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NUMERICAL APPROXIMATION OF THE PREISACH MODEL FOR HYSTERESIS (*)

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Abstract — *The classical Preisach model for hysteresis is first recalled Under suitable assumptions, it corresponds to a continuous hysteresis operator* $\mathbf{F}_\mu$ *from* $C^0([0,T])$ *into itself This model has a natural geometrical interpretation, which allows to construct a simple approximation procedure for* $\mathbf{F}_\mu$

An initial and boundary value problem for the equation

$$\frac{\partial}{\partial t}[u + \mathbf{F}_\mu(u)] - \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = f \quad in\, \Omega \times\,]0, T[$$

(with Ω *domain of* $\mathbf{R}^d$, $d \geqslant 1$*) is then considered The existence of a weak solution is proven by means of an approximation procedure, based on the simultaneous discretization of both the operators* $\dfrac{\partial}{\partial t}$ *and* $\mathbf{F}_\mu$

Finite element space approximations and two different time discretizations, one by backward differences and another by a linearization technique, are then introduced The stability of all these schemes is then proved and their implementation is discussed

Resume — *D'abord on va rappeler le modele classique de Preisach pour l'hysteresis En faisant des hypotheses convenables, cela donne lieu a une application d'hysteresis* $\mathbf{F}_\mu$ *continue de* $C^0([0,T])$ *dans le même espace Ce modele a une interpretation geometrique naturelle, qui permet de construire un simple schema d'approximation pour* $\mathbf{F}_\mu$

On considere alors un probleme aux valeurs initiales et aux limites pour l'equation

$$\frac{\partial}{\partial t}[u + \mathbf{F}_\mu(u)] - \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = f \quad dans \quad \Omega \times\,]0, T\, [$$

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où Ω est un domaine de $\mathbf{R}^d$, $d \geq 1$. L'existence d'une solution faible est démontrée en utilisant une méthode d'approximation qui est basée sur la discrétisation simultanée des opérateurs $\frac{\partial}{\partial t}$ et $\mathbf{F}_\mu$.

On introduit alors des approximations aux éléments finis et deux discrétisations en temps, l'une avec des différences régressives et l'autre par une technique de linéarisation On démontre la stabilité des schémas et on en discute l'implémentation

1. INTRODUCTION

Hysteresis arises when a time-dependent variable $w(t)$ is not uniquely determined by the value of another variable $u(t)$ at the same instant $t \in [0, T]$, but instead $w(t)$ depends on the evolution of u in the whole time interval $[0, t]$. This setting leads to the introduction of a *causal* operator $\mathbf{F}$, which allows to represent the previous memory effect in the form

$$w(t) = [\mathbf{F}(u)](t) . \tag{1.1}$$

In order to exclude different memory effects like viscosity, we shall require that $\mathbf{F}$ be *rate-independent*, i.e., that in (1.1) $w(t)$ depends just on the range of u in $[0, t]$ and on the order in which these values are assumed, not on its velocity. We shall name *hysteresis operator* any such causal and rate-independent operator.

The mathematical aspects of hysteresis have been the object of an accurate research of Krasnosel'skii, Pokrowskii and of other Soviet mathematicians (see [4]). The main results obtained by the second author of the present paper were reviewed in [11].

The models of hysteresis operators are not many, and among them that ascribed to the physicist Preisach [7] and dating back to 1933 seems to be the most useful. The basic idea of this model consists in representing a hysteretic material (a ferromagnetic body, e.g.) by means of a mixture of elements characterized by especially simple hysteresis loops of the form outlined in figure 1.

These hysteresis elements are assumed as non-interacting among them ; accordingly this construction is known as the *independent domain model.* If μ_ρ is the measure of the density of the element f_ρ corresponding to the couple of thresholds $\rho := (\rho_1, \rho_2)$, then the overall dependence is given by the *Preisach operator*

$$\mathbf{F}_\mu(u) := \int_{\rho_1 < \rho_2} f_\rho(u)\, d\mu_\rho \tag{1.2}$$

and corresponds to a fairly general hysteresis loop (see *fig.* 2).

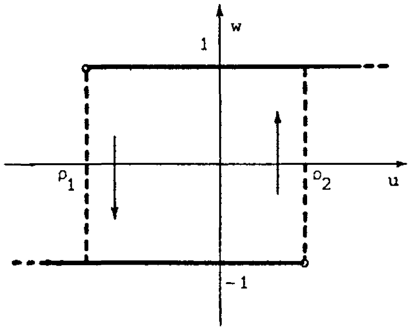

Figure 1. — **The thresholds ρ_1 and ρ_2 are any couple of real numbers such that $\rho_1 < \rho_2$. At $u = \rho_1$ ($u = \rho_2$, respectively) w jumps from 1 to -1 (from -1 to 1, respectively), but not conversely. The (discontinuous) *elementary hysteresis operator* $u \to w$ corresponding to the couple $\rho := (\rho_1, \rho_2)$ will be denoted by f_ρ.**

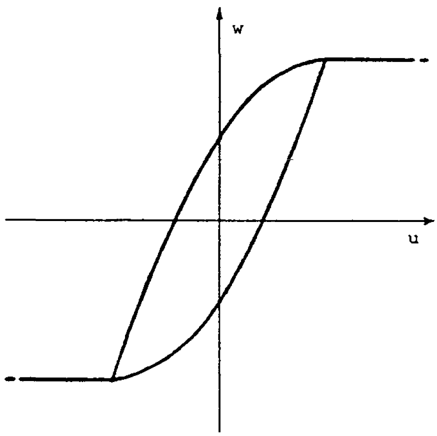

Figure 2. — **By averaging a continuum of elementary hysteresis loops as in figure 1, a fairly general hysteresis behaviour is obtained. If the density μ of the hysteresis elements has no concentrated masses, then the overall cycle is continuous.**

In this paper in section 2 we give a precise definition of the Preisach operator F_μ and review its main mathematical properties ; for more details we refer to [1, 4, 10]. We shall also outline the main geometrical properties of this model ; these will be used in section 3 for the study of convenient *numerical* approximations of F_μ, following a scheme proposed in [3].

