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# On the approximation of a quasilinear mixed problem

# ON THE APPROXIMATION OF A QUASILINEAR MIXED PROBLEM (*)

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Abstract. — *We analyze in this paper a finite difference scheme for a quasilinear equation arising from the inverse problem of determination of transonic blade profiles for turbomachineries. The existence of the solution for the finite difference scheme as well as its convergence to the solution of the original equation are established for a small data which corresponds to a subsonic flow. Various numerical results are presented for a subsonic flow as well as for a transonic flow.*

Résumé. — *On analyse dans cet article un schéma de différence finie pour une équation quasilinéaire apparaissant lors du calcul du problème inverse de détermination de profils d'aubes transsoniques pour les turbomachines. On établit l'existence de la solution approchée ainsi que sa convergence vers la solution du problème originel pour des données petites correspondant à un écoulement subsonique. Plusieurs résultats numériques sont alors présentés pour un écoulement subsonique ainsi que transsonique.*

## 0. INTRODUCTION

The main object of this article is to present a numerical investigation of a quasilinear mixed equation. This equation governs the flow of a perfect and isentropic fluid, obtained when solving the inverse problem of determination of transonic blade profiles for turbomachineries, with the Mach number distributions prescribed along the suction and the pressure side of the blade profile and the upstream Mach number as well as the inlet and outlet flow angles given as data.

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This equation was established for a fluid verifying the exact isentropicity law : $p/\rho^\gamma = $ cst (where $\gamma$ is the ratio of specific heats ($\approx 1.4$), $p$ and $\rho$ are respectively the pressure and the density of fluid) after transformation of the physical domain to the plane defined by the streamlines and the potential lines of fluid.

The unknowns of this equation are the velocity, the Mach number and the density — the two last quantities are given as algebraic functions of the velocity by virtue of St-Venant's relations for isentropic fluids (see [2], [4]).

The boundary conditions for the velocity are of mixed type, namely, we use :

— Dirichlet conditions on the profile and for the upstream and downstream conditions.
— Periodic conditions on the rest of the boundary.

We get the Dirichlet conditions on the velocity by application of the St-Venant's relations on the Mach number distributions on the profile and from the data of the upstream Mach number. Actually, by application of the flow conservation through the blade row, we obtain the downstream boundary condition on the velocity. For more details about this physical problem, we refer to [2] and [4].

This equation with Dirichlet boundary condition is a specific example of the general framework considered in [5]. One can find in [5] a throughout study of the existence and regularity of solution for this kind of quasilinear elliptic equations. Due to the mixed boundary condition considered here, the equation is not included in the general framework studied in [5]. Consequently, different techniques from that of [5] are used for some parts of the proof in this paper.

We consider here a finite difference approximation of the equation. For the analysis of the problem, we use the variational framework for finite differences as in J. Céa [1] (see also R. Temam [7]). The discrete functional space is chosen to be the space of the step functions which allow the integration by parts. We can then write the discrete system as a Galerkin approximation for the variational formulation of the problem. This kind of approximations is referred in [7] as an external approximation of subspace of $H^1(\Omega)$. The analysis for the discrete system involves more difficulties than in the continuous case. For example, the construction of a suitable test function is not as straight forward as in [5].

The paper is organized as follows :

In Section 1, we introduce the definition of the weak solutions and the strong solutions for our equation. We prove that, if a weak solution is smooth enough (in $H^2(\Omega)$), then it is actually a strong solution. In Section 2, we present a variational formulation for a finite difference scheme. We then introduce in Section 3 a family of modified problem for the discrete system

which is not *a priori* well defined. We prove by passing through the family of modified problem that the discrete system admits at least one solution and further more all the solutions are bounded by the prescribed data on the boundary. In Section 4, we prove the strong convergence of the solution for the discrete system to a weak solution of the original equation. Finally, we explain briefly how the nonlinear finite difference scheme is implemented and we also present two numerical results for the inverse problem.

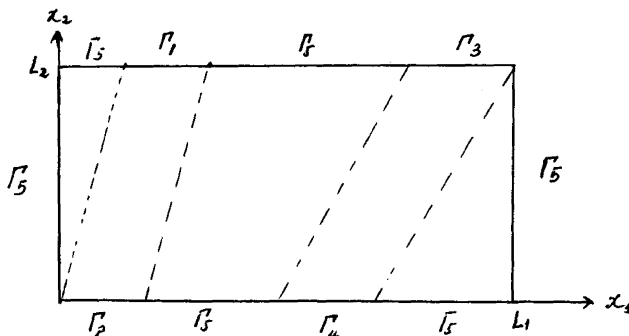## 1. A FUNCTIONAL SETTING OF THE EQUATION

The equation describing a transonic flow in the above environment is the following

$$(\mathscr{P}) \begin{cases} -\dfrac{\partial^2 u}{\partial x_2^2} - \dfrac{1 - M^2(u)}{\rho^2} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{1 - M^2}{u} \left( \dfrac{\partial u}{\partial x_2} \right)^2 + \\ \qquad\qquad\qquad\qquad + \dfrac{1 + \gamma M^4(u)}{\rho^2 u} \left( \dfrac{\partial u}{\partial x_1} \right)^2 = 0 \text{ , in } \Omega \\ u\big|_{\Gamma_5} = g, \; u\big|_{\Gamma_1} = u\big|_{\Gamma_2}, \; u\big|_{\Gamma_3} = u\big|_{\Gamma_4}, \; \dfrac{\partial u}{\partial x_2}\bigg|_{\Gamma_1} = \dfrac{\partial u}{\partial x_2}\bigg|_{\Gamma_2}, \; \dfrac{\partial u}{\partial x_2}\bigg|_{\Gamma_3} = \dfrac{\partial u}{\partial x_2}\bigg|_{\Gamma_4} \end{cases}$$

with

$$M(u) = u \sqrt{\dfrac{2}{\gamma + 1} \dfrac{1}{1 - \dfrac{\gamma - 1}{\gamma + 1} u^2}}, \quad \rho(u) = \left( 1 - \dfrac{\gamma - 1}{\gamma + 1} u^2 \right)^{\frac{1}{\gamma - 1}}$$

where $\Omega$ is the rectangle $[0, L_1] \times [0, L_2]$ and $\partial\Omega = \bigcup_{i=1}^{5} \Gamma_i$ (see figure below).

The system $(\mathscr{P})$ can be reformulate to the following conservative form (see [5] for detail) :

$$(\mathscr{P}') \begin{cases} -\dfrac{\partial^2 u}{\partial x_2^2} - \dfrac{\partial}{\partial x_1} f(u) \dfrac{\partial u}{\partial x_1} + F(u, \nabla u) = 0, \text{ in } \Omega \\[2mm] u\big|_{\Gamma_5} = g, \; u\big|_{\Gamma_1} = u\big|_{\Gamma_2}, \; u\big|_{\Gamma_3} = u\big|_{\Gamma_4}, \; \dfrac{\partial u}{\partial x_2}\bigg|_{\Gamma_1} = \dfrac{\partial u}{\partial x_2}\bigg|_{\Gamma_2}, \dfrac{\partial u}{\partial x_2}\bigg|_{\Gamma_3} = \dfrac{\partial u}{\partial x_2}\bigg|_{\Gamma_4} \end{cases}$$

with

(1.1)
$$f(u) = \frac{1 - M^2(u)}{\rho^2} \, ,$$

$$F(u, \nabla u) = \frac{1 + M^2(u)}{u} \left[ \left( \frac{\partial u}{\partial x_2} \right)^2 + f(u) \left( \frac{\partial u}{\partial x_1} \right)^2 \right] \, .$$

For fixed $0 < \alpha < K < 1$, we can show by a simple computation that

(1.2)
$$1 > f(\alpha) = f(u) \geqslant f(K) > 0 \, , \quad \forall u \in [\alpha, K]$$

and

(1.3)
$$1 < H(K) \leqslant H(u) = \frac{1 + M^2(u)}{u} \leqslant H(\alpha) \, , \quad \forall u \in [\alpha, K] \, .$$

Let

$$\mathscr{H}_p = \left\{ u \in H^1(\Omega) : u\big|_{\Gamma_1} = u\big|_{\Gamma_2}, \, u\big|_{\Gamma_3} = u\big|_{\Gamma_4} \right\} \, .$$

For each $\phi \in H^1(\Omega) \cap C(\bar{\Omega})$, we define

$$\mathscr{H}_{p, \phi} = \left\{ u \in \mathscr{H}_p : u\big|_{\Gamma_5} = \phi \right\}$$

which is a convex set of $\mathscr{H}_p$. In particular, $\mathscr{H}_{p, 0}$ is a subspace of $H^1(\Omega)$.

