M2AN. MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 23, nº 4 (1989), p. 627-647

<http://www.numdam.org/item?id=M2AN_1989__23_4_627_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE
(Vol. 23, n° 4, 1989, p. 627-647)

A NONCONFORMING FINITE ELEMENT METHOD OF UPSTREAM TYPE APPLIED TO THE STATIONARY NAVIER-STOKES EQUATION (*)

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Communicated by P G. CIARLET

Abstract. — *We present a nonconforming finite element method with an upstream discretization of the convective term for solving the stationary Navier-Stokes equations. The existence of at least one solution of the discrete problem and the convergence of subsequences of such solutions to a solution of the Navier-Stokes equations are established In addition, under certain assumptions on the data, uniqueness of the solution can be guarenteed and error estimates of the approximate solution are given Moreover, some favourable properties of the discrete algebraic system are discussed*

Resumé — *Nous présentons une méthode non conforme d'éléments finis avec une discrétisation décentrée amont du terme de convection pour la résolution des équations de Navier-Stokes stationnaires On prouve l'existence d'une solution au moins du problème discret et la convergence des sous-suites de telles solutions vers une solution des équations de Navier-Stokes stationnaires En outre on peut sous certaines hypothèses sur les données garantir l'unicité et on donne alors des estimations d'erreur de la solution approximative En outre on discute quelques propriétés importantes du système algébrique discret*

1. INTRODUCTION

The Navier-Stokes equations for viscous, incompressible flow problems have been the object of considerable research efforts. Because of its great flexibility finite element methods have received considerable attention, both from a theoretical and computational point of view. In general one uses finite elements of higher-order shape functions in order to get better approximations of velocity and pressure fields. However, this can be guaranteed, at least theoretically, only for sufficiently smooth solutions of

(*) Received in July 1987, revised in May 1988
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the considered problem. Moreover, the use of higher-order shape functions causes computational costs which can be too expensive for the problem under consideration. Therefore we propose a finite element method with lower-order shape functions. Taking into consideration the dominate influence of the convective term in the case of a higher Reynolds number, we shall use a special upstream discretization of this term.

In this paper we propose a method combining a P_1-P_0 nonconforming finite element method due to Crouzeix and Raviart [2] with an upstream discretization of the convective term which has been applied by Ohmori and Ushijima [9] in case of a scalar convection diffusion problem. The method in [2] proposed for the Stokes problem was extended to stationary Navier-Stokes equations in [7]. But the results concerning the nonconforming elements are stated without proof. An extension to time-dependent Navier-Stokes equations was done in [6].

A similar upwinding technique was first introduced in [8] to solve the Neutron transport equation. For solving the Navier-Stokes equations in terms of stream function and vorticity, this technique was applied in [3] and analyzed in [5].

The plan of the paper is the following. In Section 2 we introduce the notations used in the subsequent sections. The finite element method for the approximate solution is presented in Section 3. Section 4 contains a discussion of the properties of the algorithm and in Section 5 we give existence and convergence results for the discrete solutions.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, Ω is supposed to be a convex polygon in R^2 with boundary Γ. Let n be the unit outer normal to Ω. D_j, $j = 1, 2$ denotes the differential operator $\frac{\partial}{\partial x_j}$ and often we will use the summation convention, that one has to take the sum over an index occuring twice in some term. For a scalar function s on a measurable subset $G \subset \Omega$, let $\|s\|_{k,p,G}$ and $|s|_{k,p,G}$ be the usual norm and seminorm on the Sobolev space $W^{k,p}(G)$ [1], respectively. Then for a vectorvalued function $v = (v_1, v_2)$ belonging to $(W^{k,p}(G))^2$ we will use the norm

$$\|v\|_{k,p,G} = \sum_{i=1}^{2} \|v_i\|_{k,p,G}$$

and the semi-norm

$$|v|_{k,p,G} = \sum_{i=1}^{2} |v_i|_{k,p,G} \, .$$

In this paper we consider the stationary Navier-Stokes problem for incompressible flows, i.e. we have to find the velocity field $u = (u_1, u_2)$ and the pressure p such that

$$\begin{aligned} -\nu\,\Delta u + u_i\,D_i\,u + \operatorname{grad} p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned} \tag{2.1}$$

where ν denotes the constant inverse Reynolds number and f a given body force. In order to write (2.1) in a weak form we introduce the notations

$$V = (H_0^1(\Omega))^2 \tag{2.2}$$

$$Q = L_0^2(\Omega) = \left\{ v \in L^2(\Omega) : \int_\Omega v\,dx = 0 \right\} \tag{2.3}$$

$(.,.)$ inner product in $L^2(\Omega)$ and $(L^2(\Omega))^2$, respectively (the meaning becomes clear from the context)

$$a(u, v) = \int_\Omega D_j\,u_i\,D_j\,v_i\,dx \qquad u, v \in V \tag{2.4}$$

$$b(u, v, w) = \int_\Omega u_i\,D_i\,v_j\,w_j\,dx \quad u, v, w \in V\,. \tag{2.5}$$

Then the variational form of (2.1) reads :

Find $(u, p) \in V \times Q$ such that

$$\begin{aligned} \nu a(u, v) + b(u, u, v) - (p, \operatorname{div} v) &= (f, v) \quad \forall v \in V \\ (q, \operatorname{div} u) &= 0 \qquad\quad \forall q \in Q\,. \end{aligned} \tag{2.6}$$

It is well known that (2.6) admits at least one solution which is unique provided that $\nu^{-2}\|f\|$ is sufficiently small [4].

3. FINITE ELEMENT APPROXIMATION OF UPSTREAM TYPE

For solving the continuous problem (2.6) approximately, we will combine a nonconforming finite element method due to Crouzeix/Raviart, Temam [2, 11] with an upstream discretization of the convective term which has been applied by Ohmori, Ushijima in case of a scalar convection-diffusion problem [9].

Let $\{\tau_h\}$ be a family of triangulations of Ω into triangles K with

$$\bar{\Omega} = \bigcup_{K \in \tau_h} \bar{K}\,,$$

which is assumed to be regular in the usual sense, and let h_K be the diameter of the triangle K. We also assume that the inverse assumption on the mesh

$$\frac{h}{h_K} \leqq C \quad \forall K \in \bigcup_h \tau_h$$

is fulfilled $\left(h = \max\limits_{K \in \tau_h} h_K\right)$.

