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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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ON THE NUMERICAL SOLUTION OF THE FIRST BIHARMONIC EQUATION (*)

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Abstract — *We consider a mixed finite element discretization of the biharmonic problem Following Glowinski and Pironneau the original indefinite linear system is transformed into a positive definite one for the unknown boundary value* $\lambda = \Delta u \mid_{\partial\Omega}$. *This system is solved by a conjugate gradient method We establish a preconditioning and prove that the number of iteration steps required for a given accuracy is independent of the mesh size.*

Résumé. — *On considere une méthode d'éléments finis mixtes pour le problème de Dirichlet de l'opérateur biharmonique Comme Glowinski, Pironneau, on transforme le problème original, qui est indéfini, en un problème défini positif pour la trace* $\lambda = \Delta u \mid_{\partial\Omega}$. *Ce problème est résolu par la méthode du gradient conjugué On établit une méthode de préconditionnement et on démontre que le nombre d'itérations pour réduire l'erreur d'un facteur fixe ne dépend pas du paramètre de discrétisation.*

1. INTRODUCTION

We consider the numerical solution of the biharmonic equation

$$(1.1) \qquad \Delta^2 u = f \quad \text{in} \quad \Omega\,, \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\,,$$

where Ω is a convex, polygonal domain in the plane. Suppose that the boundary value $\lambda^* = \Delta u \mid_{\partial\Omega}$ is known. Then (1.1) is splitted into two separated Poisson equations. An initial guess $\lambda^{(0)}$ for the boundary value may be iteratively improved using the following procedure for $k = 0, 1, 2, \ldots$

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Given $\lambda^{(k)}$. Then solve

$$(1.2)\qquad \begin{aligned} \Delta\phi^{(k)} &= f \quad \text{in} \quad \Omega\,, & \phi^{(k)}|_{\partial\Omega} &= \lambda^{(k)} \\ \Delta u^{(k)} &= \phi^{(k)} \quad \text{in} \quad \Omega\,, & u^{(k)}|_{\partial\Omega} &= 0 \\ \lambda^{(k+1)} &= \lambda^{(k)} + \delta \frac{\partial u^{(k)}}{\partial n}\,, & \delta &> 0\,. \end{aligned}$$

This method is known as coupled equation approach in the finite differences context (see e.g. [7]). In the framework of finite elements the discrete analogue of (1.2) was first studied by Ciarlet & Glowinski [6], see also [3] for numerical experiments, and further improved by Glowinski & Pironneau [8].

Given λ, denote by $(\phi_\lambda, u_\lambda)$ the solution of (1.2*a*, *b*) with right hand side $f = 0$. A linear mapping L is defined by

$$\lambda \mapsto (\phi_\lambda, u_\lambda) \mapsto -\frac{\partial u_\lambda}{\partial n}\,.$$

Glowinski & Pironneau [8] observed that the operator L is $H^{-1/2}(\Gamma)$-elliptic for a smooth domain. The corresponding discrete operator L_h reflects this property. Specifically, the matrix L_h is positive definite and the spectral condition number $\kappa(L_h) = \lambda_{\max}(L_h)/\lambda_{\min}(L_h)$ grows as h^{-1} [8], where h is a mesh parameter. The discrete system is solved by the method of conjugate gradients [1].

In order to speed up the convergence, Glowinski & Pironneau have already suggested to use the $H^{-1/2}(\Gamma)$-ellipticity for preconditioning. Following this idea we will provide a preconditioning matrix C_h, such that the resulting condition number becomes independent of the mesh size. The matrix C_h is based on the inverse of the square root of a discretization of $-d^2/ds^2$ with homogeneous boundary conditions on each line segment Γ_k of $\partial\Omega$.

The proof, which is postponed to the last sections, has the following structure. First, we will generalize the properties of the continuous operator, mentioned above, to the case of a convex, polygonal domain. Here, the dual spaces $H^{-1/2}(\Gamma_i)$ of $H^{1/2}_{00}(\Gamma_i)$, Γ_i being a line segment of $\partial\Omega$, are involved. Specifically, we distinguish between $\sum_i H^{-1/2}(\Gamma_i)$ and $H^{-1/2}(\Gamma)$. Next, we will show that the properties of L carry over to the discrete operator L_h. Finally, we will prove that the inner product $\lambda_h^T C_h \lambda_h$ induces a norm which is equivalent to the $\sum_i H^{-1/2}(\Gamma_i)$-norm on the finite element space.

Numerical experiments which confirm the theoretical results are included.

2. PRELIMINARIES

We study a finite element discretization, which is based on the mixed variable formulation of (1.1) :

$$\text{find } (\phi, u) \in H^1(\Omega) \times H_0^1(\Omega) \quad \text{such that}$$

$$\begin{aligned} &\int_\Omega \phi\psi \, dx - \int_\Omega \nabla u \, \nabla \psi \, dx = 0 , && \forall \psi \in H^1(\Omega) \\ &\int_\Omega \nabla \phi \, \nabla v \, dx = - \int_\Omega f v \, dx , && \forall v \in H_0^1(\Omega) . \end{aligned} \tag{2.1}$$

In the numerical solution, the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ are replaced by finite dimensional subspaces $X_h \subset H^1(\Omega)$ and $X_{0h} = X_h \cap H_0^1(\Omega)$. Specifically, let X_h be the finite element space of continuous, piecewise linear polynomials on the given regular triangulation $\mathcal{T}_h$ of Ω. Let R_h denote the p-dimensional subspace of X_h spanned by those basis functions, which are associated with nodes on the boundary. Then

$$X_h = X_{0h} \oplus R_h . \tag{2.2}$$

Identifying each finite element function via the nodal basis with the associated coefficient vector, the discrete problem which corresponds to (2.1) is written in matrix-vector notation as

$$\begin{pmatrix} M_{11} & M_{12} & B_0 \\ M_{21} & M_{22} & -T^T \\ B_0 & -T & 0 \end{pmatrix} \begin{pmatrix} \varphi_h \\ u_h \end{pmatrix} = \begin{pmatrix} 0 \\ -f_h \end{pmatrix} . \tag{2.3}$$

The square-matrix B_0 represents the discretization of the Poisson equation with Dirichlet boundary condition.