**Hypothesis on $g$**

$$\text{(H1)} \quad \begin{cases} g \in H^1(\Omega) \cap C(\bar{\Omega}), \, 0 < \alpha \leqslant g \leqslant K < 1 \\ \mathscr{H}_{p, g} \neq 0 \text{ and } s \in \mathscr{H}_{p, g} \, . \end{cases}$$

DÉFINITION 1 : *We say that $u \in H^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of problem $(\mathscr{P})$ if*

(i) $0 < \inf_{\Omega} \mathrm{essu} \leqslant \sup_{\Omega} \mathrm{essu} < 1$ ;

(ii) for every $v \in \mathcal{H}_{p,0} \cap L^{\infty}(\Omega)$,

$$(1.4) \qquad \left( \frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_2} \right) + \left( f(u) \frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1} \right) + (F(u, \nabla u), v) = 0 \; ;$$

(iii) $u - s \in \mathcal{H}_{p,0}$.

where $(.,.)$ is the scalar product of $L^2(\Omega)$ and we denote hereafter $\|.\| = (.,.)^{1/2}$.

DÉFINITION 2 : *We say that $u$ is a strong solution of problem $(\mathcal{P})$ if $u \in H^2(\Omega)$ and $u$ satisfies the system $(\mathcal{P})$.*

PROPOSITION 1 : *If $u$ is a weak solution of (1-4) and moreover $u \in H^2(\Omega)$. Then, $u$ is a strong solution of the problem $(\mathcal{P})$.*

*Proof:* Since $u \in H^2(\Omega)$, we can integrate by parts in (1.4) :

$$(1.5) \quad - \left( \frac{\partial^2 u}{\partial x_2^2}, v \right) - \left( \frac{\partial}{\partial x_1} f(u) \frac{\partial u}{\partial x_1}, v \right) + (F(u, \nabla u), v) =$$

$$= - \int_{\partial \Omega} \left( f(u) \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right) \cdot \mathbf{n} v \, d\Gamma \; .$$

Since $D(\Omega)$ is dense in $\mathcal{H}_{p,0} \cap L^{\infty}(\Omega)$, we derive that

$$- \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial}{\partial x_1} f(u) \frac{\partial u}{\partial x_1} + F(u, \nabla u) = 0, \text{ in } \Omega$$

This relation and (1-5) imply that

$$- \int_{\partial \Omega} \left( f(u) \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right) \cdot \mathbf{n} v \, d\Gamma = 0 \; , \quad \forall v \in \mathcal{H}_{p,0} \cap L^{\infty}(\Omega) \; .$$

Then, by choosing appropriate function $v$ in $\mathcal{H}_{p,0} \cap L^{\infty}(\Omega)$, we deduce easily

$$\left. \frac{\partial u}{\partial x_2} \right|_{\Gamma_1} = \left. \frac{\partial u}{\partial x_2} \right|_{\Gamma_2}, \quad \left. \frac{\partial u}{\partial x_2} \right|_{\Gamma_3} = \left. \frac{\partial u}{\partial x_2} \right|_{\Gamma_4} . \; \P$$

We set hereafter $\langle A(u) w, v \rangle = - \left( \frac{\partial w}{\partial x_2}, \frac{\partial v}{\partial x_2} \right) - \left( f(u) \frac{\partial w}{\partial x_1}, \frac{\partial v}{\partial x_1} \right) .$

## 2. A FINITE DIFFERENCE SCHEME

We will use a finite difference scheme to approximate the problem $(\mathcal{P})$. Let us first introduce some notations.

For the sake of simplicity, we will take the uniform discretization over $\Omega$ and we assume the discretization step in both direction to be the same, i.e. $\Delta x_1 = \Delta x_2 = h$. We will consider a subsequence $\{h\} \to 0$ (still denoted by $\{h\}$) such that the points $A_i$ $(i = 1, ..., 6)$ are among the set of grid points.

For each $h$, we define our computational domain $\Omega_h$ by extending the original domain $\Omega$ along the periodical lines as showed in figure below.
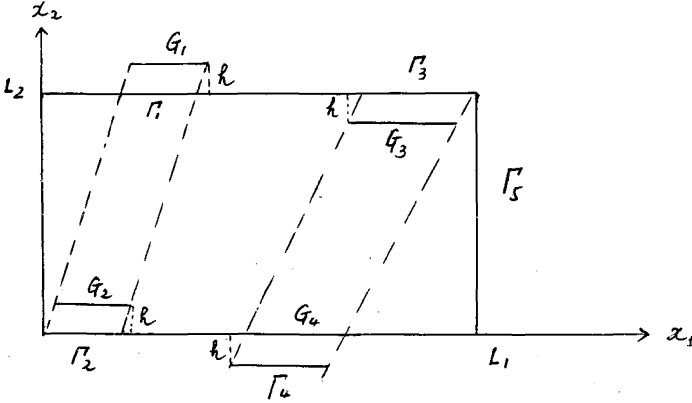


The set of discretization points on $\Gamma_i$, $G_i$ $(i = 1, ..., 4)$ will be denoted respectively by $\Gamma'_i$, $G'_i$ $(i = 1, ..., 4)$. We denote $N_1 = \dfrac{L_1}{h}$, $N_2 = \dfrac{L_2}{h}$ and let

$$M^h = \{x_{nm} = (nh, mh) : 0 \leqslant n \leqslant N_1, 0 \leqslant m \leqslant N_2\} \cup G'_1 \cup G'_4$$

$$M^h_0 = \{x_{nm} = (nh, mh) : 1 \leqslant n \leqslant N_1 - 1, 1 \leqslant m \leqslant N_2 - 1\} \cup$$

$$G'_1 \cup G'_4 \bigcup_{i=1}^{4} \Gamma'_i$$

where $G'_1 = \{x_{n, N_2 + 1} : n = M_1, ..., M_2\}$ and $G'_4 = \{x_{n, -1} : n = M_3, ..., M_4\}$.

For each

$$Q = (x_1, x_2), \text{ let } r(Q) = \left[x_1 - \frac{h}{2}, x_1 + \frac{h}{2}\right) \times \left[x_2 - \frac{h}{2}, x_2 + \frac{h}{2}\right)$$

and $W_Q(x)$ the characteristic function over $r(Q)$. We then define a set of step functions $\mathscr{X}^h$ by

$$\mathscr{X}^h = \left\{ u_h = \sum_{Q \in M^h} u_h(Q) \, W_Q(x) : \begin{array}{c} u_h(x) \in R \text{ with} \\ u_h\big|_{G'_1} = u_h\big|_{G'_2}, \, u_h\big|_{G'_3} = u_H\big|_{G'_4} \\ u_h\big|_{\Gamma'_1} = u_h\big|_{\Gamma'_2}, \, u_h\big|_{\Gamma'_3} = u_h\big|_{\Gamma'_4} \end{array} \right\}$$

where $x = (x_1, x_2)$.