Then in section 4 we deal with the boundary and initial value problem governed by the archetypical partial differential equation with hysteresis

$$\frac{\partial}{\partial t}[u + \mathbf{F}_\mu(u)] - \Delta u = f \quad \left(\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}\right) \tag{1 3}$$

in a bounded domain Ω of $\mathbf{R}^d$ $(d \geq 1)$, we shall report the existence result proven in [10], and outline a proof based on one of the approximations of the operator $\mathbf{F}_\mu$ introduced in section 3

In section 5 we introduce two finite element approximations of the equation (1 3) Here for the time discretization both backward differences and a linearization technique are proposed The latter is in the spirit of the discrete time relaxation scheme introduced for the Stefan problem in [9], and of the nonlinear Chernoff formula suggested by semigroup theory, cf [5] We also show the stability of these schemes and discuss their numerical implementation

Finally, in the appendix we present a convenient construction of the approximated hysteresis operator $\mathbf{F}_\mu^\delta$

2. THE PREISACH MODEL

2.1. The Preisach operator

For any couple $\rho = (\rho_1, \rho_2) \in \mathbf{R}^2$, with $\rho_1 < \rho_2$, we introduce the *elementary hysteresis operator* (or *relay operator*) $f_\rho . C^0([0, T]) \times \{-1, 1\} \to BV(0, T)$ (Banach space of functions $[0, T] \to \mathbf{R}$ with bounded total variation) For any $u \in C^0([0, T])$ and any $\xi = -1$ or 1, the function $z = f_\rho(u, \xi) : [0, T] \to \{-1, 1\}$ is defined as follows

$$z(0) = \begin{cases} -1 & \text{if } u(0) \leq \rho_1 \\ \xi & \text{if } \rho_1 < u(0) < \rho_2 \\ 1 & \text{if } u(0) \geq \rho_2 , \end{cases} \tag{2 1}$$

$\forall t \in \,]0, T]$, setting $X_t = \{\tau \in \,]0, t] . u(\tau) = \rho_1 \text{ or } \rho_2\}$,

$$\begin{cases} \text{if } X_t = \varnothing , & \text{then } z(t) = z(0), \\ \text{if } X_t \neq \varnothing & \text{and } u(\max X_t) = \rho_1, \quad \text{then } z(t) = -1, \\ \text{if } X_t \neq \varnothing & \text{and } u(\max X_t) = \rho_2, \quad \text{then } z(t) = 1 \end{cases} \tag{2 2}$$

The operator f_ρ actually acts into $BV(0, T)$, indeed after a possible jump from -1 to 1 (or conversely), the function z has another jump only after u has gone from ρ_2 to ρ_1 (or conversely), and the number of oscillations of u between ρ_1 and ρ_2 is necessarely finite, as u is uniformly continuous in $[0, T]$

The properties of the operators f_ρ's have been studied in [1, 4, 10] ; here we recall just the most important ones :

Causality.

$$\left\{\begin{array}{l} \forall \bar{t} \in \,]0, T], \quad \forall (u_i, \xi) \in C^0([0,T]) \times \{-1, 1\} \quad (i = 1, 2), \\ \text{if } u_1 = u_2 \text{ in } [0, \bar{t}\,], \quad \text{then } [f_\rho(u_1, \xi)](\bar{t}\,) = [f_\rho(u_2, \xi)](\bar{t}\,), \end{array}\right. \tag{2.3}$$

Rate independence.

$$\left\{\begin{array}{l} \forall s : [0, T] \to [0, T] \quad \text{monotone homeomorphism,} \\ \forall (u, \xi) \in C^0([0,T]) \times \{-1, 1\}, \quad \forall t \in [0, T], \\ [f_\rho(u, \xi)](t) = [f_\rho(u \circ s^{-1}, \xi)](s(t)) ; \end{array}\right. \tag{2.4}$$

The two previous properties mean that f_ρ is a *hysteresis operator.*

Piecewise monotonicity.

$$\left\{\begin{array}{l} \forall (u, \xi) \in C^0([0,T]) \times \{-1, 1\}, \quad \forall t_1, t_2 \in [0, T] (t_1 < t_2), \\ \text{if } u \text{ is either non-decreasing or non-increasing in } [t_1, t_2], \\ \text{then so is also } f_\rho(u, \xi) \text{ in the same interval}. \end{array}\right. \tag{2.5}$$

We shall denote by $\boldsymbol{P}$ the half-plane of the admissible thresholds ρ_1, ρ_2 ; i.e., $\mathbf{P} := \{(\rho_1, \rho_2) \in \mathbf{R}^2 : \rho_1 < \rho_2\}$. Let μ be a finite measure over $\mathbf{P}$. We shall denote by $\mathbf{S}$ the family of μ-measurable functions $\mathbf{P} \to \{-1, 1\}$. We can now introduce the so-called *Preisach operator* associated to μ :

$$\forall (u, \xi) \in C^0([0,T]) \times \mathbf{S}, \quad \forall t \in [0, T],$$

$$[\mathbf{F}_\mu(u, \xi)](t) := \int_{\mathbf{P}} [f_\rho(u, \xi_\rho)](t)\, d\mu_\rho . \tag{2.6}$$

It is easy to see that also $\mathbf{F}_\mu$ fulfils the properties (2.3) and (2.4), namely it is a *hysteresis operator ;* moreover if $\mu \geqslant 0$ then $\mathbf{F}_\mu$ fulfils also (2.5).

We then recall an important result.

PROPOSITION 1 [1, 10] : *If*

$$\left\{\begin{array}{l} \mu \textit{ has no masses either concentrated in points} \\ \textit{or along any segment parallel to the axes } \rho_1 \textit{ or } \rho_2 , \end{array}\right. \tag{2.7}$$

then

$$\forall (u, \xi) \in C^0([0,T]) \times \mathbf{S}, \quad \mathbf{F}_\mu(u, \xi) \in C^0([0,T]), \tag{2.8}$$

$$\left\{\begin{array}{l} \forall \xi \in \mathbf{S}, \textit{ the operator } C^0([0,T]) \to C^0([0,T]) : u \to \mathbf{F}_\mu(u, \xi) \\ \textit{is continuous with respect to the uniform topology}. \end{array}\right. \tag{2.9}$$

Conversely (2.8) *entails* (2.7).

2.2. Geometrical properties of the Preisach model

Let us fix any $(u, \xi) \in C^0([0, T]) \times \mathbf{S}$ and set $z_\rho := f_\rho(u, \xi_\rho)$ μ-a.e. in $\mathbf{P}$. For any $t \in [0, T]$,

$$\begin{cases} \text{if} \quad \rho_1 \geq u(t), \quad \text{then } z_\rho(t) = -1, \\ \text{if} \quad \rho_2 \leq u(t), \quad \text{then } z_\rho(t) = 1, \\ \text{if} \quad \rho_1 < u(t) < \rho_2, \quad \text{then } z_\rho(t) \text{ depends on } \xi_\rho \text{ and on } u_{\mid [0,t]}\,. \end{cases} \tag{2.10}$$

(see *fig.* 3).

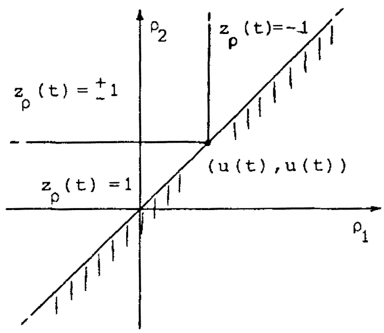

Figure 3.

Hence as u increases in time, the boundary of $A^+(t) := \{\rho \in \mathbf{P} : z_\rho(t) = 1\}$ moves up ; as u decreases the boundary of $A^-(t) := \{\rho \in \mathbf{P} : z_\rho(t) = -1\}$ moves to the left (see *fig.* 4, 5).