We denote by B_i, $1 \leqq i \leqq N$, the midpoints of inner edges and by B_i, $N+1 \leqq i \leqq N+M$, the midpoints of edges lying on the boundary Γ. Now we define the finite dimensional spaces V_h and Q_h for V and Q, respectively, by

$$\begin{aligned} (3.1)\quad V_h = & \{v \in (L^2(\Omega))^2 : v|_K \in (P_1(K))^2\ \forall K \in \tau_h, v \text{ is continuous} \\ & \text{at } B_i, 1 \leqq i \leqq N, v(B_i) = 0 \quad \text{for} \quad N+1 \leqq i \leqq N+M\} \\ (3.2)\quad Q_h = & \{q \in L_0^2(\Omega) : q|_K \in P_0(K) \quad \forall K \in \tau_h\} \end{aligned}$$

where $P_m(K)$, $m = 0, 1$, denotes the set of all polynomials on K with degree not greater than m.

Because of $V_h \not\subset V$, we have to extend the divergence operator, the bilinear form a and the trilinear form b, respectively.

For u, v, $w \in V + V_h$ and $q \in L^2(\Omega)$ we define these extensions by an elementwise calculation of the corresponding integrals such that

$$(3.3)\qquad (q, \operatorname{div}_h u) = \sum_K \int_K q \operatorname{div} u \, dx$$

$$(3.4)\qquad a_h(u, v) = \sum_K \int_K D_j u_i \, D_j v_i \, dx$$

$$(3.5)\qquad b_h(u, v, w) = \sum_K \int_K u_i \, D_i v_j \, w_j \, dx .$$

It is well known [2] that $\|\cdot\|_h$ with

$$(3.6)\qquad \|u\|_h = (a_h(u, u))^{1/2}$$

is a norm on V_h.

In [11] instead of (3.5) the trilinear form

$$(3.7)\qquad \bar{b}_h(u, v, w) = \frac{1}{2} \sum_K \int_K (u_i \, D_i v_j \, w_j - u_i \, v_j \, D_i w_j) \, dx$$

was used which can be regarded as an extension of $b(u, v, w)$ too, because of

$$b(u, v, w) = \frac{1}{2} \int_{\Omega} (u_i D_i v_j w_j - u_i v_j D_i w_j) \, dx$$

(3.8)

$$\forall u, v, w \in V \quad \text{with} \quad \operatorname{div} u = 0 .$$

Moreover, $\bar{b}_h$ satisfies the skew-symmetric property

$$\bar{b}_h(u, v, w) = -\bar{b}_h(u, w, v) \quad \forall u, v, w \in V_h ,$$

which is useful in the analysis of existence and convergence. In the case of small value of ν, one needs a suitable discretization of the convective part $b(u, u, v)$ of (2.6) in order to avoid instabilities and numerical oszillations, respectively. Therefore we will define a modified discretization of upstream type $\tilde{b}_h$ of b following the lines of [9].

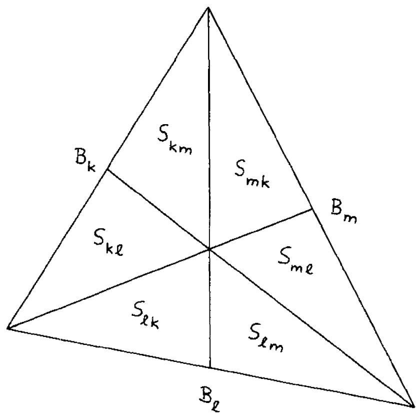

Figure 1.

Let each triangle K be devided into six barycentric fragments S_{ij} $i, j \in \{k, l, m\}$, $i \neq j$, as it is indicated in figure 1. Then, for each node B_l, $l = 1, \dots, N + M$, we define a lumped region R_l by

(3.9)
$$R_l = \bigcup_{k \in \Lambda_l} S_{lk} ,$$

where Λ_l denotes the set of all indices k, for which B_l and B_k are neighbour nodes. Furthermore, let Λ_{lk} be defined by

(3.10)
$$\Lambda_{lk} = \partial S_{lk} \cap \partial S_{kl}$$

and let n^{lk} be the unit outer normal to R_l, which is associated with the part Γ_{lk} or ∂R_l In a similar way as in [9] we can derive the following upstream discretization $\tilde{b}_h$ of the trilinear form b

$$(3\ 11)\quad \tilde{b}_h(u, v, w) = \\ = \sum_{l=1}^{N+M} \sum_{k \in \Lambda_l} \int_{\Gamma_{lk}} u_i\, n_i^{lk}\, d\gamma (1 - \lambda_{lk}(u))(v_j(B_k) - v_j(B_l))\, w_j(B_l)$$

with

$$(3\ 12)\qquad \lambda_{lk}(u) = \begin{cases} 1 & \text{if } \displaystyle\int_{\Gamma_{lk}} u_i\, n_i^{lk}\, d\gamma \geqq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now our discretization of (2 6) reads

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$(3\ 13)\quad \begin{aligned} \nu a_h(u_h, v) + \tilde{b}_h(u_h, u_h, v) - (p_h, \operatorname{div}_h v) &= (f, v) \quad \forall v \in V_h \\ (q, \operatorname{div}_h u) &= 0 \qquad\quad \forall q \in Q_h \end{aligned}$$

Remark 3 1 Contrary to $\bar{b}_h$ of (3 7), in our discretization (3 13), $\tilde{b}_h$ is not a trilinear form on V_h^3 Actually, the mapping $(u, v, w) \to \tilde{b}_h(u, v, w)$ is linear in v and w only

4. SOME PROPERTIES OF THE PROPOSED METHOD

In order to establish results concerning existence and convergence of solutions of (3 13) we derive some properties of the mapping $\tilde{b}_h . V_h^3 \to R$ First of all let us define the lumping operator L_h and the space W_h

For a given $v \in V_h$, the lumping operator L_h is defined by

$$(4\ 1)\qquad (L_h v)(x) = v(B_l) \quad \forall x \in R_l, \quad l = 1, \quad , N + M$$

Furthermore, let us define the space

$$(4\ 2)\qquad W_h = \{v \in V_h \quad (q, \operatorname{div}_h v) = 0 \quad \forall q \in Q_h\}$$

