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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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A FINITE ELEMENT APPROXIMATION OF THREE DIMENSIONAL MOTION OF A BINGHAM FLUID (*)

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Abstract. — *In this paper, we approximate solutions of an initial-boundary value problem associated with the motion of a Bingham fluid in a three dimensional domain. The method of approximation consists of the backward Euler scheme in the time variable and conforming piecewise linear finite elements in the space variables augmented by the penalty method. The convergence of this scheme is proved under a mild assumption on the data. Error estimates are also obtained when the data satisfy restrictive assumptions*

Résumé. — *Dans cet article nous approximons les solutions d'un problème aux conditions aux limites et valeurs initiales, associé au mouvement d'un fluide de Bingham dans un domaine tridimensionnel. La méthode de discrétisation se compose d'un schéma d'Euler en temps et d'éléments finis conformes linéaires par morceaux en espace avec pénalisation La convergence de ce schéma est démontrée moyennant une hypothèse faible sur les données Des estimations d'erreur sont aussi obtenues lorsque les données satisfont des hypothèses restrictives supplémentaires*

0. INTRODUCTION

The purpose of this paper is to discuss a certain finite element method to approximate solutions of an initial-boundary value problem associated with the motion of a Bingham fluid in a three dimensional domain. According to Duvaut and Lions [4], [5], the initial-boundary value problem is formulated as

$$\left(\frac{\partial u}{\partial t}, w - u \right) + a(u, w-u) + b(u, u, w) + \\ + J(w) - J(u) \geq (f, w-u) \quad \text{in} \quad (0, T), \quad (0.1)$$

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for each test function w such that $\nabla . w = 0$ in Ω and $w = 0$ on $\partial\Omega$,

$$\nabla . u = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{0.2}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times [0, T], \tag{0.3}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega. \tag{0.4}$$

Here, Ω is a bounded convex domain in R^3 with smooth boundary $\partial\Omega$, $u(x, t)$ denotes the velocity of the fluid and $f(x, t)$ stands for external force. The density, the yield limit and the viscosity are assumed to be positive constants. In particular, the density is taken to be one. We employ the notation :

$$a(u, w) = \sum_{i,j=1}^{3} 2\mu \int_\Omega D_{ij}(u)\, D_{ij}(w)\, dx, \; \mu = \text{viscosity}$$

$$D_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

$$J(u) = 2g \int_\Omega D_{II}(u)^{1/2}\, dx, \quad g = \text{yield limit}$$

$$D_{II}(u) = \frac{1}{2}\sum_{i,j=1}^{3} D_{ij}(u)^2,$$

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i\, dx,$$

$(,)$ = a scalar product which will be defined in the next section.

The conservation of momentum is expressed by (0.1) and the condition of incompressibility is given by (0.2). The above initial-boundary value problem was investigated for the first time in [4]. The numerical approximation of this problem has been the subject of numerous works : [1], [6], [9], [10] and [11]. However, the numerical study has been restricted to the case of either two dimensional domain or laminar flow in a cylindrical pipe. No attempt seems to have been made in the past on the fully discretized approximation of time-dependent solutions in a three dimensional domain. The major difficulty in a three dimensional domain is that weak solutions are not regular enough to give a meaning to the first term in (0.1). In fact, weak solutions are supposed to satisfy a weaker version of (0.1) ; see [4], [5]. This lack of regularity persists in the limit of a sequence of approximate solutions even if we use smooth data. This is due to the fact that sufficient a priori estimates for the fully discretized approximate solutions cannot be obtained even with smooth data. In this paper, we shall approximate strong solutions obtained in [13] and [14]. The above mentioned difficulty in the numerical approximation can be avoided by showing that the limit of approximate solutions coincides with the strong solution. For approximation, we employ the backward Euler scheme with respect to the time variable and conforming piecewise linear finite elements with respect to the space variables. The use of piecewise linear finite elements is attributed to the limitation on the regularity of solutions. The lack of regularity inhibits the use of higher-order finite elements, particularly when error estimates are sought. While the computation is substantially simplified by using linear finite elements, it is difficult to incorporate the divergence-free condition (even an approximate condition) into the approximate function class of linear finite elements. Therefore, we use the penalty method to deal with the incompressibility condition. Our scheme is similar to one of those discussed in [16], except that we use finite elements instead of finite differences. To maximize the regularity of solutions, the boundary of the space domain has to be sufficiently smooth, and we exclude polyhedral domains which are typically used for the numerical approximation of the Navier-Stokes equations. The region near boundary cannot be filled with tetrahedra. But we can use Hölder's inequality to estimate the error arising in this region since the measure of this region can be made arbitrarily small. One may use isoparametric finite elements to take care of the curved boundary. However, this does not improve our result due to the inherent lack of regularity of solutions.

In section 2, we prove the convergence of our scheme under the same assumption on the data as in [14]. In section 3, we obtain further regularity of solutions for the purpose of obtaining error estimates. In section 4, we analyze the error between the true solution and the approximate solution under the same assumption on the data as in [13]. Our result on the error estimate is not as strong as the known result for the Navier-Stokes equation ; see, e.g., [12] among others. This is due to the fact that we cannot raise the regularity of solutions to that of solutions of the Navier-Stokes equations.

1. NOTATIONS AND PRELIMINARIES

Throughout this paper, Ω denotes a bounded convex domain in R^3 with smooth boundary $\partial\Omega \in C^4$ and we shall retain all the notations defined in the introduction. We also employ the following notation :

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad \text{for} \quad i = 1, 2, 3, \quad \Delta = \sum_{i=1}^{3} \partial_i^2,$$
$$\nabla = (\partial_1, \partial_2, \partial_3), \quad |f| = (f_1^2 + f_2^2 + f_3^2)^{1/2}$$

and

$$\nabla . f = \sum_{i=1}^{3} \partial_i f_i , \quad \text{for} \quad f = (f_1, f_2, f_3) .$$

When E is a Banach space, $L^r(0, T ; E)$ is the set of all E-valued strongly measurable L^r functions on $[0, T]$ with the obvious norm. $C(I ; E)$ is the set of all E-valued continuous functions on the interval I.

We introduce the following function spaces :

$$S = \{\phi \in C_0^\infty(\Omega)^3 : \nabla . \phi = 0 \text{ in } \Omega\} ,$$

$$W^{m,r}(\Omega) = \left\{v \in L^r(\Omega) : \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} v \in L^r(\Omega), 1 \leq \alpha_1 + \alpha_2 + \alpha_3 \leq m\right\} .$$

For $f \in W^{m,r}(\Omega)^3$, $\| f \|_{W^{m,r}}$ denotes the norm of f in $W^{m,r}(\Omega)^3$,

$$W_0^{m,r}(\Omega) = \text{the completion of } C_0^\infty(\Omega) \text{ in } W^{m,r}(\Omega) ,$$

$$W^{-m,r'}(\Omega) = \text{the dual of } W_0^{m,r}(\Omega) , \quad \text{where} \quad \frac{1}{r'} + \frac{1}{r} = 1 , \quad 1 \leq r < \infty ,$$

X_r = the completion of S in $L^r(\Omega)^3$, $1 < r < \infty$, $V = W_0^{1,2}(\Omega)^3 \cap X_2$, V' = the dual of V. $(,)$ stands for the usual inner product in $L^2(\Omega)$.

When $f, g \in L^2(\Omega)^3$, $(f, g) = \sum_{i=1}^{3} \int_\Omega f_i \, g_i \, dx$. $(,)$ also denotes the duality pairing between $W_0^{1,2}(\Omega)^3$ and $W^{-1,2}(\Omega)^3$ or between V and V'. The meaning of $(,)$ will be clear from the context. One can characterize X_r by

$$X_r = \{v \in L^r(\Omega)^3 : \nabla . v = 0 \text{ in } \Omega \text{ and the normal component of } v \text{ vanishes on } \partial\Omega\} .$$

We let P_r denote the projection from $L^r(\Omega)^3$ onto X_r and write the Stokes operator as

$$A_r = - P_r \Delta , \quad \text{for} \quad 1 < r < \infty ,$$

with the domain,

$$D(A_r) = W^{2,r}(\Omega)^3 \cap W_0^{1,r}(\Omega)^3 \cap X_r .$$

As in [13], G denotes the set of all $v \in V$ such that for some $H \in L^2(\Omega)^3$,

$$a(v, w - v) + b(v, v, w) + J(w) - J(v) \geq (H, w - v) \qquad (1.1)$$

holds for every $w \in V$. It was shown that G is dense in V and $G \subset W_0^{1,6}(\Omega)^3$.

We define

$$J_\eta(v) = 2g \int_\Omega (\eta + D_{II}(v))^{1/2}\, dx\,, \quad \eta > 0\,,$$

and denote by $J'_\eta(.)$ the Gâteaux differential of $J_\eta(.)$. Then, we find that for every $v,\ w \in W_0^{1,2}(\Omega)^3$

$$(J'_\eta(v), w) = g \sum_{i,j=1}^{3} \int_\Omega \frac{D_{ij}(v)\, D_{ij}(w)}{\sqrt{\eta + D_{II}(v)}}\, dx \tag{1.2}$$

and

$$(J'_\eta(v) - J'_\eta(w), v - w) \geq 0\,, \tag{1.3}$$

which follows from

$$\sum_{m=1}^{n} \left(\frac{a_m}{\sqrt{\eta + |a|^2}} - \frac{b_m}{\sqrt{\eta + |b|^2}} \right)(a_m - b_m) \geq 0 \tag{1.4}$$

for every $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in R^n$. It is easy to see that for all $\eta > 0$ and $v \in W^{1,1}(\Omega)^3$

$$\left\| \frac{D_{ij}(v)}{\sqrt{\eta + D_{II}(v)}} \right\|_{L^\infty} \leq \sqrt{2}\,, \quad i, j = 1, 2, 3 \tag{1.5}$$

and

$$|J_\eta(v) - J(v)| \leq M\sqrt{\eta}\,, \tag{1.6}$$

where M is a positive constant depending only on g and Ω. We also need

LEMMA 1.1 : *If* $v \in W^{2,2}(\Omega)^3$, $\varphi \in C(\bar{\Omega})$ *and* $\varphi \geq 0$, *then for* $l = 1, 2, 3$,

$$\sum_{i,j=1}^{3} \int_\Omega \varphi \left(\partial_l \frac{D_{ij}(v)}{\sqrt{\eta + D_{II}(v)}} \right) \partial_l D_{ij}(v)\, dx \geq 0\,. \tag{1.7}$$

Proof: Since $C^2(\bar{\Omega})^3$ is dense in $W^{2,2}(\Omega)^3$, it is enough to show that (1.7) is true for all $v \in C^2(\bar{\Omega})^3$. Since $\partial\Omega$ is smooth, each $v \in C^2(\bar{\Omega})^3$ can be extended to a function $\tilde{v} \in C^2(R^3)^3$. φ can be extended to $\tilde{\varphi}$ such that $\tilde{\varphi} = \varphi$ for $x \in \bar{\Omega}$ and $\tilde{\varphi} = 0$ for $x \notin \bar{\Omega}$. The integral in (1.7) can be evaluated over R^3 after replacing φ and v by $\tilde{\varphi}$ and $\tilde{v}$. Then, we approximate ∂_l by a finite difference in the x_l-direction and arrive at the inequality by means of (1.4).

The following property of $a(,)$ will be used

$$C_1 \|v\|^2_{W_0^{1\,2}} \leq a(v, v) \leq C_2 \|v\|^2_{W_0^{1\,2}} \quad \text{for all} \quad v \in W_0^{1\,2}(\Omega)^3 \qquad (1\ 8)$$

where C_1 and C_2 are positive constants depending only on Ω and μ

Next we introduce finite dimensional approximate function spaces which consist of conforming piecewise linear finite elements Let us denote a 3-simplex by K and associate the following two numbers with K

h_K = diameter of K,

ρ_K = diameter of the greatest ball contained in K

We denote by $\mathcal{T}_h$ a finite set of simplices K such that

(i) $K \subset \bar{\Omega}$,

(ii) If K_1 and $K_2 \in \mathcal{T}_h$, then $K_1 = K_2$ or $K_1 \cap K_2$ is empty or $K_1 \cap K_2$ is exactly a complete m-face of K_1 and K_2, where $0 \leq m \leq 2$,

(iii) $\sup_{K \in \mathcal{T}_h} h_K = h$

We next write

$$\Phi_h = \bigcup_{K \in \mathcal{T}_h} K,$$

Σ_h = the set of all vertices of $\mathcal{T}_h$,

Σ_h^0 = all points of Σ_h which belong to the interior of Φ_h, and make the following assumptions on the family $\{\mathcal{T}_h\}_h$

(i) $h \to 0$

(ii) there is a positive constant δ such that

$$\sup \left\{ h_K/\rho_K \quad K \in \bigcup_h \mathcal{T}_h \right\} \leq \delta ,$$

(iii) $\Sigma_h - \Sigma_h^0 \subset \partial\Omega$

Let us define

W_h = the space of continuous functions on Φ_h which are linear on each simplex $K \in \mathcal{T}_h$

$W_{0h} = \{v \in W_h \quad v = 0 \text{ on the boundary of } \Phi_h\}$

Each element of W_{0h} can be extended to $\bar{\Omega} - \Phi_h$ by 0 so that W_{0h} is the space of continuous functions on $\bar{\Omega}$ which are linear on each simplex $K \in \mathcal{T}_h$ and vanish outside Φ_h It is evident that $W_{0h} \subset W_0^{1\,p}(\Omega)$, for any $1 \leq p < \infty$ We next define the interpolation operator $r_h \quad C(\bar{\Omega}) \to W_h$ by

$$(r_h v)(x) = v(x) \quad \text{for each} \quad x \in \Sigma_h \qquad (1\ 9)$$

Hence, if v belongs to $C(\bar{\Omega})$ and vanishes on $\partial\Omega$, then $r_h v \in W_{0h}$ It follows from Theorem 3 1 5 [3] that for all $v \in W_0^{1\,p}(\Omega)$, $3 < p < \infty$,

$$\|r_h v\|_{W_0^{1\,p}(\Omega)} \leq C \|v\|_{W_0^{1\,p}(\Omega)} , \qquad (1\ 10)$$

where C is a positive constant independent of v and h. Let $\mathcal{T}_h^*$ be any subset of $\mathcal{T}_h$ and set $\Phi_h^* = \bigcup_{K \in \mathcal{T}_h^*} K$. Again by Theorem 3.1.5 [3], we find that for all $v \in W^{1,p}(\Omega)$, $3 < p < \infty$,

$$\|v - r_h v\|_{W^{m,p}(\Phi_h^*)} \leq C h^{1-m} \|v\|_{W^{1,p}(\Phi_h^*)}, \quad m = 0, 1, \tag{1.11}$$

and, for all $v \in W^{2,2}(\Omega)$,

$$\|v - r_h v\|_{W^{m,2}(\Phi_h^*)} \leq C h^{2-m} \|v\|_{W^{2,2}(\Phi_h^*)}, \quad m = 0, 1, \tag{1.12}$$

where C denotes positive constants independent of v, h and Φ_h^*. Finally we set $V_h = (W_{0h})^3$ and use the same notation r_h for the interpolation operator : $C(\bar{\Omega})^3 \to (W_h)^3$ defined by

$$(r_h v)(x) = v(x), \quad \text{for each } x \in \Sigma_h . \tag{1.13}$$

2. APPROXIMATION OF SOLUTIONS

We first review some results on the existence of solutions of (0.1) through (0.4) obtained in [14].