For every $\phi \in \mathcal{H}_{p,\phi} \cap C(\Omega)$, we define

$$\mathcal{X}_{\phi}^{h} = \left\{ u_h(x) \in \mathcal{X}^h : u_h \big|_{\Gamma_5} = \phi \big|_{\Gamma_5} \right\}$$

and in particular

$$\mathcal{X}_{0}^{h} = \left\{ u_h(x) \in \mathcal{X}^h : u_h \big|_{\Gamma_5} = 0 \right\} .$$

We will look for the discrete unknown function $u_h$ in $\mathcal{X}_{s}^{h}$. We note that $\mathcal{X}^h$ and $\mathcal{X}_{0}^{h}$ are subspace of $L^2(R^2)$.

Let $u_{ab} = u(ah, bh)$, we then introduce the following difference operators

$$\delta_1 u_{nm} = \frac{u_{n+1,m} - u_{n-1,m}}{2h} , \quad \delta_2 u_{nm} = \frac{u_{n,m+1} - u_{n,m-1}}{2h}$$

$$\delta_1^+ u_{nm} = \frac{u_{n+1,m} - u_{,m}}{h} , \quad \delta_2^- u_{nm} = \frac{u_{n,m-1} - u_{n,m}}{h}$$

$$\delta = (\delta_1, \delta_2) .$$

We can readily check the following discrete Poincaré inequality.

LEMMA 1 :

(2.1) $$\|u\| \leq c \|\delta u\| , \quad \forall u \in \mathcal{X}_{0}^{h} .$$

Consequently, if we denote $\|\delta u\| = \left( \int_{R^2} (\delta u)^2 \, dx \right)^{1/2}$, then $\|\delta u\|$ is a norm on $\mathcal{X}_{0}^{h}$ equivalent to the norm $\|\!\|u\|\!\| = \|u\| + \|\delta u\|$.

By using these notations, we define our finite difference approximation of the problem $\mathcal{P}$ as follows

$(\mathcal{P}_h)$ $\begin{cases} - \delta_2 \delta_2 u_h(Q) - \delta_1 f(u_h(Q)) \delta_1 u_h(Q) + F_h(u_h(Q), \delta u_h(Q)) = 0 , \\ \forall Q \in M_0^h \, u_h \in \mathcal{X}_{s}^{h} . \end{cases}$

According to the definition of $\mathcal{X}^h$, we have

$$u\left( Q + \left( 0, \frac{h}{2} \right) \right) = u(Q + (0, h)) \quad \text{and} \quad u\left( Q - \left( 0, \frac{h}{2} \right) \right) = u(Q) .$$

Therefore the unknowns of $(\mathcal{P}_h)$ are $\{u_h(Q) : Q \in M_0^h\}$. Note that we used the center differences for the second order term while the upwind differences was used for the first order non-linear term.

It is obvious that $(\mathcal{P}_h)$ is equivalent to the following variational problem :

(2.2) $\begin{cases} (\delta_2 \delta_2 u_h - \delta_1 f(u_h) \delta_1 u_h, v_h) + (F_h(u_h, \delta u_h), v_h) = 0 , \\ \forall v_h \in \mathcal{X}_{0}^{h} \, u_h \in \mathcal{X}_{s}^{h} . \end{cases}$

If we denote

$$\langle A_h(u_h)\, v_h, w_h \rangle = (\delta_2 v_h, \delta_2 w_h) + (f(u_h)\, \delta_1 v_h, \delta_1 w_h)\,.$$

We can easily check by integration by parts (see [7]) that (2.2) is equivalent to

$$(\mathscr{P}_h')\quad \begin{cases} \langle A_h(u_h)\, u_h, v_h \rangle + (F_h(u_h, \delta u_h), v_h) = 0\,, \\ \forall v_h \in \mathscr{X}_0^h \; u_h \in \mathscr{X}_s^h\,. \end{cases}$$

## 3. A FAMILY OF MODIFIED PROBLEMS

As in [5], [6], we will introduce a family of modified problems for diverse reasons, among them are :

— $A_h$, $F_h$ are only well defined for $|u| \in (0, 1)$ ;
— we do not know *a priori* if the solution $u_h$ of $(\mathscr{P}_h)$ satisfies $|u_h| \in (0, 1)$.

Let us introduce now the following truncation functions

$$\tilde{f}(u) = \begin{cases} f(K)\,, & u \geqslant K \\ f(u)\,, & \alpha \leqslant u \leqslant K \\ f(\alpha)\,, & u \leqslant \alpha \end{cases}$$

$$\tilde{F}_h(u, \delta u) = \begin{cases} F_h(K, \delta u)\,, & u \geqslant K \\ F_h(u, \delta u)\,, & \alpha \leqslant u \leqslant K \\ F_h(\alpha, \delta u)\,, & u \leqslant \alpha \end{cases}$$

and

$$F_{\varepsilon, h}(u, \delta u) = h_\varepsilon(u - \alpha)\, \frac{\tilde{F}_h(u, \delta u)}{1 + \varepsilon \tilde{F}_h(u, \delta u)}$$

where $h_\varepsilon(x)$ is a continuous function on $R$ defined by

$$h_\varepsilon(x) = \begin{cases} 1\,, & x \geqslant \alpha \\ \text{linear}\,, & 0 \leqslant x \leqslant \alpha \\ 0\,, & x \leqslant 0\,. \end{cases}$$

It is clear that, if $u(x)$ is a step function, then $F(u)$, $F_{\varepsilon, h}(u, \delta u)$ are also step functions. We set

$$(3.1)\quad \langle \tilde{A}_h(u_h)\, v_h, w_h \rangle = (\delta_2 v_h, \delta_2 w_h) + (\tilde{f}(u_h)\, \delta_1 v_h, \delta_1 w_h)\,.$$

We derive from (1.2), (1.3) and the definition of $F_{\varepsilon, h}$ that

(3.2)     $(\tilde{A}_h(u)\,v, v) \geqslant \min\,\{1, f(K)\}\,\|\delta v\|^2 = f(K)\,\|\delta v\|^2\,,\quad \forall v \in \mathscr{X}^h$

(3.3)     $\left|(\tilde{A}_h(u)\,v, w)\right| \leqslant \max\,\{1, f(\alpha)\}\,\|\delta v\|\,\|\delta w\|$

$\qquad\qquad\qquad = \|\delta v\|\,\|\delta w\|\,,\quad \forall v, w \in \mathscr{X}^h$

(3.4)     $0 \leqslant f_{\varepsilon, h}(x, y) \leqslant \dfrac{1}{\varepsilon}\,,\quad \forall x, y \in R\,.$

We then introduce a family of modified problems defined as follows :

$(\mathscr{P}_{\varepsilon, h})$ $\qquad \begin{cases} (\tilde{A}_h(u_{\varepsilon, h})u_{\varepsilon, h}, v) + (F_{\varepsilon, h}(u_{\varepsilon, h}, \delta u_{\varepsilon, h}), v) = 0\,, \\[2mm] \forall v \in \mathscr{X}_0^h\, u_{\varepsilon, h} \in \mathscr{X}_s^h\,. \end{cases}$

LEMMA 2 : *The problem* $(\mathscr{P}_{\varepsilon, h})$ *admits at least one solution.*

Before proving lemma 2, we introduce first two small lemmas.

