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ASYMPTOTIC BEHAVIOUR OF STRONGLY DAMPED NONLINEAR HYPERBOLIC EQUATIONS

by Piotr BILER [1]

In this lecture we present some results on the exponential decay in time of solutions of the Cauchy problem and on their continuous dependence on parameters for nonlinear hyperbolic equations of the following type

$$u''(t) + \varepsilon Au'(t) + \alpha Au(t) + \beta A^2 u(t) + G(u(t)) = 0 . \qquad (1)$$

This equation is considered in a real Hilbert space $(H, |\,.\,|)$; $u : [0, \infty) \to H$ is a continuous function, $\alpha, \beta \geqslant 0$, for a moment $\varepsilon = 1$, A is a selfadjoint positive operator with its domain W compactly embedded in H. The least eigenvalue of A is $\zeta = \min \{(Av, v) : |v| = 1\}$ and we denote by $\|\,.\,\|$ the norm on the domain of $A^{1/2}$, $\|v\|^2 = (Av, v)$. The nonlinear term $G : W \to H$ is the Gateaux derivative of a convex functional $J : W \to [0, \infty)$, $J(0) = 0$: $(G(u), v) = \lim_{h \to 0} \frac{1}{h} (J(u + hv) - J(u))$. We suppose that G is locally Lipschitz on W: $\|G(u) - G(v)\| \leqslant C(\|u\|, \|v\|)\ |A(u - v)|$ with $C(0, 0) = 0$ in order to exclude linear terms in G, C continuous and $\int_0 C(s, 0)\, s^{-1}\, ds < \infty$. Moreover in Theorems 1, 2, G is assumed to satisfy the condition

$$2\, J(u) + (G(u), u) \leqslant 2\, \zeta^{-1} (G(u), Au) . \qquad (2)$$

A motivation of (1) goes from the mathematical physics where such equations arise as the models of internally damped vibrations of beams or plates and as strongly damped generalized nonlinear Klein-Gordon

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equations. In these applications A is an elliptic partial differential operator on a bounded domain Ω in $\mathbb{R}^N$ with homogeneous Dirichlet or Neumann conditions at the smooth boundary, say $A = -\Delta$, $W = H^2(\Omega) \cap H_0^1(\Omega) \Subset H = L^2(\Omega)$.

In the first class of our examples the nonlinearity is of nonlocal form $G(u) = m(\|u\|^2)\,Au$, where the increasing function m is locally Lipschitz, $m(0) = 0$, $m(s) \geqslant cs$ for some $c > 0$ and all $s \geqslant 0$. This corresponds to the functional

$$J(u) = \frac{1}{2}\int_0^{\|u\|^2} m(s)\,ds = \frac{1}{2} M\left(\int_\Omega |\nabla u|^2\right), \qquad M' = m, M(0) = 0\,,$$

typical in continuum mechanics. The condition (2) is verified as one has $\int_0^x m(s)\,ds \leqslant x m(x)$ and $(G(u), Au) \geqslant \zeta(G(u), u)$.

Another examples of nonlinearities are local ones with $G(u) = j'(u)$, $J(u) = \int_\Omega j(u)$, where j is a positive convex even function of class C^2, $j(0) = 0$, satisfying certain restrictions on growth and $2\,j(s) + sj'(s) \leqslant 2\,(k(s))^2$ with $k'(s) = (j''(s))^{1/2}$. This condition implies (2) since

$$\begin{aligned} 2\int j(u) + \frac{2}{\zeta}\int j'(u)\,\Delta u + \int uj'(u) &= \\ &= \int \left[2\,j(u) - \frac{2}{\zeta}\,j''(u)|\nabla u|^2 + uj'(u)\right] \\ &= \int \left[2\,j(u) - \frac{2}{\zeta}\,|\nabla(k(u))|^2 + uj'(u)\right] \\ &\leqslant \int \,[2\,j(u) - 2(k(u))^2 + uj'(u)]\,. \end{aligned}$$

In particular power like terms $G(u) = |u|^r u$ with $0 < r \leqslant 2$ for $N \leqslant 3$ and $0 < r \leqslant 2/(N-2)$ for $N > 3$ are admissible.

Due to the strong damping term Au' all the solutions of (1) converge to zero exponentially so the dynamics of (1) is very simple. However for certain values of parameters α, β one observes a phenomenon of very regular decay to zero which is fairly common for the solutions of equations and systems of parabolic type, *cf.* [6], [7], [3], but less expected for hyperbolic problems.

First we recall a slight generalization of the result in [4], *cf.* also [5], on the optimal decay rates.

