

MIGUEL LOBO

EUGENIA PEREZ

**Asymptotic behaviour of an elastic body with a
surface having small stuck regions**

M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 22, n° 4 (1988), p. 609-624

http://www.numdam.org/item?id=M2AN_1988__22_4_609_0

© AFCET, 1988, tous droits réservés.

L'accès aux archives de la revue « M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>



**ASYMPTOTIC BEHAVIOUR OF AN ELASTIC BODY
WITH A SURFACE HAVING SMALL STUCK REGIONS (*)**

Miguel LOBO ⁽¹⁾ and Eugenia PEREZ ⁽¹⁾

Communicated by E. SANCHEZ-PALENCIA

Abstract. — We shall consider an elastic body, with a surface which is partially stuck to a fixed plane. The stuck region has a diameter ε and the distance between two neighbouring zones is η . The critical size of these zones is given by the relation $\varepsilon = O(\eta^2)$. In this case the limit behaviour of the body is described by a boundary condition which is intermediate between the perfect stuck and unstuck cases.

Résumé. — On considère un corps élastique dont la surface est partiellement encastrée dans un plan fixe. La région encastrée est de diamètre ε et la distance entre deux zones voisines est η . La taille critique de ces zones est donnée par la relation $\varepsilon = O(\eta^2)$. Dans ce cas, le comportement limite du corps est décrit par une condition aux limites intermédiaire entre le cas de l'encastrement parfait et le cas « libre ».

1. INTRODUCTION

In this paper we study by means of the techniques of formal asymptotic analysis (Eckhaus [6], Sanchez-Palencia [14]), the asymptotic behaviour of an elastic body. A part Σ of its surface $\partial\Omega$ is partially stuck to the plane $x_3 = 0$. The size of the stuck zones is $O(\varepsilon)$ and the distance between them is $O(\eta)$, where $\varepsilon, \eta(\varepsilon)$ are parameters such that $\eta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$.

This problem belongs to a large class of boundary homogenization problems which, for some operators, have already been studied by authors like Sanchez-Palencia [14], Sanchez-Palencia & Sanchez-Hubert [15], Lobo & Perez [10].

We study the manner in which these stuck zones influence the displacements and the stresses, when ε is small, by calculating a relation between ε

(*) Received in June 1987.

⁽¹⁾ Departamento de Ecuaciones Funcionales, Facultad de Ciencias Avenida de los Castros s/n, 39005 Santander, Spain.

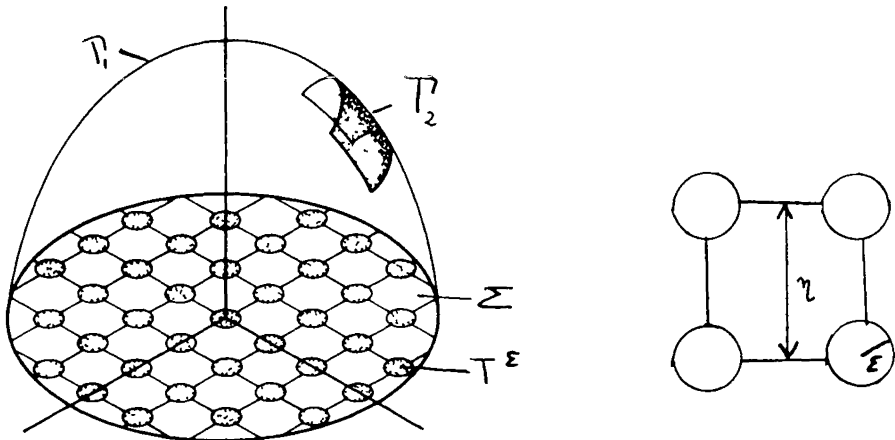
and η so that the limit behaviour will be distinct from the extreme cases, i.e. where all the surface is stuck to the plane, or all the surface is unstuck. The « critical size » of these zones is given by the relation $\varepsilon = O(\eta^2)$. For this size, the boundary conditions that we find in the limit problem, give us a relation between the stresses and the displacements from a « matrix of capacities » obtained from the solution of the « local problem ».

We also study this local problem (see section 4) posed in the semi-space \mathbb{R}^{3+} , finding its solution as the convolution of a distribution with the Green Tensor.