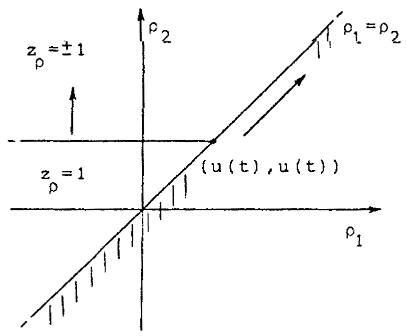

Figure 4. — As *u* increases in *t*, $A^+(t)$ moves up.

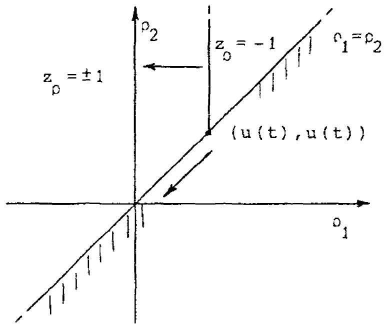

Figure 5. — As *u* decreases in *t*, $A^-(t)$ moves to the left.

Moreover for any ρ, $\rho' \in \mathbf{P}$,

$$\begin{cases} \text{if } \rho \in A^+(t) \text{ and } \rho' \leqslant \rho \quad (\text{i.e., } \rho_1' \leqslant \rho_1, \rho_2' \leqslant \rho_2), \\ \qquad \text{then } \rho' \in A^+(t), \\ \text{if } \rho \in A^-(t) \text{ and } \rho \leqslant \rho', \quad \text{then } \rho' \in A^-(t). \end{cases} \tag{2.11}$$

Hence the boundary $B(t) := \partial A^+(t) \cap \partial A^-(t)$ is a *maximal antimonotone graph,* which intersects the straight line $\rho_1 = \rho_2$ in the point $(u(t), u(t))$ (see *fig.* 6).

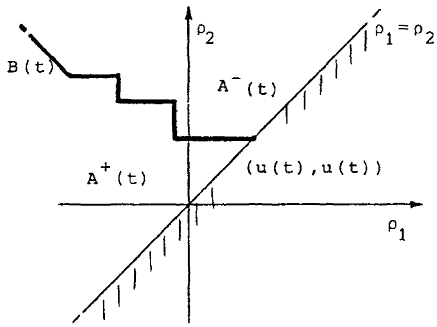

Figure 6.

At any t, $B(t)$ determines the function $\rho \to z_\rho(t)$, hence also $[\mathbf{F}_\mu(u, \xi)](t)$. Thus $B(t)$ represents the state of the system at time t. As u evolves in time, by the construction outlined above new segments parallel to the axes ρ_1 and ρ_2 appear in $B(t)$; hence $B(t)$ contains any segment non-parallel to the axes only if it was already present in $B(0)$. In particular for a *virgin* ferromagnetic material, namely for a system which never experienced any magnetization process, $B(0) := \{\rho \in \mathbf{P} : \rho_1 + \rho_2 = 0\}$.

3. APPROXIMATION OF THE PREISACH OPERATOR

In this section we introduce an approximation of the Preisach operator $\mathbf{F}_\mu$ which will also be convenient for the numerical approximation of the problem at the partial derivatives of the next section.

First, we approximate the measure μ by following a procedure already proposed in [3]. We suppose that the finite measure μ over $\mathbf{P}$ has *compact support,* i.e.

$$\operatorname{supp} \mu \subset S := \{(\rho_1, \rho_2) \in \mathbf{P} : -L \leqslant \rho_1, \rho_2 \leqslant L\} \qquad (L : \text{constant} > 0)\,, \quad (3.1)$$

and that μ is *symmetric* with respect to the straight line $\rho_1 + \rho_2 = 0$. These two conditions mean, respectively, that the maximal hysteresis loop in the (u, w)-plane is bounded and symmetric with respect to the origin. Thus, by setting

$$H := \frac{1}{2}\,\mu(\operatorname{supp} \mu)\,, \qquad (3.2)$$

the maximal hysteresis loop is contained in the set $\{(u, w) \in \mathbf{R}^2 : -L \leqslant u \leqslant L,\ -H \leqslant w \leqslant H\}$; moreover $w(u) = H$ or $-H$ if $u \geqslant L$ or $u \leqslant -L$, respectively (see *fig.* 2 and *fig.* 9).

For any $u \in C^0([0, T])$ and any $t > 0$, if $A^+(t)$ and $A^-(t)$ are defined as in section 2, we have

$$w(t) := [\mathbf{F}_\mu(u, \xi)](t) = -H + \mu(A^+(t)) = H - \mu(A^-(t))\,. \quad (3.3)$$

Due to the piecewise monotonicity property (2.5), as the *control* u increases or decreases in time from $u(t)$, the *exit* w moves from $w(t)$ along a monotone curve in the (u, w)-plane. This curve corresponds to the following function

$$F_t(u(t) + \delta u) := w(t) + \begin{cases} + \mu(A^-(t) \cap \{(\rho_1, \rho_2) \in \mathbf{P} : \rho_2 \leqslant u(t) + \delta u\}) & \text{if } \delta u \geqslant 0 \\ - \mu(A^+(t) \cap \{(\rho_1, \rho_2) \in \mathbf{P} : \rho_1 \geqslant u(t) + \delta u\}) & \text{if } \delta u < 0\,. \end{cases} \quad (3.4)$$

Assuming that μ is *absolutely continuous* with respect to the two-dimensional Lebesgue measure λ, from the Radon-Nikodym theorem we get that

$$\exists r \in L^1(S), \quad r \geqslant 0, \quad \text{such that} \quad \mu(A) = \int_A r(\rho)\, d\lambda\,, \qquad (3.5)$$

for any measurable set $A \subset \mathbf{P}$.

Let $M \geqslant 1$ be any integer and set $\delta := L/M$; let us consider the uniform decomposition $\{B_{ij}\}$ of the triangle S shown in figure 7, where the B_{ij}'s are defined as follows

$$B_{ij} := \{(\rho_1, \rho_2) \in S : (j-1)\delta \leqslant \rho_1 + L \leqslant j\delta,\ (i-1)\delta \leqslant L - \rho_2 \leqslant i\delta\},$$
$$i, j \in \{i, j = 1, \dots, 2M : i + j \leqslant 2M + 1\} . \quad (3.6)$$

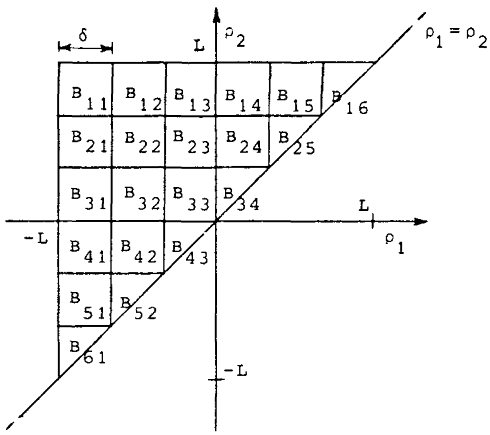

Figure 7. — Decomposition of the support of μ.

