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APPROXIMATION THEORIES FOR INERTIAL MANIFOLDS (*)

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0. INTRODUCTION

During the last few years it has been shown that some infinite dimensional nonlinear dissipative evolutionary equations have inertial manifolds. This discovery has profound significance in the study of the long-time behavior of the solutions of these equations for the following reasons :

- The inertial manifold $\mathfrak{M}$ is a positively invariant finite dimensional manifold in the ambient infinite dimensional phase space, and the given evolutionary equation reduces to a finite dimensional ordinary differential equation, an ODE, on $\mathfrak{M}$.
- Every attractor, including the global attractor, lies in $\mathfrak{M}$.
- Every solution of the nonlinear evolutionary equation is tracked at a exponential rate by a solution on $\mathfrak{M}$. This means that there is an $\eta > 0$ such that for every solution $u(t)$ of the original evolutionary system, there is a solution $v(t)$ on $\mathfrak{M}$ such that

$$\|u(t) - v(t)\| \leq K e^{-\eta t}, \quad t > 0, \tag{0}$$

where K depends on $u(0)$.

In some models the decay rate η appearing above is very large. When this happens the solutions on the inertial manifold also give useful information about the short-time behavior of an arbitrary solution $u(t)$, *provided* $u(0)$ is near $\mathfrak{M}$.

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Because the existence of an inertial manifold implies that the dynamics of the original evolutionary equation is completely described by a finite dimensional ODE, *with no error,* this should lead to substantial improvements in the computational efficiency of numerical methods used to study the evolutionary equation. In order to realize this efficiency, it is important to find good algorithms for approximating the inertial manifolds. The main objective in this paper is to examine several approximation theories for inertial manifolds. Since every existence theory is a potential spawning ground for an approximation theory, we begin with a brief review of the three known classes of existence theories for inertial manifolds.

The first existence theory uses the Lyapunov-Perron method, which is based on the variation of constants formula. While the Lyapunov-Perron method is very useful for deriving properties of inertial manifolds (in addition to proving existence), it is not a very promising arena for finding a good approximation theory. The main fault of the Lyapunov-Perron method is that it uses *backward* integration of the evolutionary equation. Since the backward integration is in the « unstable » direction of the evolutionary equation, one will encounter a blow-up of the solutions, which in turn is an inherent source of computational inefficiency.

The second class of existence theories use the Hadamard method, or the graph transform method. The basic idea here is to start with some initial approximation to the inertial manifold. This initial approximation is an easily computed manifold of the correct dimension, call it $\mathfrak{M}_0$. One then lets the dynamics of the given evolutionary equation act on $\mathfrak{M}_0$, thereby obtaining a set $\mathfrak{M}_t$ at each time $t > 0$. One then proves, under suitable hypotheses of course, that each $\mathfrak{M}_t$ is representable as the graph of some function [1], that the limit

$$\lim_{t \to \infty} \mathfrak{M}_t = \mathfrak{M}$$

exists, and that $\mathfrak{M}$ is the inertial manifold.

Approximation theories based on the Hadamard method will be better than theories based on the Lyapunov-Perron method because one is integrating forward in time, i.e., in the stable direction. Because of inequality (0) one expects that

$$\mathfrak{M}_\tau \approx \mathfrak{M} ,$$

for an appropriate $\tau > 0$. Approximation theories based on the Hadamard method try to approximate $\mathfrak{M}_\tau$. Such approximations can be easily

(1) Hence the term *graph transform* method.

implemented when τ is small, or when the constant η in (0) is large. The Euler-Galerkin method, which is introduced in Foias, Sell and Titi (1988) and described in Section 3 below, is an illustration of a Hadamard-type approximation. If the convergence of $\mathfrak{M}_t$ to $\mathfrak{M}$ is slow, then the Hadamard-type approximation theories will require the time parameter τ to be large in order to get good approximations. We expect that in these situations, one will get better approximations by using the following alternative.

The third method for proving the existence of inertial manifolds is based on the method of elliptic regularization which Sacker (1964, 1965, 1969) used in the study of finite dimensional invariant manifolds. The extension of the Sacker method to infinite dimensional dynamical systems is presented in Fabes, Luskin and Sell (1988) and Luskin and Sell (1988). A description of the main ideas of this method is presented in Sections 4-5 below.