2. SETTING OF THE PROBLEM

Let Ω be a bounded open domain of \mathbb{R}^3 situated in the semi-space $x_3 > 0$ with a Lipschitz boundary $\partial\Omega$, and its part $\Sigma = \partial\Omega \cap \{x_3 = 0\}$ is assumed to be non-empty. Let $\Gamma = \partial\Omega - \bar{\Sigma}$ and Γ_1, Γ_2 be two open domains in Γ such that Γ_2 has a positive measure and $\bar{\Gamma} = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ (cf. fig.).

Let T^1 denote a bounded open domain with a smooth boundary in the plane $\{x_3 = 0\}$, containing the origin, and T^ε denotes its homothetic εT^1 , where the quantity ε is a positive parameter, which we will make tend to zero. In order to simplify, if there is no ambiguity, we shall also use T^ε to denote any domain obtained by translation of the previous domain in the plane $\{x_3 = 0\}$.



Let $\eta(\varepsilon)$ be an increasing function of ε which tends to zero when $\varepsilon \rightarrow 0$ and such that its inverse, $\varepsilon(\eta)$, is infinitely small with respect to η , that is $\varepsilon = o(\eta)$.

For a fixed ε we construct in the plane $x_3 = 0$ a grid of squares whose vertices are the points $(n\eta, m\eta, 0)$, $n, m \in \mathbb{Z}$, and let $n(\varepsilon)$ be the number of the T^ε , centered on the vertices of the grid $\{\tilde{x}_i\}_{i=1}^{n(\varepsilon)}$, contained in Σ (cf. fig.) and Γ_ε the union of these zones. We have

$$n(\varepsilon) \simeq \frac{\mu(\Sigma)}{\eta^2}, \mu(\Sigma) = \text{surface measure of } \Sigma.$$

The geometric configuration in the plane $x_3 = 0$ is analogous to that described by authors like Sanchez-Palencia [14], Murat [12].

Given the function $\underline{f} = (f_1, f_2, f_3) \in (L^2(\Omega))^3$, consider the following boundary value problem :

$$(2.1) \quad \frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i = 0 \quad \text{in } \Omega, \quad i = 1, 2, 3$$

$$(2.2) \quad u_i^\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon$$

$$(2.3) \quad \sigma_{ij}^\varepsilon n_j = 0 \quad \text{on } \Sigma - \bar{\Gamma}_\varepsilon$$

$$(2.4) \quad u_i^\varepsilon = 0 \quad \text{on } \Gamma_2, \sigma_{ij}^\varepsilon n_j = 0 \quad \text{on } \Gamma_1$$

where

$$(2.5) \quad \sigma_{ij}^\varepsilon = a_{ijkl} e_{kh}(\underline{u}^\varepsilon), \quad e_{kh}(\underline{v}) = \frac{1}{2} \left(\frac{\partial v_h}{\partial x_k} + \frac{\partial v_k}{\partial x_h} \right).$$

We have used the usual notation for the displacements $\underline{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ and the stresses σ_{ij}^ε , and the convention of repeated indices has been adopted. The boundary value problem (2.1)-(2.5) is the problem of elasticity for a isotropic homogeneous material with coefficients of elasticity a_{ijkl} under the action of the forces \underline{f} . The boundary conditions express the fact that the body Ω is fixed by the parts Γ_2 and $\Gamma_\varepsilon = \cup_{\varepsilon} T^\varepsilon$ of the boundary, leaving the rest free.

We study the asymptotic behaviour of this problem when $\varepsilon \rightarrow 0$.

For each fixed ε the problem (2.1)-(2.5) has an equivalent variational formulation :

Find $\underline{u}^\varepsilon \in V^\varepsilon$ that verifies the equation :

$$(2.6) \quad \int_{\Omega} \sigma_{ij}^\varepsilon e_{ij}(\underline{v}) \, dx = \int_{\Omega} f_i v_i \, dx \quad \forall \underline{v} \in V^\varepsilon$$

where $\sigma_{ij}^\varepsilon = a_{ijkl} e_{kh}(\underline{u}^\varepsilon)$ and V^ε is the space completed of $(\mathcal{D}^\varepsilon(\bar{\Omega}))^3$ with the norm of $(H^1(\Omega))^3$, where

$$(2.7) \quad \mathcal{D}^\varepsilon(\bar{\Omega}) = \left\{ u \in \mathcal{C}^\infty(\bar{\Omega}) \text{ such that } u|_{\Gamma_\varepsilon} = 0, u|_{\Gamma_2} = 0 \right\}.$$

Problem (2.6) has a unique solution $\underline{u}^\varepsilon$ in the space V^ε . By Korn's inequality we have the boundedness

$$(2.8) \quad \|\underline{u}^\varepsilon\|_{(H^1(\Omega))^3} \leq C$$

where C is a constant independent of ε .