LEMMA 3 : *For each* $u_\varepsilon \in \mathscr{X}_s^h$, *there exists an unique solution* $w_\varepsilon \in \mathscr{X}_s^h$ *for the variational inequality*

(3.5)   $(\delta_2\,w_\varepsilon, \delta_2(v - w_\varepsilon)) + (\tilde{f}(u_\varepsilon)\,\delta_1\,w_\varepsilon, \delta_1(v - w_\varepsilon)) +$

$\qquad\qquad\qquad + (F_{\varepsilon, h}(u_\varepsilon, \delta u_\varepsilon), v - w_\varepsilon) \geqslant 0\,,\quad \forall v \in \mathscr{X}_s^h\,.$

*Proof* :

(i) *Existence* : It is clear that $\mathscr{X}_s$ is a closed convex set of $\mathscr{X}^h$. For fixed $u_\varepsilon \in \mathscr{X}_s$, we define a continuous bilinear form $a(.,.)$ on $\mathscr{X}_s^h \times \mathscr{X}^h$ by

$$a(w, v) = (\delta_2\,w, \delta_2\,v) + (\tilde{f}(u_\varepsilon)\,\delta_1\,w, \delta_1\,v)\,.$$

This form then define a linear continuous function, denoted by $A_1$, from $\mathscr{X}_s^h$ into $(\mathscr{X}^h)'$ which is the dual space of $\mathscr{X}^h$, namely

$$\langle A_1\,w, v \rangle = a(w, v)\,,\quad \forall w \in \mathscr{X}_s^h\,,\quad \forall v \in \mathscr{X}^h\,.$$

Moreover from (3.2), we derive that there exists a constant $c$ such that for all $w, v \in \mathscr{X}_s^h$

(3.6)     $\langle A_1\,w - A_1\,v, w - v \rangle \geqslant f(K)\,\|\delta(w - v)\|^2 \geqslant 0\,.$

Thus, $A_1$ is monotone.

Now we define the operator $B : \mathscr{X}_s^h \to (\mathscr{X}^h)'$ by setting

$$Bu = A_1\,u + \tilde{F}_\varepsilon(u_\varepsilon, \delta u_\varepsilon)\,.$$

Before investigating the properties of $B$, we define $r_h$ as the operator of the projection in $\mathcal{H}_p$ onto $\mathcal{X}^h$, namely, for $u \in \mathcal{H}_p$

$$(3.7) \qquad \begin{cases} r_h u \in \mathcal{X}^h \text{ and} \\[2mm] r_h u(Q) = \dfrac{1}{h^2} \displaystyle\int_{W_Q(x)} u(x)\, dx, \quad \forall Q \in M^h. \end{cases}$$

It can be proven that (see [7])

$$(3.8) \qquad \| \delta r_h u \| \leqslant \| \nabla u \|, \quad \forall u \in \mathcal{H}_p.$$

One can now easily check that $B$ enjoys the following properties :

(i) $B$ is monotone, i.e.

$$\langle Bw - Bv, w - v \rangle \geqslant f(K) \| \delta(w - v) \|^2, \quad \forall w, v \in \mathcal{X}_s^h.$$

(ii) $B$ is coercive in the following sense

$$\frac{\langle Bw - Br_h s, w - r_h s \rangle}{\| w - r_h s \| + \| \delta w - r_h s \|} \to +\infty \ (as \ \| w \| + \| \delta w \| \to +\infty), \quad \forall w \in \mathcal{X}_s^h.$$

In fact, the last assertion is a consequence of property (i) and (2.1), by observing that $w - r_h s \in \mathcal{X}_0^h, \ \forall w \in \mathcal{X}_s^h$.

(iii) $B$ is continuous on finite dimensional subspace.

We conclude from the corollary 1.8 of [3] (p. 86) that there exists at least one solution for (3.5).

For the sake of simplicity, we will still use $s$ to denote $r_h s$ in the sequel.

(ii) *Uniqueness* : If $w_1$, $w_2$ are two solutions of (3.5), then

$$-\langle Bw_1, v - w_1 \rangle \leqslant 0 \quad \text{and} \quad -\langle Bw_2, v - w_2 \rangle \leqslant 0.$$

The sum of these two inequalities leads to

$$\langle Bw_1 - Bw_2, w_1 - w_2 \rangle \leqslant 0$$

we then derive from (i) that $\delta(w_1 - w_2) = 0$ in $\Omega$. This and the Poincaré inequality (2.1) imply that $w_1 = w_2$. ¶

LEMMA 4 : *Let $u_\varepsilon \in \mathcal{X}_s^h$. Then the unique solution $w_\varepsilon \in \mathcal{X}_s^h$ of the problem (3.5) is the unique solution of the following problem*

$$(3.9) \qquad a(w_\varepsilon, v) + (F_{\varepsilon, h}(u_\varepsilon, \delta u_\varepsilon), v) = 0, \quad \forall v \in \mathcal{X}_0^h.$$

*Proof*: By observing that $v$, $w_\varepsilon \in \mathscr{X}_s^h$ implies $v - w_\varepsilon \in \mathscr{X}_0^h$, we conclude that

$$\langle Bw_\varepsilon, v - w_\varepsilon \rangle \geqslant 0 , \quad \forall v \in \mathscr{X}_s^h$$

is equivalent to

$$\langle Bw_\varepsilon, v \rangle \geqslant 0 , \quad \forall v \in \mathscr{X}_0^h .$$

Since $\mathscr{X}_0^h$ is a subspace of $L^2(R^2)$, the last inequality is actually an equality.⁋

*Proof of the lemma* 2 : Lemma 4 ensures that we can define an operator $T : \mathscr{X}_s \to \mathscr{X}_s$ by $w_\varepsilon = Tu_\varepsilon$. Let us prove that $T$ maps a ball (in $\mathscr{X}_s^h$) $B(M)$ to $B(M)$.

Let $v_\varepsilon = w_\varepsilon - s \in \mathscr{X}_0^h$. By definition of $w_\varepsilon$, we have

$$a(w_\varepsilon, v_\varepsilon) + (F_{\varepsilon, h}(u_\varepsilon, \delta u_\varepsilon), v_\varepsilon) = 0$$

which can be written as

$$a(w_\varepsilon, w_\varepsilon) = - (F_{\varepsilon, h}(u_\varepsilon, \delta u_\varepsilon), v_\varepsilon) + a(w_\varepsilon, s) .$$

Using (3.2), (3.3), (3.4) and lemma 1, we derive that

$$\begin{aligned}
f(K)\|\delta w_\varepsilon\| &\leqslant \varepsilon^{-1} \int_{R^2} |w_\varepsilon - s| dx + \|\delta w_\varepsilon\| \|\delta s\| \\
&\leqslant \varepsilon^{-1} m(\Omega)^{1/2} \|w_\varepsilon - s\| + \|\delta w_\varepsilon\| \|\delta s\| \\
&\leqslant \varepsilon^{-1} m(\Omega)^{1/2} \|\delta(w_\varepsilon - s)\| + \|\delta w_\varepsilon\| \|\delta s\| \\
&\leqslant \frac{f(K)}{2} \|\delta w_\varepsilon\|^2 + c_1(\|\delta s\|, \Omega, \varepsilon)
\end{aligned}$$

where $m(\Omega)$ is the mesure of $\Omega$.

Therefore

$$(3.10) \qquad\qquad \|\delta w_\varepsilon\| \leqslant \left( \frac{2 c_1}{f(K)} \right)^{1/2} = M .$$

It means that the operator T maps the ball $B(M)$ in $\mathscr{X}_s^h$ to $B(M)$. Furthermore, since $\mathscr{X}^h$ is finite dimensional, we readily check that

$$(3.11) \qquad \|\delta(u_n - u_0)\| \to 0 \Rightarrow |\tilde{f}(u_n) - \tilde{f}(u_0)| \to 0 .$$

Now let $w_n = Tu_n$ and $w_0 = Tu_0$, then we have from (3.7) :

$$\langle A_h(u_n) w_n, v \rangle + \langle F_{\varepsilon, h}(u_n, \delta u_n), v \rangle = 0 , \quad \forall \in \mathscr{X}_0^h .$$

The subtraction of these two equations leads to

$$(3.12) \quad \langle A_h(u_0)(w_n - w_0), v \rangle = \langle (A_h(u_0) - A_h(u_n)) w_n, v \rangle +$$
$$+ \langle F_{\varepsilon, h}(u_0, \delta u_0) - F_{\varepsilon, h}(u_n, \delta u_n), v \rangle .$$

