : What have we achieved by theorem 2 ?

To use the local estimates we first should ensure that dist $(\Omega', (\Gamma'')^0 \cup (\Gamma'')^+) \to 0$ for $\varepsilon, h \to 0$. This can be guaranteed if we assume : $\varepsilon^l \leqslant h$ with a certain positive constant l. The assumption $\varepsilon^l \leqslant h$ also is sufficient to ensure that

$$g = \min \left(h^{\frac{2}{q}-1}, \varepsilon^{-\frac{1}{2}} \right) \sigma^{\frac{S}{4}+1} h^{-1} \|u_h - I_h u\|_{0,\infty,\Omega''} \quad \text{is small :}$$

Indeed, ensuring that $h^\beta \|u_h - I_h u\|_{0,\infty,\Omega''}$ is bounded with a certain (e.g., positive !) constant β and making C_{11} (and with that also S) sufficiently large, for each desired positive K we can obtain $g = O(\varepsilon^K)$.

Finally, $\|u\|_{2,p,\Omega''}$ has to be bounded. For this, we mention following known fact : under certain assumptions, where the most essential of which are $\Omega'' \subset \Omega''' \subset \Omega$, dist $(\Omega'', (\Gamma''')^0) \geq C \sqrt{\varepsilon} |\ln \varepsilon|$, dist $(\Omega'', (\Gamma''')^+) \geq C\varepsilon |\ln \varepsilon|$ and (H6) for Ω''', for sufficiently smooth data the norms $\|u\|_{k,2,\Omega''}$ and with it (if $k > 3$) also $\|u\|_{2,\infty,\Omega''}$ are bounded uniformly with respect to ε (for exact assumptions, see e.g. [4].

Thus we can, somewhat simplified, say :

If $\varepsilon^l \leq h$, if $\Omega' \subset \Omega$ is a domain where boundary layers and additional « numerical boundary layers » are excluded and if the triangulation is of parallelogram type then the local error $\|u_h - I_h u\|_{0,\infty,\Omega'}$ is of order h^κ with an arbitrary $\kappa < 1$.

If the triangulation is only in the neighbourhood of the boundary layer of parallelogram type we get at least the order $h^\kappa \varepsilon^{-\frac{1}{2}}$.

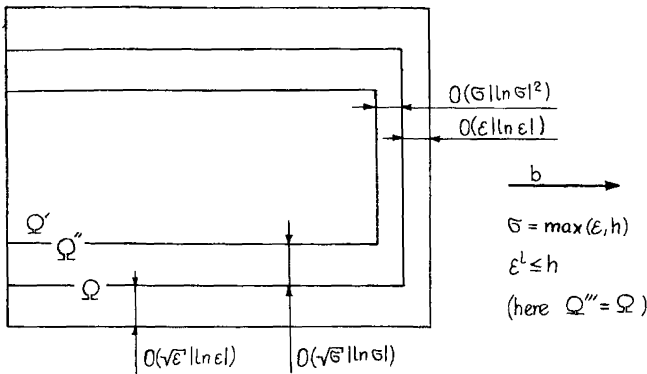For a typical situation see following figure :

Figure 5.

Remark 2 : In [6] the results of theorem 2 are written for weakly coupled systems

$$-\varepsilon \Delta u^i + b^i(x) \nabla u^i + \sum_{j=1}^{m} c^{ij}(x) u^j = f^i(x), \quad i = 1(1)m.$$

For it, the most essential changes are :

Let c be the $m \times m$-Matrix with the elements c^{ij}.

Assumptions (H1), (H2) are replaced by their (natural) generalizations :

(H1') c is of non-negative type, that means

(i) $\qquad\qquad c^{ij} \leq 0 \qquad$ for $\qquad i \neq j$

(ii) there is a vector $s \in R^m$, $s > 0$ s.t.

$$\sum_j c^{ij}(x)\, s^j \geqslant 0 \qquad \forall i = 1\,(1)\,m\,.$$

(H2′) The smallest eigenvalue of $c + c^T$-diag (div b^i) is positive.
More restrictive is the following assumption :

(H7) The directions of the b^i do not differ « too strong » from each other :
for all $i = 1\,(1)\,m$ the $(\Gamma'')^-$-boundary corresponding to the b^i is the same
(couplings between $(\Gamma'')^0$- and $(\Gamma'')^+$-boundaries may appear).