One can easily see that in our case $v \in V_h$ belongs to W_h if and only if $\operatorname{div}_h v|_K = 0 \quad \forall K \in \tau_h$, i e W_h is the space of discrete-divergence-free functions in V_h

Now we have the following

LEMMA 1 : *It holds the estimate*

$$\tilde{b}_h(u, v, v) \geqq 0 \quad \forall u \in W_h \quad \forall v \in V_h . \tag{4.3}$$

Proof: Writing $\tilde{b}_h$ for $u, v, w \in V_h$ in the form

$$\tilde{b}_h(u, v, w) = \tilde{b}_h^1(u, v, w) + \tilde{b}_h^2(u, v, w) \tag{4.4}$$

with

$$\tilde{b}_h^1(u, v, w) = \tag{4.5}$$

$$= \sum_{l=1}^{N+M} \sum_{k \in \Lambda_l} \int_{\Lambda_{lk}} u_i\, n_i^{lk}\, d\gamma (\lambda_{lk}(u)\, v_j(B_l) + (1 - \lambda_{lk}(u))\, v_j(B_k))\, w_j(B_l)$$

$$\tilde{b}_h^2(u, v, w) = - \sum_{l=1}^{N+M} \sum_{k \in \Lambda_l} \int_{\Gamma_{lk}} u_i\, n_i^{lk}\, d\gamma\, v_j(B_l)\, w_j(B_l) , \tag{4.6}$$

we obtain in an analogous way as in [9, Lemma 3]

$$\tilde{b}_h(u, v, v) + \frac{1}{2} \tilde{b}_h^2(u, v, v) = \tag{4.7}$$

$$= \sum_{l=1}^{N+M} \sum_{k \in \Lambda_l} \int_{\Gamma_{lk}} \frac{1}{2} u_i\, n_i^{lk}\, d\gamma (v_j(B_l) - v_j(B_k))^2 \left(\lambda_{lk} - \frac{1}{2}\right) \geqq 0 .$$

Using the fact that

$$\sum_{k \in \Lambda_l} \int_{\Gamma_{lk}} u_i\, n_i^{lk}\, d\gamma = 0 \quad \forall u \in W_h , \quad l = 1, \dots, N$$

one can easily verify that $\tilde{b}_h^2(u, v, v) = 0$ for $u \in W_h$, $v \in V_h$. Together with (4.7) this proves (4.3). □

The next statement implies the continuity of $\tilde{b}_h$ on V_h^3.

LEMMA 2 : *There exists a constant $c > 0$ independent of h, such that*

$$\left|\tilde{b}_h(u, v, w) - \tilde{b}_h(u^0, v, w)\right| \leqq C \left\|u - u^0\right\|_h \|v\|_h \|w\|_h \tag{4.8}$$

holds for all u, u^0, v, $w \in V_h$.

Proof: Let us define the set of indicies

$$I = \{(l, k) : l = 1, \dots, N + M, k \in \Lambda_l\} . \tag{4.9}$$

Then, we can write

$$(4.10)\qquad \left|\tilde{b}_h(u, v, w) - \tilde{b}_h(u^0, v, w)\right| \leqq \left|\sum_{(l,k)\in I} p_{lk}\right| + \left|\sum_{(l,k)\in I} q_{lk}\right|$$

with

$$(4.11)\qquad p_{lk} = \int_{\Gamma_{lk}} (u_i - u_i^0)\, n_i^{lk}\, d\gamma (1 - \lambda_{lk}(u))(v_j(B_k) - v_j(B_l))\, w_j(B_l)$$

and

$$(4.12)\qquad q_{lk} = \int_{\Gamma_{lk}} u_i^0\, n_i^{lk}\, d\gamma (\lambda_{lk}(u^0) - \lambda_{lk}(u))(v_j(B_k) - v_j(B_l))\, w_j(B_l)\,.$$

Using the fact, that $D_i v_j$, $i, j = 1, 2$ is a constant on $S_{lk} \cup S_{kl}$ for all $(l, k) \in I$, we can estimate for u, u^0, v, $w \in V_h$

$$\begin{aligned} |p_{lk}| &\leqq \text{mes}\, \Gamma_{lk} \|u - u^0\|_{0,\infty,S_{lk}}\, h |v|_{1,\infty,S_{lk}}\, \|w\|_{0,\infty,S_{lk}} \\ &\leqq hC_1 h^{-1/2} \|u - u^0\|_{0,4,S_{lk}}\, C_2 |v|_{1,2,S_{lk}}\, C_1 h^{-1/2} \|w\|_{0,4,S_{lk}} \end{aligned}$$

where C_1 and C_2 are the constants of inverse inequalities which are independent of h, l and k. This implies

$$\begin{aligned} \left|\sum_{(l,k)\in I} p_{lk}\right| &\leqq C \sum_{(l,k)\in I} \|u - u^0\|_{0,4,S_{lk}}\, |v|_{1,2,S_{lk}}\, \|w\|_{0,4,S_{lk}} \\ &\leqq C \left(\sum_{(l,k)\in I} \|u - u^0\|_{0,4,S_{lk}}^4\right)^{1/4} \left(\sum_{(l,k)\in I} |v|_{1,2,S_{lk}}^2\right)^{1/2} \\ &\quad \left(\sum_{(l,k)\in I} \|w\|_{0,4,S_{lk}}^4\right)^{1/4} \\ &= C \|u - u^0\|_{0,4,\Omega}\, \|v\|_h\, \|w\|_{0,4,\Omega}\,. \end{aligned}$$

If we apply the estimate

$$(4.13)\qquad \|z\|_{0,p,\Omega} \leqq C(p, \Omega)\, \|z\|_h \quad \forall z \in V_h\,,$$

which can be proven for $1 \leqq p < \infty$ in the two-dimensional case along the lines of Rannacher and Heywood ([6, Proof of (4.36)]), we obtain

$$(4.14)\qquad \left|\sum_{(l,k)\in I} p_{lk}\right| \leqq C \|u - u^0\|_h\, \|v\|_h\, \|w\|_h\,.$$