With respect to the decomposition (2.2) we write φ_h as $\varphi_h^T = (\varphi_{h0}, \lambda_h)$. After eliminating the variables φ_{h0} and u_h in (2.3) we obtain a positive definite linear system

$$L_h \lambda_h = b_h , \tag{2.4}$$

where

$$L_h = (T^T B_0^{-1}, I_p) \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} B_0^{-1} T \\ I_p \end{pmatrix} . \tag{2.5}$$

Since the matrix L_h is only given implicitly, we prefer an iterative procedure for the solution of (2.4), especially the method of conjugate gradients. Given $\lambda_h \in R_h$, the evaluation $L_h \lambda_h$ requires the solution of two discrete

Poisson equations. For this purpose a multigrid algorithm [9] is well suited (see [4, 11]).

We will be concerned with preconditioning techniques in the application of the cg-algorithm. Given a positive definite matrix C_h, the condition number of L_h with respect to C_h is given by $\kappa(C_h^{-1} L_h)$. Specifically, let $0 < \alpha_h < \beta_h$ be constants such that

$$\alpha_h \lambda_h^T C_h \lambda_h \leqslant \lambda_h^T L_h \lambda_h \leqslant \beta_h \lambda_h^T C_h \lambda_h , \tag{2.6}$$

then $\kappa(C_h^{-1} L_h) \leqslant \beta_h/\alpha_h$. We will provide a preconditioning matrix C_h such that the constants in (2.6) are independent of the mesh size h.

The finite element solution $\varphi_{\lambda_h} \in X_h$ of

$$\int_\Omega \nabla\varphi_{\lambda_h} \nabla v_h \, dx = 0 \qquad \forall v_h \in X_{0h} , \qquad \varphi_{\lambda_h} - \lambda_h \in X_{0h} , \tag{2.7}$$

is called discrete harmonic. The coefficient vector is given by $\varphi_{\lambda_h} = \begin{pmatrix} B_0^{-1} & T \\ & I_p \end{pmatrix} \lambda_h$. Hence, from (2.5) it follows that

$$\lambda_h^T L_h \lambda_h = \left\| \varphi_{\lambda_h} \right\|_{L_2(\Omega)}^2 . \tag{2.8}$$

Therefore, in order to prove (2.6) we will be concerned with a priori estimates of the L_2-norm of discrete harmonic functions in terms of their boundary values.

3. THE PRECONDITIONING MATRIX

When using the method of conjugate gradients for the solution of the linear system $L_h x = b$, the number of iteration steps required for a given accuracy grows as $\sqrt{\kappa(L_h)} = O(h^{-1/2})$. In order to speed up the convergence, preconditioning techniques have turned out to be useful.

We shall now construct a preconditioning matrix C_h such that the condition number $\kappa(C_h^{-1} L_h)$ is bounded independently of the mesh size. Since Ω is assumed to be polygonal, the boundary Γ of Ω consists of a finite number of straight lines Γ_k, $1 \leqslant k \leqslant r$. Let p_k denote the number of the interior nodes of Γ_k. The number of all boundary nodes is equal to $p = \sum_{k=1}^{r} (p_k + 1)$.

The $p_k \times p_k$-matrices

$$D_k = \text{tridiag}\,[-1, 2, -1] \tag{3.1}$$

correspond to the usual three-point approximation of the differential operator $-d^2/dx^2$ with homogeneous Dirichlet boundary conditions on Γ_k. The eigenvectors and eigenvalues of these Toeplitz-matrices are explicitly given by

$$\theta_l^{(k)} = \left(\frac{2}{p_k+1}\right)^{1/2} \left[\sin\left(l\,\frac{j\pi}{p_k+1}\right)\right]_{j=1}^{p_k}$$

and

$$\lambda_l^{(k)} = 4\sin^2\left(\frac{l\pi}{p_k+1}\right).$$

Therefore, D_k admits the factorization

$$D_k = Q_k\,\Lambda_k\,Q_k^T,$$

where $Q_k = [\theta_1^{(k)}, \theta_2^{(k)}, \dots, \theta_{p_k}^{(k)}]$ is unitary and $\Lambda_k = \operatorname{diag}(\lambda_1^{(k)}, \lambda_2^{(k)}, \dots, \lambda_{p_k}^{(k)})$. The powers D_k^s, $s \in \mathbb{R}$, are defined by

$$D_k^s = Q_k\,\Lambda_k^s\,Q_k^T. \tag{3.2}$$

Using Fast Fourier Transform (FFT), the evaluation of $D_k^s\,x$ requires only $O(p_k \ln p_k)$ arithmetic operations [16], provided that $p_k = s \cdot 2^t$ with s being small.

We shall also need the tridiagonal $p_k \times p_k$-mass matrices

$$M_k = \left(\int_{\Gamma_k} \psi_i\,\psi_j\,dx\right)_{ij=1}^{p_k} \tag{3.3}$$

on T_k, where $\psi_i(x)$ denotes the piecewise linear nodal basis function, which satisfies $\psi_i(x_j^k) = \delta_{ij}$, $1 \leqslant i \leqslant p_k$ for the nodes x_j^k on Γ_k. Set

$$C_k := M_k\,D_k^{-1/2}\,M_k. \tag{3.4}$$

For preconditioning, we choose the $p \times p$ matrix C_h, which has block diagonal form :

$$C_h = \begin{bmatrix} C_1 & & & & & & \\ & h^2 & & & & & \\ & & C_2 & & & & \\ & & & h^2 & & & \\ & & & & \ddots & & \\ & & & & & C_r & \\ & & & & & & h^2 \end{bmatrix}. \tag{3.5}$$

The application of the preconditioning (3.5) requires for $k = 1, 2, \ldots, r$ two real sine transformations and the solution of two linear systems with the tridiagonal mass matrix M_k. Since M_k is spectrally equivalent to I_{p_k}, one might expect at first glance, that $M_k D_k^{-1/2} M_k$ and $D_k^{-1/2}$ are spectrally equivalent, too. Indeed, if the meshpoints are distributed equidistantly, then the associated matrices

$$D_k = \text{tridiag}\,[-1, 2, -1]\,, \quad M_k = \frac{h_k}{6}\,\text{tridiag}\,[1, 4, 1]\,,$$

$h_k = 1/(p_k + 1)$, have the same eigenvectors. Thus, the matrices M_k and $D_k^{-1/4}$ commute, i.e.

$$M_k D_k^{-1/2} M_k = D_k^{-1/4} M_k^2 D_k^{-1/4}\,,$$

and we obtain

$$\frac{1}{9} D_k^{-1/2} \leqslant \frac{1}{h_k^2} M_k D_k^{-1/2} M_k \leqslant D_k^{-1/2}\,.$$

However, if the meshpoints on Γ_k are not distributed equidistantly, then D_k and M_k do not commute. In this case the conjecture is not always true, as is illustrated by the following example.