5. PROOFS OF LEMMA 8 AND 9

First some notations and some properties needed below.
We denote for all i

$$\Psi_i = \Psi_\mu(P_i)$$
$$z_i = z_h(P_i) \qquad \text{for all} \qquad z_h \in V_h$$

P_{iT}, $i = 1, 2, 3$ - edges of the triangle $T \in \mathscr{C}_h$.
λ_{iT}, barycentric coordinates corresponding to the P_{iT}.
φ_i, basis functions of the V_h with $\varphi_i(P_j) = \delta_{ij}$.

$$E_i = \operatorname{supp} \varphi_i$$
$$l_{ij} = \overrightarrow{P_i P_j}$$
$$\gamma_{ij} = \operatorname{meas} \Gamma_{ij}$$

j^* (for triangulations of parallelogram type, for a given interior node
P_i and $j \in \Lambda_i$) is chosen such that $j^* \in \Lambda_i$ and that P_{j^*}, P_i, P_j lie on the
same straight line.
T_{ij}, one triangle $\in \mathscr{C}_h$ containing P_i, P_j (for our purpose it is not essential
which of these two triangles we choose).

$$\Omega(> \sigma) = \Omega_{\mu-1} \cap U\left\{ T \in \mathscr{C}_h \,\middle|\, \Psi_\mu(x) > \sigma^{\frac{s}{2}} \quad \forall x \in T \right\}$$
$$\Omega(\leqslant \sigma) = \Omega_{\mu-1} \backslash \Omega(> \sigma)\,.$$

We use the following **properties** :

$$\Psi_\mu(x) \leqslant C\sigma^{\frac{s}{2}} \qquad \text{for all} \qquad x \in \Omega\,(\leqslant \sigma)\,. \tag{5.1}$$

For triangulations of parallelogram type for all vectors $y \in R^2$, all interior
nodes P_i it holds

$$\sum_{j \in \Lambda_i} n_{ij}\, \gamma_{ij}(l_{ij}\, y) = 2 \operatorname{meas} D_i\, y\,. \tag{5.2}$$

For all $1 \leqslant q \leqslant \infty$, sufficiently large γ, S there hold for all $z_h \in V_h$

$$\left| I_h(\Psi_\mu z_h) \right|_{0, q} \leqslant C \left\| z_h \right\|_{0, q, \Psi_\mu} + C\sigma^{\frac{S}{2}} \left\| z_h \right\|_{0, q, \Omega_{\mu-1}} \tag{5.3}$$

$$\left| I_h(\Psi_\mu z_h) \right|_{1, q} \leqslant C \frac{1}{\gamma\sigma} \left\| z_h \right\|_{0, q, \Psi_\mu} + \left| z_h \right|_{1, q, \Psi_\mu} + \sigma^{\frac{S}{2}} h^{-2} \left| z_h \right|_{0, q, \Omega_{\mu-1}} \tag{5.4}$$

$$\left| \tilde{z}_h \right|_{0, q, \tilde{\Psi}} \leqslant C \left| z_h \right|_{0, q, \tilde{\Psi}} + C\sigma^{\frac{S}{2}} \left| z_h \right|_{0, q, \Omega_{\mu-1}} \tag{5.5}$$

$$\left| z_h \right|_{k, q, \tilde{\Psi}} \leqslant C \left| z_h \right|_{k, q, \Psi} + C\sigma^{\frac{S}{2}} \left| z_h \right|_{0, q, \Omega_{\mu-1}} \tag{5.6}$$

(5.1) follows from the construction of ψ_μ by use of Taylor's formula, (5.2) is an elementary calculation and also (5.3)-(5.6) follow rather straightforward if the splitting $\Omega_{\mu-1} = \Omega(> \sigma) \cup \Omega(\leqslant \sigma)$ is used.

Proof of lemma 8 :

A) By virtue of (5.1) it follows that both $\left| l_h(z_h, I_h(\Psi_\mu z_h)) \right|_{\Omega(\leqslant \sigma)}$ (i.e. the integral defining l_h restricted to $\Omega(\leqslant \sigma)$) and $\|\| z_h \|\|_{\Psi_\mu, \Omega(\leqslant \sigma)}$ can be estimated by $C\sigma^{\frac{S}{2}+1} h^{-2} \| z_h \|_{0, 2, \Omega_{\mu-1}}^2$.

Thus in the remaing investigations we may restrict us to $\Omega(> \sigma)$, i.e. in the following we will assume $\Psi_\mu(x) > \sigma^{\frac{S}{2}}$.