We then take $v = w_n - w_0$ in (3.12), by using (3.2), (3.10) and (3.11), we derive :

$$f(K) \| \delta (w_n - w_0) \|^2 \leqslant \langle A_h(u_0)(w_n - w_0), w_n - w_0 \rangle$$
$$\leqslant | \langle (A_h(u_0) - A_h(u_n)) w_n, w_n - w_0 \rangle |$$
$$+ | \langle F_{\varepsilon, h}(u_0, \delta u_0) - F_{\varepsilon, h}(u_n, \delta u_n), w_n - w_0 \rangle |$$
$$\to 0 (\| \delta (u_n - u_0) \| \to 0)$$

*i.e.*

$$\| \delta (Tu_n - Tu_0) \| = \| \delta (w_n - w_0) \| \to 0 (\| \delta (u_n - u_0) \| \to 0) .$$

That means T is continuous from $\mathscr{X}_s^h$ to $\mathscr{X}_s^h$. We can then apply the Browder's fixed point Theorem which ensures that there exists at least one solution $u_{\varepsilon, h}$ for the problem (3.9). Evidently, $u_{\varepsilon, h}$ is also a solution of problem $(\mathscr{P}_{\varepsilon, h})$. ¶

LEMMA 5 : *We assume (H1). Then every solution $u_{\varepsilon, h}$ of $(\mathscr{P}_{\varepsilon, h})$ satisfies*

$$\alpha \leqslant u_{\varepsilon, h} \leqslant K .$$

*Proof* : We define two functions $(u_{\varepsilon, h} - \alpha)_-$ and $(u_{\varepsilon, h} - K)_+$ by

$$(u_{\varepsilon, h} - \alpha)_- (x) = \begin{cases} u_{\varepsilon, h}(x) - \alpha , & \text{if} \quad u_{\varepsilon, h}(x) \leqslant \alpha \\ 0, & \text{if} \quad u_{\varepsilon, h}(x) > \alpha \end{cases}$$

$$(u_{\varepsilon, h} - K)_+ (x) = \begin{cases} u_{\varepsilon, h}(x) - K, & \text{if} \quad u_{\varepsilon, h}(x) \geqslant K \\ 0, & \text{if} \quad u_{\varepsilon, h}(x) < K . \end{cases}$$

Since $u_{\varepsilon, h | \Gamma_5} = s$, we derive that $(u_{\varepsilon, h} - \alpha)_-$ and $(u_{\varepsilon, h} - K)_+$ belong to $\mathscr{X}_0^h$.

We now take $v = (u_{\varepsilon, h} - \alpha)_-$ in $(\mathscr{P}_{\varepsilon, h})$, by the definition of $u_{\varepsilon, h}$ and $(u_{\varepsilon, h} - \alpha)_-$, we derive

$$(F_{\varepsilon, h}(u_{\varepsilon, h}, \delta u_{\varepsilon, h}), (u_{\varepsilon, h} - \alpha)_-) = 0$$

Therefore

$$(\tilde{A}_h(u_{\varepsilon, h}) u_{\varepsilon, h}, (u_{\varepsilon, h} - \alpha)_-) = 0$$

i.e.

$$\int_\Omega \left\{ \delta_2\, u_{\varepsilon,h}\, \delta_2 (u_{\varepsilon,h} - \alpha)_- + \tilde{f}(\tilde{u}_{\varepsilon,h})\, \delta_1\, u_{\varepsilon,h}\, \delta_1 (u_{\varepsilon,h} - \alpha)_- \right\} = 0 \ .$$

This can be written as

$$\int_{u_{\varepsilon,h} \leqslant \alpha} \left\{ \delta_2 (u_{\varepsilon,h} - \alpha)_-\, \delta_2 (u_{\varepsilon,h} - \alpha)_- + \right.$$

$$\left. + \tilde{f}(\tilde{u}_{\varepsilon,h})\, \delta_1 (u_{\varepsilon,h} - \alpha)_-\, \delta_1 (u_{\varepsilon,h} - \alpha)_- \right\} = 0 \ .$$

We derive from this inequality and the relation (3.2) that

$$\delta (u_{\varepsilon,h} - \alpha)_-\, (x) = 0 \ .$$

We then derive from (2.1) that

$$(u_{\varepsilon,h} - \alpha)_-\, (x) = 0 \ , \text{ i.e. } u_{\varepsilon,h} \geqslant \alpha \ .$$

Similarly, we take $v = (u_{\varepsilon,h} - K)_+$ in $(\mathscr{P}_{\varepsilon,h})$, since $(u_{\varepsilon,h} - K)_+ \geqslant 0$ and $F_\varepsilon \geqslant 0$, we find

$$\int_\Omega \left\{ \delta_2\, u_{\varepsilon,h}\, \delta_2 (u_{\varepsilon,h} - K)_+ + \tilde{f}(\tilde{u}_{\varepsilon,h})\, \delta_1\, u_{\varepsilon,h}\, \delta_1 (u_{\varepsilon,h} - K)_+ \right\} = 0$$

which implies

$$\delta (u_{\varepsilon,h} - K)_+ = 0, \text{ i.e. } u_{\varepsilon,h} \leqslant K \ .$$

The proof is complete. ¶

From lemma 2, we deduce that there exists $u_h \in \mathscr{X}_s^h$ such that

$$u_{\varepsilon,h} \to u_h \text{ (when } \varepsilon \to 0 \text{) and } \alpha \leqslant u_h \leqslant K \ .$$

Since the problems $(\mathscr{P}_h)$ and $(\mathscr{P}_{\varepsilon,h})$ are both finite dimensional, we can directly pass to the limit $(\varepsilon \to 0)$ in $(\mathscr{P}_{\varepsilon,h})$ by noting that

$$\begin{cases} \langle \tilde{A}_h(u)\, v, w \rangle = \langle A_h(u)\, v, w \rangle \\ F_{\varepsilon,h}(u, \delta u) \to F_h(u, \delta u)(\varepsilon \to 0) \ , \end{cases} \text{ for } \alpha \leqslant u \leqslant K$$

we derive that $u_h$ is a solution of $(\mathscr{P}_h)$. We have then proved the following theorem.

THEOREM 1 : *Under the assumption* (H1). *The problem* $(\mathscr{P}_h)$ *admits at least one solution and every solution* $u_h$ *of* $(\mathscr{P}_h)$ *satisfies* $\alpha \leqslant u_h \leqslant K$. ¶

## 4. A STRONG CONVERGENCE RESULT

Our aim now is to pass to the limit $(h \to 0)$ in $(\mathscr{P}_h^i)$. Due to the complexity of the nonlinear term, we need a strong convergence result for $\{\delta u_h\}$.

We recall first that :

Given $\nu_1, \nu_2 > 0$, there exists $\sigma \in C^1(R)$ such that (see [6])

$$(4.1) \qquad \begin{cases} \nu_1 \sigma'(t) - \nu_2 |\sigma(t)| = 1 , & \forall t \in R \\ \sigma(0) = 0 . \end{cases}$$

Actually, $\sigma(t)$ is explicitly given by

$$\begin{cases} \sigma(t) = \dfrac{1}{\nu_2} \left( \exp \left( \dfrac{\nu_2 t}{\nu_1} \right) - 1 \right) , & t \geq 0 \\ \sigma(t) = -\sigma(-t) , & t \leq 0 . \end{cases}$$

We consider now a discrete function $\sigma_h(u_h - s)$ in $\mathscr{X}_0^h$ defined by

$$\sigma_h(u_h - g)(Q) = \sigma(u_h(Q) - g(Q)) , \quad \forall Q \in M^h$$

where $\sigma$ is the function defined in lemma 3 with $\nu_1 = 1$ and $\nu_2 = H(\alpha)$. We derive from Theorem 1 and (4.1) that there exists $c_2 > 0$ such that

$$(4.2) \qquad \sigma'(u_h - s)(x) \leq c_2, \ |\sigma(u_h - s)(x)| \leq c_2 , \quad \forall x \in R^2, \forall h$$