We approximate r by means of the piecewise constant functions r^δ, defined by

$$r^\delta(\rho) := r_{ij} \quad \text{if} \quad \rho \in B_{ij} , \tag{3.7}$$

where

$$r_{ij} := \frac{1}{\lambda(B_{ij})} \int_{B_{ij}} r(\rho)\, d\lambda = \frac{\mu(B_{ij})}{\lambda(B_{ij})} \tag{3.8}$$

(note that if $r \in C^0(\bar{S})$, then we can take $r_{ij} := r(y_{ij})$, where y_{ij} is the barycenter of B_{ij}). Consequently μ is approximated by the measures μ^δ defined by

$$\mu^\delta(A) := \int_A r^\delta(\rho)\, d\lambda, \quad \text{for any measurable set } A \subset \mathbf{P} , \tag{3.9}$$

and the Preisach operator $\mathbf{F}_\mu$ is approximated by the operators $\mathbf{F}_\mu^\delta$, defined as in (2.6) with μ replaced by μ^δ.

Notice that in the time and space finite element approximations of the equation (1.3), we shall represent the state of the system at each node of the mesh by means of an antimonotone graph B in the Preisach plane. Thus, in view of the developments of section 5, it is crucial to store and update B in an efficient way. This topic will be discussed in the appendix. In particular, we shall approximate B with a graph B^{δ} lying on the reticulation of $\mathbf{P}$ (see *fig.* 8).

Now we consider the monotone curve F^{δ} in the (u, w)-plane defined in (3.4), which describes the evolution of the exit w as the control u increases or decreases monotonically in time. Notice that, as for the graph B, in section 5 we shall need a curve F^{δ} for any node of the mesh. We approximate this curve by means of a piecewise linear curve $\bar{F}^{\delta}$ which interpolates F^{δ} at the points in which the control u is equal to $j\delta$, $j = -M, \ldots, 0, \ldots, M$. The first and the last segments of F^{δ} are horizontal and correspond to the values $w = -H$ and H, respectively (see *fig.* 9). Thus, all the informations concerning $\bar{F}^{\delta}$ can be stored in a real Fortran vector of dimension $2M - 1$, as we shall see in the appendix.

4. PARTIAL DIFFERENTIAL EQUATIONS WITH HYSTERESIS TERMS

Let Ω be a bounded domain of $\mathbf{R}^d$ $(d \geqslant 1)$. Let

$$u_0 \in L^2(\Omega)\,; \quad w_0 \in L^2(\Omega\,; \mathbf{S})\,; \quad f \in L^2(Q)\,. \tag{4.1}$$

We consider the following problem :

PROBLEM (**P**) : *To find* $u \in L^2(0, T\,; H^1(\Omega))$ *such that*

$$u(x, .) \in C^0([0, T]), \quad u(x, 0) = u_0(x) \quad \text{a.e. } in\ \Omega\,, \tag{4.2}$$

and, setting

$$w(x, t) = [\mathbf{F}_{\mu}(u(x, .), w_0(x))](t) \quad \forall t \in [0, T], \quad \text{a.e. } in\ \Omega\,, \tag{4.3}$$

then

$$\frac{\partial}{\partial t}\int_{\Omega}(u + w)\,v\,dx + \int_{\Omega}\nabla u\,.\,\nabla v\,dx =$$

$$= \int_{\Omega} fv\,dx \quad \forall v \in H^1(\Omega), \quad \text{a.e. } in\ ]0, T[\,. \tag{4.4}$$

Remark 1 :

(i) (4.4) corresponds to the equation

$$\frac{\partial}{\partial t}(u + w) - \Delta u = f \quad \text{in } \mathcal{D}'(Q) \tag{4.5}$$

and to the boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times\ ]0, T[\,, \tag{4.6}$$

where $\partial/\partial\nu$ denotes the exterior normal derivative.

(ii) By (4.2) and (4.3), the initial condition for equation (4.4) is

$$[u + w](x, 0) = u_0(x) + \int_{\mathbf{P}} [f_\rho(u(x, .), w_{0,\rho}(x))](0)\, d\mu_\rho \quad \text{a.e. in } \Omega\,. \tag{4.7}$$

THEOREM 1 [10] : *Assume that* (2.7) *holds,* $\mu \geq 0$ *and*

$$u_0 \in H^1(\Omega)\,. \tag{4.8}$$

Then problem (**P**) *has at least one solution such that*

$$u \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))\,. \tag{4.9}$$

Outline of the proof: In order to take advantage of the property (2.5) of *piecewise monotonicity,* it seems necessary to use some time discretization technique, at least for the hysteresis operator $\mathbf{F}_\mu$. In the scheme we shall outline, we shall also discretize the partial differential equation in time.

(i) *Approximation by time discretization.* We consider the sequence of hysteresis operators $\mathbf{F}_\mu^\delta$ obtained from $\mathbf{F}_\mu$ by replacing the measure μ with μ^δ ($\delta := L/M$), as described in section 3. We recall that, as $\delta \to 0$, we have

$$\forall (u, \xi) \in C^0([0, T]) \times \mathbf{S}\,, \quad \mathbf{F}_\mu^\delta(u, \xi) \to \mathbf{F}_\mu(u, \xi) \quad \text{uniformly in } [0, T]\,. \tag{4.10}$$

Now we fix any integer number K, set $\tau := T/K$ and introduce the following time-discretized problem :

PROBLEM ($\mathbf{P}_{\delta,\tau}$): *To find* $U^k \in H^1(\Omega)$, $k = 1, \dots, K$, *such that, setting*

$$u^{\delta,\tau}(x, t) := \textit{linear time-interpolate of } \{U^k(x)\}_{k=0,\dots,K} \tag{4.11}$$
$$(\textit{with } U^0 := u_0)\,, \quad \text{a.e. } \textit{in } \Omega\,,$$

$$W^k(x) := [\mathbf{F}_\mu^\delta(u^{\delta,\tau}(x, .), w_0(x))](k\tau) \quad \text{a.e. } \textit{in } \Omega\,, \tag{4.12}$$

then

$$\frac{1}{\tau}\int_\Omega (U^k - U^{k-1} + W^k - W^{k-1})\, v\, dx + \int_\Omega \nabla U^k \,.\, \nabla v\, dx = \int_\Omega \bar{f}^k\, v\, dx \tag{4.13}$$

$$\forall v \in H^1(\Omega)\,, \quad \textit{where} \quad \bar{f}^k(x) := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} f(x, t)\, dt \quad \text{a.e. } \textit{in } \Omega\,.$$

For any K, problem $(\mathbf{P}_{\delta,\tau})$ has a unique solution and can be solved step by step. Indeed, let us fix any $k = 1, \ldots, K$, and assume that $U^0, \ldots, U^{k-1}$ are known a.e. in Ω. Then by the causality of the operator $\mathbf{F}^{\delta}_{\mu}$, for almost any $x \in \Omega$, $W^k(x)$ depends just on $U^k(x)$ and on x ; namely there exists a function $F^k : \mathbf{R} \times \Omega \to \mathbf{R}$ such that

$$W^k(x) = F^k(U^k(x), x) \quad \text{a.e. in } \Omega . \tag{4.14}$$

Moreover, a.e. in Ω, the (unknown) function $u^{\delta,\tau}(x, .)$ is linear in $[(k-1)\tau, k\tau]$; hence it is either non-increasing or non-decreasing ; then, by the *piecewise monotonicity* (in the sense of (2.5)) of $\mathbf{F}^{\delta}_{\mu}$, the function $F^k(., x)$ is non-decreasing, a.e. in Ω. This allows to solve (4.13) by means of a standard procedure.