B) Elementary transformations lead to

$$b_h(z_h, I_h(\Psi_\mu z_h)) = T_1 + T_2 + T_3$$

where

$$T_1 = -\frac{1}{2} \sum_i z_i^2 \Psi_i \sum_{j \in \Lambda_i} \beta_{ij}$$

$$T_2 = \frac{1}{4} \sum_{i,j} (\sqrt{\Psi_i} z_i - \sqrt{\Psi_j} z_j)^2 \times$$

$$\times \left\{ |\beta_{ij}| - \beta_{ij} \left(\frac{1}{\sqrt{\Psi_j}} - \frac{1}{\sqrt{\Psi_i}} \right) ((1 - \lambda_{ij}) \sqrt{\Psi_i} + \lambda_{ij} \sqrt{\Psi_j}) \right\}$$

$$T_3 = \frac{1}{4} \sum_{i,j} \beta_{ij} (z_i^2 \Psi_i + z_j^2 \Psi_j) \times$$

$$\times \left(\frac{1}{\sqrt{\Psi_j}} - \frac{1}{\sqrt{\Psi_i}} \right) ((1 - \lambda_{ij}) \sqrt{\Psi_i} + \lambda_{ij} \sqrt{\Psi_j}).$$

C) We have for sufficiently small h

$$T_1 + (\tilde{c}\tilde{z}_h, \tilde{\Psi}_\mu \tilde{z}_h) \geqslant \frac{\alpha_0}{2} \|z_h\|_{0,2,\Psi_\mu}^2 , \tag{5.7}$$

since (using (2.3))

$$T_1 + (\tilde{c}\tilde{z}_h, \tilde{\Psi}_\mu \tilde{z}_h) = -\frac{1}{2} \sum_i z_i^2 \Psi_i \left(\sum_{j \in \Lambda_i} \beta_{ij} - \int_{D_i} \operatorname{div} b \, dx \right) +$$

$$+ \left(\left(c - \frac{1}{2} \operatorname{div} b \right) \tilde{z}_h, \tilde{\Psi}_\mu \tilde{z}_h \right) + ((\tilde{c} - c) \tilde{z}_h, \tilde{\Psi}_\mu \tilde{z}_h)$$

$$- Ch(\|b\|_{C^{l+1}} + \|c\|_{C^{0+1}})(\tilde{z}_h, \tilde{\Psi}_\mu \tilde{z}_h) + \left(\left(c - \frac{1}{2} \operatorname{div} b \right) \tilde{z}_h, \tilde{\Psi}_\mu \tilde{z}_h \right)$$

(H1) with $\alpha_0 > 0$ now implies (5.7).

D) Estimation of T_2 and T_3. For the sake of simplicity we restrict us in the first instance to a subdomain $G \subset \Omega_{\mu-1}$ that is away from $\Gamma_{\mu-1}^0 \cup \Gamma_{\mu-1}^+$ except of one smooth part $\tilde{\Gamma}$ of $\Gamma_{\mu-1}^+$.

So we have $\Psi_\mu = I_h(\Phi(t))^2$ with $t(x) = \operatorname{dist}(x, \tilde{\Gamma})$.

By Taylor's formula we get for sufficiently large γ, S, for all $t_1, t_2 > t_0$, $|t_1 - t_2| \leqslant h$

$$\frac{\Phi(t_2)}{\Phi(t_1)} - 1 = \left(\frac{t_2 - t_1}{\gamma\sigma} + \vartheta \right) \Phi_{12}^* + O(\sigma^S) . \tag{5.8}$$

In formula (5.8) (and in the following) ϑ denotes an arbitrary function with $|\vartheta| \leqslant C \left| \frac{h}{\gamma\sigma} \right|^2$. For $t_2, t_1 \leqslant S\gamma\sigma |\ln \sigma| + t_0 + h$ is $\Phi_{12}^* = 1$, otherwise Φ_{12}^* may be an arbitrary value $\frac{1}{\Phi(t)} - 1$ with $|t - t_1| \leqslant h$ or $|t - t_2| \leqslant h$.

By use of (5.8) we can estimate

$$\left| \left(\frac{1}{\sqrt{\Psi_j}} - \frac{1}{\sqrt{\Psi_i}} \right) ((1 - \lambda_{ij}) \sqrt{\Psi_i} + \lambda_{ij} \sqrt{\Psi_j}) \right| \leqslant$$

$$\leqslant \max \left(\left| 1 - \sqrt{\frac{\Psi_i}{\Psi_j}} \right|, \left| 1 - \sqrt{\frac{\Psi_j}{\Psi_i}} \right| \right) \leqslant 1 , \tag{5.9}$$

thus $T_2 \geqslant 0$.

For the investigation of T_3 we remark that, by help of (5.8) and with the notation $t_i = t(P_i)$,

$$\left(\frac{1}{\sqrt{\Psi_j}} - \frac{1}{\sqrt{\Psi_i}} \right) ((1 - \lambda_{ij}) \sqrt{\Psi_i} + \lambda_{ij} \sqrt{\Psi_j}) =$$

$$= \left(\frac{\sqrt{\Psi_i}}{\sqrt{\Psi_j}} - 1 \right) \left(1 + \lambda_{ij} \left(\frac{\sqrt{\Psi_j}}{\sqrt{\Psi_i}} - 1 \right) \right)$$

$$= \left(\frac{t_i - t_j}{\gamma \sigma} + \vartheta \right) \Phi_{ij}^* + O(\sigma^S) . \tag{5.10}$$