DEFINITION 2.1 : *A function $u(x, t)$ is called a solution of* (0.1) *through* (0.4) *on an interval* $[0, T)$ *if*

(i) $u \in L^2(0, T; V)$ *and* $\partial_t u \in L^2(0, T; V')$,
(ii) (0.1) *is satisfied for every* $w \in V$ *for almost all* $t \in (0, T)$,
(iii) $u(x, 0) = u_0(x)$.

THEOREM A. [14] : *Suppose that* $3 < r < \infty$, $u_0(x) \in X_r$ *and* $f \in L^\infty(0, T; W^{-1,r}(\Omega)^3)$. *Then, there is a unique solution* $u(x, t)$ *on an interval* $[0, T^*)$, $0 < T^* \leq T$. *Furthermore,* $u \in C([0, T^*]; X_r)$.

Our purpose is to approximate the above solution by the backward Euler scheme in the time variable and conforming piecewise linear finite elements in the space variables. Let us fix $3 < r \leq 6$, $u_0(x) \in X_r$ and $f \in L^\infty(0, T; L^2(\Omega)^3)$. Let us divide the interval $[0, T]$ into N intervals of equal length $k = T/N$, and consider the following finite dimensional problem : for given u_h^{m-1} in V_h, find u_h^m in V_h satisfying

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + a(u_h^m, v_h) + \tilde{b}(u_h^{m-1}, u_h^m, v_h) + (J_\eta'(u_h^m), v_h) + \\ + \frac{1}{\varepsilon}(\nabla \cdot u_h^m, \nabla \cdot v_h) = (f^m, v_h), \tag{2.1}$$

for every $v_h \in V_h$. Here, we take

$$u_h^0 = \text{the orthogonal projection of } u_0(x) \text{ onto } V_h \text{ in } L^2(\Omega)^3 , \quad (2.2)$$

$$f^m = \frac{1}{k} \int_{(m-1)k}^{mk} f(t)\, dt, \quad m = 1, \ldots, N , \quad (2.3)$$

and define

$$\tilde{b}(u, v, w) = \frac{1}{2} \sum_{i,j=1}^{3} \left(\int_\Omega u_i (\partial_i v_j) w_j \, dx - \int_\Omega u_i \, v_j (\partial_i w_j) \, dx \right) . \quad (2.4)$$

It is obvious that $\tilde{b}(u, v, v) = 0$, for every $u \in L^3(\Omega)^3$ and $v \in W_0^{1,2}(\Omega)^3$, and that $\tilde{b}(u, v, w) = b(u, v, w)$, for every $u \in V$ and $v, w \in W_0^{1,2}(\Omega)^3$. By virtue of the properties of $J'_\eta(.)$ and Lemma 4.3 in [15, p. 53], there is a unique u_h^m in V_h of (2.1). We now set

$$u_h(t) = u_h^m, \quad \text{for } (m-1)k \leq t < mk, \quad m = 1, \ldots, N, \quad (2.5)$$

$$\tilde{u}_h(t) = u_h^{m-1}, \quad \text{for } (m-1)k \leq t < mk, \quad m = 1, \ldots, N, \quad (2.6)$$

$$f_k(t) = f^m, \quad \text{for } (m-1)k \leq t < mk, \quad m = 1, \ldots, N . \quad (2.7)$$

We also define a piecewise linear V_h-valued function $w_h(t)$ on $[0, T]$ such that

$$w_h(mk) = u_h^m, \quad m = 0, \ldots, N , \quad (2.8)$$

and w_h is linear on $[(m-1)k, mk]$, $m = 1, \ldots, N$. The convergence of our numerical scheme is stated as

THEOREM 2.2 : *Let $\{h, k, \eta, \varepsilon\}$ be an arbitrary sequence of quadruplets such that $\{h, k, \eta, \varepsilon\} \to \{0, 0, 0, 0\}$ and $\frac{h}{\sqrt{\varepsilon}} \to 0$. Then*

$$u_h, \tilde{u}_h, w_h \to u \text{ weak} * \text{in } L^\infty(0, T^* ; L^2(\Omega)^3) , \quad (2.9)$$

$$u_h, \tilde{u}_h, w_h \to u \text{ strongly in } L^2(0, T^* ; L^2(\Omega)^3) , \quad (2.10)$$

$$u_h \to u \text{ strongly in } L^2(0, T^* ; W_0^{1,2}(\Omega)^3) , \quad (2.11)$$

where u is the solution on the interval $[0, T^*)$ of Theorem A above.

Proof: Substituting u_h^m for v_h in (2.1), we obtain

$$\frac{1}{2} \|u_h^m\|_{L^2}^2 - \frac{1}{2} \|u_h^{m-1}\|_{L^2}^2 + \frac{1}{2} \|u_h^m - u_h^{m-1}\|_{L^2}^2 + ka(u_h^m, u_h^m) + $$
$$+ \frac{k}{\varepsilon} \|\nabla \cdot u_h^m\|_{L^2}^2 \leq k |(f^m, u_h^m)|, \quad m = 1, \ldots, N , \quad (2.12)$$

from which it follows that

$$\max_{1 \leq m \leq N} \|u_h^m\|_{L^2}^2 + \sum_{m=1}^{N} \|u_h^m - u_h^{m-1}\|_{L^2}^2 + \sum_{m=1}^{N} ka(u_h^m, u_h^m) + \sum_{m=1}^{N} \frac{k}{\varepsilon} \|\nabla \cdot u_h^m\|_{L^2}^2 \leq M, \quad (2.13)$$

where M denotes a positive constant independent of h, k, ε and η. (2.13) implies that $\{u_h\}$ is bounded in $L^\infty(0, T; L^2(\Omega)^3)$ and $L^2(0, T; W_0^{1,2}(\Omega)^3)$. We borrow an idea from [18] to show that $\{w_h(t)\}$ is precompact in $L^2(0, T; L^2(\Omega)^3)$. Using the notation w_h, u_h, $\tilde{u}_h$ and f_k, (2.1) can be rewritten as

$$(\partial_t w_h, v_h) + a(u_h, v_h) + \tilde{b}(\tilde{u}_h, u_h, v_h) + (J_\eta'(u_h), v_h) + \frac{1}{\varepsilon}(\nabla \cdot u_h, \nabla \cdot v_h) = (f_k, v_h) \quad (2.14)$$

for every $v_h \in V_h$, for almost all $t \in [0, T]$. From this, we derive

$$(w_h(t+\delta) - w_h(t), v_h) = -\int_t^{t+\delta} a(u_h(s), v_h)\, ds - \int_t^{t+\delta} \tilde{b}(\tilde{u}_h(s), u_h(s), v_h)\, ds - \int_t^{t+\delta} (J_\eta'(u_h(s)), v_h)\, ds - \int_t^{t+\delta} \left(\frac{1}{\varepsilon} \nabla \cdot u_h(s), \nabla \cdot v_h\right) ds + \int_t^{t+\delta} (f_k(s), v_h)\, ds \quad (2.15)$$

for all $t \in [0, T-\delta]$, $0 < \delta < T$, and all $v_h \in V_h$. By means of (2.13), we can estimate the right-hand side of (2.15) :

$$\left| \int_t^{t+\delta} a(u_h(s), v_h)\, ds \right| \leq M \|v_h\|_{W_0^{1,2}} \delta^{1/2}, \quad (2.16)$$

$$\left| \int_t^{t+\delta} \tilde{b}(\tilde{u}_h(s), u_h(s), v_h)\, ds \right| \leq M \int_t^{t+\delta} \|v_h\|_{W_0^{1,2}} \|u_h(s)\|_{W_0^{1,2}} \|\tilde{u}_h(s)\|_{L^2}^{1/2} \|\tilde{u}_h(s)\|_{W_0^{1,2}}^{1/2} ds \leq M \|v_h\|_{W_0^{1,2}} \delta^{1/4}, \quad \text{for } t \geq k, \quad (2.17)$$

$$\left| \int_t^{t+\delta} (J_\eta'(u_h(s)), v_h)\, ds \right| \leq M \|v_h\|_{W_0^{1,2}} \delta, \quad \text{by (1.5)}, \quad (2.18)$$

$$\left| \int_t^{t+\delta} \left(\frac{1}{\varepsilon} \nabla \cdot u_h(s), \nabla \cdot v_h \right) ds \right| \leqslant M \left\| \frac{1}{\sqrt{\varepsilon}} \nabla \cdot v_h \right\|_{L^2} \delta^{1/2} , \quad (2.19)$$

$$\left| \int_t^{t+\delta} (f_k(s), v_h)\, ds \right| \leqslant M \|v_h\|_{L^2} \delta , \quad (2.20)$$

where M denotes positive constants independent of h, k, ε, η and δ. We substitute $w_h(t+\delta) - w_h(t)$ for v_h in (2.15) through (2.20) and integrate over $[k, T-\delta]$ to arrive at

$$\int_k^{T-\delta} \|w_h(t+\delta) - w_h(t)\|_{L^2}^2\, dt \leqslant M\delta^{1/4} , \quad (2.21)$$

for all $0 < \delta < \min(T-k, 1)$, where M is a positive constant independent of h, k, ε, η and δ. Let us set

$$\tilde{w}_h(t) = \begin{cases} w_h(t), & \text{for } k \leqslant t < T, \\ 0, & \text{for } 0 \leqslant t < k . \end{cases} \quad (2.22)$$

Then, it follows from (2.13) that

$$\int_0^T \|\tilde{w}_h\|_{W_0^{1,2}}\, dt \leqslant M,$$

M being a positive constant independent of h, k, ε and η , (2.23)

which, combined with (2.21), implies that the sequence $\{\tilde{w}_h(t)\}$ is precompact in $L^2(0, T; L^2(\Omega)^3)$; see [18]. In the meantime, we derive from (2.13)

$$\int_0^T \|w_h(t) - \tilde{w}_h(t)\|_{L^2}^2\, dt \leqslant Mk , \quad (2.24)$$

$$\int_0^T \|u_h(t) - w_h(t)\|_{L^2}^2\, dt \leqslant Mk , \quad (2.25)$$

$$\int_0^T \|u_h(t) - \tilde{u}_h(t)\|_{L^2}^2\, dt \leqslant Mk , \quad (2.26)$$

where M denotes positive constants independent of h, k, ε and η. By virtue of (2.24), the sequence $\{w_h(t)\}$ is also precompact in $L^2(0, T; L^2(\Omega)^3)$. Let $\{h, k, \varepsilon, \eta\}$ be any sequence of quadruplets such that $\{h, k, \varepsilon, \eta\} \to \{0, 0, 0, 0\}$ and $\frac{h}{\sqrt{\varepsilon}} \to 0$. Then, on account of (2.13), (2.25) and (2.26), there is a subsequence $\{h', k', \varepsilon', \eta'\} \to \{0, 0, 0, 0\}$ with $\dfrac{h'}{\sqrt{\varepsilon'}} \to 0$ such that for some function u,

$$u_{h'} \to u \quad \text{weakly in } L^2(0, T ; W_0^{1,2}(\Omega)^3) , \tag{2.27}$$

$$u_{h'}, \tilde{u}_{h'}, w_{h'} \to u \quad \text{weak} * \text{ in } L^\infty(0, T ; L^2(\Omega)^3) , \tag{2.28}$$

$$u_{h'}, \tilde{u}_{h'}, w_{h'} \to u \text{ strongly in } L^2(0, T ; L^2(\Omega)^3) , \tag{2.29}$$

$$w_{h'}(t) \to u(t) \quad \text{strongly in } L^2(\Omega)^3 \tag{2.30}$$

for all $t \in E \subset [0, T]$, where measure $([0, T] - E) = 0$. By (2.13), we notice that

$$\int_0^T \|\nabla \cdot u_h\|_{L^2}^2 \, dt \leqslant \varepsilon M , \tag{2.31}$$

where M is a positive constant independent of h, k, ε and η. Consequently, we find

$$u \in L^2(0, T ; V) . \tag{2.32}$$

We shall prove that the above function u coincides with the solution of Theorem A above. Let us choose any $v \in C^1([0, T] ; W^{2,p}(\Omega)^3 \cap V)$, $3 < p < \infty$. Then, $r_h v \in C^1([0, T] ; V_h)$, where V_h is equipped with the norm of $W_0^{1,p}(\Omega)^3$. Substituting $r_h v - u_h$ for v_h in (2.14) and integrating over $[0, t]$, $0 < t \leqslant T$, we have

$$\begin{aligned} \int_0^t (\partial_t w_h, r_h v - u_h) \, ds + \int_0^t a(u_h, r_h v - u_h) \, ds \\ + \int_0^t \tilde{b}(\tilde{u}_h, u_h, r_h v) \, ds + \int_0^t J_\eta(r_h v) \, ds \\ - \int_0^t J_\eta(u_h) \, ds + \frac{1}{\varepsilon} \int_0^t (\nabla \cdot u_h, \nabla \cdot (r_h v - u_h)) \, ds \\ \geqslant \int_0^t (f_k, r_h v - u_h) \, ds . \end{aligned} \tag{2.33}$$