To estimate the second sum in (4.10), we split the set I of indicies into

$$I^+ = \left\{(l, k) \in I : |(u_i^0 n_i^{lk})(P_{lk})| > \|u - u^0\|_{0,\infty,S_{lk}}\right\}$$

and $I^- = I \backslash I^+$, where P_{lk} denotes the midpoint of Γ_{lk}. For $(l, k) \in I^+$ we have

$$\left| \left((u_\iota - u_\iota^0) n_\iota^{lk} \right)(P_{lk}) \right| \leqq \| u - u^0 \|_{0, \infty, S_{lk}} < \left| (u_\iota^0 n_\iota^{lk})(P_{lk}) \right| ,$$

which implies

$$\operatorname{sign} \left((u_\iota n_\iota^{lk})(P_{lk}) \right) = \operatorname{sign} \left((u_\iota^0 n_\iota^{lk})(P_{lk}) \right)$$

and consequently, since u is linear on Γ_{lk},

$$\lambda_{lk}(u) = \lambda_{lk}(u^0) \quad \text{for} \quad (l, k) \in I^+ . \tag{4.15}$$

(4.15) yields $\sum_{(l, k) \in I^+} q_{lk} = 0$. For $(l, k) \in I^-$ we have

$$|q_{lk}| \leqq \operatorname{mes} \Gamma_{lk} \| u - u^0 \|_{0, \infty, S_{lk}} h | v |_{1, \infty, S_{lk}} \| w \|_{0, \infty, S_{lk}} .$$

Thus, we obtain in an analogous way as for p_{lk} the estimate (4.14) also for $\sum_{(l, k) \in I^-} q_{lk}$, which completes the proof of (4.8). □

To prove our convergence result in Section 5, we need for arbitrary $\kappa \in (0, 1)$ the inequality

$$\| v \|_{0, \infty, \Omega} \leqq C_\kappa h^{-\kappa} \| v \|_h \quad \forall v \in V_h \tag{4.16}$$

which is a consequence of (4.13) and the inverse inequality

$$\| v \|_{0, \infty, \Omega} \leqq C_\kappa h^{-\kappa} \| v \|_{0, p, \Omega} \quad \forall v \in V_h$$

with $\kappa = 2/p$.

Now we will estimate the difference between the two different discretizations of the convective term b.

LEMMA 3 : *There exists a constant C independent of h, such that the estimate*

$$\left| b_h(u, v, w) - \tilde{b}_h(u, v, w) \right| \leqq C_\kappa h^{1-\kappa} \| u \|_h \| v \|_h \| w \|_h \tag{4.17}$$

holds for all $u, v, w \in V_h$ *and* $\kappa \in (0, 1)$.

Proof: We decompose b_h into

$$b_h(u, v, w) = b_h^1(u, v, w) + b_h^2(u, v, w) \tag{4.18}$$

with

$$b_h^1(u, v, w) = \sum_K \int_K D_i(u_i v_j) w_j \, dx \tag{4.19}$$

and

$$b_h^2(u, v, w) = -\sum_K \int_K D_i u_i v_j w_j \, dx . \tag{4.20}$$

Using the decomposition (4.4)-(4.6) of $\tilde{b}_h$ we may write

$$b_h(u, v, w) - \tilde{b}_h(u, v, w) = Y_1 + Y_2 + Y_3 \tag{4.21}$$

with

$$Y_1 = b_h^1(u, v, w - L_h w) , \tag{4.22}$$

$$Y_2 = b_h^1(u, v, L_h w) - \tilde{b}_h^1(u, v, w) , \tag{4.23}$$

$$Y_3 = b_h^2(u, v, w) - \tilde{b}_h^2(u, v, w) . \tag{4.24}$$

At first let us estimate Y_1 by

$$\begin{aligned} |Y_1| &\leqq \sum_K \left| \int_K (D_i u_i v_j + u_i D_i v_j)(w_j - L_h w_j) \, dx \right| \\ &\leqq \sum_K (|u|_{1,2,K} \|v\|_{0,\infty,K} + \|u\|_{0,\infty,K} |v|_{1,2,K}) \|w - L_h w\|_{0,2,K} \\ &\leqq (\|u\|_h \|v\|_{0,\infty,\Omega} + \|u\|_{0,\infty,\Omega} \|v\|_h) h \|w\|_h . \end{aligned}$$

Using (4.16) we obtain

$$|Y_1| \leqq C_\kappa h^{1-\kappa} \|u\|_h \|v\|_h \|w\|_h . \tag{4.25}$$

To estimate Y_2 we start with the first sum Y_{21} in

$$\begin{aligned} Y_2 = \sum_{l=1}^{N} \int_{\Gamma_l} [\![u_i n_i^l v_j]\!]_{\Gamma_l} d\gamma \, w_j(B_l) + \sum_{(l,k) \in I} \int_{\Gamma_{lk}} u_i n_i^{lk} (\lambda_{lk}(u)(v_j - v_j(B_l)) \\ + (1 - \lambda_{lk}(u))(v_j - v_j(B_k))) w_j(B_l) \, d\gamma \end{aligned}$$

where Γ_l denotes the edge containing the node B_l, $[\![z]\!]_{\Gamma_l}$ the jump of z along Γ_l, n^l the unit normal vector on Γ_l and I the index set defined by (4.9). To be more specific, let K_1, K_2 be the two triangles with the common edge

Γ_l and let n^l be directed outward with respect to K_1. Then, the jump is defined by $[\![z]\!]_{\Gamma_l} = z|_{K_1} - z|_{K_2}$. Obviously it holds

$$\int_{\Gamma_l} [\![u_i\, n_i^l\, v_j]\!]_{\Gamma_l}\, d\gamma\, w_j(B_l) =$$

$$= \int_{\Gamma_l} [\![(u_i - L_h u_i)\, n_i^l (v_j - L_h v_j)]\!]_{\Gamma_l}\, d\gamma\, w_j(B_l)$$

$$\leqq \text{mes}\,(\Gamma_l) \sum_{m=1}^{2} h|u|_{1,\infty,K_m}\, h|v|_{1,\infty,K_m}\, \|w\|_{0,\infty,\Omega}\,.$$

By means of inverse inequalities we obtain

$$\int_{\Gamma_l} [\![u_i\, n_i^l\, v_j]\!]_{\Gamma_l}\, d\gamma\, w_j(B_l) \leqq Ch\|w\|_{0,\infty,\Omega} \sum_{m=1}^{2} |u|_{1,2,K_m}\, |v|_{1,2,K_m}\,.$$