Example : Consider the matrices

$$D = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

with $a \gg 1$. The diagonal dominant matrix M has the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Set $x = (1, -2)^T$. Then

$$x^T Dx = a + 4\,, \qquad x^T MDMx = 9\,.$$

Now we state our main result.

THEOREM 3.1 : *Let Ω be convex and let C_h be the preconditioning matrix* (3.5). *Then the condition number $\kappa(C_h^{-1} L_h)$ of L_h with respect to C_h is bounded independently of the mesh size h.*

We finish this section with the following remark. The condition number $\kappa(C_h^{-1} L_h)$ depends on the interior angles ω_k of the polygon Ω and grows as $\max \left\{ \dfrac{1}{\sin^2 \omega_k} \right\}$. Angles, which are close to zero or π spoil the condition number. Specifically, Theorem 3.1 does not apply if $\Omega = \Omega_h$ is the polygonal approximation of a smooth domain. In that case another preconditioning is appropriate : Consider on the whole boundary Γ the three-point approximation D_h of $I - d^2/dx^2$ with periodical boundary conditions [10] and set $C_h = M_h D_h^{-1/2} M_h$.

4. FURTHER PRECONDITIONING. THE BIHARMONIC EQUATION IN A SQUARE

The preconditioning matrix (3.5) has block diagonal form, each block C_k corresponding to a line segment Γ_k. Therefore, the condition number of L_h with respect to the block diagonal part of L_h is again independent of the mesh size. In general, solving a linear system with the block diagonal part of L_h is still expensive. However, if Ω is the unit square, then the block diagonal part is easy to invert, as we will describe in the remainder of this section. Thus, in this special case, another preconditioning matrix is available.

We approximate the biharmonic problem by piecewise linear elements on a subdivision of Ω into Courant's triangles K_h of length h, $h = 1/(n+1)$. Using the quadrature rule

$$\int_{K_h} \varphi(x)\, dx \approx \frac{\text{meas}\,(K_h)}{3} (\varphi(x_1) + \varphi(x_2) + \varphi(x_3)),$$

when evaluating integrals, the mass matrix M in (2.3) is replaced by the diagonal matrix $\tilde{M} = 2\,\text{diag}\,(M)$. The same discretization results when the 13-point finite difference approximation is used. Inserting $\tilde{M}$ into (2.5) and neglecting the equations corresponding to the four corner points, we obtain

$$L_h = \frac{1}{2} I + T B_0^{-2} T^T. \tag{4.1}$$

We decompose the boundary space R_h as

$$R_h = R_h^1 \oplus R_h^2,$$

where R_h^1 is spanned by those basis functions, which are associated with nodes on the lower and upper part of the boundary and R_h^2 is defined analogously. With respect to this decomposition the $(4n \times 4n)$-matrix L_h has 2×2-block structure. The preconditioning by the block diagonal part diag (L_{11}, L_{22}) is investigated. Each block L_{ii} corresponds to the biharmonic problem with Δu rather than u_n specified on two opposite sides of the square Ω. Bjørstad [2] has observed that this problem is easy to solve, since separation of the variables is possible. Assuming that Δu is specified at the left and right part of the boundary, we choose a row-wise ordering of the nodes. The resulting linear equations of the full problem have the following structure

$$(4.2)\qquad \begin{pmatrix} h^2 I_{n^2} & 0 & B_0 \\ 0 & \frac{1}{2} h^2 I_{2n} & -U^T \otimes I_n \\ B_0 & -U \otimes I_n & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -h^2 f_h \end{pmatrix},$$

with $U = [e_1, e_n]$. After eliminating the variable φ, we obtain the positive definite system

$$Au = h^4 f_h ,$$

with

$$(4.3)\qquad A = B_0^2 + 2(UU^T \otimes I) .$$

Since the inverse of L_{11} can be expressed via the inverse of the matrix A as

$$(4.4)\qquad L_{11}^{-1} = \frac{2}{h} [2(U^T \otimes I) A^{-1}(U \otimes I) - I] ,$$

we will study the solution of a linear system with the matrix A. The discretization B_0 of the two-dimensional Laplacian on the unit square can be expressed via the approximation of the one-dimensional Laplacian

$$D = \text{tridiag } [-1, 2, -1] ,$$

as

$$B_0 = I \otimes D + D \otimes I ,$$

with $I = I_n$. Using the spectral decomposition of D, i.e.

$$D = Q \Lambda Q$$

with $\Lambda = \text{diag }(\lambda_i)$ and $Q = [\theta_1, \theta_2, \dots, \theta_n]$ as defined in § 3, the matrix A can be written as

$$\begin{aligned} A &= I \otimes D^2 + 2(D \otimes D) + D^2 \otimes I + 2(UU^T \otimes I) \\ &= (I \otimes Q)[I \otimes \Lambda^2 + 2(D \otimes \Lambda) + (D^2 \otimes I) + 2(UU^T \otimes I)](I \otimes Q) , \end{aligned}$$

each block corresponding to a row. Using the permutation P, which converts the row-wise ordering into a column-wise ordering, Bjørstad [2] obtains

$$(4.5)\qquad \begin{aligned} A = (I \otimes Q) P [\Lambda^2 \otimes I + 2(\Lambda \otimes D) + I \otimes D^2 + 2(I \otimes UU^T)] \times \\ \times P (I \otimes Q) =: (I \otimes Q) PSP (I \otimes Q) . \end{aligned}$$

The matrix $S = \operatorname{diag}(S_i)$ is block diagonal with $(n \times n)$-matrices

$$S_i = \lambda_i^2 I + 2\lambda_i D + D^2 + 2\, UU^T$$

having bandwidth $d = 2$ and therefore being easily invertable.

Inserting (4.5) into (4.4), we obtain

$$L_{11}^{-1} = \frac{2}{h}\left[2(U^T \otimes Q)\, PS^{-1} P(U \otimes Q) - I\right].$$

Therefore, the evaluation of $L_{11}^{-1} r_1$ involves four sine transformations of length n and the solution of a linear system with the $n^2 \otimes n^2$-matrix S, which is pentadiagonal.

5. NUMERICAL RESULTS

We will provide some numerical results for the biharmonic equation on the unit square. We use Courant's triangulation with triangles of length h, h, $\sqrt{2}\,h$. In order to determine the unknown boundary value $\lambda = \Delta u\,|_{\partial\Omega}$, we solve the system of linear equations

$$L_h \lambda_h = b_h \tag{5.1}$$

using the method of conjugate gradients. The evaluation of L_h requires the solution of two discrete Poisson equations. Since Ω is the unit square, we have used Buneman's algorithm for this purpose.