The first term can be written as

$$\begin{aligned} \int_0^t (\partial_t w_h, r_h v - u_h) \, ds = -\frac{1}{2} \|w_h(t)\|_{L^2}^2 + \\ + \frac{1}{2} \|u_h^0\|_{L^2}^2 + (w_h(t), r_h v(t)) - (u_h^0, r_h v(0)) \\ - \int_0^t (w_h, \partial_t r_h v) \, ds + \int_0^t (\partial_t w_h, w_h - u_h) \, ds . \end{aligned} \tag{2.34}$$

But, for $mk \leq s < (m+1)k$, $m = 0, \ldots, N-1$,

$$(\partial_t w_h(s), w_h(s) - u_h(s)) = \left(\frac{s - mk}{k} - 1\right) \frac{1}{k} \|u_h^{m+1} - u_h^m\|_{L^2}^2 \leq 0\,, \quad (2.35)$$

and consequently,

$$\int_0^t (\partial_t w_h, w_h - u_h)\, ds \leq 0\,, \quad (2.36)$$

which together with (2.33) and (2.34) yields

$$\begin{aligned} &-\frac{1}{2} \|w_h(t)\|_{L^2}^2 + \frac{1}{2} \|u_h^0\|_{L^2}^2 + (w_h(t), r_h v(t)) \\ &- (u_h^0, r_h v(0)) - \int_0^t (w_h, \partial_t r_h v)\, ds \\ &+ \int_0^t a(u_h, r_h v - u_h)\, ds + \int_0^t \tilde{b}(\tilde{u}_h, u_h, r_h v)\, ds \\ &+ \int_0^t J_\eta(r_h v)\, ds - \int_0^t J_\eta(u_h)\, ds + \frac{1}{\varepsilon} \int_0^t (\nabla \cdot u_h, \nabla \cdot r_h v)\, ds \\ &\geq \int_0^t (f_k, r_h v - u_h)\, ds\,, \end{aligned} \quad (2.37)$$

for all $t \in [0, T]$. Since $v \in C^1([0, T]\,; W^{2,p}(\Omega)^3 \cap V)$, it is easy to see that, for any $t \in [0, T]$,

$$\begin{aligned} \left| \int_0^t \frac{1}{\varepsilon} (\nabla \cdot u_h, \nabla \cdot r_h v)\, ds \right| &\leq \int_0^T \frac{1}{\varepsilon} \left| (\nabla \cdot u_h, \nabla \cdot (r_h v - v)) \right| ds, \\ &\text{by (1.12) and (2.13)}, \\ &\leq M \frac{h}{\sqrt{\varepsilon}} \to 0, \quad \text{since } \frac{h}{\sqrt{\varepsilon}} \to 0\,. \end{aligned} \quad (2.38)$$

We now consider (2.37) for a subsequence $\{h', k', \varepsilon', \eta'\}$ for which (2.27) through (2.30) hold. With the aid of (2.27), (2.29) and

$$\lim_{h \to 0} r_h v = v \quad \text{in the norm of } C^1([0, T]\,; W_0^{1,p}(\Omega)^3), \quad 3 < p < \infty\,, \quad (2.39)$$

we find

$$\int_0^t \tilde{b}(\tilde{u}_{h'}, u_{h'}, r_{h'} v)\, ds \to \int_0^t \tilde{b}(u, u, v)\, ds = \int_0^t b(u, u, v)\, ds \quad (2.40)$$

as $\{h', k', \varepsilon', \eta'\} \to \{0, 0, 0, 0\}$. We also notice that as $\{h', k', \varepsilon', \eta'\} \to \{0, 0, 0, 0\}$,

$$\underline{\lim} \int_0^t a(u_{h'}, u_{h'})\, ds \geq \int_0^t a(u, u)\, ds\,, \tag{2.41}$$

$$\lim \int_0^t J_{\eta'}(r_{h'} v)\, ds = \int_0^t J(v)\, ds\,, \tag{2.42}$$

$$\underline{\lim} \int_0^t J_{\eta'}(u_{h'})\, ds \geq \int_0^t J(u)\, ds\,. \tag{2.43}$$

Consequently, we conclude that for all $t \in E$; see (2.30),

$$\begin{aligned} &-\frac{1}{2} \|u(t)\|_{L^2}^2 + \frac{1}{2} \|u_0\|_{L^2}^2 + (u(t), v(t)) \\ &- (u_0, v(0)) - \int_0^t (u, \partial_t v)\, ds + \int_0^t a(u, v - u)\, ds + \int_0^t b(u, u, v)\, ds \\ &+ \int_0^t J(v)\, ds - \int_0^t J(u)\, ds \geq \int_0^t (f, v - u)\, ds \end{aligned} \tag{2.44}$$

holds for each $v \in C^1([0, T]\,; W^{2,p}(\Omega)^3 \cap V)$, $p > 3$. Here we also used the fact that $u_h^0 \to u_0$ strongly in $L^2(\Omega)^3$. Next we choose any $v \in L^2(0, T\,; V) \cap C([0, T]\,; X_3)$ such that $\partial_t v \in L^2(0, T\,; V')$. Then, there is a sequence $\{v_n\}$ in $C^1([0, T]\,; W^{2,p}(\Omega)^3 \cap V)$, $p > 3$, such that as $n \to \infty$,

$$v_n \to v \quad \text{strongly in } L^2(0, T\,; V) \text{ and } C([0, T]\,; X_3)\,, \tag{2.45}$$

$$\partial_t v_n \to \partial_t v \quad \text{strongly in } L^2(0, T\,; V')\,. \tag{2.46}$$

Hence, (2.44) holds for every $v \in L^2(0, T\,; V) \cap C([0, T]\,; X_3)$ with $\partial_t v \in L^2(0, T\,; V')$, for all $t \in E$. One can show the existence of such a sequence as follows. Let $v_\delta(t) = v\left(\frac{T}{T + 2\delta}(t + \delta)\right)$, for each $\delta > 0$. Then, $v_\delta \in L^2(-\delta, T + \delta\,; V) \cap C([-\delta, T + \delta]\,; X_3)$ and $\partial_t v_\delta \in L^2(-\delta, T + \delta\,; V')$. Furthermore, $v_\delta \to v$ strongly in $L^2(0, T\,; V) \cap C([0, T]\,; X_3)$ and $\partial_t v_\delta \to \partial_t v$ strongly in $L^2(0, T\,; V')$ as $\delta \to 0$. Next we set

$$v_{\delta, m}(t) = \int_{-\frac{\delta}{2}}^{T + \frac{\delta}{2}} \rho_{1/m}(t - s)\, v_\delta(s)\, ds \tag{2.47}$$

where $\rho_{1/m}(.)$ is the Friedrichs mollifier. Then, $v_{\delta, m} \in C^1(-\infty, \infty\,; V)$ and, for each fixed δ, $v_{\delta, m} \to v_\delta$ strongly in $L^2(0, T\,; V) \cap C([0, T]\,; X_3)$,

$\partial_t v_{\delta, m} \to \partial_t v_\delta$ strongly in $L^2(0, T; V')$ as $m \to \infty$. Thus, it is enough to show that $C^1([0, T]; W^{2,p}(\Omega)^3 \cap V)$, $p > 3$, is dense in $C^1([0, T]; V)$. But this follows from the fact that $W^{2,p}(\Omega)^3 \cap V$ is dense in V.

We now proceed to prove that u is the same as the solution of Theorem A denoted by $\bar{u}$ on the interval $[0, T^*]$. Substituting $\bar{u}$ for v in (2.44), we have

$$\begin{aligned} &-\frac{1}{2}\|u(t)\|_{L^2}^2 - \frac{1}{2}\|u_0\|_{L^2}^2 + (u(t), \bar{u}(t)) \\ &- \int_0^t (u, \partial_t \bar{u})\, ds + \int_0^t a(u, \bar{u} - u)\, ds \\ &+ \int_0^t b(u, u, \bar{u})\, ds + \int_0^t J(\bar{u})\, ds - \int_0^t J(u)\, ds \\ &\geqslant \int_0^t (f, \bar{u} - u)\, ds, \end{aligned} \tag{2.48}$$

for all $t \in E \cap [0, T^*]$. In the meantime, $\bar{u}$ satisfies, according to Definition 2.1,

$$\begin{aligned} (\partial_t \bar{u}, u - \bar{u}) + a(\bar{u}, u - \bar{u}) + b(\bar{u}, \bar{u}, u) + J(u) - J(\bar{u}) \geqslant \\ \geqslant (f, u - \bar{u}), \quad \text{for almost all } t \in (0, T^*), \end{aligned} \tag{2.49}$$

which yields

$$\begin{aligned} &-\frac{1}{2}\|\bar{u}(t)\|_{L^2}^2 + \frac{1}{2}\|u_0\|_{L^2}^2 + \int_0^t (\partial_t \bar{u}, u)\, ds \\ &+ \int_0^t a(\bar{u}, u - \bar{u})\, ds + \int_0^t b(\bar{u}, \bar{u}, u)\, ds \\ &+ \int_0^t J(u)\, ds - \int_0^t J(\bar{u})\, ds \geqslant \int_0^t (f, u - \bar{u})\, ds, \end{aligned} \tag{2.50}$$

for all $t \in [0, T^*]$. Adding (2.48) and (2.50), we obtain

$$\begin{aligned} \|u(t) - \bar{u}(t)\|_{L^2}^2 + 2\int_0^t a(u - \bar{u}, u - \bar{u})\, ds \leqslant \\ \leqslant 2\int_0^t |b(\bar{u}, \bar{u}, u) + b(u, u, \bar{u})|\, ds, \end{aligned} \tag{2.51}$$

for all $t \in E \cap [0, T^*]$. Meanwhile, we can estimate

$$|b(\bar{u}, \bar{u}, u) + b(u, u, \bar{u})| \leqslant M\|u\|_{L^r}\|u - \bar{u}\|_{L^2}^{2/s}\|u - \bar{u}\|_V^{1 + \frac{3}{r}}, \tag{2.52}$$

where $\frac{2}{s}+\frac{3}{r}=1$, $r>3$, and M is a positive constant depending only on Ω. Consequently, we arrive at

$$\|u(t)-\bar{u}(t)\|_{L^2}^2 \leqslant M \int_0^t \|u-\bar{u}\|_{L^2}^2 \, ds \tag{2.53}$$

for all $t \in E \cap [0, T^*]$, M being a positive constant, from which it follows that

$$u \equiv \bar{u}, \quad \text{on } [0, T^*]. \tag{2.54}$$

By the uniqueness of solution $\bar{u}$, u_h converges to u in the sense of (2.9) and (2.10) for the whole sequence $\{h, k, \varepsilon, \eta\}$.

Next we shall prove that u_h converges to u strongly in $L^2(0, T^*; W_0^{1,2}(\Omega)^3)$ by slightly modifying an argument due to [17]. We first show that $u_h \to u$ strongly in $L^2(0, \tilde{T}; W_0^{1,2}(\Omega)^3)$ for any $0<\tilde{T}<T^*$. Since u_h converges to u strongly in $L^2(0, T^*; L^2(\Omega)^3)$, there is a subsequence $\{h', k', \varepsilon', \eta'\} \to \{0,0,0,0\}$ such that $u_{h'}(t)$ converges to $u(t)$ strongly in $L^2(\Omega)^3$ for almost all $t \in [0, T^*]$. For convenience, we shall use the notation h, k, ε, η for $h', k', \varepsilon', \eta'$. Choose any $0<\tilde{T}<T^*$. Then, there is $\tilde{T}<t^*<T^*$ such that $u_h(t^*) \to u(t^*)$ strongly in $L^2(\Omega)^3$. Let N^* be an integer such that $(N^*-1)\,k \leqslant t^* < N^* k$. It follows from (2.1) that

$$\begin{aligned}\frac{1}{2}\|u_h^{N^*}\|_{L^2}^2 - \frac{1}{2}\|u_h^0\|_{L^2}^2 + \frac{1}{2}\sum_{m=1}^{N^*}\|u_h^m - u_h^{m-1}\|_{L^2}^2 \\ + \sum_{m=1}^{N^*} k a(u_h^m, u_h^m) + \sum_{m=1}^{N^*} k(J'_\eta(u_h^m), u_h^m) \\ + \frac{1}{\varepsilon}\sum_{m=1}^{N^*} k\|\nabla \cdot u_h^m\|_{L^2}^2 = \sum_{m=1}^{N^*} k(f^m, u_h^m).\end{aligned} \tag{2.55}$$

Following [17], we can define for each h a function $u_h^+ \in L^2(0, T; V_h)$ such that as $h \to 0$,

$$u_h^+ \to u \quad \text{strongly in } L^2(0, T^*; W_0^{1,2}(\Omega)^3). \tag{2.56}$$

We then consider the expression

$$\chi_h = \int_0^{kN^*} a(u_h - u_h^+, u_h - u_h^+)\, dt, \tag{2.57}$$

which can be split into

$$\chi_h^1 = \int_0^{kN^*} a(u_h^+, u_h^+)\, dt ,$$

$$\chi_h^2 = -2 \int_0^{kN^*} a(u_h, u_h^+)\, dt ,$$

$$\chi_h^3 = \int_0^{kN^*} a(u_h, u_h)\, dt .$$

Using (2.55), we can rewrite χ_h^3 as

$$\begin{aligned} \chi_h^3 = & -\frac{1}{2} \|u_h^{N^*}\|_{L^2}^2 + \frac{1}{2} \|u_h^0\|_{L^2}^2 - \frac{1}{2} \sum_{m=1}^{N^*} \|u_h^m - u_h^{m-1}\|_{L^2}^2 \\ & - \sum_{m=1}^{N^*} k(J_\eta'(u_h^m), u_h^m) - \frac{1}{\varepsilon} \sum_{m=1}^{N^*} k\|\nabla \cdot u_h^m\|_{L^2}^2 \\ & + \sum_{m=1}^{N^*} k(f^m, u_h^m) . \end{aligned} \tag{2.58}$$

Hence, as $\{h, k, \varepsilon, \eta\} \to \{0, 0, 0, 0\}$,

$$\begin{aligned} \overline{\lim}\, \chi_h \leqslant & -\frac{1}{2} \|u(t^*)\|_{L^2}^2 + \frac{1}{2} \|u_0\|_{L^2}^2 - \int_0^{t^*} a(u, u)\, dt \\ & + \int_0^{t^*} (f, u)\, dt - \underline{\lim}\, 2g \int_0^{t^*} \int_\Omega D_{II}(u_h)(\eta + D_{II}(u_h))^{-1/2}\, dx\, dt . \end{aligned} \tag{2.59}$$