Thus, using (4.16) we can estimate

$$\begin{aligned} Y_{21} &\leqq 2\, Ch\|w\|_{0,\infty,\Omega} \sum_{K\in\tau_h} |u|_{1,2,K}\, |v|_{1,2,K} \\ &\leqq 2\, Ch\|w\|_{0,\infty,\Omega}\, \|u\|_h\, \|v\|_h \\ &\leqq C_\kappa h^{1-\kappa}\, \|u\|_h\, \|v\|_h\, \|w\|_h\,. \end{aligned}$$

Now, let us consider the second sum in Y_2 denotes by Y_{22}. If we take into consideration that $\Gamma_{lk} = \Gamma_{kl}$, $\lambda_{lk}(u) = 1 - \lambda_{kl}(u)$ and $n^{lk} = -n^{kl}$, we get

$$\begin{aligned} Y_{22} = \frac{1}{2} \sum_{(l,k)\in I} \int_{\Gamma_{lk}} & u_i\, n_i^{lk}(\lambda_{lk}(u)(v_j - v_j(B_l)) + \\ & + (1 - \lambda_{lk}(u))(v_j - v_j(B_l)))(w_j(B_l) - w_j(B_k))\, d\gamma\,. \end{aligned}$$

Since u, v and w are linear on Γ_{lk}, we can estimate

$$\begin{aligned} |Y_{22}| &\leqq \frac{1}{2} \sum_{(l,k)\in I} \text{mes}\,\Gamma_{lk}\|u\|_{0,\infty,S_{lk}}\, h|v|_{1,\infty,S_{lk}}\, h|w|_{1,\infty,S_{lk}} \\ &\leqq \frac{1}{2} \sum_{(l,k)\in I} h\|u\|_{0,\infty,S_{lk}}\, C_2|v|_{1,2,S_{lk}}\, C_2|w|_{1,2,S_{lk}} \\ &\leqq Ch\, \|u\|_{0,\infty,\Omega}\, \|v\|_h\, \|w\|_h\,, \end{aligned}$$

where C_1 and C_2 are again the constants of inverse inequalities which are independent of h, l and k. Together with (4.16) and the estimate for Y_{21} we receive

$$|Y_2| \leqq C_\kappa h^{1-\kappa}\|u\|_h\, \|v\|_h\, \|w\|_h\,. \tag{4.26}$$

Finally, we have to estimate Y_3 Using the identity

$$\tilde{b}_h^2(u, v, w) = b_h^2(u, L_h v, L_h w) \quad \forall u, v, w \in V_h$$

we get

$$Y_3 = b_h^2(u, v - L_h v, w) + b_h^2(u, L_h v, w - L_h w) \tag{4 27}$$

It is easy to check that the lumping operator L_h satisfies

$$\|L_h v\|_{0\,\infty\,K} = \|v\|_{0\,\infty\,K} \quad \forall v \in V_h, \quad K \in \tau_h \tag{4 28}$$

and

$$\|v - L_h v\|_{0\,2\,K} \leqq h|v|_{1\,2\,K} \quad \forall v \in V_h, \quad K \in \tau_h \tag{4 29}$$

Thus, from (4 27), the definition (4 20) of b_h^2 and (4 16) we obtain

$$\begin{aligned} |Y_3| &\leqq \sum_K |u|_{1\,2\,K}\, h|v|_{1\,2\,K}\, \|w\|_{0\,\infty\,K} + \\ &\quad + |u|_{1\,2\,K}\, \|v\|_{0\,\infty\,K}\, h|w|_{1\,2\,K} \\ &\leqq h\|u\|_h\, \|v\|_h\, \|w\|_{0\,\infty\,\Omega} + h\|u\|_h\, \|v\|_{0\,\infty\,\Omega}\, \|w\|_h \\ &\leqq C_\kappa h^{1-\kappa}\, \|u\|_h\, \|v\|_h\, \|w\|_h, \end{aligned} \tag{4 30}$$

which completes together with (4 25), (4 26) the proof of (4 17) □

5. EXISTENCE AND CONVERGENCE OF THE DISCRETE SOLUTIONS

In this section we study solvability of the discrete problem (3 13) and convergence properties of its solutions to a solution of the continuous problem (2 6)

It can be shown that our nonconforming finite element discretization fulfills the discrete LBB-condition, i e there is a constant $\alpha > 0$, independent of h, such that

$$\sup_{v \in V_h} \frac{(p, \operatorname{div}_h v)}{\|v\|_h} \geqq \alpha \|p\|_{0\,2\,\Omega} \quad \forall p \in Q_h \tag{5 1}$$

Therefore, it is possible to separate the problem of finding a solution (u_h, p_h) of (3 13) into one for determing u_h and another one for determing p_h with a known u_h [4] The discrete velocity field u_h solves the problem

Find $u_h \in W_h$ such that

$$\nu a_h(u_h, v) + \tilde{b}_h(u_h, u_h, v) = (f, v) \quad \forall v \in W_h \tag{5 2}$$

where W_h denotes the space of discrete-divergence-free functions defined in (4.2).

THEOREM 1 : *Assume that* $f \in (L^2(\Omega))^2$. *Then there exists at least one solution* $(u_h, p_h) \in V_h \times Q_h$ of (3.13).

Proof: Let $P : W_h \to W_h$ be the mapping defined by

$$a_h(Pv, w) = \nu a_h(v, w) + \tilde{b}_h(v, v, w) - (f, w)$$

for all $v, w \in W_h$. Then, if k is sufficiently large, from Lemma 1 we conclude for $\|v\|_h = k$

$$\begin{aligned} a_h(Pv, v) &\geqq \nu a_h(v, v) - (f, v) \\ &\geqq \|v\|_h \left(\nu \|v\|_h - C(2, \Omega) \|f\|_{0,2,\Omega}\right) > 0 \end{aligned}$$

where $C(2, \Omega)$ is the constant from (4.13). In order to show the continuity of P we apply Lemma 2

$$\begin{aligned} \|Pv - Pw\|_h^2 &= \nu a_h(v - w, Pv - Pw) + \tilde{b}_h(v, v, Pv - Pw) - \\ &\quad - \tilde{b}_h(w, w, Pv - Pw) \\ \|Pv - Pw\|_h^2 &\leqq \nu \|v - w\|_h \|Pv - Pw\|_h + \tilde{b}_h(v, v - w, Pv - Pw) \\ &\quad + \tilde{b}_h(v, w, Pv - Pw) - \tilde{b}_h(w, w, Pv - Pw) \\ &\leqq (\nu + C(\|v\|_h + \|w\|_h)) \|v - w\|_h \|Pv - Pw\|_h \end{aligned}$$

and obtain for bounded v and w

$$\|Pv - Pw\|_h \leqq C \|v - w\|_h .$$

Then, by means of [11, II Lemma 1.4] we obtain the existence of at least one solution $u_h \in W_h$ of (5.2). The existence of a unique $p_h \in Q_h$ such that the pair (u_h, p_h) fulfills (3.13) follows in the usual way from (5.1) [4]. □