I. INERTIAL MANIFOLDS

The type of equation we study can be reduced to an abstract evolutionary equation of the form

$$u' + Au = F(u), \quad u(0) = u_0 \tag{1}$$

on a Hilbert space H. We will assume that A is a self adjoint operator defined on a dense domain $\mathscr{D} = \mathscr{D}(A) \subset H$ and that A is positive with compact resolvent. This means that $-A$ generates an analytic semigroup e^{-At}, and that the fractional powers A^{α}, are defined for all $\alpha \geqslant 0$, see Pazy (1983). Furthermore, for every α, $0 < \alpha \leqslant 1$, there is a constant M_{α} such that

$$|e^{-At}x - x| \leqslant M_{\alpha} t^{\alpha} |A^{\alpha} x| , \quad x \in \mathscr{D}(A^{\alpha}) . \tag{2}$$

The nonlinear term F is assumed to be a C^1-function

$$F : \mathscr{D}(A) \to \mathscr{D}(A^{\beta}) ,$$

where $0 < \beta \leqslant 1$ is fixed, satisfying the following two properties :

(A) There is a constant C_0 such that

$$|A^{\beta} F(u)| \leqslant C_0 , \quad u \in \mathscr{D}(A) .$$

(B) There is a constant C_1 such that the Gateaux derivative $DF(u)$ satisfies

$$|A^{\beta} DF(u) v| \leqslant C_1 |Av| , \quad u, v \in \mathscr{D}(A) . \tag{3}$$

Because of (3) the function F satisfies a global Lipschitz condition, i.e.,

$$|A^{\beta}[F(u_1) - F(u_2)]| \leqslant C_1 |Au_1 - Au_2| \tag{4}$$

for all u_1, $u_2 \in \mathscr{D}(A)$.

We also assume that $F : \mathscr{D}(A^{1-\beta}) \to H$ is locally Lipschitz continuous. This implies that for $u_0 \in \mathscr{D}(A^{1-\beta})$ there is a unique mild solution of (1). We will represent this solution as $S(t)\, u_0$, where

$$S(t)\, u_0 = e^{-At}\, u_0 + \int_0^t e^{-A(t-s)}\, F(S(s)\, u_0)\, ds\,.$$

This solution is a classical solution for $t > 0$, and when $u_0 \in \mathscr{D}(A)$, it is differentiable for $0 \leqslant t$.

Finally we assume that F has bounded support, i.e., there is a constant $\rho > 0$ such that

$$F(u) = 0\,, \quad \text{when} \quad |Au| > \rho\,.$$

We will not describe in detail how nonlinear (parabolic-type) partial differential equations are reformulated as an abstract evolutionary equation of the type described above. Such reformulations can be found in the two recent books by Hale (1988) and Temam (1988), and in the papers Foias, Sell, and Temam (1986), Mallet-Paret and Sell (1988), and Constantin, Foias, Nicolaenko, and Temam (1988, 1989). An important feature in these problems is that the original equation is dissipative. This means that there is a bounded set $B \subset H$ such that for every $u_0 \in \mathscr{D}(A^{1-\beta})$ there is a time $T = T(u_0)$ such that $S(t)\, u_0 \in B$ for all $t > T$. Since the operator A has compact resolvent, the dissipative property implies that there is a global attractor $\mathfrak{A}$ for (1) and that $\mathfrak{A}$ is compact and invariant, see Billotti and La Salle (1971). Furthermore, $\mathfrak{A}$ has finite Hausdorff dimension, see Mallet-Paret (1976), Foias and Temam (1979), and Mañé (1981). The 2D Navier-Stokes equation, the Kuramoto-Sivaskinsky equations, the Cahn-Hilliard equations, and many reaction diffusion equations can be reduced to (1) with the given properties on F. In each case the reduction step involves a modification of the nonlinearities of the given partial differential equation outside of some neighborhood of the global attractor. This modification is a common feature in handling such equations. We will not describe the modification here, but instead refer the reader to the references cited above.

A subset $\mathfrak{M} \subset H$ is said to be an **inertial manifold** for (1) if $\mathfrak{M}$ satisfies the following four conditions :

(A) $\mathfrak{M}$ is a finite dimensional Lipschitz manifold in H.

(B) $\mathfrak{M}$ is smooth, i.e., $\mathfrak{M}$ is of class C^1.

(C) $\mathfrak{M}$ is positively invariant, i.e., if $u_0 \in \mathfrak{M}$ then $S(t)\,u_0 \in \mathfrak{M}$ for all $t > 0$.

(D) $\mathfrak{M}$ is exponentially attracting, i.e., there is a $\mu > 0$ such that for every $u_0 \in H$ there is a constant $K = K(u_0)$ such that

$$\text{dist}\,(S(t)\,u_0, \mathfrak{M}) \leqslant K\,e^{-ut}\,, \quad t \geqslant 0\,.$$

The smoothness of $\mathfrak{M}$, which is not a part of the definition of an inertial manifold as presented in Foias, Sell and Temam (1986), is an important property and it will be used below. The smoothness of the inertial manifold is not a major issue. Most theories which yield the existence of a Lipschitz manifold $\mathfrak{M}$ also imply the smoothness of $\mathfrak{M}$, see Chow, Lu, and Sell (1988) and Mallet-Paret and Sell (1988).