Meanwhile, we set $w = 2u$ in (0.1) to see

$$\begin{aligned} \frac{1}{2} \|u(t^*)\|_{L^2}^2 - \frac{1}{2} \|u_0\|_{L^2}^2 + \int_0^{t^*} a(u, u)\, dt + \int_0^{t^*} J(u)\, dt \geqslant \\ \geqslant \int_0^{t^*} (f, u)\, dt . \end{aligned} \tag{2.60}$$

Combining the above two inequalities, we get

$$\begin{aligned} \overline{\lim}\, \chi_h & \leqslant \int_0^{t^*} J(u)\, dt - \underline{\lim}\, 2g \int_0^{t^*} \int_\Omega D_{II}(u_h)(\eta + D_{II}(u_h))^{-1/2}\, dx\, dt \\ & \leqslant \int_0^{t^*} J(u)\, dt - \underline{\lim} \int_0^{t^*} J(u_h)\, dt \leqslant 0 \end{aligned} \tag{2.61}$$

since

$$\|D_{II}(u_h)(\eta + D_{II}(u_h))^{-1/2} - D_{II}(u_h)^{1/2}\|_{L^\infty} \leqslant \sqrt{\eta} M \tag{2.62}$$

for almost all t, M being a positive constant independent of h, k, ε and η. This proves $u_h \to u$ strongly in $L^2(0, \tilde{T}; W_0^{1,2}(\Omega)^3)$ for a certain subsequence $\{h', k', \varepsilon', \eta'\} \to \{0, 0, 0, 0\}$. By the uniqueness of u, this convergence is true for the whole sequence $\{h, k, \varepsilon, \eta\}$. If $T^* < T$, then we can extend $u(t)$ to an interval $[0, T^* + \delta)$ for some small $\delta > 0$, since $u(t) \in C([0, T^*]; X_r)$. We then apply the above result so that $u_h \to u$ strongly in $L^2(0, T^*; W_0^{1,2}(\Omega)^3)$. If $T^* = T$, then we extend $f(t)$ by setting $f(t) = 0$ for $t > T$ so that $f \in L^\infty(0, \infty; L^2(\Omega)^3)$. According to (2.1), we can define u_h^m for $m = 1, 2, \ldots, 2N$ and consequently, u_h is extended to $[0, 2T]$. In the meantime, u can be also extended to a larger interval $[0, T + \delta)$. We can again apply the above result to get the strong convergence in $L^2(0, T; W_0^{1,2}(\Omega)^3)$.

The proof of Theorem 2.2 is now complete.

3. REGULARITY OF TIME-DISCRETE SOLUTIONS

In this section, we first establish interior regularity of stationary solutions and, based on this, we obtain more regularity of time-discrete approximate solutions than in [14].

THEOREM 3.1 : *Let v be a unique solution in V of*

$$\underline{a}(v, \underline{w} - v) + \underline{J_\eta}(w) - \underline{J_\eta}(\underline{v}) \geq (\underline{H}, w - v) \tag{3.1}$$

for all $w \in V$, where $H \in L^2(\Omega)^3$ is given. Let Ω_δ be the set $\{x \in \Omega$: distance $(x, \partial\Omega) > \delta\}$. Then

$$\|v\|_{W^{2,2}(\Omega_\delta)} \leq \frac{C}{\delta} (\|H\|_{L^2} + M), \tag{3.2}$$

where C and M are positive constants independent of δ, η and H.

Before proceeding to the proof, we recall that if $f \in W^{m,p}(\Omega)^3$, $\|f\|_{W^{m,p}}$ always denotes the norm in $W^{m,p}(\Omega)^3$. If the norm is taken over a space domain other than Ω, it will be denoted explicitly.

Proof: It was shown in [13] that there are a unique function v_λ in $D(A^2)$ and $p_\lambda \in W^{1,2}(\Omega)$ with the condition $\int_\Omega p_\lambda \, dx = 0$ such that

$$-\mu \, \Delta v_\lambda - \lambda\mu \, \Delta A \, v_\lambda - g \, \partial_j \frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}} + \nabla p_\lambda = H, \tag{3.3}$$

where we write $A = A_2$ which was defined in Section 1, and μ and g are the viscosity and the yield limit, respectively. The third term is a vector function which is represented by its i-th component. Throughout this section, we adopt the summation convention on repeated indices. In fact, the case $\lambda = \eta$ was discussed in [13], but the same analysis can be applied to the case $\lambda \neq \eta$ and we have

$$\|v_\lambda\|_V^2 + \lambda \|Av_\lambda\|_{L^2}^2 \leq C \|H\|_{L^2}^2 \tag{3.4}$$

and

$$\|v_\lambda\|_{W^{1,6}} + \|p_\lambda\|_{L^6} \leq C(\|H\|_{L^2} + M), \tag{3.5}$$

where C and M stand for positive constants independent of λ, η and H. Next let us write

$$Av_\lambda = -\Delta v_\lambda + \nabla q_\lambda$$

where q_λ is a scalar function. It then follows from (3.4) that

$$\lambda \|\Delta v_\lambda\|_{L^2}^2 + \lambda \|\nabla q_\lambda\|_{L^2}^2 \leq C \|H\|_{L^2}^2, \tag{3.6}$$

C being a positive constant independent of λ, η and H. Furthermore, according to Cattabriga's theorem [2], $v_\lambda \in D(A^2)$ implies that $v_\lambda \in W^{4,2}(\Omega)^3$ and $\nabla q_\lambda \in W^{2,2}(\Omega)^3$ since $\partial\Omega \in C^4$. This regularity is necessary to justify the manipulations in obtaining the estimates below. We next construct a function $\varphi_\delta(x) \in C^3(\bar{\Omega})$ such that, for all sufficiently small $\delta > 0$,

$$\varphi_\delta = 1 \quad \text{on } \Omega_\delta \quad \text{and } \varphi_\delta = 0 \quad \text{on } \partial\Omega, \tag{3.7}$$

$$|\partial_i \varphi_\delta(x)|^2 \leq \frac{C}{\delta^2} \varphi_\delta(x), \quad \text{for all } x \in \Omega, i = 1, 2, 3, \tag{3.8}$$

$$|\partial_i \partial_j \varphi_\delta(x)|^2 \leq \frac{C}{\delta^4} \varphi_\delta(x), \quad \text{for all } x \in \Omega, i, j = 1, 2, 3 \tag{3.9}$$

$$|\partial_i \partial_j \partial_l \varphi_\delta(x)| \leq \frac{C}{\delta^3}, \quad \text{for all } x \in \Omega, i, j, l = 1, 2, 3, \tag{3.10}$$

where C is a positive constant independent of δ. Let $d(x) =$ distance $(x, \partial\Omega)$ for $x \in \bar{\Omega}$. Since $\partial\Omega \in C^3$, which is enough for φ_δ, there is a positive number ζ such that $d(x) \in C^3(\Gamma_\zeta)$, where $\Gamma_\zeta = \{x \in \bar{\Omega} : d(x) \leq \zeta\}$; see [7]. We next define a function $\varphi(s) \in C^3([0, \infty))$ such that

$$\varphi(s) = \begin{cases} s^4, & \text{for } 0 \leq s < \frac{1}{2} \\ 1, & \text{for } s \geq 1, \end{cases} \tag{3.11}$$

and $\left(\frac{1}{2}\right)^4 \leq \varphi(s) \leq 1$, for $\frac{1}{2} \leq s \leq 1$, and then, set for $0 < \delta < \zeta$, $x \in \bar{\Omega}$,

$$\varphi_\delta(x) = \begin{cases} \varphi\left(\frac{1}{\delta} d(x)\right) & \text{if } d(x) \leq \delta, \\ 1, & \text{if } d(x) > \delta . \end{cases} \tag{3.12}$$

Then, it is easy to see that this $\varphi_\delta(x)$ satisfies (3.7) through (3.10). We shall need the following lemma.

LEMMA 3.2 : *If* u $\in W_0^{1,2}(\Omega)$, *then for all sufficiently small* $\delta > 0$,

$$\|u\partial_i\partial_j\varphi_\delta\|_{L^2} \leq \frac{C}{\delta}\|u\|_{W_0^{1,2}} \tag{3.13}$$

$$\|u\partial_i\partial_j\partial_l\varphi_\delta\|_{L^2} \leq \frac{C}{\delta^2}\|u\|_{W_0^{1,2}}, i, j, l = 1, 2, 3 . \tag{3.14}$$

This is an easy consequence of the well-known fact :

LEMMA 3.3 : *If* $u \in W_0^{1,2}(\Omega)$, *then*

$$\int_{\Omega - \Omega_\delta} |u|^2 \, dx \leq C\delta^2 \|u\|^2_{W_0^{1,2}}, \tag{3.15}$$

C being a positive constant independent of δ *and u.*

Let us multiply both sides of (3.3) by $-\varphi_\delta \Delta v_\lambda$ and integrate over Ω :

$$\mu \int_\Omega \varphi_\delta |\Delta v_\lambda|^2 \, dx + \lambda\mu \int_\Omega (\Delta A v_\lambda) \cdot \varphi_\delta \Delta v_\lambda \, dx +$$
$$+ g \int_\Omega \left(\partial_j \frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}}\right) \varphi_\delta \Delta v_{\lambda i} \, dx - \int_\Omega \nabla p_\lambda \cdot \varphi_\sigma \Delta v_\lambda \, dx$$
$$= -\int_\Omega H \cdot \varphi_\delta \Delta v_\lambda \, dx , \quad \text{where} \quad v_\lambda = (v_{\lambda 1}, v_{\lambda 2}, v_{\lambda 3}) . \tag{3.16}$$

We shall consider each term of (3.16). Below, C and M stand for positive constants independent of δ, λ, η and H :

$$\lambda \int_\Omega (\Delta A v_\lambda) \cdot \varphi_\delta \Delta v_\lambda \, dx =$$
$$= -\lambda \int_\Omega (\partial_i A v_\lambda) \cdot (\partial_i \varphi_\delta) \Delta v_\lambda \, dx - \lambda \int_\Omega (\partial_i A v_\lambda) \cdot \varphi_\delta \partial_i \Delta v_\lambda \, dx$$
$$= \lambda \int_\Omega (A v_\lambda) \cdot (\Delta\varphi_\delta) \Delta v_\lambda \, dx + \lambda \int_\Omega (A v_\lambda) \cdot (\partial_i \varphi_\delta) \partial_i \Delta v_\lambda \, dx$$
$$+ \lambda \sum_{i=1}^{3} \int_\Omega \varphi_\delta |\partial_i \Delta v_\lambda|^2 \, dx - \lambda \int_\Omega \partial_i \nabla q_\lambda \cdot \varphi_\delta \partial_i \Delta v_\lambda \, dx . \tag{3.17}$$

$$\lambda\left|\int_\Omega (Av_\lambda)\cdot(\Delta\varphi_\delta)\,\Delta v_\lambda\,dx\right| \leq \frac{C}{\delta^2}\,\|H\|_{L^2}^2 \quad \text{by (3.6) and (3.9)}. \tag{3.18}$$

$$\lambda\left|\int_\Omega Av_\lambda\cdot(\partial_i\varphi_\delta)\,\partial_i\Delta v_\lambda\,dx\right| \leq$$

$$\leq \lambda\int_\Omega |Av_\lambda|^2\,|\nabla\varphi_\delta|^2\,\frac{1}{\varphi_\delta}\,dx + \frac{\lambda}{4}\sum_{i=1}^{3}\int_\Omega \varphi_\delta\,|\partial_i\Delta v_\lambda|^2\,dx$$

$$\leq \frac{C}{\delta^2}\,\|H\|_{L^2}^2 + \frac{\lambda}{4}\sum_{i=1}^{3}\int_\Omega \varphi_\delta\,|\partial_i\Delta v_\lambda|^2\,dx\,,\quad \text{by (3.4) and (3.8)}. \tag{3.19}$$

$$\lambda\left|\int_\Omega \partial_i\nabla q_\lambda\cdot\varphi_\delta\,\partial_i\Delta v_\lambda\,dx\right| = \lambda\left|\int_\Omega \partial_i q_\lambda(\partial_j\varphi_\delta)\,\partial_i\Delta v_{\lambda j}\,dx\right|,$$

since $\nabla\cdot v_\lambda = 0$,

$$\leq \lambda\int_\Omega |\nabla q_\lambda|^2\,\frac{1}{\varphi_\delta}\,|\nabla\varphi_\delta|^2\,dx + \frac{\lambda}{4}\sum_{i=1}^{3}\int_\Omega \varphi_\delta\,|\partial_i\Delta v_\lambda|^2\,dx$$

$$\leq \frac{C}{\delta^2}\,\|H\|_{L^2}^2 + \frac{\lambda}{4}\sum_{i=1}^{3}\int_\Omega \varphi_\delta\,|\partial_i\Delta v_\lambda|^2\,dx\,,\quad \text{by (3.6) and (3.8)}. \tag{3.20}$$

$$\int_\Omega\left(\partial_j\frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}}\right)\varphi_\delta\,\Delta v_{\lambda i}\,dx = -\int_\Omega \frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}}\,(\partial_j\varphi_\delta)\,\Delta v_{\lambda i}\,dx - \\ - \int_\Omega \frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}}\,\varphi_\delta\,\partial_j\Delta v_{\lambda i}\,dx\,. \tag{3.21}$$

$$-\int_\Omega \frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}}\,\varphi_\delta\,\partial_j\Delta v_{\lambda i}\,dx = -\int_\Omega \frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}}\,\varphi_\delta\,\Delta D_{ij}(v_\lambda)\,dx \\ = \int_\Omega \frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}}\,(\partial_l\varphi_\delta)\,\partial_l D_{ij}(v_\lambda)\,dx \\ + \int_\Omega\left(\partial_l\frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}}\right)\varphi_\delta\,\partial_l D_{ij}(v_\lambda)\,dx\,. \tag{3.22}$$