In order to study the convergence properties of the solutions (u_h, p_h) of (3.13) we introduce the embedding operator $I_h : V + V_h \to (L^2(\Omega))^6$ defined on each element K by

$$(I_h v)(x) = (v(x), \operatorname{grad} v(x)) \quad \forall x \in \mathring{K} .$$

As a consequence of inequality (4.13) the embedding operator I_h is continuous uniformly in h, i.e. there is a constant $C > 0$ such that

$$\|I_h v\| \leqq C \|v\|_h \quad \forall v \in V + V_h . \tag{5.3}$$

THEOREM 2 *Let* $\{(u_h, p_h)\}$ *be a sequence of solutions of the discrete problem* (3 13) *where* h *tends to zero Then there exists a subsequence* $\{(u_h, p_h)\}$ *and an element* (u, p) *belonging to* $V \times Q$ *such that* $I_h u_h$ *converges to* $(u, \operatorname{grad} u)$ *in* $(L^2(\Omega))^6$, p_h *converges to* p *weakly in* $L^2(\Omega)$ *and the pair* (u, p) *is a solution of the continuous problem* (2 6) *Moreover, if* (u, p) *belongs to* $(H^2(\Omega))^2 \times H^1(\Omega)$ *the pressure* p_h *converges to* p *also strong in* $L^2(\Omega)$

Proof Following the lines of Temam [11], we only have to modify some details, which result from replacing the discretization $\bar{b}_h$ defined in (3 7) by our upstream discretization $\tilde{b}_h$ defined in (3 11) Therefore, we will only mention the important steps of the proof

Setting $v_h = u_h$ in (5 2) we obtain from Lemma 1 the a priori estimate

$$\|u_h\|_h \leqq \nu^{-1} C(2, \Omega) \|f\|_{0\,2\,\Omega} \tag{5 4}$$

By means of the discrete LBB-condition (5 1) we conclude from (3 13) and (5 4) for all h

$$\|p_h\|_{0\,2\,\Omega} \leqq C$$

such that the sequence $\{I_h u_h, p_h\}$ is uniformly bounded in $(L^2(\Omega))^7$ Consequently, we are able to select a subsequence being weakly convergent For simplicity, we will denote this subsequence again by $\{(u_h, p_h)\}$ The weak limit (u, p) of $\{(u_h, p_h)\}$ belongs to the space $W \times Q$ (*cf* [11]) where

$$W = \{v \in V . \operatorname{div} v = 0\}$$

In order to show that (u, p) is a solution of the continuous problem we introduce the restriction operator $r_h . V \to V_h$ and $r_h \; W \to W_h$, respectively, which is defined by

$$(r_h v)(B_j) = \frac{1}{\operatorname{mes} \Gamma_j} \int_{\Gamma_j} v \, ds$$

and consider (3 13) for v replaced by $r_h v$ with $v \in (C_0^\infty(\Omega))^2$ As in [11] it holds

$$\left.\begin{aligned} a_h(u_h, r_h v) &\to a(u, v) \\ (p_h, \operatorname{div}_h r_h v) &\to (p, \operatorname{div} v) \\ (f, r_h v) &\to (f, v) \end{aligned}\right\} \quad \text{for} \quad h \to 0 \quad \forall v \in (C_0^\infty(\Omega))^2$$

and we have to verify

$$\tilde{b}_h(u_h, u_h, r_h v) \to b(u, u, v) \quad \text{for} \quad h \to 0 \quad \forall v \in (C_0^\infty(\Omega))^2 \tag{5 6}$$

Analogously to the proof of Lemma 3.3 in [11, II.3) we can show that

(5.7) $b_h(u_h, u_h, r_h v) \to b(u, u, v)$ for $h \to 0$ $\forall v \in (C_0^\infty(\Omega))^2$.

Using Lemma 3 we can estimate

$$\left|\tilde{b}_h(u_h, u_h, r_h v) - b_h(u_h, u_h, r_h v)\right| \leqq C_\kappa h^{1-\kappa}\|u_h\|_h^2 \, \|r_h v\|_h .$$

From (5.4) and the fact that $I_h(r_h v - v)$ tends to zero in the norm of $(L^\infty(\Omega))^6$ (*cf.* [11]) we see that $\|u_h\|_h$ and $\|r_h v\|_h$, are uniformly bounded. Thus, (5.7) implies (5.6) and the weak limit (u, p) fulfills

$$\begin{aligned} \nu a(u, v) + b(u, u, v) - (p, \operatorname{div} v) &= (f, v) \quad \forall v \in (C_0^\infty(\Omega))^2 \\ (q, \operatorname{div} u) &= 0 \qquad\quad \forall q \in Q . \end{aligned}$$

Since $C_0^\infty(\Omega)$ is a dense subset of $H_0^1(\Omega)$, (u, p) is a solution of the continuous problem (2.6).

Now we prove the strong convergence of $I_h(u_h - u)$ in $(L^2(\Omega))^6$. For this we consider

$$X_h = a_h(u_h - r_h u, u_h - r_h u) = \|u_h - r_h u\|_h^2 \geqq 0 .$$

Since u_h fulfills (5.2), we obtain

$$\begin{aligned} X_h &= a_h(u_h, u_h) - 2\, a_h(u_h, r_h u) + a_h(r_h u, r_h u) \\ &= \frac{1}{\nu}\left\{(f, u_h) - \tilde{b}_h(u_h, u_h, u_h)\right\} - 2\, a_h(u_h, r_h u) + a_h(r_h u, r_h u) \end{aligned}$$

and with lemma 1

$$X_h \leqq \frac{1}{\nu}(f, u_h) - 2\, a_h(u_h, r_h u) + a_h(r_h u, r_h u) . \tag{5.8}$$

The right hand side of (5.8) for $h \to 0$ converges to

$$\frac{1}{\nu}(f, u) - a(u, u) = \frac{1}{\nu} b(u, u, u) = 0 ,$$

which implies $\|u_h - r_h u\|_h \to 0$ for $h \to 0$. The triangle inequality concludes the proof of the strong convergence of $I_h(u_h - u)$ to zero in $(L^2(\Omega))^6$.