LEMMA 6 : *Let*

$$\begin{cases} g_\alpha(t, v) = \dfrac{\sigma(t) - \sigma(v)}{t - v} - \dfrac{H(\alpha)}{2} [\sigma(t) + \sigma(v)] \\ g_\alpha(t, t) = \lim_{v \to t} g_\alpha(t, v) . \end{cases}$$

*Then for any fixed $\alpha \in (0, 1)$, there exists $K \in (\alpha, 1)$ such that*

$$g_\alpha(t, v) \geq \frac{1}{2} \quad \text{for} \quad \begin{cases} \alpha - K \leq v \leq K - \alpha \\ 2(\alpha - K) \leq t - v \leq 2(K - \alpha) . \end{cases}$$

*Proof:* We note that

$$g_\alpha(t, t) = \sigma'(t) - H(\alpha) \sigma(t) \geq 1$$

especially $g_\alpha(0, 0) = 1$. The lemma then follows by noting that $g_\alpha(t, v)$ is a continuous function of $(t, v)$. ¶

Before passing to the limit in $(\mathscr{P}'_h)$, let us prove first a stability result in $\mathscr{X}^h$.

LEMMA 7 : *We assume* (H1) *and* (H2) $\alpha \leqslant s(x) \leqslant K$ *with* $K$ *defined in lemma* 6. *Then*

$$(4.3) \qquad \| \delta u_h \| \leqslant \frac{2 c_2 f(\alpha)}{f(K)}, \quad \forall h.$$

*Proof:* For $v \in \mathscr{X}^h$, we have

$$(4.4) \quad \delta_2 v = \frac{1}{h} \sum_{Q \in M^h} v(Q) \left[ W_Q \left( x + \left( 0, \frac{h}{2} \right) \right) - W_Q \left( x - \left( 0, \frac{h}{2} \right) \right) \right]$$

$$= \frac{1}{h} \sum_{Q \in M^h} v(Q) [ W_{Q - \left( 0, \frac{h}{2} \right)}(x) - W_{Q + \left( 0, \frac{h}{2} \right)}(x) ]$$

$$= \frac{1}{h} \sum_{Q \in M^h} [ v(Q) - v(Q - (0, h)) ] W_{Q - \left( 0, \frac{h}{2} \right)}(x)$$

$$= \sum_{Q \in M^h} \delta_2^- v(Q) W_{Q - \left( 0, \frac{h}{2} \right)}(x).$$

Replacing $v$ by $\sigma_h(u_h - s)$ in (4.4) :

$$(4.5) \quad \delta_2 \sigma_h(u_h - s) =$$

$$= \frac{1}{h} \sum_{Q \in M^h} [ \sigma((u_h - s)(Q)) - \sigma((u_h - s)(Q - (0, h))) ] W_{Q - \left( 0, \frac{h}{2} \right)}(x)$$

$$= \sum_{Q \in M^h} \frac{\sigma((u_h - s)(Q)) - \sigma((u_h - s)(Q - (0, h)))}{u(Q) - s(Q) - u(Q - (0, h)) + s(Q - (0, h))} \times$$

$$\times [\delta_2^- u(Q) - \delta_2^- s(Q)] W_{Q - \left( 0, \frac{h}{2} \right)}(x).$$

We derive from (4.3)

$$(4.6) \qquad \| \delta_2 v \|^2 = (\delta_2 v, \delta_2 v) = h^2 \sum_{Q \in M^h} | \delta_2^- v(Q) |^2.$$

By using (1.3) and (4.4), we obtain

$$(4.7) \quad (H(u_h) | \delta_2 u_h |^2, \sigma(v)) =$$

$$= \sum_{Q \in M^h} (\delta_2^- v(Q))^2 W_{Q - \left( 0, \frac{h}{2} \right)}(x) \sum_{Q \in M^h} H(u_h(Q)) \sigma(v(Q)) W_Q(x)$$

$$\leqslant H(\alpha) h^2 \sum_{Q \in M^h} \frac{\sigma(v)(Q) + \sigma(v)(Q - (0, h))}{2} (\delta_2^- u(Q))^2.$$

Let $t = (u_h - s)(Q)$, $v = (u_h - s)(Q - (0, h))$, under the assumption (H2), we have

$$\begin{cases} \alpha - K \leqslant v \leqslant K - \alpha \\ 2(\alpha - K) \leqslant t - v \leqslant 2(K - \alpha) . \end{cases}$$

Then by using lemma 6 and (4.4)-(4.7), we derive

$$(4.8) \quad (\delta_2 u_h, \delta_2 \sigma_h(u_h - s)) + (H(u_h) | \delta_2 u_h |^2, \sigma(u_h - s)) \geqslant \frac{1}{2} \| \delta_2 u_h \|^2 -$$

$$- h^2 \sum_{Q \in M^h} \frac{\sigma((u_h - s)(Q)) - \sigma((u_h - s)(Q - (0, h)))}{u(Q) - s(Q) - u(Q - (0, h)) + s(Q - (0, h))} \times$$

$$\times | \delta_2^- u(Q) - \delta_2^- s(Q) |$$

$$= \frac{1}{2} \| \delta_2 u_h \|^2 - h^2 \sum_{Q \in M^h} \sigma'((u_h - s) \eta_Q) | \delta_2^- u(Q) - \delta_2^- s(Q) |$$

(from (4.2) and the Schwarz inequality)

$$\geqslant \frac{1}{2} \| \delta_2 u_h \|^2 - c_2 \| \delta_2 s \| \| \delta_2 u_h \|$$

with $\eta_Q \in [Q - (0, h), Q]$.

Similarly

$$(4.9) \quad (\delta_1 u_h, f(u_h) \delta_1 \sigma_h(u_h - s)) + (H(u_h) f(u_h) | \delta_1 u_h |^2, \sigma(u_h - s))$$

$$\geqslant \frac{f(K)}{2} \| \delta_1 u_h \|^2 - f(\alpha) h^2 \sum_{Q \in M^h} \sigma'((u_h - s) \zeta_Q) | \delta_1^- u(Q) - \delta_1^- s(Q) |$$

$$\geqslant \frac{f(K)}{2} \| \delta_1 u_h \|^2 - c_2 f(\alpha) \| \delta_1 s \| \| \delta_1 u_h \|$$

with $\zeta_Q \in [Q - (h, 0), Q]$.

Finally, by taking $v = \sigma_h(u_h - g)$ in $(\mathscr{P}_h')$, and using (4.8) and (4.9), we find

$$\frac{f(K)}{2} \| \delta u_h \|^2 \leqslant c_2 f(\alpha) \| \delta s \| \| \partial u_h \|$$

i.e.

$$\| \delta u_h \| \leqslant \frac{2 c_2 f(\alpha)}{f(K)} . \quad ¶$$

We conclude from lemma 7 that (see [7] for details) there exists a function $U \in \mathscr{H}_{p, s} \cap L^\infty(R^2)$, $\alpha \leqslant U \leqslant K$ and a subsequence of $\{h\}$, still denoted by $\{h\}$ such that (see [7])

$$(4.10) \qquad\qquad u_h - U \to 0$$

$$(4.11) \qquad\qquad \nabla \delta u_h - \delta r_h U \to 0$$

weakly in $L^2(R^2)$ and

$$(4.12) \qquad\qquad u_h - U \to 0 \text{ a.e. in } R^2 .$$

We are going to establish an strong convergence for $u_h$ which is essential for passing to the limit in $(\mathscr{P}'_h)$. For each $h$, we denote $U^h = r_h U \in \mathscr{X}^h_s$ (see (3.7)), we can prove as in [7] that

$$(4.13) \qquad\qquad \|\delta U^h - \nabla U\| \to 0 .$$

We are now in position to prove

LEMMA 8 : *Under the assumption* (H1) *and* (H2), *we have* $\|\delta(u_h - U^h)\| \to 0$.