According to Lemma 1.1,

$$\int_\Omega\left(\partial_l\frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}}\right)\varphi_\delta\,\partial_l D_{ij}(v_\lambda)\,dx \geq 0\,. \tag{3.23}$$

Since

$$(\partial_i \varphi_\delta)\, \partial_j \partial_l v_\lambda = \partial_j \partial_l ((\partial_i \varphi_\delta)\, v_\lambda) - \\ - (\partial_j \partial_l \partial_i \varphi_\delta)\, v_\lambda - (\partial_i \partial_l \varphi_\delta)\, \partial_j v_\lambda - (\partial_j \partial_i \varphi_\delta)\, \partial_l v_\lambda$$

and, for each i, j, l,

$$\| \partial_j \partial_l ((\partial_i \varphi_\delta)\, v_\lambda) \|_{L^2}^2 \leq C \| \Delta ((\partial_i \varphi_\delta)\, v_\lambda) \|_{L^2}^2 , \tag{3.24}$$

we deduce

$$\sum_{i,j,l=1}^{3} \| (\partial_i \varphi_\delta)(\partial_j \partial_l v_\lambda) \|_{L^2}^2 \leq C \int_\Omega |\nabla \varphi_\delta|^2 \, |\Delta v_\lambda|^2 \, dx$$
$$+ C \sum_{i,j=1}^{3} \int_\Omega (\partial_i \partial_j \varphi_\delta)^2 \, |\nabla v_\lambda|^2 \, dx + C \sum_{i,j,l=1}^{3} \int_\Omega (\partial_i \partial_j \partial_l \varphi_\delta)^2 \, |v_\lambda|^2 \, dx$$
$$\leq \frac{C}{\delta^2} \int_\Omega \varphi_\delta |\Delta v_\lambda|^2 \, dx + \frac{C}{\delta^4} \|H\|_{L^2}^2 ,$$
$$\text{by (3.4), (3.8), (3.9) and (3.14)} . \tag{3.25}$$

Consequently, we have from (1.5), (3.21) through (3.25),

$$- g \int_\Omega \left(\partial_j \frac{D_{ij}(v_\lambda)}{\sqrt{\eta + D_{II}(v_\lambda)}} \right) \varphi_\delta \, \Delta v_{\lambda i} \, dx \leq$$
$$\leq \frac{\mu}{4} \int_\Omega \varphi_\delta |\Delta v_\lambda|^2 \, dx + \frac{C}{\delta^2} + \frac{C}{\delta^2} \|H\|_{L^2} . \tag{3.26}$$

Next we find

$$\int_\Omega \nabla p_\lambda \cdot \varphi_\delta \, \Delta v_\lambda \, dx = - \int_\Omega p_\lambda (\partial_i \varphi_\delta)\, \Delta v_{\lambda i} \, dx$$

since $\nabla \cdot v_\lambda = 0$,

$$\leq C (\|H\|_{L^2} + M) \left(\int_\Omega (|\nabla \varphi_\delta|)^2 \, |\Delta v_\lambda|^2 \, dx \right)^{1/2} , \quad \text{by (3.5)}$$
$$\leq \frac{\mu}{4} \int_\Omega \varphi_\delta |\Delta v_\lambda|^2 \, dx + \frac{C}{\delta^2} (\|H\|_{L^2}^2 + M), \quad \text{by (3.8)} . \tag{3.27}$$

Now we combine (3.17) through (3.27) to conclude

$$\int_\Omega \varphi_\delta |\Delta v_\lambda|^2 \, dx \leq \frac{C}{\delta^2} (\|H\|_{L^2}^2 + \|H\|_{L^2} + M), \tag{3.28}$$

from which it follows that

$$\| \varphi_\delta \, \Delta v_\lambda \|_{L^2} \leq \frac{C}{\delta} \, (\| H \|_{L^2} + M) \tag{3 29}$$

Since $\varphi_\delta \, v_\lambda \in W_0^{1\ 2}(\Omega)^3 \cap W^{2\ 2}(\Omega)^3$, we have

$$\| \varphi_\delta \, v_\lambda \|_{W^{2\ 2}} \leq C \, \| \Delta (\varphi_\delta \, v_\lambda) \|_{L^2} ,$$

by (3 29)

$$\leq \frac{C}{\delta} \, (\| H \|_{L^2} + M) + C \, \| \nabla \varphi_\delta \|_{L^\infty} \, \| v_\lambda \|_{W_0^{1\ 2}} + C \, \| v_\lambda \, \Delta \varphi_\delta \|_{L^2} ,$$

by (3 4), (3 8) and (3 13),

$$\leq \frac{C}{\delta} \, (\| H \|_{L^2} + M) \tag{3 30}$$

With the aid of (3 4), we can extract a subsequence still denoted by $\{v_\lambda\}$ such that

$$v_\lambda \to v^* \quad \text{weakly in } V \text{ as } \lambda \to 0 \text{ for some function } v^* \tag{3 31}$$

In the meantime, each v_λ satisfies

$$a(v_\lambda, w - v_\lambda) + \lambda \mu (A v_\lambda, A w - A v_\lambda) + \\ + J_\eta(w) \quad J_\eta(v_\lambda) \geq (H, w - v_\lambda) , \tag{3 32}$$

for all $w \in S$

By virtue of

$$\overline{\lim_{\lambda \to 0}} \, \lambda (A v_\lambda, A w - A v_\lambda) \leq \overline{\lim_{\lambda \to 0}} \, \lambda (v_\lambda, A^2 w) = 0 , \tag{3 33}$$

$$\underline{\lim_{\lambda \to 0}} \, J_\eta(v_\lambda) \geq J_\eta(v^*) , \tag{3 34}$$

and

$$\underline{\lim_{\lambda \to 0}} \, a(v_\lambda, v_\lambda) \geq a(v^*, v^*) , \tag{3 35}$$

we can pass $\lambda \to 0$ in (3 32) to obtain

$$a(v^*, w - v^*) + J_\eta(w) - J_\eta(v^*) \geq (H, w - v^*) , \\ \text{for all } w \in S , \tag{3 36}$$

and hence, for all $w \in V$ By the uniqueness of solution of (3 1), $v^* \equiv v$ Consequently, (3 30) yields (3 2), and the proof is complete

Next we consider the regularity of solutions to the problem : for given $u^{m-1} \in V$, find $u^m \in V$ and $p^m \in L^2(\Omega)$ such that

$$\frac{1}{k}(u^m - u^{m-1}, w) + a(u^m, w) + b(u^{m-1}, u^m, w) + \\ + (J'_\eta(u^m), w) + (\nabla p^m, w) = (f^m, w) \quad (3.37)$$

for all $w \in W_0^{1,2}(\Omega)^3$. Here, $k = T/N$, $u^0 \in V$ and $f \in C([0, T]; L^2(\Omega)^3)$ are given data, and $f^m = f(km)$, $m = 1, \dots, N$. Existence and uniqueness of u^m can be easily shown together with the estimates

$$\max_{1 \leqslant m \leqslant N} \|u^m\|_{L^2} \leqslant M \quad (3.38)$$

$$\sum_{m=1}^{N} k\|u^m\|_V^2 \leqslant M, \quad (3.39)$$

where M denotes positive constants independent of k and η. Since $V \subset X_6$, we can use the result of [14] with $r = 6$ to conclude that there are positive numbers $T_1 \leqslant T$ and σ_1 independent of k and η such that

$$\|u^m\|_{L^6} \leqslant M, \quad M \text{ being a positive constant independent of } k \text{ and } \eta, \quad (3.40)$$

provided $k \leqslant \sigma_1$ and $km \leqslant T_1$. Let us substitute $\frac{1}{k}(u^m - u^{m-1})$ for w in (3.37) :

$$\left\|\frac{1}{k}(u^m - u^{m-1})\right\|_{L^2}^2 + \frac{1}{2k}a(u^m, u^m) - \frac{1}{2k}a(u^{m-1}, u^{m-1}) \\ + \frac{1}{2k}a(u^m - u^{m-1}, u^m - u^{m-1}) \leqslant \frac{1}{k}J_\eta(u^{m-1}) - \frac{1}{k}J_\eta(u^m) \\ + \|f^m\|_{L^2}\left\|\frac{1}{k}(u^m - u^{m-1})\right\|_{L^2} + \left|b\left(u^{m-1}, u^m, \frac{1}{k}(u^m - u^{m-1})\right)\right|, \quad (3.41)$$

for $m = 1, \dots, N$. We proceed to estimate the last term of (3.41) by writing

$$u^m = x^m + y^m + z^m \quad (3.42)$$

where $x^m \in V$, $y^m \in V$ and $z^m \in V$ are solutions of

$$a(x^m, v) = -(J'_\eta(u^m), v), \quad \text{for all } v \in V, \quad (3.43)$$

$$a(y^m, v) = -b(u^{m-1}, u^m, v) + (f^m, v), \quad \text{for all } v \in V, \quad (3.44)$$

$$a(z^m, v) = \left(-\frac{1}{k}(u^m - u^{m-1}), v\right), \quad \text{for all } v \in V, \quad (3.45)$$

respectively. With the help of Cattabriga's theorem [2] and (3.40), we find

$$\|y^m\|_{W^{1,3}} \leqslant M\ , \quad \text{for all } m = 1, \dots, N^* \quad \text{where} \quad kN^* \leqslant T_1\ , \tag{3.46}$$

and

$$\|x^m\|_{W^{1,3}} \leqslant M\ , \quad \text{for all } m = 1, \dots, N\ , \tag{3.47}$$

where M denotes positive constants independent of k and η. We next consider the operator $\Lambda : q \to \psi$ where $\psi \in V$ is the solution of $a(\psi, v) = (q, v)$ for all $v \in V$. Then, Λ is a bounded linear operator from $L^3(\Omega)^3$ into $V \cap W^{2,3}(\Omega)^3$, and also from $W^{-1,3}(\Omega)^3$ into $V \cap W^{1,3}(\Omega)^3$ according to Cattabriga's theorem [2]. Therefore, Λ is a bounded linear operator from $[L^3(\Omega)^3, W^{-1,3}(\Omega)^3]_{1/2}$ into $[W^{2,3}(\Omega)^3, W^{1,3}(\Omega)^3]_{1/2} \cap V$, where $[\,,]_{1/2}$ is an intermediate space defined by the complex interpolation method. We note that $[W^{2,3}(\Omega)^3, W^{1,3}(\Omega)^3]_{1/2} = H^{3/2,3}(\Omega)^3$, where $H^{s,p}(\Omega)$ stands for the space of Bessel potentials restricted to Ω. Since $[L^3(\Omega)^3, W^{-1,3}(\Omega)^3]_{1/2}$ is the dual of $[W_0^{1,3/2}(\Omega)^3, L^{3/2}(\Omega)^3]_{1/2}$ and $[W_0^{1,3/2}(\Omega)^3, L^{3/2}(\Omega)^3]_{1/2} = H^{1/2,3/2}(\Omega)^3$ is densely imbedded into $L^2(\Omega)^3$, we see that $L^2(\Omega)^3$ is imbedded into $[L^3(\Omega)^3, W^{-1,3}(\Omega)^3]_{1/2}$; see [19]. Therefore, we have

$$\|\Lambda q\|_{H^{3/2,3}} \leqslant M \|q\|_{L^2}\ , \quad \text{for all } q \in L^2(\Omega)^3 \tag{3.48}$$

M being a positive constant. Consequently, we obtain

$$\|z^m\|_{H^{3/2,3}} \leqslant M \left\| \frac{1}{k} (u^m - u^{m-1}) \right\|_{L^2}, \quad \text{for } m = 1, \dots, N. \tag{3.49}$$

In the meantime, we have by (3.40), (3.46) and (3.47),

$$\|z^m\|_{L^3} \leqslant \|u^m\|_{L^3} + \|y^m\|_{L^3} + \|x^m\|_{L^3} \leqslant M\ , \quad m = 1, \dots, N^* \tag{3.50}$$

and thus, by interpolation,

$$\|z^m\|_{W^{1,3}} \leqslant M \left\| \frac{1}{k} (u^m - u^{m-1}) \right\|_{L^2}^{2/3}, \quad m = 1, \dots, N^*\ . \tag{3.51}$$

It now follows from (3.46), (3.47) and (3.51) that

$$\begin{aligned} &\left| b\left(u^{m-1}, u^m, \frac{1}{k} (u^m - u^{m-1})\right) \right| \leqslant \\ &\leqslant M \|u^{m-1}\|_{L^6} \|u^m\|_{W^{1,3}} \left\| \frac{1}{k} (u^m - u^{m-1}) \right\|_{L^2} \\ &\leqslant M \left(\left\| \frac{1}{k} (u^m - u^{m-1}) \right\|_{L^2} + \left\| \frac{1}{k} (u^m - u^{m-1}) \right\|_{L^2}^{5/3} \right), \quad m = 1, \dots, N^*\ , \end{aligned} \tag{3.52}$$

which, together with (3.41), yields

$$\frac{1}{2}k\left\|\frac{1}{k}(u^m-u^{m-1})\right\|_{L^2}^2+\frac{1}{2}a(u^m,u^m)-\frac{1}{2}a(u^{m-1},u^{m-1})+$$
$$+\frac{1}{2}a(u^m-u^{m-1},u^m-u^{m-1})\leq J_\eta(u^{m-1})-J_\eta(u^m)+kM\,,\quad(3.53)$$

for $m=1,\ldots,N^*$, M being a positive constant independent of k and η. Hence, we obtain

$$\sum_{m=1}^{N^*}k\left\|\frac{1}{k}(u^m-u^{m-1})\right\|_{L^2}^2+a(u^{N^*},u^{N^*})\leq$$
$$\leq a(u^0,u^0)+2J_\eta(u^0)+M\,,\quad(3.54)$$

which is valid for any $N^*\geq 1$ such that $kN^*\leq T_1$. In view of our purpose, we may assume $0<\eta\leq 1$. Then, we derive from (3.45) and (3.54),

$$\sum_{m=1}^{N^*}k\|z^m\|_{W^{2,2}}^2\leq M\quad(3.55)$$

and

$$\|u^m\|_V\leq M\,,\quad m=0,1,\ldots,N^*\,,\quad(3.56)$$

for all $N^*\geq 1$ such that $kN^*\leq T_1$, where M denotes positive constants independent of k and η. By means of (3.46), (3.47) and (3.55), we can estimate

$$\sum_{m=1}^{N^*}k\|u^m\|_{W^{1,3}}^2\leq M\,,\quad(3.57)$$

and hence, by recalling (3.44) and using Cattabriga's theorem,

$$\sum_{m=1}^{N^*}k\|y^m\|_{W^{2,2}}^2\leq C\sum_{m=1}^{N^*}k\|f^m\|_{L^2}^2+$$
$$+C\sum_{m=1}^{N^*}k\|u^{m-1}\|_V^2\,\|u^m\|_{W^{1,3}}^2\leq M\,,\quad\text{by (3.56)}\,.\quad(3.58)$$