THEOREM 1 : *Let* $u : [0, \infty) \to H$ *be any weak solution of* (1) *satisfying the energy equations*

$$\frac{d}{dt}|u'|^2 + 2\|u'\|^2 + \alpha \frac{d}{dt}\|u\|^2 + \beta \frac{d}{dt}|Au|^2 + 2(G(u), u') = 0 , \quad (3)$$

$$2\frac{d}{dt}(u', u) - 2|u'|^2 + \frac{d}{dt}\|u\|^2 + 2\alpha\|u\|^2 + 2\beta|Au|^2 + 2(G(u), u) = 0 , \quad (4)$$

$$2\frac{d}{dt}(u', Au) - 2\|u'\|^2 + \frac{d}{dt}|Au|^2 + 2\alpha|Au|^2 + 2\beta\|Au\|^2 + 2(G(u), Au) = 0 . \quad (5)$$

(They are obtained formally by taking the inner product of (1) *with* $2u'$, $2u$, $2Au$ *respectively and a justification of this procedure is done using the Galerkin approximations.)*

i) *If* $\beta > 1/4$ *or* $0 \leq \beta < 1/4$ *and* $\zeta < 2\alpha/(1 - 2\beta^{1/2})$ *or* $\beta = 1/4$ *and* $\alpha > 0$, *then* $\|u(t)\|^2 = 0(\exp(-\zeta t))$ *when* t *tends to* $+\infty$.

ii) *If* $0 < \beta < 1/4$ *and* $2\alpha/(1 - 2\beta^{1/2}) \leq \zeta \leq \alpha/(\beta^{1/2} - 2\beta)$, *then* $|u(t)|^2 = 0(\exp(-2\alpha t/(1 - 2\beta^{1/2})))$.

iii) *If* $0 < \beta < 1/4$ *and* $\zeta > \alpha/(\beta^{1/2} - 2\beta)$, *then* $|u(t)|^2 = 0(\exp(-\zeta t(1 - (1 - 4\beta - 4\alpha/\zeta)^{1/2})))$.

iv) *If* $\beta = 0$ *and* $\zeta \geq 2\alpha$, *then* $|u(t)|^2 = 0(\exp(-2\alpha t))$.

The optimal character of these estimates is immediately seen for the linear equation with $G \equiv 0$. It has the special solutions of the form $u(t) = \mathrm{Re}\,(\exp(-\sigma t))\,z$, where $Az = \xi z$, $\zeta \leq \xi$, with $\sigma = \sigma(\xi)$ satisfying $\sigma^2 - \xi\sigma + \alpha\xi + \beta\xi^2 = 0$. Their norms (squared) decay to zero like $\exp(-2t\,\mathrm{Re}\,\sigma)$. The minimal decay rate is therefore equal to $\inf\{2\,\mathrm{Re}\,\sigma(\xi) : \zeta \leq \xi \in \mathrm{sp}\,(A)\}$ and these are exactly the exponents in i)-iv). Note that $\mathrm{Re}\,\sigma(\xi)$ is not monotone with respect to ξ in certain cases so taking the inf is important : certain higher modes are less damped than the lower ones. In the proof of Theorem 1 ([4], [5]) different linear combinations of (3), (4), (5) are used. They lead to differential inequalities of Gronwall type and after integration to estimates like $|u' + Au - \rho u|^2 = 0(\exp(-2\rho t))$ with some $0 < \rho \leq \zeta/2$, which are of parabolic type, so they can be easily integrated once more to get the results above.

In some cases more interesting estimates from below of $|u|$ and $|u'|$ may be given — namely if $\mathrm{Re}\,\sigma(\xi)$ is a linear function of ξ. These cases are described in

THEOREM 2 :

i) *If* $0 < \beta < 1/4$, $\alpha = 0$, *then* $\|u'(t)\|$, $\|Au(t)\| = 0(\exp(-t\Lambda_\infty))$ *with* *some* $\Lambda_\infty = \frac{1}{2}(1 \pm (1 - 4\beta)^{1/2})\xi$ *where* $\xi \geqslant \zeta$ *is an eigenvalue of* A. *Moreover*

$$\exp(t\Lambda_\infty)\Big[u'(t) + \frac{1}{2}(1 + (1 - 4\beta)^{1/2})Au(t),$$

$$u'(t) + \frac{1}{2}(1 - (1 - 4\beta)^{1/2})Au(t)\Big]$$

tends to a vector in $H \times H$ *with exactly one nonzero component which is an eigenvector of* A *(if the initial data were not zero).*

ii) *If* $\beta > 1/4$ *and* $\alpha \geqslant 0$ *or* $\beta \geqslant 1/4$ *and* $\alpha > 0$, *then* $N(t) = \left\|u' + \frac{1}{2}Au\right\|^2 + (\beta - 1/4)\|Au\|^2 + \alpha|Au|^2$ *is equivalent to a multiple of* $\exp(-t\Lambda_\infty)$ *with an eigenvalue* Λ_∞ *of* A.

In the first case (1) reduces to a parabolic system in the new variables $v = u' + \mu u$, $w = u' + \nu u$, $\mu + \nu = 1$, $\mu\nu = \beta$, and the proof uses the results from [7].

In the second case such reduction is not possible, moreover one can show that none of the terms in $N(t)$ is equivalent to an exponential ; contrary, they oscillate. Here the result follows once the convergence of the quotient $N(t)/\left(\left|u' + \frac{1}{2}Au\right|^2 + (\beta - 1/4)|Au|^2 + \alpha\|u\|^2\right)$ to Λ_∞ when t tends to $+\infty$ is established.