3. ASYMPTOTIC EXPANSIONS

When $\varepsilon \rightarrow 0$, it is evident that a boundary layer phenomenon take place on Σ due to the geometric structure of the problem. We shall proceed to apply the techniques of asymptotic matched expansions.

3.1. Outer expansion

By virtue of the estimate (2.8) we postulate an « outer expansion », for the displacements and the stresses respectively of type :

$$(3.1) \quad \begin{aligned} \underline{u}^\varepsilon &= \underline{u}^0 + \varepsilon \underline{u}^1 + \varepsilon^2 \underline{u}^2 \dots \\ \sigma_{ij}^\varepsilon &= \sigma_{ij}^0 + \varepsilon \sigma_{ij}^1 + \varepsilon^2 \sigma_{ij}^2 + \dots \end{aligned}$$

These expansions are considered valid in $\{x_3 > 0\} \cap \bar{\Omega}$. We can postulate that the convergences of $\underline{u}^\varepsilon$ to \underline{u}^0 and σ_{ij}^ε to σ_{ij}^0 will take place in $(H^1(\Omega_r))^3$ and $L^2(\Omega_r)$ respectively, $\Omega_r = \{x_3 > r\} \cap \Omega, \forall r > 0$.

From the validity of expansions (3.1), from boundedness (2.8) and finally as an application of Rellich's Theorem we can deduce the following conditions for \underline{u}^0 :

$$(3.2) \quad \frac{\partial \sigma_{ij}^0}{\partial x_j} + f_i = 0 \quad \text{in } \Omega, \quad i = 1, 2, 3$$

$$(3.3) \quad \sigma_{ij}^0 n_j = 0 \quad \text{on } \Gamma_1$$

$$(3.4) \quad u_i^0 = 0 \quad \text{on } \Gamma_2$$

where

$$(3.5) \quad \sigma_{ij}^0 = a_{ijkl} e_{kh}(\underline{u}^0), \quad \underline{u}^0 = (u_1^0, u_2^0, u_3^0).$$

Equations (3.2), (3.3) and (3.4) are satisfied in $\mathcal{D}'(\Omega)$, $H^{-1/2}(\Gamma_1)$, and $H^{1/2}(\Gamma_2)$ respectively.

Remark 3.1 : The boundary conditions satisfied by \underline{u}^0 on the manifold Σ , will be obtained as a consequence of applying the matching relations with local expansions (see, for example Eckhaus [6], Sanchez-Palencia [14]). ■

For the local asymptotic study in a neighbourhood of Σ , we suppose that at each geometric center \tilde{x} of the stuck zones a boundary layer is found.

3.2. Local expansion

We carry out an enlargement in the neighbourhood of the geometric center \tilde{x} of each zone T^ε by changing the variable :

$$(3.6) \quad y = \frac{x - \tilde{x}}{\varepsilon}$$

where

$$x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3), \quad \tilde{x} = (x_1, \tilde{x}_2, 0).$$

This enlargement transforms T^ε of size $O(\varepsilon)$ into T^1 of size $O(1)$ and the closest centers are at a distance η/ε that tends to ∞ when $\varepsilon \rightarrow 0$.