Next we can use Theorem 3.1 by putting

$$H=-\frac{1}{k}(u^m-u^{m-1})+f^m+u_j^{m-1}\,\partial_j u^m$$

to obtain

$$\|u^m\|_{W^{2,2}(\Omega_\delta)}\leq$$
$$\leq\frac{C}{\delta}\left(M+\left\|\frac{1}{k}(u^m-u^{m-1})\right\|_{L^2}+\|f^m\|_{L^2}+\|u^{m-1}\|_V\,\|u^m\|_{W^{1,3}}\right)\,.\quad(3.59)$$

Combining (3.54) through (3.59), we have

$$\sum_{m=1}^{N^*} k\|x^m\|^2_{W^{2,2}(\Omega_\delta)} \leq \frac{M}{\delta^2}, \tag{3.60}$$

where M is a positive constant independent of δ, k and η. Finally, according to Cattabriga's theorem, (1.5) yields

$$\|x^m\|_{W^{1,p}} \leq C_p, \quad \text{for any } 1 < p < \infty, \quad m = 1, \ldots, N, \tag{3.61}$$

where C_p is a positive constant depending only on p. We have proved

THEOREM 3.2 : *There are positive numbers $T_1 \leq T$ and σ_1 independent of k and η such that the solution u^m of* (3.37) *can be written as $u^m = x^m + y^m + z^m$, where y^m and z^m satisfy* (3.58) *and* (3.55), *respectively, and x^m satisfies* (3.60) *and* (3.61), *provided $0 < k \leq \sigma_1$, $kN^* \leq T_1$ and $0 < \eta \leq 1$.*

4. ERROR ESTIMATES

To obtain error estimates, we need solutions more regular than those in Section 2. Here we assume $u_0 \in G$, $f \in C([0, T]; L^2(\Omega)^3)$ and $\partial_t f \in L^2(0, T; W^{-1,2}(\Omega)^3)$. Then, according to Theorem 3.2 in [13], there is a unique solution u of (0.1) to (0.4) on some interval $[0, T_2)$, $0 < T_2 \leq T$. Furthermore,

$$u \in L^\infty(0, T_2; W_0^{1,6}(\Omega)^3) \cap C([0, T_2]; V) \tag{4.1}$$

$$\partial_t u \in L^2(0, T_2; V) \cap L^\infty(0, T_2; X_2). \tag{4.2}$$

Our intent is to estimate the error between the true solution u and the approximate solution u_h constructed by the scheme (2.1) and (2.5). For this purpose, it is necessary to introduce two intermediary functions which bridge the gap between u and u_h. The first one is the solution u_η of the regularized problem with $0 < \eta \leq 1$:

$$(\partial_t u_\eta, w - u_\eta) + a(u_\eta, w - u_\eta) + b(u_\eta, u_\eta, w) + \\ + J_\eta(w) - J_\eta(u_\eta) \geq (f, w - u_\eta),$$

$$\text{for every } w \in V, \quad \text{for almost all } t \in (0, T), \tag{4.3}$$

$$u_\eta \in L^2(0, T; V), \ \partial_t u_\eta \in L^2(0, T; V), \tag{4.4}$$

$$u_\eta(x, 0) = u_{0\eta}(x) \quad \text{in} \quad \Omega. \tag{4.5}$$

The second one is the time-discrete approximate solution u_k constructed in [14] ; see also (3.37). Our error estimates consist of the following three estimates.

(Estimate I) Estimate the difference between u and u_η.
(Estimate II) Estimate the difference between u_η and u_k.
(Estimate III) Estimate the difference between u_k and u_h.

4.1. Estimate I

First of all, we have to choose $u_{0\eta}$ in (4.5).

LEMMA 4.1 : *Let $u_0 \in G$. Then, there is $u_{0\eta} \in V \cap W_0^{1,6}(\Omega)^3$ such that*

$$\|u_0 - u_{0\eta}\|_V \leq C\eta^{1/4}, \quad C \text{ being a positive constant depending only on } \Omega, \tag{4.6}$$

$$a(u_{0\eta}, v - u_{0\eta}) + b(u_{0\eta}, u_{0\eta}, v) + J_\eta(v) - J_\eta(u_{0\eta}) \geq \geq (H_\eta, v - u_{0\eta}), \quad \text{for all } v \in V, \tag{4.7}$$

where $H_\eta \in L^2(\Omega)^3$ with

$$\|H_\eta\|_{L^2} \leq M, \tag{4.8}$$

and M is a positive constant independent of η.

Proof: By the same argument as in [13], there is a unique function $u_{0\eta}$ in $V \cap W_0^{1,6}(\Omega)^3$ satisfying

$$a(u_{0\eta}, v - u_{0\eta}) + J_\eta(v) - J_\eta(u_{0\eta}) \geq \geq (H, v - u_{0\eta}) - b(u_0, u_0, v - u_{0\eta}) \tag{4.9}$$

for all $v \in V$, where H is the function associated with u_0 in the definition of G. Here the norm of $u_{0\eta}$ in $W_0^{1,6}(\Omega)^3$ depends on the L^2-norm of H and $u_{0j}\, \partial_j u_0$, and is independent of η. Meanwhile, u_0 satisfies

$$a(u_0, u_{0\eta} - u_0) + J(u_{0\eta}) - J(u_0) \geq \geq (H, u_{0\eta} - u_0) - b(u_0, u_0, u_{0\eta} - u_0) \tag{4.10}$$

since $u_{0\eta} \in V$. Substituting u_0 for v in (4.9) and adding the above two inequalities, we have

$$a(u_{0\eta} - u_0, u_{0\eta} - u_0) \leq C\sqrt{\eta}, \quad \text{by (1.6)}, \tag{4.11}$$

where C is a positive constant depending only on Ω and g. Next we set $H_\eta = H - u_{0j}\partial_j u_0 + u_{0\eta j}\, \partial_j u_{0\eta}$. Then, $u_{0\eta}$ satisfies (4.6) and (4.7) with this H_η, which satisfies (4.8).

The assertion on the solution u_η of (4.3), (4.4) and (4.5) is

PROPOSITION 4.2 : *There is a unique solution* u_η *of* (4.3), (4.4) *and* (4.5) *on some interval* $[0, T_3)$, $0 < T_3 \leq T$, *with* $u_{0\eta}$ *constructed in the above lemma. Furthermore,*

$$u_\eta \in L^\infty(0, T_3 ; W_0^{1,6}(\Omega)^3) \cap C([0, T_3] ; V), \tag{4.12}$$

$$\partial_t u_\eta \in L^2(0, T_3 ; V) \cap L^\infty(0, T_3 ; X_2), \tag{4.13}$$

$$\partial_t^2 u_\eta \in L^2(0, T_3 ; V'). \tag{4.14}$$

Here, T_3 *and the norm of* u_η *in the function class of* (4.12), (4.13) *are independent of* $0 < \eta \leq 1$, *and*

$$\| \partial_t^2 u_\eta \|_{L^2(0, T_3, V')} \leq M \eta^{-1/2}, \tag{4.15}$$

where M *is a positive constant independent of* $0 < \eta \leq 1$.

The proof of the above assertion is the same as that of Theorem 3.2 in [13]. In the process of this proof, the estimate of u_η in the norm of the function class in (4.12) and (4.13) follows. It is also seen that T_3 can be chosen independently of η on account of (4.8). To derive (4.14) and (4.15), we note that (4.3) implies

$$(\partial_t u_\eta, w) + a(u_\eta, w) + b(u_\eta, u_\eta, w) + (J_\eta'(u_\eta), w) = \\ = (f, w), \quad \text{for every } w \in V, \tag{4.16}$$

for almost all $t \in (0, T_3)$. By virtue of (4.12) and (4.13), it is apparent that $\partial_t^2 u_\eta \in L^2(0, T_3 ; V')$ and that for all $0 < \eta \leq 1$,

$$\left\| \partial_t \frac{D_{ij}(u_\eta)}{\sqrt{\eta + D_{II}(u_\eta)}} \right\|_{L^2(0, T_3, L^2(\Omega))} \leq M \eta^{-1/2}. \tag{4.17}$$

Hence, (4.15) is obtained. We are now ready to estimate the difference between u and u_η. Let us set

$$T_4 = \min(T_2, T_3). \tag{4.18}$$

Substitute u_η for w in (0.1) and u for w in (4.3), and add the resulting inequalities :

$$\frac{1}{2} \frac{d}{dt} \| u - u_\eta \|_{L^2}^2 + a(u - u_\eta, u - u_\eta) \leq | b(u_\eta, u_\eta, u) + b(u, u, u_\eta) | \\ + | J_\eta(u_\eta) - J(u_\eta) | + | J(u) - J_\eta(u) |, \quad \text{for almost all } t \in (0, T_4), \tag{4.19}$$

which, combined with (1.6), (4.1) and (4.12), yields

$$\frac{d}{dt}\|u-u_\eta\|_{L^2}^2+a(u-u_\eta,u-u_\eta)\leq M(\sqrt{\eta}+\|u-u_\eta\|_{L^2}^2), \quad (4.20)$$

for almost all $t\in(0,T_4)$, M being a positive constant independent of $0<\eta\leq 1$. From this and (4.6), we deduce

$$\sup_{t\in[0,T_4]}\|u(t)-u_\eta(t)\|_{L^2}^2\leq M\eta^{1/2}, \quad (4.21)$$

$$\int_0^{T_4}\|u-u_\eta\|_V^2\,dt\leq M\eta^{1/2}, \quad (4.22)$$

where M denotes positive constants independent of $0<\eta\leq 1$. In fact, we shall need the discretized version of the above estimates later on :

$$\|u(km)-u_\eta(km)\|_{L^2}^2\leq M\eta^{1/2}, \quad \text{for} \quad m=1,\dots,N^*, \quad (4.23)$$

$$\sum_{m=1}^{N^*}k\|u(km)-u_\eta(km)\|_V^2\leq M(\eta^{1/2}+k^2) \quad (4.24)$$

where $kN^*\leq T_4$, and M denotes positive constants independent of k and $0<\eta\leq 1$. (4.24) follows from (4.22) and the fact that the norm of $\partial_t(u-u_\eta)$ in $L^2(0,T_4;V)$ is bounded uniformly in $0<\eta\leq 1$.

4.2. Estimate II

Let us define

$$u_k(t)=u^m, \quad \text{for} \quad (m-1)k\leq t<mk, \quad m=1,\dots,N, \quad (4.25)$$

where u^m's are determined through (3.37) with $u^0=u_{0\eta}$, which is the same as in Proposition 4.2. According to [14], a solution of (4.3), (4.4) and (4.5) can be obtained as a limit of the sequence $\{u_k\}$. By the uniqueness of solution, this solution has to coincide with u_η of Proposition 4.2 on the time interval $[0,T_5]$, where

$$T_5=\min(T_1,T_4). \quad (4.26)$$

Here, $[0,T_1]$ is the time interval associated with the estimate (3.40) where the solution exists according to the method of [14]. T_4 was defined by (4.18). We shall estimate the difference between u_k and u_η on the interval $[0,T_5]$ by using the argument of [8]. Since (4.13) and (4.14) imply

$\partial_t u_\eta \in C([0, T_3]; X_2)$ possibly after a modification on a set of measure zero, it follows from (4.3) that

$$\left(\frac{1}{k}(u_\eta(t_m) - u_\eta(t_{m-1})), v\right) + a(u_\eta(t_m), v) + b(u_\eta(t_{m-1}), u_\eta(t_m), v) + \\ + (J'_\eta(u_\eta(t_m)), v) = (f(t_m), v) + (R^m, v), \quad \text{for all } v \in V, \quad (4.27)$$

where $t_m = mk$, $m = 0, 1, \ldots, N^*$, $kN^* \leqslant T_5$ and

$$R^m = -\partial_t u_\eta(t_m) + \frac{1}{k}(u_\eta(t_m) - u_\eta(t_{m-1})) + \\ + \sum_{j=1}^{3}(u_{\eta j}(t_{m-1}) - u_{\eta j}(t_m))\, \partial_j u_\eta(t_m) \cdot \quad (4.28)$$

We next set

$$e^m = u_\eta(t_m) - u^m, \quad \text{for} \quad m = 0, 1, \ldots, N^*, kN^* \leqslant T_5 . \quad (4.29)$$

Subtracting (3.37) from (4.27), we have

$$(e^m - e^{m-1}, v) + ka(e^m, v) + kb(e^{m-1}, u_\eta(t_m), v) \\ + kb(u^{m-1}, e^m, v) + k(J'_\eta(u_\eta(t_m)) - J'_\eta(u^m), v) \\ - k(R^m, v), \quad \text{for} \quad m = 1, \ldots . N^*, kN^* \leqslant T_5 . \quad (4.30)$$

Putting $v = e^m$, we obtain

$$\frac{1}{2}\|e^m\|_{L^2}^2 - \frac{1}{2}\|e^{m-1}\|_{L^2}^2 + \\ + \frac{1}{2}\|e^m - e^{m-1}\|_{L^2}^2 + ka(e^m, e^m) \leqslant k|b(e^{m-1}, u_\eta(t_m), e^m)| \\ + k|(R^m, e^m)|, \quad \text{for} \quad m = 1, \ldots, N^*, kN^* \leqslant T_5 . \quad (4.31)$$

Since the norm of u_η in $C([0, T_3]; V)$ is bounded uniformly in $0 < \eta \leqslant 1$, we estimate

$$|b(e^{m-1}, u_\eta(t_m), e^m)| \leqslant M\|e^{m-1}\|_V^{1/2}\, \|e^{m-1}\|_{L^2}^{1/2}\, \|e^m\|_V \quad (4.32)$$

for $m = 1, \ldots, N^*$, $kN^* \leqslant T_5$ and, following [8],

$$\|R^m\|_{V'}^2 \leqslant Mk \int_{t_{m-1}}^{t_m} (\|\partial_t^2 u_\eta\|_{V'}^2 + \|\partial_t u_\eta\|_V^2)\, dt , \quad (4.33)$$

which yields by (4.13) and (4.15),

$$\sum_{m=1}^{N^*} k\|R^m\|_{V'}^2 \leqslant Mk^2\left(\frac{1}{\eta}+M\right), \tag{4.34}$$

where M denotes positive constants independent of k and $0<\eta\leqslant 1$. Combining (4.31), (4.32) and (4.34), we derive, for all N^* such that $kN^*\leqslant T_5$,

$$\|e^{N^*}\|_{L^2}^2 + \sum_{m=1}^{N^*}\|e^m - e^{m-1}\|_{L^2}^2 + \sum_{m=1}^{N^*} ka(e^m, e^m) \leqslant$$
$$\leqslant M\sum_{m=1}^{N^*} k\|e^{m-1}\|_{L^2}^2 + Mk^2\left(\frac{1}{\eta}+M\right), \tag{4.35}$$

where M denotes positive constants independent of k and $0<\eta\leqslant 1$. According to Lemma 2.4 of [8], (4.35) yields

$$\max_{0\leqslant m\leqslant N^*}\|e^m\|_{L^2}^2 + \sum_{m=1}^{N^*} k\|e^m\|_V^2 \leqslant Mk^2\left(\frac{1}{\eta}+M\right) \tag{4.36}$$

for $kN^*\leqslant T_5$, where M denotes positive constants independent of k and $0<\eta\leqslant 1$.