The methods for finding inertial manifolds begin with a splitting of the Hilbert space H into two parts $\hat{P}H$ and $\hat{Q}H$, where $\hat{P}$ is an orthogonal projection on H with finite dimensional range and $\hat{Q} = I - \hat{P}$. The prototypical choice for this splitting occurs when $\hat{P}$ is the orthogonal projection onto Span $\{w_1, \dots, w_M\}$, where w_i is the i-th eigenvector of A with associated eigenvalue λ_i, and

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_n \to \infty\,.$$

The usual existence theories for inertial manifolds seek to realize $\mathfrak{M}$ as the graph of a smooth function

$$\Phi : \hat{P}H \to \hat{Q}H\,.$$

We shall say more about the properties of $\mathfrak{M}$ and Φ later. One should note that with $\hat{P}$ as described above, then for any Lipschitz mapping $\Phi : \hat{P}H \to \hat{Q}H$, its graph is an M-dimensional Lipschitz manifold in H.

The list of references on inertial manifolds is growing rapidly, and any attempt to cite all such papers is bound to have some omissions. The following papers will be included in the ultimate « complete » list : Chow, Lu, and Sell (1988), Constantin (1988), Constantin, Foias, Nicolaenko, and Temam (1988, 1989), Doering, Gibbon, Holm, and Nicolaenko (1988), Foias, Nicolaenko, Sell, and Temam (1988), Foias, Sell and Temam (1986), Foias, Sell and Titi (1988), Ghidaglia (1988), Hale and Sell (1988), Henry (1981), Jolly (1988), Kamaev (1981), Mallet-Paret and Sell (1987), Mañé (1977), Marion (1988), Mora (1983), Mora and Solà-Morales (1987, 1988), Nicolaenko, Scheurer, and Temam (1987), Sell and You (1988), and Taboada (1988).

In this lecture we will present three methods for approximating inertial manifolds. All of these methods can be viewed as *modified Galerkin* approximations. In order to present a uniform framework for viewing these approximation theories, we describe next the Galerkin and modified Galerkin methods.

II. MODIFIED GALERKIN APPROXIMATIONS

The classical theory of Galerkin approximations for nonlinear evolutionary equation (1) can be best described by first fixing two integers $M \geqslant 1$ and $N \geqslant 1$, and letting P and Q denote the orthogonal projection onto

$$\operatorname{Span}\{w_1, \ldots, w_M\} \quad \text{and} \quad \operatorname{Span}\{w_{N+1}, \ldots, w_{N+M}\},$$

respectively. Next let $R = I - P - Q$. For $u \in H$ set $p = Pu$, $q = Qu$, $r = Ru$. By applying P, Q and R to (1) we obtain the equivalent system :

$$\begin{cases} p' + APp = PF(p+q+r), & \dim p = M \\ q' + AQq = QF(p+q+r), & \dim q = N \\ r' + ARr = RF(p+q+r), & \dim r = \infty \end{cases} \tag{5}$$

where we have used the commutativity relationships $PA = AP$, $QA = AQ$ and $RA = AR$, which hold on $\mathcal{D}(A)$.

The **classical Galerkin approximation** of (1), or equivalently of (5), involves setting certain terms in (5) equal to 0. Thus the $(M+N)$-dimensional Galerkin approximation is formed by setting $r = 0$ in (5) and thereby « obtaining »

$$\begin{cases} p' + APp = PF(p+q), & \dim p = M, \\ q' + AQq = QF(p+q), & \dim q = N, \end{cases} \tag{6}$$

while the M-dimensional Galerkin approximation is

$$p' + APp = PF(p), \quad \dim p = M, \tag{7}$$

i.e., set $r = 0$ and $q = 0$.

Let us concentrate on (6) for the moment. If it happens that

$$RF(p+q) = RF(p+q+r)|_{r=0} \equiv 0,$$

then the $(M+N)$-dimensional system (6) describes the dynamics of (5) on the invariant manifold

$$r = 0.$$

Of course, this rarely happens. However it is oftentimes the case that $RF(p+q)$ is small. In fact, the *raison d'être* behind the Galerkin approximations is the following :

The function $RF(p+q)$ is small and the dynamics of (6) *is a good approximation to the dynamics of* (5), *provided $(M+N)$ is sufficiently large. One expects, and there are theories which prove, that the approximation gets better as the dimension $(M+N)$ gets larger. In particular, for N large, the $(M+N)$-dimensional system* (6) *is expected to generate a better approximation to the dynamics of* (5) *than the M-dimensional system* (7).