By help of (5.10) and (2.3) we can estimate

$$T_3 = \frac{1}{4} \sum_{i, \Psi_i < 1} z_i^2 \psi_i \sum_{j \in \Lambda_i} (b(P_i) n_{ij} \gamma_{ij} + O(h^3)) \left\{ \left(\frac{t_i - t_j}{\gamma \sigma} + \vartheta \right) \Phi_{ij}^* + O(\sigma^S) \right\}$$

$$\geqslant \frac{1}{4} \sum_{i, \Psi_i < 1} z_i^2 \Psi_i \, b(P_i) \, R_i - \frac{\alpha_0}{4} \| \tilde{z}_h \|_{0, 2, \tilde{\Psi}_\mu}^2 \tag{5.11}$$

where

$$R_i = \Phi_i^* \sum_{j \in \Lambda_i} n_{ij} \gamma_{ij} \left(\frac{t_i - t_j}{\gamma \sigma} + \vartheta \right) \tag{5.12}$$

$$\Phi_i^* = 1 \quad \text{for} \quad \Phi(t_i) \leqslant \frac{1}{2} , \quad \Phi_i^* = \frac{1}{\Phi(t_i)} - 1 \quad \text{for} \quad \Phi(t_i) > \frac{1}{2} . \tag{5.13}$$

Now, for the further treatment of R_i, the parallelogram type of our triangulation is used. From (5.2) with $y = \nabla t(P_i)$ we get

$$b(P_i) R_i = b(P_i) \Phi_i^* \left(- \frac{2}{\gamma \sigma} \, \text{meas} \, D_i \, \nabla t(P_i) + h \vartheta \right) \tag{5.14}$$

and thus, by virtue of $b(P_i) \nabla t(P_i) \leqslant - C < 0$, for γ sufficiently large

$$b(P_i) R_i \geqslant C \Phi_i^* \frac{h^2}{\gamma \sigma} \quad \text{with} \quad C > 0 . \tag{5.15}$$

E) Estimation of $\varepsilon (\nabla z_h, \nabla I_h(\Psi_\mu z_h))$. Elementary calculation give (again for the above introduced subdomain G)

$$\varepsilon (\nabla z_h, \nabla I_h(\Psi_\mu z_h)) - \varepsilon |z_h|_{1, 2, \Psi}^2 = T_4$$

with (denoting $\Psi_{k+3, T} = \Psi_{k, T}$ for $k = 1, 2$)

$$T_4 = \frac{\varepsilon}{9} \sum_i \sum_{T \subset E_i} \left\{ (\text{meas } T) \, \Psi_i \, \nabla z_h|_T \sum_{j=1}^{3} z_{jT} \nabla \lambda_{jT} \times \right.$$

$$\left. \times \left(\frac{\Psi_{jT} - \Psi_{j+1, T}}{\Psi_i} + \frac{\Psi_{jT} - \Psi_{j+2, T}}{\Psi_i} \right) \right\} \tag{5.16}$$

From (5.8) we get for all P_j, $P_k \in T \subset E_i$

$$\left| \frac{\Psi_j - \Psi_k}{\Psi_i} \right| \leqslant C\Phi_T^* \frac{h}{\gamma\sigma} + C\sigma^S ,$$

where Φ_T^* is an arbitrary of the values Φ_i^*, Φ_j^*, Φ_k^*. Thus it follows

$$|T_4| \leqslant CT_5 + C\varepsilon\sigma^S h^{-2} \|\tilde{z}_h\|_{0,2,\tilde{\Psi}_\mu}^2 , \tag{5.17}$$

where

$$T_5 = \frac{\varepsilon h^2}{\gamma\sigma} \sum_i \Psi_i \Phi_T^* \sum_{T \subset E_i} |\nabla z_h| \, |_T \|z_h\|_{\infty, T} .$$

It holds for an arbitrary $\kappa > 0$

$$T_5 \leqslant C \left(\frac{\varepsilon\kappa}{\gamma\sigma} \|\nabla z_h\|_{0,2,\tilde{\Psi}}^2 + \frac{\varepsilon h^2}{\gamma\sigma\kappa} \sum_i \Psi_i \Phi_i^* z_i^2 \right) . \tag{5.18}$$

Setting $\kappa = \sigma \sqrt{\gamma}$ we get

$$T_5 \leqslant \frac{C}{\sqrt{\gamma}} \|\!|z_h|\!\|_{\tilde{\Psi}_\mu}^2 + \frac{1}{\sqrt{\gamma}} \frac{h^2}{\gamma\sigma} \sum_i \Psi_i \Phi_i^* z_i^2 . \tag{5.19}$$

F) Lemma 8 now follows — on the subdomain G — by combination of (5.7), (5.9), (5.11), (5.15), (5.19) for sufficiently large γ, (5.5), (5.6).