The performance of the preconditioning techniques is studied by choosing the right hand side as

$$b_h(x, y) = \begin{cases} \sin(\pi x) + \sin(\pi x h^{-1}/2) & \text{on} \quad \{(x, 0), 0 \leqslant x \leqslant 1\} \\ 0 & \text{on} \quad \partial\Omega \setminus \{(x, 0), 0 \leqslant x \leqslant 1\}, \end{cases}$$

which is a superposition of low and high frequencies. The starting value is $\lambda_h^{(0)} = 0$. The iteration is terminated, if the relative error of the residuum with respect to the Euclidian norm is less than ε, i.e.

$$R_k = \frac{\|L_h \lambda_h^{(k)} - b_h\|}{\|b_h\|} \leqslant \varepsilon .$$

Without preconditioning the number of iteration steps required to gain a given accuracy ε is bounded by $O(h^{-1/2} \log \varepsilon^{-1})$. This is confirmed by the following table.

TABLE 1

Number of cg-iterations for a given accuracy ε

mesh size h / accuracy ε	1/16	1/32	1/64
10^{-3}	7	9	12
10^{-6}	12	16	21

The next table shows the independence of the number of pcg-iterations on the mesh size h, when using the preconditioning (3.5).

TABLE 2

Number of pcg-iterations with $C_h \sim D^{-1/2}$

mesh size h / accuracy ε	1/16	1/32	1/64
$\varepsilon = 10^{-3}$	7	7	7
$\varepsilon = 10^{-6}$	12	13	13

Finally, we present the results when using the block diagonal part of L_h for preconditioning. Note, however, that this preconditioning is only available in case of a rectangular domain Ω.

TABLE 3

Number of pcg-iterations with $C_h = blockdiag\ (L_h)$

mesh size h / accuracy ε	1/16	1/32	1/64
$\varepsilon = 10^{-3}$	6	7	7
$\varepsilon = 10^{-6}$	9	10	10

6. A PRIORI ESTIMATES

The rest of this paper is concerned with the proof that the condition number $\kappa(C_h^{-1} L_h)$ is independent of the mesh size, i.e.

$$\alpha \lambda_h^T C_h \lambda_h \leq \lambda_h^T L_h \lambda_h \leq \beta \lambda_h^T C_h \lambda_h , \quad \lambda_h \in R_h \tag{6.1}$$

with α, β being independent of h. Here, C_h is the preconditioning matrix (3.5).

Let $\varphi_{\lambda_h} \in X_h$ be the discrete harmonic function with boundary value λ_h, i.e. the solution of (2.7). Using (2.8) and the notation

$$\|\lambda_h\|_{C_h} := \sqrt{\lambda_h^T C_h \lambda_h}$$

inequality (6.1) is rewritten as

$$\alpha \|\lambda_h\|_{C_h} \leq \|\varphi_{\lambda_h}\|_{L_2(\Omega)} \leq \beta \|\lambda_h\|_{C_h} . \tag{6.1'}$$

In order to prove (6.1), we consider the continuous case at first. Let

$$H_{00}^{1/2}(\Gamma_i) = [H_0^1(\Gamma_i), L_2(\Gamma_i)]_{1/2}$$

denote the interpolation space [10], and let

$$H^{-1/2}(\Gamma_i) = (H_{00}^{1/2}(\Gamma_i))'$$

denote the dual space. Set

$$\|\lambda\|_{-1/2,\Gamma} := \left(\sum_{i=1}^{r} \|\lambda\|^2_{H^{-1/2}(\Gamma_i)}\right)^{1/2} . \tag{6.2}$$

In the proof of the following theorem, we will make use of a trace theorem given in the appendix.

THEOREM 6.1 : *Let the polygonal domain $\Omega \subset \mathbb{R}^2$ be convex and $\lambda \in H^{1/2}(\Gamma)$. Then the L_2-norm of the harmonic function ϕ_λ can be estimated from below and from above by the $\|\cdot\|_{-1/2,\Gamma}$-norm of it's boundary value λ :*

$$c_0 \|\lambda\|_{-1/2,\Gamma} \leq \|\phi_\lambda\|_{L_2(\Omega)} \leq c_1 \|\lambda\|_{-1/2,\Gamma} . \tag{6.3}$$

Proof: Since $\lambda \in H^{1/2}(\Gamma)$, we have $\phi_\lambda \in H^1(\Omega)$. By partial integration we obtain for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$

$$\begin{aligned} \int_\Omega (-\Delta u)\, \phi_\lambda \, dx &= \int_\Omega \nabla u \, \nabla \phi_\lambda \, dx - \int_\Gamma \frac{\partial u}{\partial n} \lambda \, ds \\ &= - \int_\Gamma \frac{\partial u}{\partial n} \lambda \, ds . \end{aligned} \tag{6.4}$$

We will first prove the second inequality of (6.3). Let $u_\lambda \in H_0^1(\Omega)$ denote the solution of the Poisson equation (1.2*b*) with right hand side ϕ_λ. Since Ω is convex, regularity theory ensures that $u_\lambda \in H_0^1 \cap H^2(\Omega)$ and that

$$\|u_\lambda\|_{H^2(\Omega)} \leq c_2 \|\phi_\lambda\|_{L_2(\Omega)} . \tag{6.5}$$

Furthermore, the trace theorem given in the appendix states that $\frac{\partial u_\lambda}{\partial n} \in H_{00}^{1/2}(\Gamma_i)$ and that

$$\left(\sum_{i=1}^{r} \left\| \frac{\partial u_\lambda}{\partial n} \right\|^2_{H_{00}^{1/2}(\Gamma_i)}\right)^{1/2} \leq c_3 \| u_\lambda \|_{H^2(\Omega)} . \tag{6.6}$$

Inserting (6.6) and (6.5) into (6.4) yields

$$\| \phi_\lambda \|^2_{L_2(\Omega)} = \int_\Omega (-\Delta u_\lambda)\, \phi_\lambda \, dx \leq c_2\, c_3 \| \lambda \|_{-1/2,\Gamma} \; \| \phi_\lambda \|_{L_2(\Omega)} ,$$

which proves the second inequality of (6.3).