(ii) *A priori estimates.* Let us take $v = U^k - U^{k-1}$ in (4.13) and sum for $k = 1, \ldots, n$, for any fixed $n = 1, \ldots, K$. Noting that $W^{k-1}(x) = F^{k-1}(U^{k-1}(x), x) = F^k(U^{k-1}(x), x)$ a.e. in Ω, by (4.14) and by the monotonicity of F^k with respect to its first argument, we have

$$\sum_{k=1}^{n} \tau \int_{\Omega} \left(\frac{U^k - U^{k-1}}{\tau} \right)^2 dx + \frac{1}{2} \int_{\Omega} (|\nabla U^k|^2 - |\nabla U^0|^2)\, dx \leqslant$$
$$\leqslant \left[\sum_{k=1}^{n} \tau \int_{\Omega} \bar{f}^k(x)^2 \, dx \right]^{1/2} \left[\sum_{k=1}^{n} \tau \int_{\Omega} \left(\frac{U^k - U^{k-1}}{\tau} \right)^2 dx \right]^{1/2},$$

whence we get

$$\sum_{k=1}^{K} \tau \int_{\Omega} \left(\frac{U^k - U^{k-1}}{\tau} \right)^2 dx + \max_{1 \leqslant k \leqslant K} \int_{\Omega} |\nabla U^k|^2 \, dx \leqslant \text{Constant}, \tag{4.15}$$

independent of K and M ,

that is

$$\|u^{\delta,\tau}\|_{H^1(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega))} \leqslant$$
$$\leqslant \text{Constant independent of } K \text{ and } M . \tag{4.16}$$

Moreover as the measures μ^{δ}'s are uniformly bounded, we have

$$\|w^{\delta,\tau}\|_{L^\infty(Q)} \leqslant \text{Constant independent of } K \text{ and } M , \tag{4.17}$$

where

$$w^{\delta,\tau}(x, t) := \quad \text{linear time-interpolate of } \{W^k(x)\}_{k=0, \ldots, K} \tag{4.18}$$

(with $W^0 := w_0$) , a.e. in Ω .

(iii) *Limit procedure.* By (4.16) and (4.17) there exist u and w such that, possibly taking subsequences, as $(\delta, \tau) \to (0,0)$

$$u^{\delta,\tau} \to u \quad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \tag{4.19}$$

$$w^{\delta,\tau} \to w \quad \text{weakly star in } L^\infty(Q). \tag{4.20}$$

Then taking K and $M \to \infty$ in (4.13) we get (4.4). As the injection of $H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$ into $L^2(\Omega; C^0([0,T]))$ is compact, possibly extracting a further subsequence, as $(\delta, \tau) \to (0,0)$, we have

$$u^{\delta,\tau}(x,.) \to u(x,.) \quad \text{uniformly in } [0,T], \quad \text{a.e. in } \Omega; \tag{4.21}$$

then by the continuity of $\mathbf{F}_\mu^\delta$ we get

$$\mathbf{F}_\mu^\delta(u^{\delta,\tau}(x,.), w_0(x)) \to \mathbf{F}_\mu(u(x,.), w_0(x)) \quad \text{uniformly in } [0,T], \text{ a.e. in } \Omega; \tag{4.22}$$

hence by (4.20) and (4.22) we get (4.3)//.

Remark 2 :

(i) The previous proof can be used for any approximating sequence of hysteresis operators $\mathbf{F}^n$ fulfilling (4.10).

(ii) The uniqueness of the solution of problem (**P**) has been recently proven by M. Hilpert.

5. APPROXIMATION OF THE PARTIAL DIFFERENTIAL EQUATION

In this section we present two finite element approximations of the partial differential equation with hysteresis considered in section 4 and discuss their implementation.

Let us introduce some standard notations.

Let K be an integer ≥ 1, and $\tau := T/K$ be the time step; for any $k = 1, \ldots, K$, we set $\bar{z}^k := \dfrac{1}{\tau}\displaystyle\int_{(k-1)\tau}^{k\tau} z(t)\, dt$ for any integrable function on $[0,T]$, and $\partial z^k := [z^k - z^{k-1}]/\tau$ for any family $\{z^k\}_{k=0}^K$.

For any $h > 0$, let $\{\mathbf{S}_h\}_h$ be a regular family of decompositions of Ω into closed d-simplices [2]. For simplicity, we suppose that $\bar{\Omega} = \Omega_h := \bigcup_{S \in \mathbf{S}_h} S$. We define the finite element spaces we shall use

$$V_h := \left\{ \phi \in C^0(\bar{\Omega}) : \phi_{|S} \text{ is linear } \quad \forall S \in \mathbf{S}_h \right\}. \tag{5.1}$$

Denoting by $\{\phi_i\}_{i=1}^I$ the canonical base of V_h, and by $I_h : C^0(\bar{\Omega}) \to V_h$ the linear interpolant operator, we introduce the following matrices :

$$
\begin{aligned}
M &:= \left\{ \int_\Omega I_h(\phi_i\, \phi_j)\, dx \right\}_{i,j=1}^{I} =: \{m_{ij}\} \quad \text{diagonalized mass matrix} \\
K &:= \left\{ \int_\Omega \nabla\phi_i \,.\, \nabla\phi_j\, dx \right\}_{i,j,=1}^{I} = \{k_{ij}\} \quad \text{stiffness matrix}\,.
\end{aligned} \tag{5.2}
$$

Finally, for $V \in V_h$, we denote by $\mathbf{V}$ the vector of the nodal values $V_i := V(x_i)$ of V.

When a time discretization of the P.D.E. (4.13) is introduced, step by step we have to solve K elliptic problems in the unknowns U^k and W^k, which approximate $u(k\tau)$ and $w(k\tau)$ respectively. Let us fix any $k = 1, \ldots, K$ and assume that $U^0, \ldots, U^{k-1}$ are known. Then by the causality of the operator $\mathbf{F}^\delta_\mu$, $W^k(x)$ depends just on $U^k(x)$ and on x, a.e. in Ω. Namely, there exists a function $F^k = F^{k,\delta} : \mathbf{R} \times \Omega \to \mathbf{R}$ such that

$$
W^k(x) = F^k(U^k(x), x) \quad \text{a.e. in } \Omega\,; \tag{5.3}
$$

moreover, a.e. in Ω, since the linear interpolate of $U^{k-1}(x)$ and $U^k(x)$ is either non-decreasing or non-increasing, the function $F^k(.,x)$ is non-decreasing, by the piecewise monotonicity of $\mathbf{F}^\delta_\mu$ (see (2.5)). Finally we can approximate $F^k(.,x)$ with a piecewise linear function $\bar{F}^k(.,x)$ (see Sect. 3).