G) Now let G be a subdomain of $\Omega_{\mu-1}$ that has a positive distance to $\Gamma_{\mu-1}^0 \cup \Gamma_{\mu-1}^+$ except of one smooth part $\tilde{\Gamma}$ of $\tilde{\Gamma}_{\mu-1}^0$. Then we have the following changes in comparison with D), E): $t(x)$ now denotes $\frac{1}{K}$ dist $(\chi^-(x), \tilde{\Gamma})$ (see the construction of Ψ_μ).

In the construction of Φ, the definition of ϑ and in the formulas (5.8)-(5.14), (5.17), (5.18) σ is replaced by $\sqrt{\sigma}$.

Due to $b(P_i) \nabla t(P_i) \leqslant 0$ we get now instead of (5.15) $b(P_i) R_i \geqslant - Ch|\vartheta|$. Furthermore we set in (5.18) (with σ replaced by $\sqrt{\sigma}$) $\kappa = \sqrt{\sigma}$ and get instead of (5.19)

$$T_5 \leqslant \frac{C}{\gamma} \|\!|z_h|\!\|_{\tilde{\Psi}_\mu}^2 .$$

Combination of (5.7), (5.9), (5.11) (5.5), (5.6) and the analoga to (5.15), (5.19) again leads to the assertion of lemma 8.

H) It remains to investigate subdomains of $\Omega_{\mu-1}$ which contain a neighbourhood of an edge of $\Gamma_{\mu-1}^0 \cup \Gamma_{\mu-1}^+$. In this case the construction of Ψ_μ is based on a product of two functions Ψ^k.

Using the identity $ab - 1 = (a - 1) + (b - 1) + (a - 1)(b - 1)$ the term $\dfrac{\sqrt{\Psi_i}}{\sqrt{\Psi_j}} - 1$ becomes a sum of two terms of the form (5.8). Thus T_3 and T_4 become a sum of two terms of the already investigated forms of T_3 and T_4. On this way the assertion of lemma 8 follow in the whole domain $\Omega_{\mu - 1}$.

Proof of lemma 9 :

A) We prove that for all $p > 1$ there is a constant $C < \infty$ s.t. for all $v \in W^{2,p}(\Omega)$, $v_h = I_h v$, $z_h \in V_{0h}$ the following approximation properties are valid :

$$\|v - v_h\|_{j,p} \leqslant Ch^{2-j}\|v\|_{2,p}, \qquad j = 0, 1 \tag{5.20}$$

$$\|v_h - \tilde{v}_h\|_{0,p} \leqslant Ch\|v_h\|_{1,p} \tag{5.21}$$

$$\left|(cv_h, z_h) - (\tilde{c}\tilde{v}_h, \tilde{z}_h)\right| \leqslant Ch\|c\|_{C^{0+1}(\Omega)}\|v\|_{2,p}\|z_h\|_{1,p'} \tag{5.22}$$

$$\left|(b\,\nabla v_h, z_h) - b_h(v_h, z_h)\right| \leqslant Ch\|b\|_{C^{0+1}(\Omega)}\|v\|_{2,p}\|z_h\|_{1,p'}. \tag{5.23}$$

Moreover, using special symmetry properties we get for triangulations of parallelogram type

$$(v_h, \tilde{z}_h) = (\tilde{v}_h, z_h) \tag{5.24}$$

$$\left|(cv_h, z_h) - (\tilde{c}\tilde{v}_h, \tilde{z})\right| \leqslant Ch\|c\|_{C^{0+1}(\Omega)}\|v\|_{2,p}\|z_h\|_{0,p'} \tag{5.25}$$

$$\left|(b\,\nabla v_h, z_h) - b_h(v_h, z_h)\right| \leqslant Ch\|b\|_{C^{0+1}(\Omega)}\|v\|_{2,p}\|z_h\|_{0,p'} \tag{5.26}$$

(5.20) is a known interpolation property.

For (5.21) and (with a slight modification) (5.23) see [2] (there lemma 2.1, and 4.6.1).

Using the splitting

$$(cv_h, z_h) - (\tilde{c}\tilde{v}_h, \tilde{z}_h) = (cv_h, z_h - \tilde{z}_h) + (c(v_h - \tilde{v}_h), \tilde{z}_h) + ((c - \tilde{c})\tilde{v}_h, \tilde{z}) \tag{5.27}$$

(5.22) follows from (5.21) and (5.20).

For basis functions $\varphi_i \in V_h$, $\varphi_j \in V_{0h}$ and for triangulations of parallelogram type we can easily verify $(\varphi_i, \tilde{\varphi}_j) = (\tilde{\varphi}_i, \varphi_j)$. So (5.24) follows.