*Proof:* We take $v = \sigma(u_h - U^h)$ in $(\mathscr{P}'_h)$ :

$$\langle A_h(u_h) u_h, \sigma(u_h - U^h) \rangle + \langle F_h(u_h, \delta u_h), \sigma(u_h - U^h) \rangle = 0$$

which can be written as

$$\langle A_h(u_h)(u_h - U^h), \sigma(u_h - U^h) \rangle + \langle F_h(u_h, \delta(u_h - U^h)), \sigma(u_h - U^h) \rangle$$
$$= - (F_h(u_h, \delta U_h), \sigma(u_h - U^h)) - \langle A_h(u_h) U_h, \sigma(u_h - U^h) \rangle .$$

A similar computation as in the proof of lemma 7 leads to :

$$\frac{f(K)}{2} \|\delta u_h - U^h\|^2 \leqslant H(\alpha) \int_{R^2} \sigma(u_h - U^h) |\delta U^h|^2 \, dx \times$$

$$\times h^2 \sum_{Q \in M^h} \frac{\sigma((u_h - U^h)(Q)) - \sigma((u_h - U^h)(Q - (0, h)))}{u(Q) - U_h(Q) - u(Q - (0, h)) + U^h(Q - (0, h))}$$

$$\times |\delta_2^- U^h(Q) \, \delta_2^- (u_h - U^h)|$$

$$\times f(\alpha) h^2 \sum_{Q \in M^h} \frac{\sigma((u_h - U^h)(Q)) - \sigma((u_h - U^h)(Q - (h, 0)))}{u(Q) - U^h(Q) - u(Q - (h, 0)) + U^h(Q - (h, 0))}$$

$$\times |\delta_1^- U^h(Q) \, \delta_1^- (u_h - U_h)|$$

$$= H(\alpha) \int_R^2 |\sigma(u_h - U^h)| \, |\delta U^h|^2 \, dx +$$

$$+ h^2 \sum_{Q \in M^h} \sigma'((u_h - U^h) \eta_Q) |\delta_2^- U^h(Q) \, \delta_2^- (u_h - U^h)(Q)|$$

$$+ f(\alpha) h^2 \sum_{Q \in M^h} \sigma'((u_h - U^h) \zeta_Q) |\delta_1^- U^h(Q) \, \delta_1^- (u_h - U^h)(Q)|$$

$$= A_1 + A_2 + A_3$$

with $\eta_Q \in [Q - (0, h), Q]$, $\zeta_Q \in [Q - (h, 0), Q]$.

It is then sufficient to prove that

$$|A_i| \to 0 \ (h \to 0) \ i = 1, 2, 3 .$$

From (4.11), we have

(4.14)
$$\begin{cases} \sigma(u_h - U^h) \to 0 \text{ a.e. in } \Omega \\ \sigma'(u_h - U^h) \to \sigma'(0) \text{ a.e. in } \Omega . \end{cases}$$

It is clear from (4.5) and (4.6) that $\|\delta U^h\| \leq \|\tilde{\delta} U^h\|$, we then derive from (3.7) and (4.2) that

$$\int_{R^2} |\sigma(u_h - U^h)| \, |\delta U^h|^2 \leq c_3 , \quad \forall h .$$

We now apply the Lebesgue's dominating convergence theorem, by using (4.14), we obtain

$$|A_1| = H(\alpha) \int_{R^2} |\sigma(u_h - U^h)| \, |\delta U^h|^2 \to 0 .$$

Now let us deal with $A_2$.

$$A_2 = h^2 \sum_{Q \in M^h} [\sigma'((u_h - U^h) \eta_Q) - \sigma'(0) + \sigma'(0)] \times$$

$$\times |\delta_2^- U^h(Q) \delta_2^- (u_h - U^h)(Q)|$$

$$= h^2 \sum_{Q \in M^h} [\sigma'((u_h - U^h) \eta_Q) - \sigma'(0)] |\delta_2^- U^h(Q) \delta_2^- (u_h - U^h)(Q)|$$

$$+ h^2 \sum_{Q \in M^h} \sigma'(0) |\delta_2^- U^h(Q) \delta_2^- (u_h - U^h)(Q)|$$

$$\leq \max_{\eta_Q} [\sigma'((u_h - U^h) \eta_Q) - \sigma'(0)] \int_{R^2} |\delta_2 U^h \delta_2(u_h - U^h)| \, dx$$

$$+ \sigma'(0) \int_{R^2} |\delta_2 U^h \delta_2(u_h - U^h)| \, dx$$

$$= A_{21} + A_{22} .$$

We derive from (4.14) and Lebesgue's theorem that

$$A_{21} \to 0 .$$

Finally, since $\delta_2 U^h$ is bounded in $L^2(R^2)$, we derive from the weak convergence (4.11) of $\delta(u_h - U^h)$ that $A_{22}$ also tends to zero.

The treatment for $A_3$ is totally the same as for $A_2$. The proof is then complete. ¶

We deduce from (4.12) and lemma 8 that

$$(4.16) \qquad \| \, \|\delta u_h\| - \nabla U \| \leqslant \| \delta (u_h - U^h) \| + \| \delta U^h - \nabla U \| \to 0$$

which implies

$$| \delta u_h - \nabla U | \to 0 \text{ a.e. in } R^2 .$$

With the aid of this strong convergence, the passage to the limit in $(\mathscr{P}'_h)$ is immediate and we find that $U|_\Omega$ is a weak solution of problem (1.4). We have then proved

THEOREM 2 : *Under the assumption* (H1) *and* (H2), *there exists* $u(u = U|_\Omega) \in H^1(\Omega) \cap L^\infty(\Omega)$, $\alpha \leqslant U \leqslant K$ *and a subsequence of* $\{h\}$, *still noted by* $\{h\}$, *such that*

$$\| \delta u_h - \nabla u \| \to 0$$

*and* $U$ *is a weak solution of problem* (1.4).

*Remark :*

(i) The assumption (H2) is purely technical. Actually, by using the same method, we can prove directly that the problem (1.4) admits at least one solution without assuming (H2).

(ii) The numerical results presented in the next section suggest that our results hold also for a transonic flow. The theoretical justification of this result is currently under consideration.

## 5. NUMERICAL ASPECTS AND RESULTS

In this section, we do not intend to develop the physical aspects of the inverse problem, but rather to give a sketch of the numerical computation of the problem. For further descriptions of the physical aspects as well as for the existence of a closed profile for the inverse problem, the reader is referred to B. Michaux [4] and the references therein.

We now recall briefly the numerical setting of the inverse problem for the determination of transonic blade profiles.

Under the assumptions that the flow is perfect and isentropic and from the dynamical equation, curl $u = 0$ and the continuity equation, div $\rho u = 0$ we obtain the equations of the problem $(\mathscr{P})$ which govern the fluid flow in the computational domain. This equations was established after transformation of the physical domain to the plane defined by the streamlines and the potential lines of the fluid (see [4]).
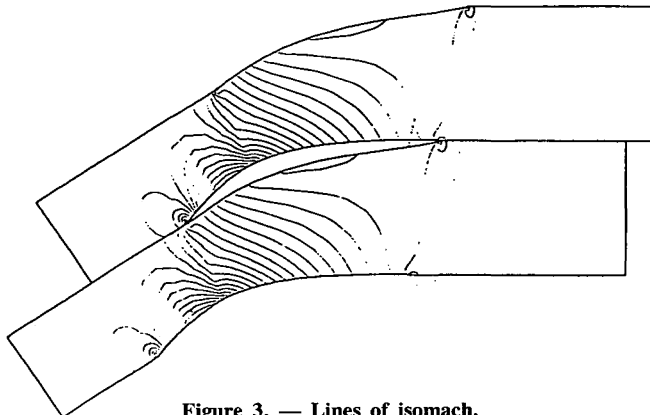
**Example 1 : Subsonic profile.**
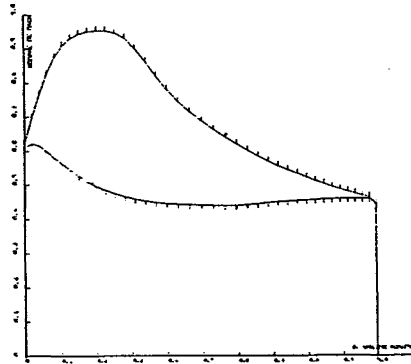