5.1. The backward difference method

First we introduce our method based on backward differences in time, already mentioned in the proof of Theorem 1.

PROBLEM $(\mathbf{P}_{\delta,\tau,h})_1$: *Given U^0 and $W^0 \in V_h$ (suitably obtained from u_0 and w_0), for any $k = 1, \ldots, K$ find U^k, $W^k \in V_h$ such that*

$$
W^k = I_h \bar{F}^k(U^k(x), x) \tag{5.4}
$$

$$
\int_\Omega I_h([\partial U^k + \partial W^k]\, \phi) + \int_\Omega \nabla U^k \,.\, \nabla \phi = \int_\Omega \bar{f}^k\, \phi \quad \forall \phi \in V_h\,. \tag{5.5}
$$

Remark 3 : At each node x_i of the mesh, the discrete initial data U^0_i and W^0_i correspond to an antimonotone curve B^δ_i in the Preisach plane (see Sect. 3.2).

Denoting the function $\bar{F}^k$ in the node x_i by

$$
F^k_i(.) := \bar{F}^k(., x_i)\,, \tag{5.6}
$$

from (5.4) we have

$$W_i^k = F_i^k(U_i^k)\,; \tag{5.7}$$

hence the discrete problem can be written in matrix form as follows :

$$\mathbf{M}[\mathbf{U}^k + \mathbf{W}^k] + \tau\mathbf{K}\mathbf{U}^k = \mathbf{M}[\mathbf{U}^{k-1} + \mathbf{W}^{k-1}] + \tau\mathbf{f}^k := \mathbf{Q}^k \tag{5.8}$$

where $f_i^k = \int_\Omega \bar{f}^k \phi_i$.

It is easily seen that (5.8) is a system of nonlinear algebraic equations associated to a continuous and uniformly monotone operator ; thus it has one and only one solution [6, p. 167].

The stability of this scheme can be proved similarly to the stability of the time discretized problem $(\mathbf{P}_{\delta,\tau})$ set in section 4 (see (4.15)).

The solution $\mathbf{U}^k$ can be computed by using the following nonlinear S.O.R. method :

Let $0 < \omega < 2$ be fixed ; by choosing $\boldsymbol{\xi}^{k,0} := \mathbf{U}^{k-1}$, compute $\boldsymbol{\xi}^{k,n}$ for $n = 1, 2, \dots$ by setting $\xi_i^{k,n} := \omega\alpha + (1-\omega)\,\xi_i^{k,n-1}$, for any $i = 1, \dots, I$, where α is the solution of the following algebraic equation

$$m_{ii}[\alpha + F_i^k(\alpha)] + \tau k_{ii}\,\alpha = Q_i^k - \sum_{j=1}^{i-1} k_{ij}\,\xi_j^{k,n} - \sum_{j=i+1}^{I} k_{ij}\,\xi_j^{k,n-1} =: p_i^{k,n}\,, \tag{5.9}$$

that is

$$\alpha = ([m_{ii} + \tau k_{ii}]\,I + m_{ii}\,F_i^k)^{-1}\,(p_i^{k,n})\,. \tag{5.10}$$

We stress that the function $([m_{ii} + \tau k_{ii}]\,I + m_{ii}\,F_i^k)^{-1}$ is piecewise linear and depends on the function F_i^k, which must be computed and stored at each time step and in each node of the mesh (see Sect. 3 and appendix). Thus, this method seems quite expensive because of the large amount of memory necessary to store all these informations. On the contrary, the cost of the updating of the graphs B_i^k and of the corresponding curves F_i^k is less than the cost of the solution of the nonlinear system (5.8). In the next section we shall introduce a linearization scheme in order to overcome the difficulties of the large memory occupation and of the nonlinearity of the discrete problem.

Remark 4 : By setting

$$\mathbf{V}^k := \mathbf{U}^k + \mathbf{W}^k \tag{5.11}$$

and

$$\beta_i^k := (I + F_i^k)^{-1} \tag{5.12}$$

(notice that β_i^k is a non-decreasing and Lipschitz continuous function such that $(\beta_i^k)' \leq 1$), we have

$$U_i^k = \beta_i^k(V_i^k) \tag{5.13}$$

and the nonlinear system (5.8) can be equivalently rewritten as follows

$$\mathbf{M}\mathbf{V}^k + \tau \mathbf{K}\mathbf{U}^k = \mathbf{Q}^k . \tag{5.14}$$

By using this simple transformation, it is possible to treat numerically also hysteresis functionals with discontinuities (see [8]).

5.2. A linearized method

We shall introduce a linearized method based on a time-relaxation of the hysteresis relation. More precisely, we replace (5.3) with

$$W^k - W^{k-1} + (F^k)^{-1} (W^k) \ni U^{k-1} . \tag{5.15}$$

This allows to formulate the following algorithm.

PROBLEM $(\mathbf{P}_{\delta,\tau,h})_2$: *Given U^0 and $W^0 \in V_h$ (suitably obtained from u_0 and w_0), for any $k = 1, \ldots, K$, find U^k and $W^k \in V_h$ such that*

$$W^k - W^{k-1} + I_h(\bar{F}^k)^{-1} (W^k(x), x) \ni U^{k-1} \quad \text{in } \Omega , \tag{5.16}$$

$$\int_\Omega I_h([\partial U^k + \partial W^k]\, \phi) + \int_\Omega \nabla U^k \cdot \nabla \phi = \int_\Omega \bar{f}^k \phi \quad \forall \phi \in V_h . \tag{5.17}$$

By using the position (5.6), the discrete problem can be written in matrix form as follows :

$$W_i^k = (I + (F_i^k)^{-1})^{-1} (U_i^{k-1} + W_i^{k-1}) \quad i = 1, \ldots, I , \tag{5.18}$$

$$[\mathbf{M} + \tau \mathbf{K}]\, \mathbf{U}^k = \mathbf{M}[\mathbf{U}^{k-1} - \mathbf{W}^k + \mathbf{W}^{k-1}] + \tau \mathbf{f}^k . \tag{5.19}$$

It is easily seen that (5.19) is a positive definite linear system, and (5.18) is just a node by node algebraic equation, which involves the evaluation of a piecewise linear function obtained from the hysteresis operator. Thus, problem $(\mathbf{P}_{\delta,\tau,h})_2$ has one and only one solution.

The solution of this scheme is obviously faster than the solution of the nonlinear one, because one can take advantage of very efficient linear solvers for (5.19). Moreover, to make the algebraic correction (5.18) node by node, one does not need to compute and store simultaneously all the functions F_i^k ; thus, the present scheme uses much less memory than the previous one.