The **modified Galerkin approximations** begin with (5) and, as a first step, one sets $r=0$ to obtain (6). The modification now occurs in the second step. Instead of setting $q=0$ to obtain (7) one uses $q=\Phi_a(p)$ to obtain the modified equation

$$p' + APp = PF(p+\Phi_a(p)), \quad \dim p = M. \tag{8}$$

One wants to take advantage of the theory of inertial manifolds in order to determine the function $\Phi_a(p)$. The main idea behind the modified Galerkin approximations is the following :

When (5) *has an inertial manifold, then the long-time dynamics of* (5) *can be better approximated by the M-dimensional system* (8) *than by the $(M+N)$-dimensional system* (6), *for any $N \geq 1$.*

Naturally the approximation (8) is preferable in this situation.

Assume now that the system (5) has an inertial manifold $\mathfrak{M}$ and that $\mathfrak{M}=$ Graph Φ where $\Phi=(\Phi_q,\Phi_r)$ is a smooth function

$$\Phi : PH \to QH \oplus RH .$$

Then the dynamics on $\mathfrak{M}$ is completely and accurately described by the M-dimensional system

$$p' + APp = PF(p+\Phi_q(p)+\Phi_r(p)), \tag{9}$$

which is called an *inertial form* in Foias, Sell and Temam (1986). In other words, the long-time behavior of any solution $u(t)=S(t)\,u_0$ of (5) is completely determined (with *no* error) by an associated solution

$$v(t) = p(t) + \Phi_q(p(t)) + \Phi_r(p(t)),$$

where $p(t)$ is an appropriate solution of the inertial form (9). In this way the long-time dynamics of the infinite dimensional system (5) are completely and accurately described by the dynamics of the M-dimensional system (9).

Under a spectral gap condition on the eigenvalues of A, one can show that there is a constant K_1, which does not depend on N, such that

$$\|\Phi_r\|_\infty \leq K_1(\lambda_{M+N+1})^{-\beta}, \tag{10}$$

where the norm is given by

$$\|\Phi_r\|_\infty = \sup\ \{|A\Phi_r(p)| : p \in PH\}\ ,$$

see Foias, Sell and Temam (1986), and Foias, Sell and Titi (1988). Therefore if one chooses

$$\Phi_a = \Phi_q$$

for equation (8), it follows from (4) and (10) that the error term

$$\text{Error } (p) = F(p + \Phi_q(p) + \Phi_r(p)) - F(p + \Phi_a(p))$$

satisfies

$$\begin{aligned}\|\text{Error}\| &\overset{\text{def}}{=} \sup\ \{|A^\beta \text{Error } (p)| : p \in PH\} \\ &\leqslant C_1 K_1 (\lambda_{M+N+1})^{-\beta}\,.\end{aligned}$$

In the remainder of this paper we shall describe alternate choices for Φ_a. In each case we believe that the calculation of Φ_a is easier than the calculation of Φ_q, on the one hand, and under suitable hypotheses

$$\|\Phi_a - \Phi_q\|_\infty$$

is small, on the other.

III. EULER-GALERKIN APPROXIMATION

The Hadamard method is one of the methods used in the proof of the existence of inertial manifolds for (5). The idea here is to begin with the flat manifold

$$\mathfrak{M}_0 = \{u = p + q + r : q = 0, r = 0\}$$

in H and set

$$\mathfrak{M}_t = S(t)\,\mathfrak{M}_0\,.$$

Then $\mathfrak{M}_t$ is a subset of H, and under a suitable cone condition, $\mathfrak{M}_t$ can be represented as the graph of a function

$$\mathfrak{M}_t = \text{Graph } \Psi^t\ ,$$

where $\Psi^t : PH \to QH \oplus RH$ is a Lipschitz continuous function, see Mallet-

Paret and Sell (1988) and Constantin, Foias, Nicolaenko and Temam (1988). Furthermore one can show that the limit

$$\lim_{t \to \infty} \Psi^t = \Phi$$

exists, and $\mathfrak{M} = \text{Graph}\ \Phi$ is an inertial manifold. More precisely, there are constants $b > 0$, $\mu > 0$, and $K_2 > 0$ such that

$$\|\Psi^t\|_\infty \leqslant b \quad \text{and} \quad \|\Psi^t - \Phi\|_\infty \leqslant K_2\, e^{-\mu t},$$

for all $t > 0$. The value of b, μ and K_2 depend on M, the dimension of $\mathfrak{M}$. In many cases one has

$$\limsup_{M \to \infty} K_2 < +\infty$$

while

$$b \to 0 \quad \text{and} \quad \mu \to \infty$$

as $M \to \infty$, see Mallet-Paret and Sell (1988), and Foias, Sell and Titi (1988). What this implies is that for every $\varepsilon > 0$ and $\tau > 0$ there is an M_0 such that if $\dim PH \geqslant M_0$ and the spectral gap condition holds, so that $\mathfrak{M} = \text{Graph}\ \Phi$ is an inertial manifold (with bounded support), where $\Phi : PH \to QH \oplus RH$, then