We postulate an expansion in the variable y , a « local expansion », for the displacements and the stresses respectively of type :

$$(3.7) \quad \begin{aligned} \underline{u}^\varepsilon &= \underline{v}^0(y) + \varepsilon \underline{v}^1(y) + \dots \\ \sigma_{ij}^\varepsilon &= \frac{1}{\varepsilon} \sigma_{ij}^{-1} + \sigma_{ij}^0 + \varepsilon \sigma_{ij}^1 + \dots \end{aligned}$$

These expansions are assumed to be valid in any neighbourhood of the origin, and therefore, formally we have that \underline{v}^0 is the solution of problem :

$$(3.8) \quad \frac{\partial \sigma_{ij}^{-1}}{\partial y_j} = 0 \quad \text{in } \mathbb{R}^{3+}, \quad i = 1, 2, 3$$

$$(3.9) \quad v_i^0 = 0 \quad \text{on } T^1$$

$$(3.10) \quad \sigma_{ij}^{-1} n_j = 0 \quad \text{on } \mathbb{R}^2 - \bar{T}^1 \\ + \text{boundary conditions when } |y| \rightarrow \infty, \quad y_3 \geq 0$$

where \mathbb{R}^{3+} is $\{x \in \mathbb{R}^3 / x_3 > 0\}$ and

$$(3.11) \quad \sigma_{ij}^{-1} = a_{ijkl} e_{khy}(\underline{v}^0), \quad e_{khy}(\underline{v}) = \frac{1}{2} \left(\frac{\partial v_h}{\partial y_k} + \frac{\partial v_k}{\partial y_h} \right).$$

The condition at infinity, is obtained by supposing that the extended domains of validity of the outer expansion (3.1) and local expansion (3.7) for displacements overlap (we use Eckhaus' terminology [6]). In other words, we impose an asymptotic matching principle. This condition is given by :

$$(3.12) \quad \underline{v}^0(y) \rightarrow \underline{u}^0(\tilde{x}) \quad \text{when } |y| \rightarrow \infty, \quad y_3 \geq 0.$$

Remark 3.2 : The variable \tilde{x} behaves as a parameter in the problem (3.8)-(3.12). By linearity we can write the solution in the form :

$$(3.13) \quad \underline{v}^0(y) = u_k^0(\tilde{x})(\underline{e}^k - \underline{w}^k(y))$$

where u_k^0 is the k -th component of \underline{u}^0 and \underline{e}^k is the unitary vector in the direction x_k , \underline{W}^k is the solution of an problem similar to (3.8)-(3.12), the « local problem » :

$$(3.14) \quad \frac{\partial \sigma_{ij}^k}{\partial y_j} = 0 \quad \text{in } \mathbb{R}^{3+}, \quad i = 1, 2, 3$$

$$(3.15) \quad \underline{W}^k = \underline{e}^k \quad \text{on } T^1$$

$$(3.16) \quad \sigma_{ij}^k n_j = 0 \quad \text{on } \mathbb{R}^2 - \bar{T}^1$$

$$(3.17) \quad \underline{W}^k(y) \rightarrow 0, \quad \text{when } |y| \rightarrow \infty, \quad y_3 \geq 0$$

where

$$(3.18) \quad \sigma_{ij}^k = a_{ij\ell h} e_{\ell h}(W^k). \quad \blacksquare$$

4. STUDY OF THE LOCAL PROBLEM

The study of the correct statement of the problem (3.14)-(3.18) lead us to the consideration of the following functional spaces :

Let $\mathcal{D}(\bar{\mathbb{R}}^{3+})$ be the space of functions that are the restrictions to $\bar{\mathbb{R}}^{3+}$ of the elements of $\mathcal{D}(\mathbb{R}^3)$ and let $\mathcal{D}_1(\bar{\mathbb{R}}^{3+})$ be the space of functions of $\mathcal{D}(\bar{\mathbb{R}}^{3+})$ such that they are null in a neighbourhood of \bar{T}^1 . We consider on these spaces the norm defined by :

$$(4.1) \quad \|\underline{u}\|^2 = \sum_{i,j=1}^3 \|e_{ij}(\underline{u})\|_{L^2(\mathbb{R}^{3+})}^2$$

and we define the functional spaces \mathcal{V} and \mathcal{V}_1 as the completion of $(\mathcal{D}(\bar{\mathbb{R}}^{3+}))^3$ and $(\mathcal{D}_1(\bar{\mathbb{R}}^{3+}))^3$ respectively with the norm defined by (4.1).

The elements of \mathcal{V} are identified with distributions \underline{u} such that $e_{ij}(\underline{u}) \in L^2(\mathbb{R}^{3+}) \forall i, j = 1, 2, 3$; they are in $(H_{\text{loc}}^1(\mathbb{R}^{3+}))^3$ (see Duvaut & Lions [5], Temam [17]). On the other hand, the elements of \mathcal{V}_1 have null trace on T^1 .