Remark 5 : It is easy to prove that

$$(I + (F_i^k)^{-1})^{-1} = I - \beta_i^k , \tag{5.20}$$

where β_i^k is defined in (5.12). By using the position (5.11) and setting

$$Z^{k-1} \in V_h : \quad Z_i^{k-1} := \beta_i^k (V_i^{k-1}) , \tag{5.21}$$

the discrete problem can be equivalently rewritten as follows

$$[\mathbf{M} + \tau \mathbf{K}]\, \mathbf{U}^k = \mathbf{M}\mathbf{Z}^{k-1} + \tau \mathbf{f}^k \tag{5.22}$$

$$\mathbf{V}^k = \mathbf{V}^{k-1} - \mathbf{Z}^{k-1} + \mathbf{U}^k . \tag{5.23}$$

This formulation is an extension of the so-called nonlinear Chernoff formula (suggested by semigroup theory) for nonlinear parabolic equations (see, e.g. [5]).

We conclude this section by outlining the proof of the stability of the scheme $(\mathbf{P}_{\delta, \tau, h})_2$.

THEOREM 2 : *There exists a positive constant C independent of δ, τ and h such that*

$$\sum_{k=1}^{K} \tau \| \partial Z^k \|^2_{L^2(\Omega)} + \max_{1 \leq k \leq K} \| \nabla U^k \|^2_{L^2(\Omega)} + \\ + \sum_{k=1}^{K} \| \nabla [U^k - U^{k-1}] \|^2_{L^2(\Omega)} \leq C . \tag{5.24}$$

Proof: We give just a simplified proof in the case $f = 0$. From (5.22) and (5.23) we obtain

$$\int_\Omega I_h(\partial V^k \phi) + \int_\Omega \nabla U^k . \nabla \phi = 0 \quad \forall \phi \in V_h , \quad \text{for} \quad k = 1, \dots, K , \tag{5.25}$$

$$U^k = \frac{1}{2} V^k + \frac{1}{2} Z^k + \frac{1}{2} [V^k - Z^k] - [V^{k-1} - Z^{k-1}] \\ \text{for} \quad k = 0, \dots, K , \tag{5.26}$$

where $V^{-1} := V^0$, $\beta_i^{K+1} := \beta_i^K$ and β_i^0 is any non-decreasing real function such that $\beta_i^0(V_i^0) = U_i^0$. By setting

$$\alpha_i^k := I - \beta_i^k \tag{5.27}$$

(note that α_i^k is a non-decreasing and Lipschitz continuous function such that $(\alpha_i^k)' \leq 1$), we have

$$V_i^k - Z_i^k = \alpha_i^{k+1}(V_i^k) . \tag{5.28}$$

Moreover, note that

$$\beta_i^{k+1}(V_i^k) = \beta_i^k(V_i^k) , \quad \text{hence} \quad \alpha_i^{k+1}(V_i^k) = \alpha_i^k(V_i^k) . \tag{5.29}$$

We take $\phi \doteq U^k - U^{k-1}$ in (5.25) and sum for $k = 1, \dots, n$ $(n \leqslant K)$. By using (5.26), (5.21), (5.28) and (5.29), we obtain

$$\partial V_i^k\, \partial U_i^k \geqslant \frac{1}{2}(\partial V_i^k)^2 + \frac{1}{2}(\partial Z_i^k)^2 + \frac{1}{2}(\partial[V_i^k - Z_i^k])^2 - \\ - \frac{1}{2}(\partial V_i^k)^2 - \frac{1}{2}(\partial[V_i^{k-1} - Z_i^{k-1}])^2 ; \quad (5.30)$$

since

$$\int_\Omega I_h \chi = \sum_{i=1}^{I} \chi(x_i) \int_\Omega \phi_i \quad \forall \chi \in C^0(\bar{\Omega}) , \quad (5.31)$$

we have

$$\sum_{k=1}^{n} \tau \int_\Omega I_h(\partial V^k\, \partial U^k) \geqslant \frac{1}{2} \sum_{k=1}^{n} \tau \int_\Omega I_h((\partial Z^k)^2) . \quad (5.32)$$

The second term in (5.25) yields

$$\sum_{k=1}^{n} \int_\Omega \nabla U^k . \nabla[U^k - U^{k-1}] = \\ = \frac{1}{2} \int_\Omega (|\nabla U^n|^2 - |\nabla U^0|^2) + \frac{1}{2} \sum_{k=1}^{n} \int_\Omega |\nabla[U^k - U^{k-1}]|^2 . \quad (5.33)$$

The assertion (5.24) then follows by means of the well known interpolation property

$$\int_\Omega I_h(\phi^2) \geqslant \int_\Omega \phi^2 \quad \forall \phi \in V_h . \quad (5.34)$$

These a priori estimates allow us to take the limit in the discrete problem $(\mathbf{P}_{\delta,\tau,h})_2$, and to prove the convergence of the discrete solution to a solution of the continuous problem $(\mathbf{P})$, by means of the procedure outlined in section 4.

Remark 6 : Other algorithms could be proposed for the approximation of our equation with hysteresis, of course. For instance, usual linearization techniques for nonlinear parabolic equations can be used in this context. Among them, we suggest the following scheme

$$\int_\Omega I_h([\partial U^k + (\bar{F}^k)'\,(U^{k-1})\,\partial U^k]\,\phi) + \int_\Omega \nabla U^k . \nabla \phi = \\ = \int_\Omega \bar{f}^k\, \phi , \quad \forall \phi \in V_h ; \quad (5.35)$$

this one is appealing, even if one has to be careful in treating the function $(\bar{F}^k)'$, which is discontinuous in $2M+1$ points. It is easily seen that it has one and only one solution and that it is stable in the sense of problem $(\mathbf{P}_{\delta,\tau,h})_1$.

Setting $m_{ij}^k := (F_i^k)'\,(U_i^{k-1})\,m_{ij}$ and $\mathbf{M}^k := \{m_{ij}^k\}$, this scheme reduces to the following symmetric and positive definite linear system

$$[\mathbf{M} + \mathbf{M}^k + \tau\mathbf{K}]\,\mathbf{U}^k = [\mathbf{M} + \mathbf{M}^k]\,\mathbf{U}^{k-1} + \tau\mathbf{f}^k\,; \tag{5.36}$$

nevertheless, in contrast to the scheme $(\mathbf{P}_{\delta,\tau,h})_2$, its matrix has to be recomputed at each time step.

APPENDIX

Here we shall present an efficient implementation of the approximation procedure for the hysteresis operator $\mathbf{F}_\mu$ discussed in section 3.

First, we consider the storing and the updating in Fortran structure data of the antimonotone graph B in the Preisach plane. Since we approximate B with a graph B^δ lying on the reticulation of $\mathbf{P}$, we have to assume that the value u which characterises the intersection between the graph B^δ and the straight line $\rho_1 = \rho_2$ is of the form $u = U\delta$, U being an integer number. In particular, B^δ is composed of the straight line $\rho_1 + \rho_2 = 0$ up to the point $(Q\delta, -Q\delta)$, and several horizontal and vertical segments up to the point $(U\delta, U\delta)$. So,

— if $U \geqslant M$ or $U \leqslant -M$, we have $Q := -U$ or $Q := U$, and B^δ has just one horizontal or vertical segment, respectively ;

— if $|U| < M$, we have $Q := -M$, and B^δ has $N \in \{2, \dots, 2M\}$ horizontal and vertical segments.