(5.25) results again from the splitting (5.27) since by virtue of (5.24)

$$(cv_h, z_h - \tilde{z}_h) = (cv_h - I_h(cv_h), z_h) - (cv_h - I_h(cv_h), \tilde{z}_h) +$$
$$+ (I_h(cv_h) - \tilde{c}\tilde{v}_h, z_h).$$

It remains to prove (5.26). This estimate would be follow from Hölder's inequality if we had shown that for all $i = 1(1)N$

$$\left| (b \, \nabla v_h, \, \varphi_i) - b_h(v_h, \, \varphi_i) \right| \leqslant C h^{3 - \frac{2}{p}} \| v \|_{2, p, E_i} . \tag{5.28}$$

For the proof of (5.28) we may assume that $b = b(P_i) = \text{const.}$ on E_i. Indeed, let b be variable and $\hat{b}(x) = b(P_i) \; \forall x \in E_i$ (and let $\hat{\beta}_{ij}$, $\hat{\lambda}_{ij}$ and $\hat{b}_h(.,.)$ be defined correspondingly to $\hat{b}$). Then we conclude from Hölder's inequality

$$\left| ((b - \hat{b}) \, \nabla v_h, \, \varphi_i) \right| \leqslant C h^{3 - \frac{2}{p}} \| b \|_{C^{0+1}} \| v_h \|_{1, p, E_i}$$

and also

$$\left| (b_h - \hat{b}_h)(v_h, \, \varphi_i) \right| = \left| \sum_{j \in \Lambda_i} ((\beta_{ij} - \hat{\beta}_{ij})(\lambda_{ij} - 1) + \hat{\beta}_{ij}(\lambda_{ij} - \hat{\lambda}_{ij})) \, l_{ij} \, \nabla v_h|_{T_{ij}} \right|$$

$$\leqslant C h^{3 - \frac{2}{p}} \| b \|_{C^{0+1}} \| v_h \|_{1, p, E_i}$$

(for the second estimate it was used that for $\lambda_{ij} \neq \hat{\lambda}_{ij}$ we have $\text{sgn} \, \beta_{ij} \neq \text{sgn} \, \hat{\beta}_{ij}$ and so $\left| \hat{\beta}_{ij} \right| \leqslant \left| \beta_{ij} - \hat{\beta}_{ij} \right|$.)

By virtue of $\| v_h \|_{1, p, E_i} \leqslant C \| v \|_{1, p, E_i}$ it is thus clear that we in the following may restrict us to the case $b = b(P_i) = \text{const.}$ on E_i.

First we transform $(b \, \nabla v_h, \, \varphi_i)$. There holds

$$(b \, \nabla v_h, \, \varphi_i) = \frac{1}{3} \sum_{T \subset E_i} (\text{meas } T) \, b \, \nabla v_h|_T .$$

We now set $y = \frac{1}{6} \sum_{T \subset E_i} \nabla v_h|_T$ and get by use of (5.2)

$$(b \, \nabla v_h, \, \varphi_i) = \frac{1}{2} \sum_{j \in \Lambda_i} b n_{ij} \, \gamma_{ij}(l_{ij} \, y)$$

$$= \frac{1}{2} \sum_{j \in \Lambda_i} \beta_{ij} \, l_{ij}(y - \nabla v_h|_{T_{ij}}) - \frac{1}{2} \sum_{j \in \Lambda_i} \beta_{ij}(v_i - v_j) . \tag{5.29}$$

Next we transform $b_h(v_h, \, \varphi_i)$. By virtue of $\beta_{ij*} = - \beta_{ij}$, $\lambda_{ij*} = \lambda_{ij} - 1$ we get

$$b_h(v_h, \, \varphi_i) = \frac{1}{2} \sum_{j \in \Lambda_i} \{ \beta_{ij}(\lambda_{ij} - 1)(v_i - v_j) + \beta_{ij*}(\lambda_{ij*} - 1)(v_i - v_{j*}) \}$$

$$= \frac{1}{2} \sum_{j \in \Lambda_i} \beta_{ij} \, \lambda_{ij}(2 \, v_i - v_j - v_{j*}) - \frac{1}{2} \sum_{j \in \Lambda_i} \beta_{ij}(v_i - v_j) . \tag{5.30}$$