Next, let $\rho_i \in H_{00}^{1/2}(\Gamma_i)$ such that $\| \rho_i \|_{H_{00}^{1/2}(\Gamma_i)} = 1$ and

$$\| \lambda \|_{-1/2,\Gamma} \leq c_4 \int_{\Gamma_i} \lambda \rho_i \, ds . \tag{6.7}$$

By the trace theorem, there is $u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $\left.\frac{\partial u}{\partial n}\right|_{\Gamma_i} = \rho_i$, $\left.\frac{\partial u}{\partial n}\right|_{\Gamma_l} = 0$, $l \neq i$ and

$$\| u \|_{H^2(\Omega)} \leq c_5 \| \rho_i \|_{H_{00}^{1/2}(\Gamma_i)} = c_5 . \tag{6.8}$$

From (6.4) we get

$$\begin{aligned} \int_{\Gamma_i} \rho_i \, \lambda \, ds &= \int_\Omega (-\Delta u)\, \phi_\lambda \, dx \\ &\leq \| u \|_{H^2(\Omega)} \; \| \phi_\lambda \|_{L_2(\Omega)} . \end{aligned} \tag{6.9}$$

Inserting (6.9) and (6.8) into (6.7) yields the first inequality of (6.3). □

The estimate for the continuous case may be carried over to the discrete case (see also [12], [13] for similar arguments used in different contexts).

THEOREM 6.2 : *Let Ω be convex. Given $\lambda_h \in R_h$, let φ_{λ_h} be the discrete harmonic function with boundary value λ_h, i.e. the solution of (2.7). Then there are positive constants α, β, such that*

$$\begin{aligned} \alpha \left(\| \lambda_h \|_{-1/2,\Gamma} + h^{1/2} \, \| \lambda_h \|_{L_2(\Gamma)} \right) &\leq \| \varphi_{\lambda_h} \|_{L_2(\Omega)} \\ &\leq \beta \left(\| \lambda_h \|_{-1/2,\Gamma} + h^{1/2} \, \| \lambda_h \|_{L_2(\Gamma)} \right) . \end{aligned} \tag{6.10}$$

Proof: Let $\phi_{\lambda_h} \in H^1(\Omega)$ denote the harmonic function with boundary value λ_h. By the approximation properties of the finite element space X_h, we see that ([8], p. 184)

$$\|\phi_{\lambda_h} - \varphi_{\lambda_h}\|_{L_2(\Omega)} \leq c_1 h^{1/2} \|\lambda_h\|_{L_2(\Gamma)} . \tag{6.11}$$

Therefore, Theorem 6.1 implies

$$\begin{aligned} \|\varphi_{\lambda_h}\|_{L_2(\Omega)} &\leq \|\phi_{\lambda_h}\|_{L_2(\Omega)} + \|\varphi_{\lambda_h} - \phi_{\lambda_h}\|_{L_2(\Omega)} \\ &\leq \beta(\|\lambda_h\|_{-1/2,\Gamma} + h^{1/2}\|\lambda_h\|_{L_2(\Gamma)}) , \end{aligned}$$

which proves the second inequality of (6.10).

Next we will prove the first inequality of (6.10). Since the triangulation is regular, a simple scaling argument yields

$$\|\varphi_{\lambda_h}\|_{L_2(\Omega)} \geq c_2 h^{1/2} \|\lambda_h\|_{L_2(\Gamma)} . \tag{6.12}$$

Put $\eta := h^{1/2}\|\lambda_h\|_{L_2(\Gamma)} / \|\lambda_h\|_{-1/2,\Gamma}$. From (6.3) and (6.11) we conclude that

$$\begin{aligned} \|\varphi_{\lambda_h}\|_{L_2(\Omega)} &\geq \|\phi_{\lambda_h}\|_{L_2(\Omega)} - \|\varphi_{\lambda_h} - \phi_{\lambda_h}\|_{L_2(\Omega)} \\ &\geq c_3 \|\lambda_h\|_{-1/2,\Gamma} - c_1 h^{1/2}\|\lambda_h\|_{L_2(\Gamma)} \\ &= (c_3 - c_1\eta)\|\lambda_h\|_{-1/2,\Gamma} . \end{aligned} \tag{6.13}$$

Using (6.12) and (6.13) we obtain

$$\|\varphi_{\lambda_h}\|_{L_2(\Omega)} \geq \max\{c_2\eta, c_3 - c_1\eta\}\, \|\lambda_h\|_{-1/2,\Gamma} . \tag{6.14}$$

Since

$$\min_{\eta>0} \max\{c_2\eta, c_3 - c_1\eta\} = \frac{c_3 c_2}{c_1 + c_2} := \alpha > 0 ,$$

we get the result as stated. □

7. DISCRETE NORMS AND MATRIX-REPRESENTATIONS

In order to apply Theorem 6.2 for the proof of (6.1), we must verify that the norm $(\|\cdot\|^2_{-1/2,\Gamma} + h\|\cdot\|^2_{L_2(\Gamma)})^{1/2}$ is represented by the matrix C_h defined by (3.5). More precisely, we will prove that the norms $\|\cdot\|_{C_h}$ and $(\|\cdot\|^2_{-1/2,\Gamma} + h\|\cdot\|^2_{L_2(\Gamma)})^{1/2}$ are equivalent on R_h with constants being independent of the mesh size. Let $R_h^0 \subset R_h$ denote the subspace consisting of those piecewise linear functions which vanish at the corners of $\partial\Omega$. Then $v_h \in R_h^0$ if and only if $v_h \in R_h$ and $v_h|_{\Gamma_k} \in H_0^1(\Gamma_k)$, $1 \leq k \leq r$.

7.1. Matrix-Representation of the $H_{00}^{1/2}(\Gamma_k)$-norm on $R_h^0|_{\Gamma_k}$.

We denote the nodes on the line segment $\Gamma_k = \overline{P_k P_{k+1}}$ by

$$P_k = x_0^k < x_1^k < \cdots < x_{p_k}^k < x_{p_k+1}^k = P_{k+1}\,.$$

Let $v_h \in R_h^0|_{\Gamma_k}$. Then

$$|v_h|^2_{H^1(\Gamma_k)} = \sum_{l=0}^{p_k} \frac{1}{|x_{l+1}^k - x_l^k|} (v_h(x_{l+1}^k) - v_h(x_l^k))^2\,.$$

Since the triangulation is regular, i.e. $\sigma h \leq |x_{l+1}^k - x_l^k| \leq h$, the H^1-norm on $R_h^0|_{\Gamma_k}$ is equivalent to

$$\left[\frac{1}{h} \sum_{l=0}^{p_k} (v_h(x_{l+1}^k) - v_h(x_l^k))^2\right]^{1/2},$$

and the associated bilinear form is represented by the tridiagonal $p_k \times p_k$-matrix

$$\frac{1}{h} D_k = \frac{1}{h} \operatorname{tridiag}\,[-1, 2, -1]\,. \tag{7.1}$$

By interpolation we obtain the following

PROPOSITION 7.1 : *The norms*

$$\|v_h\|_{H_{00}^{1/2}(\Gamma_k)} \quad \textit{and} \quad |||\, v_h \,|||_{1/2, \Gamma_k} := \sqrt{v_h^T D_k^{1/2} v_h}$$

are equivalent on $R_h^0|_{\Gamma_k}$ with constants being independent of the mesh size h.