$$\|\Psi^\tau - \Phi\|_\infty \leqslant \varepsilon .$$

The Euler-Galerkin method, which is introduced in Foias, Sell and Titi (1988), uses the implicit Euler method for approximating Ψ^τ. The implicit Euler method for the system (5) can be summarized as follows : Let (p_0, q_0, r_0) be a given initial condition and let $(p(\tau), q(\tau), r(\tau))$ denote the corresponding solution of (5) at $t = \tau$. The implicit Euler approximation $(p_a(\tau), q_a(\tau), r_a(\tau))$ is given by letting $p_a(t)$ be the solution of

$$p' + APp = PF(p), \quad p(0) = p_0$$

and then setting

$$\begin{aligned} q_a(\tau) &= q_0 + \tau[-AQq_a(\tau) + QF(p_a(\tau) + q_a(\tau) + r_a(\tau))], \\ r_a(\tau) &= r_0 + \tau[-ARr_a(\tau) + RF(p_a(\tau) + q_a(\tau) + r_a(\tau))]. \end{aligned} \tag{11}$$

One can also describe this by asking that the « slope » of the line segment joining (p_0, q_0, r_0) to $(p_a(\tau), q_a(\tau), r_a(\tau))$ being given by evaluating the (q, r)-equations at the terminal point $(p_a(\tau), q_a(\tau), r_a(\tau))$. Next we define

$\Psi = (\Psi_1, \Psi_2) = (q_a(\tau),\ r_a(\tau))$. Since the mapping $p_0 \to p = p_a(\tau)$ is a homeomorphism of PH, the solution Ψ of (11) can be written in the form :

$$\begin{aligned} \Psi_1(p) &= (I + \tau AQ)^{-1}\,[q_0 + \tau QF(p + \Psi_1(p) + \Psi_2(p))] \\ \Psi_2(p) &= (I + \tau AR)^{-1}\,[\tau_0 + \tau RF(p + \Psi_1(p) + \Psi_2(p))]\,. \end{aligned} \tag{12}$$

The existence of a solution of the system (12) can be derived by use of the contraction mapping theorem.

In applying the implicit Euler method to estimate Ψ^{τ} we begin with $(p_0, q_0, r_0) \in \mathfrak{M}_0$, i.e., $q_0 = 0$, $r_0 = 0$. Furthermore, the first iteration of the method of successive approximations $\Phi_a = (\Phi_1, \Phi_2)$, where

$$\begin{aligned} \Phi_1(p) &= \tau(I + \tau AQ)^{-1}\, QF(p) \\ \Phi_2(p) &= \tau(I + \tau AR)^{-1}\, RF(p)\,, \end{aligned}$$

already leads to a useful approximation of the inertial manifold, see Foias, Sell and Titi (1988). This method is applied to a numerical study for the Kuramoto-Sivashinsky equation in Foias, Jolly, Kevrekides, Sell and Titi (1988).

IV. ELLIPTIC REGULARIZATION

A short time ago Sacker (1964, 1965) introduced a new method for proving the existence of invariant manifolds for finite dimensional dynamical systems This method is based on the theory of elliptic regularization of the underlying first order partial differential equation which defines the invariant manifold.

This method can be extended to the infinite dimensional systems considered here. In order to motivate the Sacker method, let us return to the situation where (5) has a inertial manifold of the form

$$\mathfrak{M} = \text{Graph}\ \Phi,$$

where $\Phi = (\Phi_q, \Phi_r)$ is a smooth function. The invariance of $\mathfrak{M}$ implies that if $p(t)$ is a solution of the inertial form (9) then

$$\Phi(p(t)) = (q(t), r(t)) = (\Phi_q(p(t)), \Phi_r(p(t)))$$

is a solution of the (q, r)-system

$$\begin{cases} q' + AQq = QF(p(t) + q + r) \\ r' + ARr = RF(p(t) + q + r)\,. \end{cases} \tag{13}$$

Since $\Phi = \Phi(p)$ is smooth we denote the derivative with respect to p by

$$D\Phi(p) = (D\Phi_q(p), D\Phi_r(p))\,.$$

The chain rule then implies that

$$\frac{\partial}{\partial t}\Phi(p(t)) = D\Phi(p(t))\frac{\partial}{\partial t}p(t).$$

By combining this fact with (9) and (13) we then obtain

$$\begin{aligned} D\Phi_q(p)(-APp + PF(p + \Phi_q(p) + \Phi_r(p))) = \\ = -AQ\Phi_q(p) + QF(p + \Phi_q(p) + \Phi_r(p)), \\ D\Phi_r(p)(-APp + PF(p + \Phi_q(p) + \Phi_r(p))) = \\ = -AR\Phi_r(p) + RF(p + \Phi_q(p) + \Phi_r(p)). \end{aligned} \tag{14}$$

For the moment, let us drop the r-equation above and set $\Phi_r = 0$ in the q-equation. Also replace $D\Phi_q$ by $\nabla\Phi_q$. One then has

$$\nabla\Phi_q(-APp + PF(p + \Phi_q)) + AQ\Phi_q = QF(p + \Phi_q), \tag{15}$$

a first order partial differential system, where $\dim \Phi_q = N$.