4.3. Estimate III

For convenience, we shall use the following terminology.

DEFINITION 4.3 : *A number is called « universal » if it is a positive constant independent of h, k, ε and η.*

As before, we assume that $0<\eta\leqslant 1$, and retain the meaning of T_5 defined by (4.26). (2.1) can be rewritten as

$$(u_h^m - u_h^{m-1}, v_h - u_h^m) + ka(u_h^m, v_h - u_h^m) +$$
$$+ k\tilde{b}(u_h^{m-1}, u_h^m, v_h - u_h^m) + kJ_\eta(v_h) - kJ_\eta(u_h^m)$$
$$+ k\left(\frac{1}{\varepsilon}\nabla\cdot u_h^m, \nabla\cdot(v_h - u_h^m)\right) \geqslant k(f^m, v_h - u_h^m), \tag{4.37}$$

for all $v_h\in V_h$, $m=1,\dots,N$, where we take

$$u_h^0 = r_h u_0, \tag{4.38}$$

$$f^m = f(km), \quad m = 1,\dots,N. \tag{4.39}$$

Here u_0 is the same as in the previous section, and r_h is the operator defined by (1.13). In the meantime, (3.37) can be rewritten as

$$(u^m - u^{m-1}, w - u^m) + ka(u^m, w - u^m) + k\tilde{b}(u^{m-1}, u^m, w - u^m) + \\ + kJ_\eta(w) - kJ_\eta(u^m) - k(p^m, \nabla \cdot (w - u^m)) \geqslant k(f^m, w - u^m) \quad (4.40)$$

for all $w \in W_0^{1,2}(\Omega)^3$, $m = 1, \dots, N$. Requiring that $\int_\Omega p^m \, dx = 0$ for each m, it follows from (3.37) and (3.54),

$$\sum_{m=1}^{N^*} k \|p^m\|_{L^2}^2 \leqslant M , \quad (4.41)$$

where $kN^* \leqslant T_5$ and M is universal. By Theorem 3.2, each $u^m \in W_0^{1,6}(\Omega)^3$ and hence, we can substitute $r_h u^m$ for v_h in (4.37) :

$$(u_h^m - u_h^{m-1}, u^m - u_h^m) + ka(u_h^m, u^m - u_h^m) + k\tilde{b}(u_h^{m-1}, u_h^m, u^m - u_h^m) + \\ + kJ_\eta(u^m) - kJ_\eta(u_h^m) + k\left(\frac{1}{\varepsilon} \nabla \cdot u_h^m, \nabla \cdot (u^m - u_h^m)\right) \geqslant \\ \geqslant k(f^m, u^m - u_h^m) + R_h^m , \quad (4.42)$$

where

$$R_h^m = (u_h^m - u_h^{m-1}, u^m - r_h u^m) + \\ + ka(u_h^m, u^m - r_h u^m) + k\tilde{b}(u_h^{m-1}, u_h^m, u^m - r_h u^m) \\ + kJ_\eta(u^m) - kJ_\eta(r_h u^m) + k\left(\frac{1}{\varepsilon} \nabla \cdot u_h^m, \nabla \cdot (u^m - r_h u^m)\right) \\ + k(f^m, r_h u^m - u^m) , \quad (4.43)$$

for $m = 1, \dots, N$. Putting $w = u_h^m$ in (4.40), we write

$$(u^m - u^{m-1}, u_h^m - u^m) + ka(u^m, u_h^m - u^m) + \\ + k\tilde{b}(u^{m-1}, u^m, u_h^m - u^m) + kJ_\eta(u_h^m) \\ - kJ_\eta(u^m) - k(p^m, \nabla \cdot (u_h^m - u^m)) \geqslant k(f^m, u_h^m - u^m) , \quad (4.44)$$

for $m = 1, \dots, N$. Adding (4.42) and (4.44), we have

$$(u_h^m - u^m, u_h^m - u^m) - (u_h^{m-1} - u^{m-1}, u_h^m - u^m) + \\ + ka(u_h^m - u^m, u_h^m - u^m) + k\tilde{b}(u_h^{m-1} - u^{m-1}, u^m, u_h^m - u^m) \\ + k\left(\frac{1}{\varepsilon} \nabla \cdot u_h^m, \nabla \cdot u_h^m\right) + k(p^m, \nabla \cdot (u_h^m - u^m)) \\ \leqslant - R_h^m , \quad \text{for} \quad m = 1, \dots, N . \quad (4.45)$$

For $m = 1, .., N^*$, where $kN^* \leqslant T_5$, we use (3 40) and integration by parts to estimate

$$\left|\tilde{b}(u_h^{m-1} - u^{m-1}, u^m, u_h^m - u^m)\right| \leqslant$$
$$\leqslant M \|u_h^{m-1} - u^{m-1}\|_{W_0^{1\,2}} \|u^m\|_{L^6} \|u_h^m - u^m\|_{W_0^{1\,2}}^{1/2} \|u_h^m - u^m\|_{L^2}^{1/2}$$
$$+ M \|u_h^m - u^m\|_{W_0^{1\,2}} \|u^m\|_{L^6} \|u_h^{m-1} - u^{m-1}\|_{W_0^{1\,2}}^{1/2} \|u_h^{m-1} - u^{m-1}\|_{L^2}^{1/2}$$
$$\leqslant \frac{1}{4} C_1 \|u_h^{m-1} - u^{m-1}\|_{W_0^{1\,2}}^2 + \frac{1}{4} C_1 \|u_h^m - u^m\|_{W_0^{1\,2}}^2$$
$$+ \Theta(\|u_h^m - u^m\|_{L^2}^2 + \|u_h^{m-1} - u^{m-1}\|_{L^2}^2), \tag{4 46}$$

where C_1 appears in (1.8), and M and Θ denote universal constants It follows from (4.45) and (4 46) that

$$\frac{1}{2} \|u_h^m - u^m\|_{L^2}^2 + \frac{1}{2} \|u_h^m - u^m - (u_h^{m-1} - u^{m-1})\|_{L^2}^2 +$$
$$+ kC_1 \|u_h^m - u^m\|_{W_0^{1\,2}}^2 + \frac{k}{\varepsilon} \|\nabla \cdot u_h^m\|_{L^2}^2$$
$$\leqslant \frac{1}{2} \|u_h^{m-1} - u^{m-1}\|_{L^2}^2 + \frac{k}{\varepsilon} \|\nabla \cdot u_h^m\|_{L^2}^2$$
$$+ \frac{\varepsilon k}{4} \|p^m\|_{L^2}^2 + \frac{1}{4} kC_1 \|u_h^{m-1} - u^{m-1}\|_{W_0^{1\,2}}^2$$
$$+ \frac{1}{4} kC_1 \|u_h^m - u^m\|_{W_0^{1\,2}}^2$$
$$+ k\Theta(\|u_h^m - u^m\|_{L^2}^2 + \|u_h^{m-1} - u^{m-1}\|_{L^2}^2) + |R_h^m| , \tag{4 47}$$

for all $m = 1, ..., N^*$, where $kN^* \leqslant T_5$ and Θ is a universal constant From now on, we assume

$$k \leqslant \sigma_2 \tag{4 48}$$

Here we choose σ_2 to be a universal constant such that $\sigma_2 \leqslant \sigma_1$ which appeared in Theorem 3.2 and $\sigma_2 \Theta \leqslant \frac{1}{4}$, where Θ is the same as in (4 47) Now (4.47) can be written as

$$\left(\frac{1}{2} - k\Theta\right) \|u_h^m - u^m\|_{L^2}^2 + \frac{3}{4} kC_1 \|u_h^m - u^m\|_{W_0^{1\,2}}^2 \leqslant$$
$$\leqslant \left(\frac{1}{2} + k\Theta\right) \|u_h^{m-1} - u^{m-1}\|_{L^2}^2 + \frac{1}{4} kC_1 \|u_h^{m-1} - u^{m-1}\|_{W_0^{1\,2}}^2 +$$
$$+ \frac{1}{4} \varepsilon k \|p^m\|_{L^2}^2 + |R_h^m| , \tag{4 49}$$

and thus, by setting $\rho = \dfrac{1 + 2k\Theta}{1 - 2k\Theta} > 1$,

$$\|u_h^m - u^m\|_{L^2}^2 + \frac{3}{2}\frac{1}{1-2k\Theta}kC_1\|u_h^m - u^m\|_{W_0^{1,2}}^2$$
$$\leq \rho\|u_h^{m-1} - u^{m-1}\|_{L^2}^2 + \frac{1}{2}\frac{1}{1-2k\Theta}kC_1\|u_h^{m-1} - u^{m-1}\|_{W_0^{1,2}}^2$$
$$+ \frac{1}{2}\frac{\varepsilon k}{1-2k\Theta}\|p^m\|_{L^2}^2 + \frac{2}{1-2k\Theta}|R_h^m|, \tag{4.50}$$

for $m = 1, \ldots, N^*$, where $kN^* \leq T_5$. Multiply both sides of (4.50) by ρ^{N^*-m}:

$$\rho^{N^*-m}\|u_h^m - u^m\|_{L^2}^2 + \frac{3}{2}\rho^{N^*-m}\frac{1}{1-2k\Theta}kC_1\|u_h^m - u^m\|_{W_0^{1,2}}^2$$
$$\leq \rho^{N^*-m+1}\|u_h^{m-1} - u^{m-1}\|_{L^2}^2$$
$$+ \frac{1}{2}\rho^{N^*-m}\frac{1}{1-2k\Theta}kC_1\|u_h^{m-1} - u^{m-1}\|_{W_0^{1,2}}^2$$
$$+ \frac{1}{2}\rho^{N^*-m}\frac{\varepsilon k}{1-2k\Theta}\|p^m\|_{L^2}^2 + \rho^{N^*-m}\frac{2}{1-2k\Theta}|R_h^m|, \tag{4.51}$$

and add over $m = 1, \ldots, N^*$:

$$\|u_h^{N^*} - u^{N^*}\|_{L^2}^2 + \sum_{m=1}^{N^*}\rho^{N^*-m}\frac{kC_1}{1-2k\Theta}\|u_h^m - u^m\|_{W_0^{1,2}}^2$$
$$\leq \rho^{N^*}\|u_h^0 - u_{0\eta}\|_{L^2}^2 + \frac{1}{2}\rho^{N^*-1}\frac{kC_1}{1-2k\Theta}\|u_h^0 - u_{0\eta}\|_{W_0^{1,2}}^2$$
$$+ \frac{1}{2}\sum_{m=1}^{N^*}\rho^{N^*-m}\frac{\varepsilon k}{1-2k\Theta}\|p^m\|_{L^2}^2 + \sum_{m=1}^{N^*}\rho^{N^*-m}\frac{2}{1-2k\Theta}|R_h^m|. \tag{4.52}$$

Recalling that $k = T/N$, $kN^* \leq T_5 \leq T$ and the assumption (4.48), we find

$$1 < \rho^m \leq \exp(CT), \quad \text{for all} \quad m = 1, \ldots, N^*, \tag{4.53}$$

where C is a universal constant.

Since $u_h^0 = r_h u_0$ and $u_0 \in V \cap W_0^{1,6}(\Omega)^3$, we obtain by means of (1.10), (1.11) and (4.6),

$$\|u_h^0 - u_{0\eta}\|_{L^2} \leq \|r_h u_0 - u_0\|_{L^2} + \|u_0 - u_{0\eta}\|_{L^2} \leq M(h + \eta^{1/4}), \tag{4.54}$$

$$\|u_h^0 - u_{0\eta}\|_{W_0^{1,2}} \leq \|r_h u_0 - u_0\|_{W_0^{1,2}} + \|u_0 - u_{0\eta}\|_{W_0^{1,2}} \leq M, \tag{4.55}$$

where M denotes universal constants. It now follows from (4.41), (4.48), (4.52) through (4.55) that for $kN^* \leqslant T_5$,

$$\max_{0 \leqslant m \leqslant N^*} \|u_h^m - u^m\|_{L^2}^2 + \sum_{m=1}^{N^*} k \|u_h^m - u^m\|_{W_0^{1,2}}^2 \leqslant$$

$$\leqslant M(h^2 + \eta^{1/2} + k + \varepsilon) + M \sum_{m=1}^{N^*} |R_h^m| , \quad (4.56)$$

where M denotes universal constants. In order to estimate the last term of (4.56), we shall derive some basic estimates. As before, let $\Phi_h = \bigcup_{K \in \mathcal{T}_h} K$.