For the treatment of the first summands of the right hand sides of (5.29) and (5.30) we will use an estimate of $(\nabla v_h|_S - \nabla v_h|_T)$ where S and T are two triangles $\subset E_i$ and S has arisen from T by a translation by a vector l (of course $|l| \leq h$). To this end we choose a function $\hat{v} \in C^2(\bar{\Omega})$ such that $\|\hat{v} - v\|_{2,p,E_i} \leq h\|v\|_{2,p,E_i}$. Then we get

$$
\begin{aligned}
\|\nabla v_h|_S - \nabla v_h|_T\|_{0,p,T} &= \|\nabla v(x+l) - \nabla v(x)\|_{0,p,T} + hO(\|v\|_{2,p,E_i}) \\
&= \|\nabla \hat{v}(x+l) - \nabla \hat{v}(x)\|_{0,p,T} + hO(\|v\|_{2,p,E_i}).
\end{aligned}
\tag{5.31}
$$

From $(k = 1, 2, \quad p < \infty)$

$$
\int_T \left| \frac{\partial}{\partial x_k} \hat{v}(x-1) - \frac{\partial}{\partial x_k} \hat{v}(x) \right|^p dx =
$$

$$
= \int_T \left| l \int_{\alpha=0}^1 \nabla\left(\frac{\partial}{\partial x_k} \hat{v}(x+\alpha l) \right) d\alpha \right|^p dx
$$

$$
\leq Ch^p \int_{\alpha=0}^1 \int_T \left| \nabla\left(\frac{\partial}{\partial x_k} \hat{v}(x+\alpha l) \right) \right|^p dx\, d\alpha
$$

$$
\leq Ch^p \sup_{\alpha \in (0,1)} \int_T \left| \nabla\left(\frac{\partial}{\partial x_k} \hat{v}(x+\alpha l) \right) \right|^p dx \leq Ch^p \|\hat{v}\|_{2,p,E_i}^p
$$

it follows

$$
\|\nabla \hat{v}(x+l) - \nabla \hat{v}(x)\|_{0,p,T} \leq Ch\|\hat{v}\|_{2,p,E_i} \leq Ch\|v\|_{2,p,E_i}.
\tag{5.32}
$$

(For $p = \infty$ (5.32) can be proven in a similar way.)
(5.31) and (5.32) yield

$$
\begin{aligned}
|\nabla v_h|_S - \nabla v_h|_T| &\leq Ch^{-\frac{2}{p}} \|\nabla v_h|_S - \nabla v_h|_T\|_{0,p,T} \\
&\leq Ch^{1-\frac{2}{p}} \|v\|_{2,p,E_i}.
\end{aligned}
\tag{5.33}
$$

Now let's turn again to the investigation of (5.29) and (5.30).
From (5.33) and the definition of y it follows

$$
\left| \sum_{j \in \Lambda_i} \beta_{ij} l_{ij}(y - \nabla v_h|_{T_{ij}}) \right| \leq C|b|_\infty h^{3-\frac{2}{p}} \|v\|_{2,p,E_i}
\tag{5.34}
$$

and

$$\left| \sum_{j \in \Lambda_i} \beta_{ij} \lambda_{ij} (2 v_i - v_j - v_{j*}) \right| = \left| \sum_{j \in \Lambda_i} \beta_{ij} \lambda_{ij} l_{ij} (\nabla v_h |_{T_{ij}} - \nabla v_h |_{T_{ij*}}) \right|$$

$$\leqslant C |b|_\infty h^{3 - \frac{2}{p}} \|v\|_{2, p, E_i} . \tag{5.35}$$

Combination of (5.29), (5.30), (5.34), (5.35) give (5.26).

B) We have for $w_h = u_h - I_h u$, for all $z_h \in V_{0h}$

$$l_h(w_h, z_h) = l(u - I_h u, z_h) + (l - l_h)(I_h u, z_h) .$$

From that, using (5.20), (5.25), (5.26) for $v_h = w_h$ and replacing z_h by $I_h(\Psi_\mu z_h)$, we get for triangulations of parallelogram type

$$
\begin{aligned}
l_h(w_h, I_h(\Psi_\mu z_h)) &\leqslant \varepsilon |u - I_h u|_{1, p, \Omega_{\mu-1}} \|I_h(\Psi_\mu z_h)\|_{1, p'} \\
&\quad + (\|b\|_\infty + \|c\|_\infty) \|u - I_h u\|_{1, p, \Omega_{\mu-1}} \|I_h(\Psi_\mu z_h)\|_{0, p'} \\
&\quad + Ch (\|b\|_{C^{0+1}} + \|c\|_{C^{0+1}}) \|u\|_{2, p, \Omega_{\mu-1}} \|I_h(\Psi_\mu z_h)\|_{0, p'} \\
&\leqslant Ch \|u\|_{2, p, \Omega_{\mu-1}} (\varepsilon \|I_h(\Psi_\mu z_h)\|_{1, p'} + \|I_h(\Psi_\mu z_h)\|_{0, p'}) .
\end{aligned}
\tag{5.36}
$$