(See *fig.* 8). We stress that the antimonotone graph corresponding to the *virgin* material (namely the straight line $\rho_1 + \rho_2 = 0$) is approximated by the piecewise horizontal/vertical curve shown in figure 8.

In order to store conveniently the graph B^δ, we use an integer Fortran vector IB (of maximal dimension $2M$) as follows :

— $IB(1) = -Q$ (resp. $= Q$) if the first segment of B^δ is horizontal (resp. vertical) (note that Q is always a negative integer number) ;

— $IB(I)$, $I = 2, \dots, N$ contains the first coordinate ρ_1 (resp. second coordinate ρ_2) of the second vertex of the $(I-1)$-segment of B^δ if it is horizontal (resp. vertical) (note that $IB(N) = U$).

(See *fig.* 8 for some examples).

The updating of the vector IB is quite simple. Just notice that if the new control $NEWU$ increases (resp. decreases) with respect to U, it cancels all

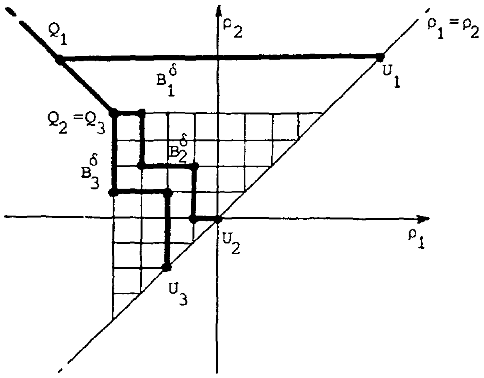

Figure 8. — The antimonotone graph B^δ in the Preisach plane : three examples ($M = 4$).

Ex. 1 : $U_1 = 6$; $N = 1$, $IB = 6$. Ex. 2 : virgin material ; $N = 5$, $IB = 4, -3, 2, -1, 0$.
Ex. 3 : $U_3 = -2$; $N = 3$, $IB = -4, 1, -2$.

the smaller second coordinates (resp. the larger first coordinates), and hence generates a final horizontal (resp. vertical) segment.

Now we deal with the construction of the piecewise monotone curve $\bar{F}^\delta$ in the (u, w)-plane defined in (3.4) (see *fig.* 9).

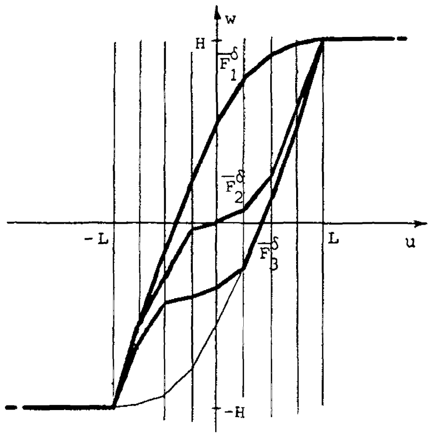

Figure 9. — Piecewise linear monotone curves in the (u, w)-plane corresponding to the graphs B_1, B_2 and B_3 shown in figure 8, in the case μ = Lebesgue measure.

All the informations concerning $\bar{F}^{\delta}$ can be stored in a real Fortran vector W of dimension $2M-1$. We recall that $(U\delta, U\delta)$ is the intersection point between the graph B^{δ} and the straight line $\rho_1 = \rho_2$.

If $U \geqslant M$ (resp. $U \leqslant -M$) the curve $\bar{F}^{\delta}$ is the upper (resp. lower) part of the hysteresis loop (see *fig.* 9). It is obviously convenient to construct and store these two curves once and for all.

The construction of $\bar{F}^{\delta}$ when $|U| \leqslant M-1$, i.e. the filling of the vector W, can be easily performed starting from the antimonotone curve B^{δ} in the Preisach plane as follows. We start from $(-M\delta, -H)$ and compute the value $W(k)$ for $k = 1, \ldots, M+U$ by means of the elements of the column k of $\{\mu_{ij}\}$ belonging to A^{+}, and $W(k)$ for $k = M+U+1, \ldots, 2M-1$ by means of the elements of the row $[2M-k+1]$ of $\{\mu_{ij}\}$ belonging to A^{-}; more precisely, by setting $W(0) = -H$, we define

$$W(k) = W(k-1) + \sum_{i \; B_{ik} \in A^{+}} \mu_{ik} \quad \text{for} \quad k = 1, \ldots, M+U$$

$$W(k) = W(k-1) + \sum_{j \; B_{2M-k+1,j} \in A^{-}} \mu_{2M-k+1,j}$$

$$\text{for} \quad k = M+U+1, \ldots, 2M-1 \,.$$

Now it is convenient to store in a real Fortran vector C (of dimension $M[2M+1]$) the following values:

$$C(k) := \sum_{l=i}^{2M-j+1} \mu_{lj}\,, \quad \text{for} \quad i, j \in \{i, j = 1, \ldots, 2M : i + j \leqslant 2M+1\}\,;$$

here $\mu_{ij} := \mu(B_{ij}) = r_{ij}\,\lambda(B_{ij})$ and $k := \operatorname{ind}(i, j) := [4M+2-j]\,[j-1]/2 + i$. More precisely, for any cell B_{ij}, the corresponding $C(\operatorname{ind}(i, j))$ is the sum of the values $\{\mu_{ij}\}$ of the column j from the row i to $2M-j+1$. By the symmetry of the measure μ with respect to the straight line $\rho_1 + \rho_2 = 0$, the sum of the $\{\mu_{ij}\}$'s of the row i from the column j to $2M-i+1$, is given by the value $C(\operatorname{ind}(j, i))$ corresponding to the cell B_{ji}.

With this position, if B_{nk} is the first cell of the column k under the graph B (resp. $B_{2M-k+1,n}$ is the first cell of the row $2M-k+1$ to the right of the graph B), then we have

$$\sum_{i \; B_{ik} \in A^{+}} \mu_{ik} = C(\operatorname{ind}(n, k))$$

$$(\text{resp.} \quad \sum_{j \; B_{2M-k+1,j} \in A^{-}} \mu_{2M-k+1,j} = C(\operatorname{ind}(n, 2M-k+1)))\,.$$

Thus the construction of the curve $\bar{F}^{\delta}$ can be easily performed by using the previous strategy

Remark 7 If the curve $\bar{F}^{\delta}$ is already known and the control variable u increased (resp decreased) reaching the value $U\delta$, we need to update only the decreasing (resp increasing) part of $\bar{F}^{\delta}$ moving from the couple $(U\delta, \bar{F}^{\delta}(U)) = (U\delta, W(M+U))$ We remind that the information about the last movement of the control variable u is given by the antimonotone graph B^{δ} in the Preisach plane , more precisely, if the last segment of B^{δ} is horizontal (resp vertical), then the control increased (resp decreased)

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