For each $\delta \geqslant 0$, we denote by $\Phi_{h,\delta}$ the union of all K such that $K \in \mathcal{T}_h$ and distance $(K, \partial\Omega) \geqslant \delta$. Then, there is a positive constant C depending only on Ω such that

$$\sup_{x \in \Omega - \Phi_{h,\delta}} \text{distance } (\partial\Omega, x) \leqslant C(h + \delta), \quad \text{for all } h \text{ and } \delta . \quad (4.57)$$

This follows from the assumption on $\mathcal{T}_h$. We can now estimate by virtue of (1.10), (1.11), (1.12) and Theorem 3.2 :

$$\sum_{m=1}^{N^*} k \|u^m - r_h u^m\|_{W_0^{1,2}}^2 \leqslant 2 \sum_{m=1}^{N^*} k \|y^m + z^m - r_h y^m - r_h z^m\|_{W_0^{1,2}}^2$$

$$+ 2 \sum_{m=1}^{N^*} k \|x^m - r_h x^m\|_{W_0^{1,2}}^2 , \quad (4.58)$$

$$\sum_{m=1}^{N^*} k \|y^m + z^m - r_h y^m - r_h z^m\|_{W_0^{1,2}}^2$$

$$\leqslant \sum_{m=1}^{N^*} k \|y^m + z^m - r_h y^m - r_h z^m\|_{W^{1,2}(\Phi_{h,\delta})}^2$$

$$+ C(h + \delta)^{2/3} \sum_{m=1}^{N^*} k \|y^m + z^m - r_h y^m - r_h z^m\|_{W^{1,6}}^2, \quad \text{by (4.57)}$$

$$\leqslant C(h^2 + (h + \delta)^{2/3}) \sum_{m=1}^{N^*} k \|y^m + z^m\|_{W^{2,2}}^2$$

$$\leqslant M(h^2 + (h + \delta)^{2/3}) \leqslant Mh^{2/3} , \quad (4.59)$$

by choosing $\delta = 0$ and assuming $h \leqslant 1$, where C's and M's are universal constants.

$$\sum_{m=1}^{N^*} k\|x^m - r_h x^m\|^2_{W_0^{1,2}} \leqslant \sum_{m=1}^{N^*} k\|x^m - r_h x^m\|^2_{W^{1,2}(\Phi_{h,\delta})}$$

$$+ C(h+\delta)^{1-2/p} \sum_{m=1}^{N^*} k\|x^m - r_h x^m\|^2_{W^{1,p}}, \quad 3 < p < \infty, \text{ by } (4.57)$$

$$\leqslant M\left(\frac{h}{\delta}\right)^2 + C_p(h+\delta)^{1-2/p}$$

$$\leqslant (M + C_p)\, h^{(2-4/p)/(3-2/p)} \tag{4.60}$$

by choosing $\delta = h^{2/(3-2/p)}$, M and C_p being universal constants. Combining these estimates, we obtain

$$\sum_{m=1}^{N^*} k\|u^m - r_h u^m\|^2_{W_0^{1,2}} \leqslant Mh^{2/3} + C_p\, h^{(2-4/p)/(3-2/p)}$$

$$\leqslant C_p\, h^{(2-4/p)/(3-2/p)}, \tag{4.61}$$

for all $0 < h \leqslant 1$ and $3 < p < \infty$. Similarly, we have

$$\sum_{m=1}^{N^*} k\|u^m - r_h u^m\|^2_{L^2} \leqslant 2 \sum_{m=1}^{N^*} k\|y^m + z^m - r_h y^m - r_h z^m\|^2_{L^2}$$

$$+ 2\sum_{m=1}^{N^*} k\|x^m - r_h x^m\|^2_{L^2}, \tag{4.62}$$

$$\sum_{m=1}^{N^*} k\|y^m + z^m - r_h y^m - r_h z^m\|^2_{L^2} \leqslant$$

$$\leqslant \sum_{m=1}^{N^*} k\|y^m + z^m - r_h y^m - r_h z^m\|^2_{L^2(\Phi_{h,\delta})}$$

$$+ \sum_{m=1}^{N^*} k\|y^m + z^m - r_h y^m - r_h z^m\|^2_{L^2(\Omega - \Phi_{h,\delta})}. \tag{4.63}$$

Using (1.12) and Theorem 3.2, we obtain

$$\sum_{m=1}^{N^*} k\|y^m + z^m - r_h y^m - r_h z^m\|^2_{L^2(\Phi_{h,\delta})} \leqslant Mh^4. \tag{4.64}$$

To estimate the other term, we set for each $h > 0$ and $\delta \geqslant 0$

$$\tilde{\Omega}_{h,\delta} = \{x \in \Omega : \text{distance } (x, \partial\Omega) \leqslant C(h+\delta)\}$$

where C is the same as in (4.57), and notice that for all $v \in W_0^{1,2}(\Omega)^3$,

$$\int_{\Omega - \Phi_{h,\delta}} |v|^2 \, dx \leq M(h+\delta)^2 \|v\|^2_{W^{1,2}(\tilde{\Omega}_{h,\delta})}, \tag{4.65}$$

M being a positive constant depending only on Ω. By the same manipulation as in (4.59), we get

$$\sum_{m=1}^{N^*} k \|y^m + z^m - r_h y^m - r_h z^m\|^2_{W^{1,2}(\tilde{\Omega}_{h,\delta})} \leq M(h+\delta)^{2/3}. \tag{4.66}$$

Hence, combining (4.65) and (4.66), we choose $\delta = 0$ so that

$$\sum_{m=1}^{N^*} k \|y^m + z^m - r_h y^m - r_h z^m\|^2_{L^2(\Omega - \Phi_{h,\delta})} \leq Mh^{8/3}, \tag{4.67}$$

which, together with (4.64), yields, assuming $h \leq 1$,

$$\sum_{m=1}^{N^*} k \|y^m + z^m - r_h y^m - r_h z^m\|^2_{L^2} \leq Mh^{8/3}, \tag{4.68}$$

where M is a universal constant. In this manner, we can also obtain

$$\begin{aligned} \sum_{m=1}^{N^*} k \|x^m - r_h x^m\|^2_{L^2} &\leq M\left(\frac{h^2}{\delta}\right)^2 + C_p (h+\delta)^{3-2/p}, \quad 3 < p < \infty, \\ &\leq (M + C_p)\, h^{(12-8/p)/(5-2/p)}, \end{aligned} \tag{4.69}$$

by choosing $\delta = h^{4/(5-2/p)}$ and assuming $h \leq 1$, M and C_p being universal constants. It follows from (4.68) and (4.69)

$$\begin{aligned} \sum_{m=1}^{N^*} k \|u^m - r_h u^m\|^2_{L^2} &\leq Mh^{8/3} + C_p\, h^{(12-8/p)/(5-2/p)} \\ &\leq C_p\, h^{(12-8/p)/(5-2/p)} \end{aligned} \tag{4.70}$$

for all $0 < h \leq 1$ and $3 < p < \infty$. We are ready to estimate $\sum_{m=1}^{N^*} |R_h^m|$.

$$\sum_{m=1}^{N^*} |(u_h^m - u_h^{m-1}, u^m - r_h u^m)| \leq$$

$$\leq M\left(\sum_{m=1}^{N^*} \|u_h^m - u_h^{m-1}\|^2_{L^2}\right)^{1/2} \frac{1}{\sqrt{k}} \left(\sum_{m=1}^{N^*} k \|u^m - r_h u^m\|^2_{L^2}\right)^{1/2},$$

by means of (2.13) and (4.70)

$$\leqslant C_p \frac{1}{\sqrt{k}} h^{(6-4/p)/(5-2/p)} . \tag{4.71}$$

Note that (2.13) is still valid with the initial condition $u_h^0 = r_h u_0$.

$$\sum_{m=1}^{N^*} k \left| a(u_h^m, u^m - r_h u^m) \right| \leqslant M \left(\sum_{m=1}^{N^*} k \| u_h^m \|_{W_0^{1,2}}^2 \right)^{1/2} \times$$

$$\times \left(\sum_{m=1}^{N^*} k \| u^m - r_h u^m \|_{W_0^{1,2}}^2 \right)^{1/2} ,$$

using (2.13) and (4.61),

$$\leqslant C_p h^{(1-2/p)/(3-2/p)} . \tag{4.72}$$

$$\sum_{m=1}^{N^*} k \left| \tilde{b}(u_h^{m-1}, u_h^m, u^m - r_h u^m) \right| \leqslant \text{using integration by parts ,}$$

$$\leqslant M \sum_{m=1}^{N^*} k \| u_h^{m-1} \|_{W_0^{1,2}} \| u_h^m \|_{W_0^{1,2}} \| u^m - r_h u^m \|_{L^3} . \tag{4.73}$$

To estimate the right-hand side, we derive from (1.10), (2.13) and (4.48),

$$\max_{0 \leqslant m \leqslant N} \| u_h^m \|_{W_0^{1,2}} \leqslant \frac{M}{\sqrt{k}} , \quad M \text{ being a universal constant .} \tag{4.74}$$

Thus, it follows that

$$\sum_{m=1}^{N^*} k \left| \tilde{b}(u_h^{m-1}, u_h^m, u^m - r_h u^m) \right| \leqslant \frac{M}{\sqrt{k}} \left(\sum_{m=1}^{N^*} k \| u_h^m \|_{W_0^{1,2}}^2 \right)^{1/2} \times$$

$$\times \left(\sum_{m=1}^{N^*} k \| u^m - r_h u^m \|_{L^2} \| u^m - r_h u^m \|_{W_0^{1,2}} \right)^{1/2}$$

by means of (2.13), (4.61) and (4.70),

$$\leqslant \frac{1}{\sqrt{k}} C_p h^{(3-2/p)/(5-2/p) + (1/2-1/p)/(3-2/p)} . \tag{4.75}$$

$$\sum_{m=1}^{N^*} k \left| J_\eta(u^m) - J_\eta(r_h u^m) \right| \leqslant \sum_{m=1}^{N^*} kM \| u^m - r_h u^m \|_{W_0^{1,2}}$$

using (4.61),

$$\leqslant C_p h^{(1-2/p)/(3-2/p)} . \tag{4.76}$$

$$\sum_{m=1}^{N^*} k\left|\left(\frac{1}{\varepsilon}\nabla \cdot u_h^m, \nabla \cdot (u^m - r_h u^m)\right)\right| \leq$$

$$\leq \frac{1}{\sqrt{\varepsilon}}\left(\sum_{m=1}^{N^*} \frac{1}{\varepsilon} k \|\nabla \cdot u_h^m\|_{L^2}^2\right)^{1/2}\left(\sum_{m=1}^{N^*} k\|u^m - r_h u^m\|_{W_0^{1,2}}^2\right)^{1/2}$$

using (2.13) and (4.61),

$$\leq \frac{1}{\sqrt{\varepsilon}} C_p h^{(1-2/p)/(3-2/p)} . \tag{4.77}$$

$$\sum_{m=1}^{N^*} k|(f^m, r_h u^m - u^m)| \leq$$

$$\leq \left(\sum_{m=1}^{N^*} k\|f^m\|_{L^2}^2\right)^{1/2}\left(\sum_{m=1}^{N^*} k\|u^m - r_h u^m\|_{L^2}^2\right)^{1/2}$$
$$\leq C_p h^{(6-4/p)/(5-2/p)}, \text{ by (4.70)} . \tag{4.78}$$

It now follows from (4.71) through (4.78) that

$$\sum_{m=1}^{N^*} |R_h^m| \leq C_p h^{(1-2/p)/(3-2/p)} + C_p \frac{1}{\sqrt{\varepsilon}} h^{(1-2/p)/(3-2/p)}$$
$$+ C_p \frac{1}{\sqrt{k}} h^{(3-2/p)/(5-2/p)+(1/2-1/p)/(3-2/p)}, \tag{4.79}$$

where $0 < h \leq 1$, and C_p denotes universal constants depending on $3 < p < \infty$. Consequently, (4.56) yields

$$\max_{0 \leq m \leq N^*} \|u_h^m - u^m\|_{L^2}^2 + \sum_{m=1}^{N^*} k\|u_h^m - u^m\|_{W_0^{1,2}}^2$$
$$\leq M(\varepsilon + k + \sqrt{\eta}) + C_p h^{(1-2/p)/(3-2/p)}$$
$$+ C_p \frac{1}{\sqrt{\varepsilon}} h^{(1-2/p)/(3-2/p)} + C_p \frac{1}{\sqrt{k}} h^{(3-2/p)/(5-2/p)+(1/2-1/p)/(3-2/p)} . \tag{4.80}$$

We combine estimates I, II and III to arrive at

THEOREM 4.4. *Let* $u_0 \in G$ *and* $f \in C([0, T]; L^2(\Omega)^3)$ *with* $\partial_t f \in L^2(0, T; W^{-1,2}(\Omega)^3)$. *Denote by* u *the strong solution obtained in* [13] *and by* u_h *the approximate solution constructed through* (2.1) *and* (2.5) *with*

$u_h^0 = r_h u_0$ and $f^m = f(km)$. Then, there are $0 < T^* \leqslant T$ and $0 < \sigma \leqslant 1$ such that for all $0 < h \leqslant 1$, $0 < k \leqslant \sigma$, $\varepsilon > 0$, $0 < \eta \leqslant 1$ we have

$$\max_{0 \leqslant m \leqslant N^*} \|u_h^m - u(km)\|_{L^2}^2 + \sum_{m=1}^{N^*} k \|u_h^m - u(km)\|_{W_0^{1,2}}^2$$

$$\leqslant M\left(\varepsilon + k + \sqrt{\eta} + \frac{k^2}{\eta}\right) + C_p h^{(1-2/p)/(3-2/p)}$$

$$+ C_p \frac{1}{\sqrt{\varepsilon}} h^{(1-2/p)/(3-2/p)} + C_p \frac{1}{\sqrt{k}} h^{(3-2/p)/(5-2/p)+(1/2-1/p)/(3-2/p)} \tag{4.81}$$

provided $kN^* \leqslant T^*$ and $3 < p < \infty$, where T^*, σ, M and C_p are positive numbers independent of h, k, ε and η, and C_p depends on p.

Remark 4.5 : If the conditions for the existence of global solutions set forth in [13] and [14] are satisfied, then $T^* = T$. For example, if $u_0 \equiv 0$, and f, $\partial_t f$ are sufficiently small in $C([0, T]; L^2(\Omega)^3)$ and $L^2(0, T; W^{-1,2}(\Omega)^3)$, respectively, then $T^* = T$.

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