Another questions concerning the equation (1) arise when one considers dependence of the solution on the parameters α, β, $\varepsilon \geqslant 0$. For instance Avrin proved in [2] that the solutions of the damped nonlinear Klein-Gordon equation tend to solutions of the conservative equation when ε tends to zero. This problem is interesting and the result not obvious since the velocity of propagation of disturbances is infinite for (1) with $\varepsilon > 0$ and finite for $\varepsilon = 0$.

We consider in [5] singular perturbation problem for (1) when the parameter β tends to zero. Here the mechanism of damping of the modes is quite different for $\beta > 0$ and $\beta = 0$. In the former case $\sigma(\xi)$ increases for large ξ and in the latter $\sigma(\xi)$ decreases to 2α (for $\xi \geqslant 4\alpha$). This gave rise to the question of the convergence of solutions when β tends to zero. Under a supplementary hypothesis on the nonlinear terms G (still satisfied by G's in the examples)

$$\begin{cases} (G(u), Au) \geqslant 0 \\ (G(u), A^2u) \geqslant -\alpha\|Au\|^2 - C(|Au|^2 + 1) \quad \textit{with a positive constant } C \end{cases} \tag{6}$$

our result reads :

THEOREM 3 : *Let $u = u(\beta)$ be the strong global solution of* (1) *with initial data $u(0) = u_0$, $u'(0) = u_1 \in W$. Then there exists a sequence $\beta_k \to 0$ and a global weak solution w of* (1) *with $\beta = 0$ such that $u(\beta_k) \to w$ in $C([0, T]; W)$ for all $T > 0$. If w is the unique solution of the limiting equation, then simply* $\lim_{\beta \to 0} u(\beta) = w$.

« Strong » means that each term in (1) is a continuous H-valued function of t. The existence and uniqueness of such solutions is proved in [1]. Our proof combines integral equation approach from [1], factorization of second order (in time) equations and higher order energy estimates similar to (3), (4), (5).

To finish we give a new result on regular exponential convergence to zero of solutions of the equation

$$u'' + \varepsilon A^p u' + Au + G(u) = 0 , \tag{7}$$

where $\varepsilon > 0$ and $p \geqslant 0$, with the same physical examples as before but with different damping terms $A^p u'$ (linear but in general of nonlocal character).

The idea of study fractional power damping operators was reminded me by Xavier Mora to whom I am much indebted.

We begin with the observation that for the linear equation (7) with $G \equiv 0$ the decay rates

$$\sigma_{1,2}(\xi) = \frac{1}{2} \varepsilon \xi^p (1 \pm (1 - 4 \varepsilon^{-2} \xi^{1-2p})^{1/2}) . \tag{8}$$

$\xi \in \mathrm{sp}\ (A)$, both strictly increase with ξ if and only if $1/2 < p < 1$ (here we consider $0 < \varepsilon \ll 1$ and $\xi \gg 1$ such that $\xi > (4 \varepsilon^{-2})^{1/(2p-1)}$). Therefore we may expect a similar result on the regular convergence to zero as in Theorem 2 in this situation only.

This is in fact true modulo a reasonable technical assumption on G below (which is similar to (2))

there is a constant C_G such that

$$2 J(u) + (G(u), u) - C_G(G(u), A^p u) \leqslant 0 . \tag{9}$$

THEOREM 4 : *Let $u(t)$ be a nonzero solution of* (7), $1/2 < p < 1$. *Then there exists $\Lambda_\infty \in \{\sigma_1(\xi) : \xi \in \mathrm{sp}\ (A)\} \cup \{\sigma_2(\xi) : \xi \in \mathrm{sp}\ (A)\}$ such that $\Lambda(t) = [(\sigma_2(A) v_1, v_1) + (\sigma_1(A) v_2, v_2)]/[|v_1|^2 + |v_2|^2]$ tends to Λ_∞ when t tends to $+\infty$. The numerator and the denominator of $\Lambda(t)$ are both equivalent to exponentials $C \exp(-2 t\Lambda_\infty)$. Here $v_k = u' + \sigma_k(A) u$, $k = 1,2$, and the operators $\sigma_k(A)$ are defined by the functions* (8) *via the standard functional calculus of selfadjoint operators.*

The ıdea of the proof ıs the followıng first one shows, usıng an energy equatıon for $|u' + \varepsilon A^p u - \rho u|^2 + \|u\|^2 - \varepsilon\rho(A^p u, u) + \rho^2|u|^2 + 2J(u)$, an estımate of the exponentıal decay of $\|u(t)\|^2$ wıth the rate 2ρ of order $1/C_G$ (certaınly not optımal) Then the equatıon (7) ıs factored to the system $v_1' + \sigma_2(A)v_1 = -G(u)$, $v_2' + \sigma_1(A)v_2 = -G(u)$ and a versıon of the result ın [7] ımplıes the desıred conclusıon

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