Proof: Let $s \geq 0$. Set

$$|||\, v_h \,|||_{s, \Gamma_k} := \left[h v_h^T \left(\frac{1}{h^2} D_k\right)^s v_h\right]^{1/2}. \tag{7.2}$$

Then the imbeddings

$$i : (R_h^0|_{\Gamma_k}, |||\cdot|||_{0, \Gamma_k}) \to L_2(\Gamma_k)$$

and
$$i : (R_h^0|_{\Gamma_k}, ||| \cdot |||_{1,\Gamma_k}) \to H_0^1(\Gamma_k)$$

are continuous with constants c_0 and c_1, resp. Therefore, the interpolation theorem [10] yields the continuity of i between the interpolated spaces. Specifically, in case $s = 1/2$ we obtain

$$(7.3) \qquad \|v_h\|_{H_{00}^{1/2}(\Gamma_k)} \leq (c_0\, c_1)^{1/2} ||| v_h |||_{\frac{1}{2},\Gamma_k}, \quad v_h \in R_h^0|_{\Gamma_k}.$$

Next, we consider the L_2-projection p_0 onto $R_h^0|_{\Gamma_k}$. Obviously $p_0 : L_2(\Gamma_k) \to (R_h^0|_{\Gamma_k}, ||| \cdot |||_{0,\Gamma_k})$ is continuous with constant c_2. Furthermore, we claim that $p_0 : H_0^1(\Gamma_k) \to (R_h^0|_{\Gamma_k}, ||| \cdot |||_{1,\Gamma_k})$ is continuous. Indeed, let $v \in H_0^1(\Gamma_k)$. Since v is a continuous function, the interpolant $J_h v \in R_h^1|_{\Gamma_k}$, $J_h v(x_l^k) = v(x_l^k)$, is well defined and

$$\|J_h v\|_{H^1(\Gamma_k)} \leq c \|v\|_{H^1(\Gamma_k)}.$$

Therefore, using approximation properties and inverse inequalities, we have

$$\begin{aligned} c_3 ||| p_0 v |||_{1,\Gamma_k} &\leq \|p_0 v\|_{H^1(\Gamma_k)} \\ &\leq \|v\|_{H^1} + \|J_h v - v\|_{H^1} + \|p_0 v - J_h v\|_{H^1} \\ &\leq c\|v\|_{H^1} + ch^{-1}(\|p_0 v - v\|_{L_2} + \|v - J_h v\|_{L_2}) \\ &\leq c\|v\|_{H^1}. \end{aligned}$$

Thus, the projection mappings are continuous and from the interpolation theorem we obtain for $s = 1/2$

$$(7.4) \qquad ||| p_0 v |||_{1/2,\Gamma_k} \leq c\|v\|_{H_{00}^{1/2}(\Gamma_k)}. \qquad \square$$

7.2. Matrix-Representation of the $H^{-1/2}(\Gamma_k)$-norm on $R_h^0|_{\Gamma_k}$.

By definition,

$$(7.5) \qquad \|u\|_{H^{-1/2}(\Gamma_k)} = \sup_{v \in H_{00}^{1/2}(\Gamma_k)} \frac{\int_{\Gamma_k} uv\, ds}{\|v\|_{H_{00}^{1/2}(\Gamma_k)}}.$$

If $u \in R_h^0|_{\Gamma_k}$, then we will see that it is sufficient to take the supremum over the subspace $R_h^0|_{\Gamma_k}$:

PROPOSITION 7.2 : *The norms*

$$\|u_h\|_{H^{-1/2}(\Gamma_k)} \quad \text{and} \quad |||u_h|||_{-1/2,\Gamma_k} := \sup_{v_h \in R_h^0|_{\Gamma_k}} \frac{\int_{\Gamma_k} u_h v_h \, ds}{|||v_h|||_{1/2,\Gamma_k}}$$

are equivalent on $R_h^0|_{\Gamma_k}$.

Proof: By (7.4) the L_2-projection

$$p_0 : H_{00}^{1/2}(\Gamma_k) \to (R_h^0|_{\Gamma_k}, |||\cdot|||_{1/2,\Gamma_k})$$

is continuous. Let $u_h \in R_h^0|_{\Gamma_k}$ and $v \in H_{00}^{1/2}(\Gamma_k)$. Then

$$\int_{\Gamma_k} u_h v \, ds = \int_{\Gamma_k} u_h p_0 v \, ds \leq c \frac{\left|\int_{\Gamma_k} u_h p_0 v \, ds\right|}{|||p_0 v|||_{1/2,\Gamma_k}} \|v\|_{H_{00}^{1/2}(\Gamma_k)} .$$

This implies

$$\|u_h\|_{H^{-1/2}(\Gamma_k)} \leq c \sup_{v_h \in R_h^0|_{\Gamma_k}} \frac{\int_{\Gamma_k} u_h v_h \, ds}{|||v_h|||_{1/2,\Gamma_k}} .$$

The reversed inequality follows from the inclusion $R_h^0|_{\Gamma_k} \subset H_{00}^{1/2}(\Gamma_k)$. □

Using the $p_k \times p_k$-mass matrix M_k and the Euclidian norm $\|\cdot\|$ we can write

$$(7.6) \qquad \begin{aligned} |||u_h|||_{-1/2,\Gamma_k} &= \sup_{v_h \in R_h^0|_{\Gamma_k}} \frac{\int_{\Gamma_k} u_h v_h \, ds}{|||v_h|||_{1/2,\Gamma_k}} \\ &= \sup_{v_h \in \mathbb{R}^{p_k}} \frac{u_h^T M_k v_h}{\|D_k^{1/4} v_h\|} \\ &= \sup_{z_h \in \mathbb{R}^{p_k}} \frac{u_h^T M_k D_k^{-1/4} z_h}{\|z_h\|} \\ &= \|D_k^{-1/4} M_k u_h\| . \end{aligned}$$