For each $k = 1, 2, 3$ we take the function $\underline{\psi}^k = (\psi_1^k, \psi_2^k, \psi_3^k)$ that satisfies :

$$(4.2) \quad \underline{\psi}^k \in \mathcal{D}(\bar{\mathbb{R}}^{3+}), \quad \underline{\psi}^k|_{\gamma(\bar{T}^1)} = \underline{e}^k$$

where $\gamma(\bar{T}^1)$ indicates a neighbourhood of \bar{T}^1 .

For each $k = 1, 2, 3$, the problem (4.2) has an equivalent variational formulation :

Find $\underline{W}^k \in \Psi^k + \mathcal{V}_1$ that satisfies the equation :

$$(4.3) \quad \int_{\mathbb{R}^{3+}} \sigma_{ij}^k e_{ij}(\underline{u}) dy = 0 \quad \forall \underline{u} \in \mathcal{V}_1 \quad \sigma_{ij}^k = a_{ij\ell h} e_{\ell h}(\underline{W}^k).$$

Problem (4.3) has a unique solution \underline{W}^k in the space $\Psi^k + \mathcal{V}_1$.

The fact that the function \underline{W}^k which fulfills equation (3.4), is a weak solution of problem (3.14)-(3.18) is a consequence of the following remarks :

\underline{W}^k satisfies the equation (3.14) in the sense of distributions.

For any smooth domain B of plane $\{x_3 = 0\}$, $\sigma_{ij}^k n_j|_B \in H^{-1/2}(B)$, $i = 1, 2, 3$. Besides, we can apply the generalized Green's formula and obtain :

$$(4.4) \quad \int_{\mathbb{R}^{3+}} \sigma_{ij}^k e_{ij}(\underline{u}) dy = \langle \sigma_{ij}^k n_j|_{y_3=0}, u_i \rangle_{H^{-1/2}(B) \times H^{1/2}(B)} \quad \forall \underline{u} \in (\mathcal{D}(\bar{\mathbb{R}}^{3+}))^3$$

where B is an open domain with a smooth boundary in the plane $\{x_3 = 0\}$ such that $(\text{sop } \underline{u}) \cap \{y_3 = 0\} \subset B$.

It can be deduced then, that $\sigma_{ij}^k n_j|_{y_3=0}$ is a distribution with compact support contained in \bar{T}^1 , belonging to $H^{-1/2}(\mathbb{R}^2)$. Thus (Lions & Magenes [9]), it is identified as an element of $H^{-1/2}(T^1)$ and we can write :

(4.5)

$$\int_{\mathbb{R}^{3+}} \sigma_{ij}^k e_{ij}(\underline{v}) dy = \langle \sigma_{ij}^k n_j|_{y_3=0}, v_i \rangle_{H^{-1/2}(T^1) \times H^{1/2}(T^1)} \quad \forall \underline{v} \in (\mathcal{D}(\bar{\mathbb{R}}^{3+}))^3.$$

Remark 4.1 : We observe that the norm defined by relation (4.1) does not provide us with information about the behaviour of the functions of space \mathcal{V} at infinity contrary to what occurs when the space \mathcal{V} is the Dirichlet space, completion of $\mathcal{D}(\bar{\mathbb{R}}^{3+})$ with the gradient norm (see Ladyzenskaya [7], Sanchez-Palencia [14]). The condition that \underline{W}^k fulfills at infinity will come as a consequence of Theorem 4.1. ■

Remark 4.2 : Once the solution \underline{W}^k of problem (4.3) is known, and taking problem (3.14)-(3.17) into account, we can consider the Neumann problem on \mathbb{R}^{3+} in the following form :

$$(4.6) \quad \frac{\partial \tau_{ij}}{\partial y_j} = 0 \quad \text{in } \mathbb{R}^{3+}, \quad i = 1, 2, 3$$

$$(4.7) \quad \tau_{ij} n_j = q_i \quad \text{on } \{y_3 = 0\}$$

$$(4.8) \quad \underline{U}(y) \rightarrow 0, \quad \text{when } |y| \rightarrow \infty, \quad y_3 \geq 0$$

where

$$(4.9) \quad \tau_{ij} = a_{ijkl} e_{khy}(\underline{U}), \quad q = (q_1, q_2, q_3), \quad q_i = \sigma_{ij}^k n_j |_{y_3=0}.$$