Figure 1. — Mach number distributions on the profile.
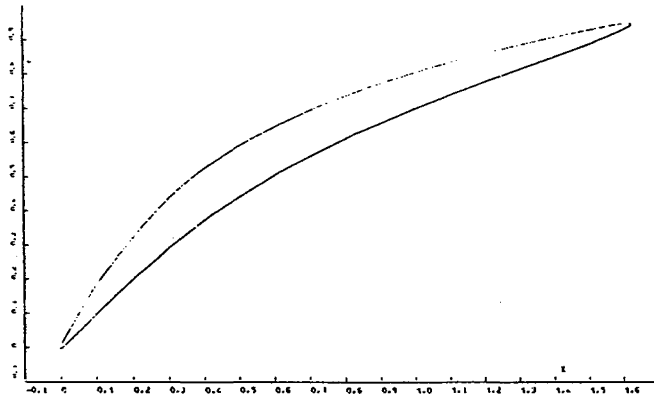


Figure 2. — Profile obtained from the computation.



Figure 3. — Lines of isomach.

From the data of the upstream Mach number, the Mach number distribution on the suction and pressure sides as well as the inlet and outlet flow angles which complete the geometry of the considered problem, we obtain the geometry of the computational domain $\Omega$ -see *fig.* 1), as well as the boundary conditions for the problem ($\mathscr{P}$).

The streamline curvatures in the physical domain as well as the angle ($\phi$) between the streamline tangent vector and the physical domain basis vector $\vec{i}$ can be determined by a function of the aerodynamic unknowns (velocity $u$, Mach number $M$ and density $\rho$). The cartesian coordinates of the blade profile are obtained by an integration of first order differential equations ; these equations are functions of angle and velocity, along the streamlines defining the profile.

In addition to the problem ($\mathscr{P}$), we have the following expression for the curvatures $\chi$ :

$$(5.1) \qquad\qquad \chi = \rho \, \frac{\partial u}{\partial x_2} \quad \text{in} \quad \bar{\Omega} .$$

Finally, the deviation that generates the blade profiles as well as their cartesian coordinates are obtained by integrations of the following equations in $\bar{\Omega}$ :

$$(5.2) \quad \begin{cases} \dfrac{\partial \phi}{\partial x_1} = \dfrac{\chi}{u} \\[2mm] \dfrac{\partial x}{\partial x_1} = \dfrac{\cos \phi}{u} \\[2mm] \dfrac{\partial y}{\partial x_1} = \dfrac{\sin \phi}{u} \\[2mm] BC : \phi(0, x_2) = \phi_1, \, x(0, x_2) = x_0(x_2), \, y(0, x_2) = y_0(x_2) \end{cases}$$

where $\phi_1$, $x_0$ and $y_0$ are physical data.

## Numerical methods

Due to the mixed type of the equation of the problem ($\mathscr{P}$) (elliptic-hyperbolic), we approximated the partial derivatives by the scheme considered in section 2, for the mesh points where the flow is subsonic ($M(u) < 1$) and by the upwind scheme with three points for the mesh points where the flow is supersonic ($M(u) > 1$). Due to the nonlinearity of the discrete problem ($\mathscr{P}_h$), to compute the aerodynamic unknowns, we used a fixed point method for the determination of the Mach number and the density, and the Newton method at each iteration of the fixed point method to compute the velocity. The periodic boundary conditions were also treated
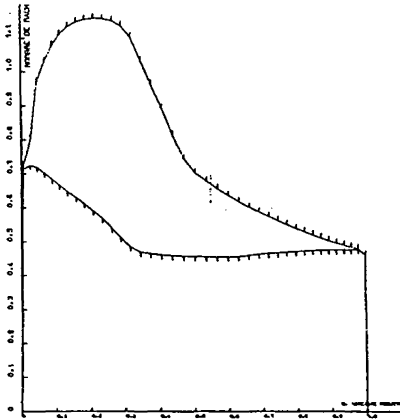
**Example 2. Transonic profile.**



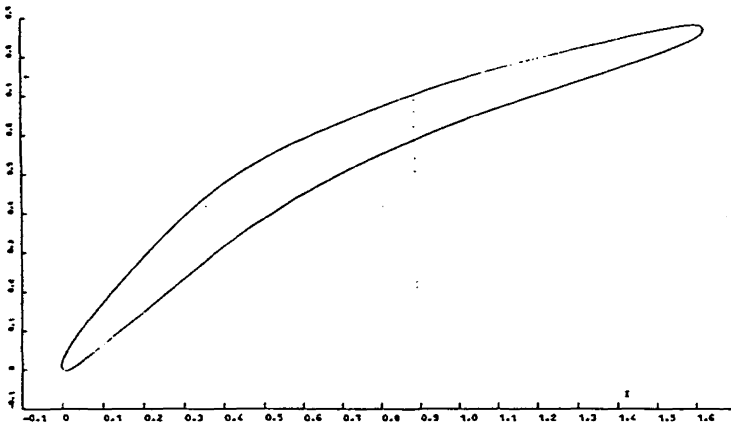Figure 4. — Mach number distributions on the profile.



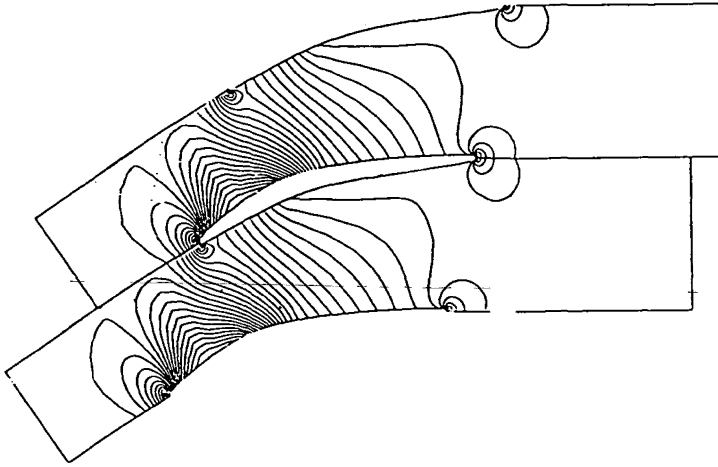Figure 5. — Profile obtained from the computation.

**Figure 6. — Lines of isomach.**

during the application of the Newton method (see [4] for more details). Finally, we integrated the first order equations (5.1) and (5.2) by the trapezoidal numerical integration rules, to get the geometry of the desired profile.

**Numerical results**

Example 1 presents the geometry of a blade profile obtained from data corresponding to a subsonic flow. For this case, the inlet and outlet angles correspond respectively to 45° 26′ and 11° 28′. The upstream Mach number is 0.7525. We present in figure 1 the distributions of Mach number on the profile. The maximal value of the Mach number on the profile is 0.95. In particular, we note that the numerical result presented in figure 3 confirms the results of theorem 1.

Example 2 corresponds to a transonic flow. The inlet and outlet angles are also respectively 45° 26′ and 11° 28′. The upstream Mach number is now 0.8525 The distributions of Mach number on the profile in this case is presented in figure 4. In this case, the maximal value of the Mach number on the profile is 1.15. We remark that the theoretical result of theorem 1 still holds in this case.

We notice finally that the numerical results of these two examples, obtained by using the actual numerical methods, corresponds well to the physical experimentation.

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