The strong convergence in $L^2(\Omega)$ of the pressure p_h in the case $(u, p) \in (H^2(\Omega))^2 \times H^1(\Omega)$ follows from (5.1) in the following way. Multiplying the equation

$$-\nu\, \Delta u + u_i\, D_i\, u + \operatorname{grad} p = f ,$$

which now holds in $(L^2(\Omega))^2$, with $v \in V_h$, integrating over K, applying Green's formula and summing-up over all finite elements K we get

$$\nu a_h(u, v) + b_h(u, u, v) - (p, \operatorname{div}_h v) = (f, v) + l(v)$$

for all $v \in V_h$ where l is defined by

$$l(v) = \sum_K \left(\nu \int_{\partial K} \frac{\partial u}{\partial n} v \, ds - \int_{\partial K} p(v \, . \, n) \, ds \right) .$$

Together with (3.13) we have for each $v \in V_h$

$$(5.9) \quad (p_h - p, \operatorname{div}_h v) = \nu a_h(u_h - u, v) + \tilde{b}_h(u_h, u_h, v) - b_h(u_h, u_h, v) + \\ + b_h(u_h, u_h, v) - b_h(u, u, v) + l(v) .$$

Using the representation

$$b_h(u_h, u_h, v) - b_h(u, u, v) = b_h(u_h - u, u, v) + b_h(u_h, u_h - u, v)$$

and taking into consideration that u_h is uniformly bounded and b_h is a continuous trilinear form on $V + V_h$ we obtain

$$|b_h(u_h, u_h, v) - b_h(u, u, v)| \leqq C \|u - u_h\|_h \|v\|_h$$

for some positive constant C independent of h. In [2] it was already shown that

$$|l(v)| \leqq Ch \|v\|_h \quad \text{for all} \quad v \in V_h .$$

Together with lemma 3 and (5.4) it follows from (5.9)

$$|(p - p_h, \operatorname{div}_h v)| \leqq (C \|u - u_h\|_h + C_\kappa h^{1-\kappa} + Ch) \|v\|_h .$$

Let $\tilde{p}_h$ be the orthogonal projection in $L^2(\Omega)$ of p on Q_h. Then by means of (5.1) we have

$$\begin{aligned} \|\tilde{p}_h - p_h\|_{0,2,\Omega} &\leqq \frac{1}{\alpha} \sup_{v \in V_h} \frac{(\tilde{p}_h - p_h, \operatorname{div}_h v)}{\|v\|_h} \\ &= \frac{1}{\alpha} \sup_{v \in V_h} \frac{(p - p_h, \operatorname{div}_h v)}{\|v\|_h} \\ &\leqq C \|u - u_h\|_h + C_\kappa h^{1-\kappa} . \end{aligned}$$

Thus, we get the estimate

$$(5.10) \quad \|p - p_h\|_{0,2,\Omega} \leqq \inf_{q \in Q_h} \|p - q\|_{0,2,\Omega} + C \|u - u_h\|_h + C_\kappa h^{1-\kappa}$$

such that for the convergent subsequence $\{u_{h'}\}$ the associated sequence $\{p_{h'}\}$ converges to the solution p in $L^2(\Omega)$. □

Now, we will study the case of sufficiently large ν in which the unique solvability of the problems (2.6) and (3.13) can be guaranteed and give a result concerning the rate of convergence.

THEOREM 3 : *Let ν be sufficiently large. Then both problems* (2.6) *and* (3.13) *have uniquely determined solutions. Moreover, if the solution* (u, p) *of* (2.6) *belongs to* $(H^2(\Omega))^2 \times H^1(\Omega)$ *the error estimate*

$$\|p - p_h\|_{0,2,\Omega} + \|u - u_h\|_h \leqq C_\kappa h^{1-\kappa} \tag{5.11}$$

with an arbitrary $\kappa \in (0, 1)$ *is satisfied.*

Proof: Let (u_1, p_1) and (u_2, p_2) be two different solutions of (3.13). From (5.2) we have for $v = u_1 - u_2 \in W_h$

$$\begin{aligned} \nu a_h(v, v) &= \tilde{b}_h(u_2, u_2, v) - \tilde{b}_h(u_1, u_1, v) \\ &= \tilde{b}_h(u_2, u_2, v) - \tilde{b}_h(u_1, u_2, v) - \tilde{b}_h(u_1, v, v) . \end{aligned}$$

Applying Lemma 1 and Lemma 2 we can estimate

$$\nu \|v\|_h^2 \leqq C \|v\|_h^2 \|u_2\|_h .$$

By means of the a priori estimate (5.4) it follows $v = 0$ if $\nu^2 \|f\|_{0,2,\Omega}^{-1}$ is sufficiently large. The relation $p_1 = p_2$ can be easily concluded from the discrete LBB-condition (5.1). In a similar way we can also prove uniqueness of the solution of problem (2.6).

In order to prove the error estimate let us consider $w = u_h - v \in W_h$ with an arbitrary $v \in W_h$. Then we have

$$\begin{aligned} \nu \|w\|_h^2 &\leqq \nu a_h(u_h - v, w) = \nu a_h(u - v, w) + \nu a_h(u_h - u, w) \\ &\leqq \nu \|u - v\|_h \|w\|_h + (f, w) - b_h(u, u, w) - \nu a_h(u, w) \\ &\quad + b_h(u_h, u_h, w) - \tilde{b}_h(u_h, u_h, w) + b_h(u, u, w) - b_h(u_h, u_h, w) . \end{aligned}$$