By construction the given function $\Phi = (\Phi_q, \Phi_r)$ has bounded support, i.e., one has

$$\Phi(p) = 0, \quad \text{when} \quad |Ap| > \rho,$$

see Foias, Sell and Temam (1986) and Chow, Lu and Sell (1988). This means that one is looking for a solution of (14) or (15) that satisfies $\Phi(p) = 0$ on $\partial\Omega_\rho$ where

$$\Omega_\rho = \{p \in PH : |Ap| < \rho\}.$$

This suggests that one might try to construct an inertial manifold by solving (15) in Ω_ρ subject to the boundary conditions mentioned above. Since F has bounded support, it follows that every boundary point of Ω_ρ is a point of strict ingress for the inertial form (9). Therefore by using a method of characteristics one should, in principle, be able to find a sufficiently regular solution of (15), provided shocks do not develop.

The first step in the Sacker method, which we formulate in terms of (15), is to replace (15) with the second order partial differential equation

$$-\varepsilon\,\Delta\Phi_q + \nabla\Phi_q(B(p, \Phi_q)) + AQ\Phi_q = QF(p + \Phi_q), \tag{16}$$

where $B(p, \Phi_q) = -APp + PF(p + \Phi_q)$. One then seeks a solution Φ_q of (16) which satisfies one of the boundary conditions

$$\Phi_q(p) = 0, \quad \text{on} \quad \partial\Omega_\rho,$$

or

$$\Phi_q(p) = 0, \quad \text{at} \quad |p| = \infty.$$

The object is to study the behavior of solutions of (16) as $\varepsilon \to 0^+$ By deriving suitable a priori bounds on the solutions of (16), bounds which are independent of ε, one can show that the limit as $\varepsilon \to 0^+$ exists and is a weak solution of (15) For the inertial manifold problem in an infinite dimensional space H, we seek a priori bounds which are independent of both ε and $N = \dim \Phi_q$ One then shows that the limit as $\varepsilon \to 0^+$ and $N \to \infty$ exists and describes an invariant manifold for the original infinite dimensional system (5)

In addition to studying the behavior of solutions of (16) as $\varepsilon \to 0^+$, the extension of the Sacker method to the study of inertial manifolds involves two mathematical issues which did not arise in Sacker (1964, 1965) The first of these is that the solution $\Phi = (\Phi_q, \Phi_r)$ has range in an infinite dimensional space Secondly the domain of Φ is PH and is no longer a compact manifold without boundary

The a priori bounds, which are independent of ε, do not come freely In order for the limit

$$\lim_{\varepsilon \to 0^+} \Phi_q$$

to be smooth, one needs assumptions on the coefficients, especially $B(p, \Phi_q)$ and $QF(p + \Phi_q)$, which prevent shock phenomena from developing in (15) Such shocks would be evident in the regularized problem (16) for small $\varepsilon > 0$ The hypotheses which guarantee that the a priori bounds be independent of ε and N are analogous to the spectral gap conditions appearing in Foias, Sell and Temam (1986), for example The following theorem is proved in Fabes, Luskin and Sell (1988)

THEOREM 1 *Let* (1) *be given satisfying the conditions stated above with* $\beta = 1$ *Then there is a constant* K, *which depends only on* C_0 *and* C_1, *such that if*

$$\lambda_{M+1} - \lambda_M > K ,$$

then there is a weak solution of Φ *of* (15) *with* $\Phi(p) = 0$ *at* $|p| = \infty$ *and such that*

$$\|\Phi\|_{W^{1\,\infty}} \leq R ,$$

where

$$\|\Phi\|_{W^{1\,\infty}} = \sup \{|A\Phi(p)| . p \in PH\} + \sup \{|AD\Phi(p)| . p \in PH\}$$

and R *depends on* C_0, C_1 *and the spectral gap* $(\lambda_{M+1} - \lambda_M)$

For equation (15) with fixed M, finite $N \geq 1$ and fixed $\varepsilon > 0$, it is possible to obtain information on the error between the solution $\Phi_{\varepsilon\, N}$ of (15) and the inertial manifold Φ for the full problem (5). These bounds will be described in the next section, where we use the Sacker method to introduce a parabolic regularization of (5).