Thus, the $H^{-1/2}(\Gamma_k)$-norm on $R_h^0|_{\Gamma_k}$ is induced by the $p_k \times p_k$-matrix

$$C_k = M_k D_k^{-1/2} M_k . \tag{7.7}$$

7.3. Matrix-Representation of the $H^{-1/2}(\Gamma_k)$-norm on $R_h|_{\Gamma_k}$.

Finally, we consider the general case that $v_h \in R_h|_{\Gamma_k}$ does not necessarily vanish at the endpoints of Γ_k. We claim that the norm $(\|v_h\|^2_{H^{-1/2}(\Gamma_k)} + h\|v_h\|^2_{L_2(\Gamma_k)})^{1/2}$ on $R_h|_{\Gamma_k}$ is induced by the $(p_k+2) \times (p_k+2)$-matrix

$$\begin{pmatrix} h^2 & & \\ & C_k & \\ & & h^2 \end{pmatrix} . \tag{7.8}$$

Here, we identify each funtion $v_h \in R_h|_{\Gamma_k}$ with the vector of nodal values $(v_0, (v_1, v_2, \dots, v_{p_k}), v_{p_k+1})^T \in \mathbb{R}^{p_k+2}$.

Using the L_2-projection p_0 onto $R_h^0|_{\Gamma_k}$, we consider the decomposition

$$v = (v - p_0 v) + p_0 v . \tag{7.9}$$

By the usual approximation properties of the finite element space $R_h^0|_{\Gamma_k}$, we obtain :

LEMMA 7.3 : Let $v \in R_h|_{\Gamma_k}$. Then

$$\|v - p_0 v\|_{H^{-1/2}(\Gamma_k)} \leq h^{1/2} \|v\|_{L_2(\Gamma_k)} . \tag{7.10}$$

Proof: Let $\phi \in H_{00}^{1/2}(\Gamma_k)$. Then

$$\int_{\Gamma_k} (v - p_0 v)\, \phi \, ds = \int_{\Gamma_k} v (\phi - p_0 \phi)\, ds$$

and
$$\|\phi - p_0 \phi\|_{L_2(\Gamma_k)} \leq c h^{1/2} \|\phi\|_{H_{00}^{1/2}(\Gamma_k)} .$$

This proves (7.10). □

Let $v^{(1)} = (v_1, v_2, \dots, v_{p_k})$ denote the vector of nodal values associated with the interior nodes of Γ_k, $v^{(2)} := (v_0, 0, \dots, 0, v_{p_k+1})^T$ and

$$M_{12} v^{(2)} := (h_0 v_0, 0, \dots, 0, h_{p_k} v_{p_k+1}) \in \mathbb{R}^{p_k} , \quad h_l = |x_{l+1}^k - x_l^k| .$$

Then the coefficient vector of $p_0 v \in R_h^0|_{\Gamma_k}$ is given by

$$p_0 v = M_k^{-1}(M_k v^{(1)} + M_{12} v^{(2)}) .$$

Thus, using (7.6)

$$\interleave p_0 v \interleave_{-1/2, \Gamma_k} = \| D_k^{-1/4} M_k v^{(1)} + D_k^{-1/4} M_{12} v^{(2)} \| . \tag{7.11}$$

The second member of the right hand side of (7.11) can be estimated using

LEMMA 7.4 : *Let* $v^{(2)} = (v_0, 0, \dots, 0, v_{p_k+1})$. *Then*

$$\| D_k^{-1/4} M_{12} v^{(2)} \| \leq h (v_0^2 + v_{p_k+1}^2)^{1/2} . \tag{7.12}$$

Proof: Hölder's inequality implies

$$\| D_k^{-1/4} M_{12} v^{(2)} \| \leq \| M_{12} v^{(2)} \|^{1/2} \, \| D_k^{-1/2} M_{12} v^{(2)} \|^{1/2} . \tag{7.13}$$

The solution u of the linear system

$$D_k u = M_{12} v^{(2)}$$

is given by

$$u_j = h_0 v_0 + \frac{h_{p_k} v_{p_k+1} - h_0 v_0}{p_k + 1} j , \quad 1 \leq j \leq p_k .$$

Therefore,

$$\begin{aligned} \| D_k^{-1/2} M_{12} v^{(2)} \|^2 &= u^T M_{12} v^{(2)} \\ &= h_0^2 v_0^2 + h_{p_k}^2 v_{p_k+1}^2 - \frac{1}{p_k + 1} (h_0 v_0 - h_{p_k} v_{p_k+1})^2 \\ &\leq c h^2 (v_0^2 + v_{p_k+1}^2) . \end{aligned} \tag{7.14}$$

Inserting (7.14) into (7.13) completes the proof. □

Collecting the previous results we obtain

PROPOSITION 7.5 : *Let* $v_h \in M_h|_{\Gamma_k}$. *Then the norms*

$$\left(\| v_h \|^2_{H^{-1/2}(\Gamma_k)} + h \| v_h \|^2_{L_2(\Gamma_k)} \right)^{1/2} , \quad \left(v_h^T \begin{pmatrix} h^2 & & \\ & C_k & \\ & & h^2 \end{pmatrix} v_h \right)^{1/2}$$

are equivalent.

Proof: Combining (7.9)-(7.12) yields

$$\|v_h\|_{H^{-1/2}(\Gamma_k)} \begin{cases} \leqslant \|D_k^{-1/4} M_k v^{(1)}\| + c_2 h \left(\sum_{j=0}^{p_k+1} v_j^2 \right)^{1/2} \\ \geqslant \|D_k^{-1/4} M_k v^{(1)}\| - c_2 h \left(\sum_{j=0}^{p_k+1} v_j^2 \right)^{1/2} . \end{cases}$$

Using the inverse inequality

$$h \left(\sum_{j=1}^{p_k} v_j^2 \right)^{1/2} \leqslant c \|D_k^{-1/4} M_k v^{(1)}\|$$

completes the proof. □

APPENDIX : A TRACE THEOREM

We will establish the trace theorem which has been used in the proof of the a priori estimates given in section 6. Let $(a, b) \subset \mathbb{R}$. The Sobolev space $H_{00}^{1/2}(a, b)$, which is defined to be the interpolation space $[L_2(a, b), H_0^1(a, b)]_{1/2}$, has an explicit representation [10] :

$$\text{(A.1)} \quad H_{00}^{1/2}(a, b) = \left\{ u \in H^{1/2}(a, b) ; \frac{u}{\sqrt{(b-x)(x-a)}} \in L_2(a, b) \right\} ,$$

the interpolation norm being equivalent to

$$\text{(A.2)} \qquad \|u\|_{H_{00}^{1/2}(a, b)} := \left(\|u\|^2_{H^{1/2}(a, b)} + \int_a^b \frac{|u(x)|^2}{(b-x)(x-a)} dx \right)^{1/2} .$$