We split the term $R = b_h(u, u, w) - b_h(u_h, u_h, w)$ into

$$\begin{aligned} R &= b_h(u, u - u_h, w) + b_h(u - u_h, u_h, w) \\ &\leqq C(\|u\|_h + \|u_h\|_h) \|u - u_h\|_h \|w\|_h \end{aligned}$$

and take into consideration that u and u_h are uniformly bounded such that

$$R \leqq C\nu^{-2} \|u - u_h\|_h \|w\|_h .$$

From the triangle inequality it follows

$$\|u - u_h\|_h \leqq \nu(\|u - v\|_h + \|w\|_h)$$
$$\leqq 2\nu\|u - v\|_h + \frac{(f, w) - b_h(u, u, w) - \nu a_h(u, w)}{\|w\|_h}$$
$$+ \frac{b_h(u_h, u_h, w) - \tilde{b}_h(u_h, u_h, w)}{\|w\|_h} + \frac{C}{\nu}\|u - u_h\|_h .$$

Now, if ν^2 is greater than C we have the estimate

$$\|u - u_h\|_h \leqq C \inf_{v \in W_h} \|u - v\|_h + \sup_{w \in W_h} \frac{|(f, w) - b_h(u, u, w) - \nu a_h(u, w)|}{\|w\|_h}$$
$$+ \sup_{w \in W_h} \frac{|b_h(u_h, u_h, w) - \tilde{b}_h(u_h, u_h, w)|}{\|w\|_h}$$

and the error is decomposed into three parts, the approximation error, the discretization error caused by the nonconforming finite element method and the error due to the upstream discretization.

The estimates of the first and second error are obtained as in [2, 11]. On the third term we apply Lemma 3 and (5.10) yields the estimation for the pressure. □

Finally, we shall give a result about the algebraic system corresponding to our discrete problem (3.13). Splitting the algebraic system by means of a pressure-velocity iteration and solving the nonlinear system by a simple iteration technique we get the linear system

$$A(\underline{u}^m)\,\underline{u}^{m+1} = F \tag{5.12}$$

where $\underline{u}^m$ denotes the m-th iterate of the vector of velocity components. We will show that under a certain assumption on the triangulation the matrix $A(\underline{u})$ is an M-matrix. To verify that $A = (a_{ij})$ is an M-matrix it is sufficient to show that

(i) $a_{ij} \leqq 0$ for $i \neq j$ and

(ii) $\exists e \geqq 0$ such that $Ae \geqq 0$ and for all $i \in \{1, \dots, n\}$ with $(Ae)_i = 0$ there exists a chain $i_0 = i,\ i_1, \dots, i_p$ such that $(Ae)_{i_p} > 0$ and $a_{i_{q-1}, i_q} < 0$ for $q = 1, \dots, p$.

Let the triangulation of Ω be of weakly acute type, i.e. the interior angles of all triangles are not greater than $\pi/2$. Moreover, let $\psi_i = (\varphi_i, 0)$,

$\psi_{i+N} = (0, \varphi_i)$, $i = 1, \dots, N$ be the basis of V_h satisfying $\varphi_i(B_j) = 0$ for $i \neq j$ and $\varphi_i(B_i) = 1$. Then the matrix $A(\underline{u})$ in (5.12) is given by

$$a_{ij} = \begin{cases} \nu a_h(\psi_j, \psi_i) + \tilde{b}_h(u, \psi_j, \psi_i) & i, j = 1, \dots, 2N, \\ 0 & \text{otherwise}. \end{cases} \tag{5.13}$$

THEOREM 4 : *Let the triangulation of* Ω *be of weakly acute type. Then the matrix* $A(\underline{u})$ *of* (5.12) *defined by* (5.13) *is an M-matrix.*

Proof: Taking into account the representation (3.11) with (3.12) we get the nonpositivity of $\tilde{b}_h(u, \psi_j, \psi_i)$ for $i \neq j$ and the nonnegativity for $i = j$. The direction of $\nabla\varphi_i$ on a triangle K corresponds to the outer normal on the boundary ∂K in the node B_i of K. Therefore, $a_h(\psi_j, \psi_i)$ is nonpositive for $i \neq j$ and negative only in the case where i, j are neighbour nodes and the angle between both edges with midpoints B_i, B_j is smaller than $\pi/2$. Consequently, the assumption (i) is fulfilled. We set $e = (1, \dots, 1)$ such that $(Ae)_i$ corresponds to the i-th row sum. Obviously, it follows that

$$(Ae)_i \geqq 0 \quad \text{for} \quad i = 1, \dots, 2N.$$

If for some $i = i_0$ $(Ae)_i = 0$ we have to construct a chain $i_0, i_1, \dots, i_p$ such that $a_{i_{q-1}, i_q} < 0$, $q = 1, \dots, p$. For this aim it is sufficient to show that

$$\int_\Omega \nabla\varphi_{i_{q-1}} \nabla\varphi_{i_q}\, dx < 0 \qquad q = 1, \dots, p. \tag{5.14}$$

Let B_i and B_j are neighbour nodes, K the triangle containing these nodes and k the third node of K. Since the triangulation is of weakly acute type we have

$$\int_K \nabla\varphi_i \nabla\varphi_j\, dx = 0 \Rightarrow \tag{5.15}$$
$$\Rightarrow \left(\int_K \nabla\varphi_i \nabla\varphi_k\, dx < 0 \text{ and } \int_K \nabla\varphi_k \nabla\varphi_j\, dx < 0 \right).$$

Therefore, starting with an index i_0 not belonging to a node of a boundary triangle we can find a chain of two or three indices satisfying (5.14) and connecting B_i with any of the four neighbour nodes. Continuing this procedure we come to a node B_i, $i = i_p$, of a boundary triangle K. Now we have

$$(Ae)_i \geqq -\nu \sum_{\substack{j \in \Lambda_i \\ B_j \in \Gamma}} \int_\Omega \nabla\varphi_j \nabla\varphi_i\, dx. \tag{5.16}$$

Every integral in the above sum is nonpositive In order to prove $(Ae)_i > 0$ we have to exclude the case that every integral in (5 16) is equal to zero In the following let us consider this case

For the boundary triangle K containing the node B_i, we denote by B_j the boundary node and by B_k the third node Because of (5 15) B_k does not belong to Γ and (5 14) holds for $q = p + 1$ and $i_{p+1} = k$ Therefore, let us take the chain which consists of the above chain with $i_p = i$ and $i_{p+1} = k$ (5 15) also implies

$$\int_\Omega \nabla\varphi_j \nabla\varphi_k \, dx < 0$$

and therefore $(Ae)_k > 0$ Consequently, the assumption (ii) holds and $A(\underline{u})$ is an M-matrix □

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