V. PARABOLIC REGULARIZATION

In this section we want to take another point of view in analyzing (15) and (16), but with the same objective in mind. The basic observation is that one can view the Laplacian term $(-\varepsilon\,\Delta\Phi)$ in (16) as a perturbation term added to (1) or (5). More precisely let $B = -\Delta$ be given on Ω_ρ with the boundary condition $\Phi = 0$ on $\partial\Omega_\rho$. The effect of adding $(-\varepsilon\,\Delta\Phi)$ to (15) is then equivalent to perturbing the (p, q, r) equations (5) to

$$\begin{cases} p' + APp = PF(p + \Phi_q + \Phi_r) \\ \Phi_q' + (AQ + \varepsilon BQ)\,\Phi_q = QF(p + \Phi_q + \Phi_r) \\ \Phi_r' + (AR + \varepsilon BR)\,\Phi_r = RF(p + \Phi_q + \Phi_r)\,. \end{cases} \tag{17}$$

where $\varepsilon > 0$. This perturbation is a parabolic regularization of the original system (5).

We shall say that $\Phi^\varepsilon = (\Phi_q^\varepsilon, \Phi_r^\varepsilon)$ is a solution of (17) if $\Phi^\varepsilon = \Phi^\varepsilon(p)$ is a function of $p \in PH$ with the following property : whenever $p(t)$ is a solution of

$$p' + APp = PF(p + \Phi_q(p) + \Phi_r(p))$$

then $(p(t), \Phi_q^\varepsilon(p(t)), \Phi_r^\varepsilon(p(t)))$ satisfies

$$\begin{cases} p(t)' + APp(t) = PF(p(t) + \Phi_q^\varepsilon(p(t)) + \Phi_r^\varepsilon(p(t))) \\ \Phi_q^\varepsilon(p(t))' + (AQ + \varepsilon BQ)\,\Phi_q^\varepsilon(p(t)) = QF(p(t) + \Phi_q^\varepsilon(p(t)) + \Phi_r^\varepsilon(p(t))) \\ \Phi_r^\varepsilon(p(t))' + (AR + \varepsilon BR)\,\Phi_r^\varepsilon(p(t)) = RF(p(t) + \Phi_q^\varepsilon(p(t)) + \Phi_r^\varepsilon(p(t)))\,. \end{cases}$$

The solution $\Phi^\varepsilon = (\Phi_q^\varepsilon, \Phi_r^\varepsilon)$ is said to be smooth if Φ_q^ε and Φ_r^ε are smooth as functions of $p \in PH$. Let $\hat{Q} = Q + R$.

The problem we address is to find a family $\Phi^\varepsilon = (\Phi_q^\varepsilon, \Phi_r^\varepsilon)$ of smooth solutions of (17) for $\varepsilon > 0$ with the property that $\Phi^\varepsilon \to \Phi^0$ (as $\varepsilon \to 0^+$) where the graph of Φ^0 is an inertial manifold of (5). Part of the problem is to describe the topology in which Φ^ε converges to Φ^0 and to estimate the difference $\|\Phi^\varepsilon - \Phi^0\|$ in a suitable norm.

Before stating our main result, it is convenient to outline our basic approach to the problem described above. We use the Lyapunov-Perron method for constructing invariant manifolds for (17), see Foias, Sell and Temam (1986), Foias, Sell and Titi (1988) and Chow, Lu and Sell (1988).

The first step in this process is to show that for $\varepsilon > 0$ the linear operator $[-(A\hat{Q} + \varepsilon B\hat{Q})]$ is the infinitesimal generator of an analytic semigroup, which we will write as $e^{-(A\hat{Q}+\varepsilon B\hat{Q})t}$, $t \geq 0$. Since we are especially interested in the behavior as $\varepsilon \to 0^+$, we will need to compare $e^{-(A\hat{Q}+\varepsilon B\hat{Q})t}$ with the limiting semigroup $e^{-A\hat{Q}t}$.

The next step, which lies at the heart of the Lyapunov-Perron method, is the construction of the infinite integral operator

$$\mathcal{T}_\varepsilon \Phi(p) = \int_{-\infty}^{0} e^{(A\hat{Q}+\varepsilon B\hat{Q})s}\, \hat{Q}F(p(s) + \Phi(p(s)))\, ds\,, \quad \varepsilon \geq 0\,,$$

where $p(t)$ is the solution of the ordinary differential equation

$$p' + APp = PF(p + \Phi(p))$$

satisfying $p(0) = p$, and $\Phi = \Phi_q + \Phi_r$. It is shown in Luskin and Sell (1988) that a fixed point Φ^ε of $\mathcal{T}_\varepsilon$ is a solution of (17) for $\varepsilon \geq 0$. Also it is shown that under a suitable spectral gap condition, $\mathcal{T}_\varepsilon$ is a strict contraction and that it has a fixed point Φ^ε. Furthermore the fixed point Φ^ε is a smooth function of p. The final step is to calculate $\|\Phi^\varepsilon - \Phi^0\|_\infty$. Our main result is the following.