In (A.1), (A.2) we have used the Sobolev space $H^{1/2}(a, b)$, which can be defined using the norm

$$\text{(A.3)} \quad \|u\|_{H^s(a, b)} := \left(\|u\|^2_{L_2(a, b)} + \int_a^b \int_a^b \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} dx\, dy \right)^{1/2}$$

with $s = 1/2$. If $H_0^s(a, b)$, $0 \leqslant s \leqslant 1$, is defined to be the completion of $C_0^\infty(a, b)$ with respect to the norm (A.3), then $H_0^{1/2}(a, b) = H^{1/2}(a, b)$ and $H_{00}^{1/2}(a, b)$ is strictly contained in $H_0^{1/2}(a, b)$ [10]. This explains the additional zero in the indexing of $H_{00}^{1/2}(a, b)$.

The functions in $H_{00}^{1/2}(a, b)$ can be extended by 0 to functions in $H^{1/2}(\mathbb{R})$. The following is contained in [10].

PROPOSITION A.1 : *Let $u \in L_2(a, b)$ and let $\tilde{u}$ denote the extension of u by zero. Then $\tilde{u} \in H^{1/2}(\mathbb{R})$ if and only if $u \in H^{1/2}_{00}(a, b)$. In addition, there are positive constants σ_1, σ_2 such that*

$$\sigma_1 \|\tilde{u}\|_{H^{1/2}(\mathbb{R})} \leq \|u\|_{H^{1/2}_{00}(a,b)} \leq \sigma_2 \|\tilde{u}\|_{H^{1/2}(\mathbb{R})} \tag{A.4}$$

for every $u \in H^{1/2}_{00}(a, b)$.

If Ω is smooth, then the trace operator

$$H^2(\Omega) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$$
$$u \mapsto \left(u, \frac{\partial u}{\partial n}\right)$$

is known to be continuous and surjective with continuous right inverse [10]. Generalizations to domains with corners are given by Yakovlev [16]. We only need a special case which is more easily established.

THEOREM A.2 : *Let Ω be a convex polygon.*

a) *Let $u \in H^1_0 \cap H^2(\Omega)$. Then $\left.\frac{\partial u}{\partial n}\right|_{\Gamma_i} \in H^{1/2}_{00}(\Gamma_i)$ and*

$$\sum_{i=1}^{r} \left\| \frac{\partial u}{\partial n} \right\|_{H^{1/2}_{00}(\Gamma_i)} \leq \alpha \|u\|_{H^2(\Omega)} .$$

b) *Let $\rho_i \in H^{1/2}_{00}(\Gamma_i)$. Then there is a function $u \in H^1_0 \cap H^2(\Omega)$ satisfying $\left.\frac{\partial u}{\partial n}\right|_{\Gamma_i} = \rho_i$ and $\beta \|u\|_{H^2(\Omega)} \leq \sum_{i=1}^{r} \|\rho_i\|_{H^{1/2}_{00}(\Gamma_i)}$.*

The constants α and β depend on the interior angles ω_i of Ω and α/β grows as $\max_{1 \leq i \leq r} \left\{ \frac{1}{\sin(\omega_i)^2} \right\}$.

Proof: Let us consider the special case $\hat{\Omega} = (0,1) \times (0,1)$ at first.

a) If $u \in H^1_0 \cap H^2(\hat{\Omega})$, then $\nabla u\big|_{\Gamma} \in H^{1/2}(\hat{\Gamma})^2$ and

$$\|\nabla u\|_{H^{1/2}(\hat{\Gamma})} \leq c \|u\|_{H^2(\hat{\Omega})} .$$

Furthermore, since the derivative $\left.\frac{\partial u}{\partial x}\right|_{\hat{\Gamma}}$ $\left(\text{and } \left.\frac{\partial u}{\partial y}\right|_{\hat{\Gamma}}, \text{resp.}\right)$ is equal to zero on two opposite sides of the boundary, the result as stated follows from Proposition A.1.

b) Let $\rho \in H^{1/2}_{00}(0,1)$. Then $\tilde{\rho} \in H^{1/2}(\mathbb{R})$, where $\tilde{\rho}$ is the extension of ρ by zero. Using the results for the half space (see e.g. [10]), there is $u \in H^2(\mathbb{R}^2_+)$ satisfying

$$u(x, 0) = 0 , \quad -\frac{\partial u}{\partial y}(x, 0) = \tilde{\rho}(x) \tag{A.5}$$

and

$$\|u\|_{H^2(\mathbb{R}_+^2)} \leq c\,\|\rho\|_{H_{00}^{1/2}(0,1)} .$$

We may assume that u vanishes outside $\left(0, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{4}{3}\right)$. For $x > 0$ we define by reflection

$$(R_1 u)(x, y) := u(x, y) - \sum_{k=1}^{2} \alpha_k u(-kx, y) ,$$

where the numbers α_1 and α_2 are determined by the two conditions

(A.6) $$\frac{\partial^j}{\partial x^j}(R_1 u)(0, y) = 0 , \quad j = 0, 1 .$$

Note that $R_1 u$ retains the boundary conditions (A.5) for $x > 0$. Using a second reflection at $\{(1, y), y \in \mathbb{R}_+\}$, we obtain a function as stated.

Now we treat the general case. By a partition of unity, we only have to consider the situation in a neighbourhood $U \cap \Omega$ of a convex corner with angle ω. With the help of an affine mapping F we are brought back to a neighbourhood $U \cap \hat{\Omega}$ of a corner with right angle. The affine mapping $F(x) = Bx + b$, $B = \begin{pmatrix} 1 & \cos\omega \\ 0 & \sin\omega \end{pmatrix}^{-1}$ leads to a correspondence

$$x \in \Omega \to \hat{x} \in \hat{\Omega} , \qquad u \to \hat{u} \circ F .$$

Normals are generally not preserved through affine mappings. However, it is easily seen that

$$\frac{\partial u}{\partial n} = \frac{\partial \hat{u}}{\partial \hat{n}} \frac{1}{\sin(\omega)} .$$

Finally, concerning the transformations of the norms we refer to ([5], p. 117). $\square$

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