Analogously, if we have a triangulation of parallelogram type only on $\Omega_{\mu-1} \backslash \Omega_\mu$, we get by (5.20), (5.22), (5.23) the weaker estimate

$$
l_h(w_h, I_h(\Psi_\mu z_h)) \leqslant Ch \|u\|_{2, p, \Omega_{\mu-1}} (\|z_h\|_{1, p', \Omega_\mu} +
$$
$$
+ \varepsilon \|I_h(\Psi_\mu z_h)\|_{1, p', \Omega_{\mu-1} \backslash \Omega_\mu} + \|I_h(\Psi_\mu z_h)\|_{0, p', \Omega_{\mu-1} \backslash \Omega_\mu}) \tag{5.37}
$$

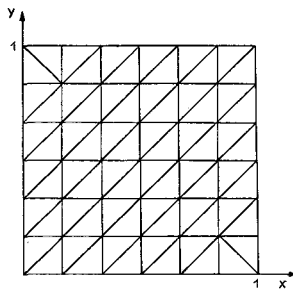Now the assertions of lemma 9 follow from (5.36), (5.37) by help of (5.3) and (5.4).

Figure 6.

6. NUMERICAL EXAMPLE

From numerical examples we got the following statements.

Generally, with classical FEM and piecewise linear approximation we get useful numerical solutions for $\varepsilon \geqslant O(h)$ and senseless solutions for $\varepsilon < O(h^2)$. The hybrid upwind-FEM gives for all ε-h-relations useful numerical solutions, in all cases the qualitative behaviour of the exact solution is given correctly. Sharp contures may be smoothened somewhat but not too much (in the provided numerical experiments boundary layers were extended by not more than $2h$), and so the theoretical result from
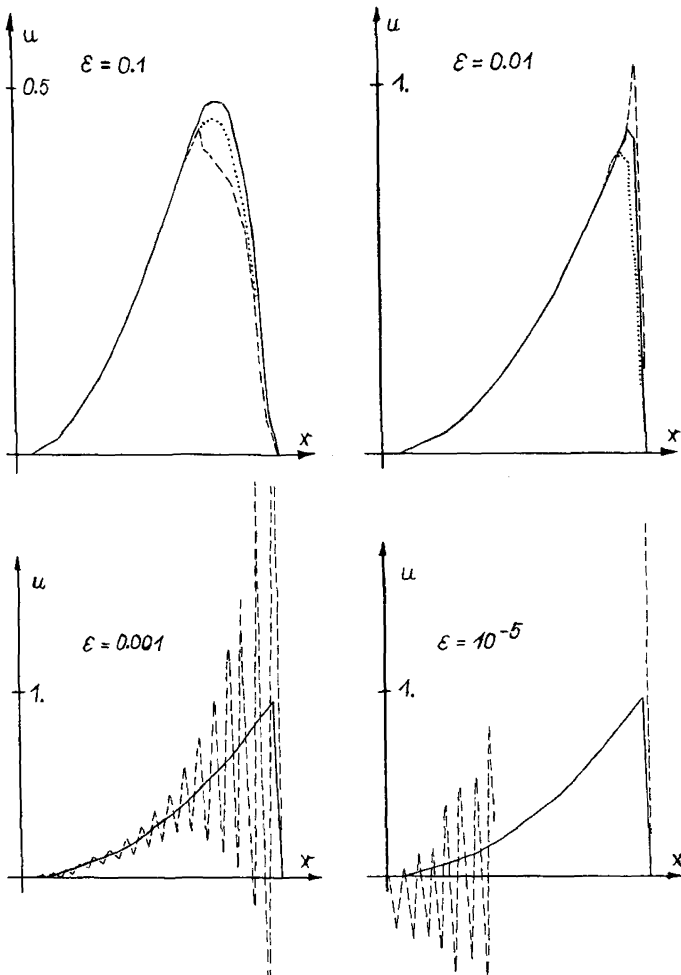

Figure 7.

theorem 2 on the thickness $\sigma|\ln\sigma|^2$ of the numerical boundary layer generally seems to be too pessimistic.

In figure 7 we see the exact solution and the numerical solutions of the problem

$$-\varepsilon\,\Delta u + u_x + u_y + 2\,u = f \qquad \text{in} \qquad \Omega = (0,1)^2$$
$$u = 0 \qquad \text{on} \qquad \partial\Omega$$

with $\qquad u_{\text{exact}} = xy\left(1 - \exp\dfrac{1-x}{\varepsilon}\right)\left(1 - \exp\dfrac{1-y}{\varepsilon}\right).$

The solutions are drawn along the line $x = y$.

The full line stands for the exact solution, the clashed line for the solution obtained by classical FEM and the dotted line for the solution obtained by the hybrid upwind-FEM.

The triangulation was of the type as you can see in figure 6, we chose 36×36 nodal points and so the stepsize h was $\dfrac{\sqrt{2}}{35}$.

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