THEOREM 2 : *Let* (1) *be given satisfying the conditions stated above. Then there are constants K_0 and K_1, depending on C_1 and β, such that, if for some M the eigenvalues of A satisfy the spectral gap condition*

$$\lambda_{M+1} - \lambda_M \geq K_1(\lambda_{M+1}^\alpha + \lambda_M^\alpha)$$

and $\lambda_M \geq K_0$, where $\alpha = 1 - \beta$, then for all $\varepsilon \geq 0$ the operator $\mathcal{T}_\varepsilon$ is a strict contraction on a suitable function space $\mathcal{F}$ and, therefore, $\mathcal{T}_\varepsilon$ has a fixed point Φ^ε in $\mathcal{F}$. Furthermore the following properties hold :

(A) $\Phi^\varepsilon : PH \to \hat{Q}H \cap \mathcal{D}(A)$ *is a smooth function with Supp* $\Phi^\varepsilon \subset \Omega_\rho$, *for* $\varepsilon \geq 0$.
(B) *There is a constant L_0, which depends on M but not on ε, such that* $\|\Phi^\varepsilon\|_\infty \leq L_0$ *for all* $\varepsilon \geq 0$.
(C) *The derivative $\nabla\Phi^\varepsilon$ satisfies* $|A\hat{Q}\,\nabla\Phi^\varepsilon(p)| \leq 1$ *for all* $\varepsilon \geq 0$.
(D) Φ^ε *is a solution of* (17) *for all* $\varepsilon \geq 0$.
(E) *There is a constant L_1, which depends on M but not on ε, such that*

$$\|\Phi^\varepsilon - \Phi^0\|_\infty \leq L_1\, \varepsilon^{1/2}\,, \quad \varepsilon \geq 0\,. \tag{18}$$

The full proof of Theorem 2 is given in Luskin and Sell (1988), so we will not reproduce it here. The derivation of (18), on the other hand, is rather simple and we can explain the main idea easily. The proof uses inequality (2) (with $\alpha = 1/2$ and applied to $e^{-\varepsilon Bt}$) and the fact that $\hat{F}(\Phi^0) \in \mathscr{D}(B^{1/2})$ where $\hat{F}(\Phi^0)(p) = F(p + \Phi^0(p))$. In order to prove that there is an L_1, which is independent of ε, such that (18) is satisfied, we let $p_\varepsilon(t)$ denote the solution of the approximate inertial form

$$p' + APp = PF(p + \Phi^\varepsilon(p))$$

that satisfies $p_\varepsilon(0) = p$. Since $\Phi^\varepsilon = \mathcal{T}_\varepsilon \Phi^\varepsilon$ for all $\varepsilon \geqslant 0$ one has

$$\begin{aligned}
\Phi^\varepsilon(p) - \Phi^0(p) &= \\
&= \int_{-\infty}^{0} [e^{(A\hat{Q} + \varepsilon B)s} \hat{Q}\hat{F}(\Phi^\varepsilon(s)(p_\varepsilon(s))) - e^{A\hat{Q}s} \hat{Q}\hat{F}(\Phi^0)(p_0(s))]\, ds \\
&= \int_{-\infty}^{0} e^{(A\hat{Q} + \varepsilon B)s} \hat{Q}[\hat{F}(\Phi^\varepsilon)(p_\varepsilon(s)) - \hat{F}(\Phi^0)(p_0(s))]\, ds \\
&\quad + \int_{-\infty}^{0} e^{A\hat{Q}s} \hat{Q}[e^{\varepsilon Bs} \hat{F}(\Phi^0)(p_0(s)) - \hat{F}(\Phi^0)(p_0(s))]\, ds .
\end{aligned}$$

After applying A to this equation, one shows that the middle integral is bounded by

$$\frac{1}{2} \|\Phi^\varepsilon - \Phi^0\|_\infty ,$$

and the last integral is bounded by

$$\int_{-\infty}^{0} e^{\lambda_{N+1}s} \left\| e^{\varepsilon Bs} \hat{F}(\Phi^0) - \hat{F}(\Phi^0) \right\| ds .$$

By using (2) we then obtain

$$\frac{1}{2} \|\Phi_\varepsilon - \Phi_0\|_\infty \leqslant C_{10}\, \varepsilon^{1/2} \int_{-\infty}^{0} e^{\lambda_{N+1}s} |s|^{1/2}\, ds \left\| B^{1/2} \hat{F}(\Phi^0) \right\| ,$$

which implies (18).

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