This problem has a weak formulation in the space \mathcal{V} :

Find $\underline{U} \in \mathcal{V}$ satisfying the equation :

$$(4.10) \quad \int_{\mathbb{R}^{3+}} \tau_{ij} e_{ij}(\underline{v}) dy = \langle q_i, v_i \rangle_{H^{-1/2}(T^1) \times H^{1/2}(T^1)}, \quad \forall \underline{v} \in (\mathcal{D}(\bar{\mathbb{R}}^{3+}))^3$$

where $\tau_{ij} = a_{ijkl} e_{khy}(\underline{U})$.

As \underline{W}^k satisfies equation (4.5), problem (4.10) has \underline{W}^k as unique solution in the space \mathcal{V} . ■

Remark 4.3 : On the other hand, problem (4.6)-(4.9) corresponds to an equilibrium state of an elastic medium limited by the plane $\{x_3 = 0\}$, with null deformations at infinity, and consequently, the only forces that act do so through $\{x_3 = 0\}$. If $\underline{q} \in (\mathcal{D}(\mathbb{R}^2))^3$ the problem is studied in Landau & Lifschitz [8] and the classic solution is given by the function :

$$(4.11) \quad \underline{U} = G * \underline{q}, \quad U_i = G_{ij} * q_j, \quad i = 1, 2, 3$$

where G is the Green tensor for the equilibrium equations of an elastic semi-infinite isotropic and homogeneous medium. This tensor is defined by the relations :

$$(4.12) \quad \begin{aligned} G_{11}(x, y, z) &= \frac{1 + \sigma}{2 \pi E} \left(\frac{2(1 - \sigma)r + z}{r(r + z)} + \frac{x^2(2r(\sigma r + z) + z^2)}{r^3(r + z)^2} \right) \\ G_{12}(x, y, z) &= \frac{1 + \sigma}{2 \pi E} \frac{xy(2r(\sigma r + z) + z^2)}{r^3(r + z)^2} \\ G_{13}(x, y, z) &= \frac{1 + \sigma}{2 \pi E} \left(\frac{xz}{r^3} - \frac{(1 - 2\sigma)x}{r(r + z)} \right) \\ G_{22}(x, y, z) &= \frac{1 + \sigma}{2 \pi E} \left(\frac{2(1 - \sigma)r + z}{r(r + z)} + \frac{y^2(2r(\sigma r + z) + z^2)}{r^3(r + z)^2} \right) \\ G_{23}(x, y, z) &= \frac{1 + \sigma}{2 \pi E} \left(\frac{yz}{r^3} - \frac{(1 - 2\sigma)y}{r(r + z)} \right) \\ G_{33}(x, y, z) &= \frac{1 + \sigma}{2 \pi E} \left(\frac{z^2}{r^3} + \frac{2(1 - \sigma)}{r} \right) \\ G_{ij} &= G_{ji}, \quad i, j = 1, 2, 3, \end{aligned}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, $(x, y, z) \in \mathbb{R}^{3+}$.

We see that the terms, G_{ij} and their derivatives $\frac{\partial G_{ij}}{\partial x_l}$, $\frac{\partial G_{ij}}{\partial x_l \partial x_m}$ are homogeneous functions of degree -1 , -2 , -3 respectively. So, the following estimates can be verified :

$\forall (x, y, z) \in \mathbb{R}^{3+}$, $i, j, l, m = 1, 2, 3$

$$(4.13) \quad \begin{aligned} |G_{ij}(x, y, z)| &\leq \frac{\text{Cte}}{r} \\ \left| \frac{\partial G_{ij}}{\partial x_l}(x, y, z) \right| &\leq \frac{\text{Cte}}{r^2} \\ \left| \frac{\partial G_{ij}}{\partial x_l \partial x_m}(x, y, z) \right| &\leq \frac{\text{Cte}}{r^3} \end{aligned}$$

where Cte does not depend on (x, y, z) . ■

Remark 4.4 : Let $q \in H^{-1/2}(T^1)$, then the function $W = q * G_{ij}$ is defined in \mathbb{R}^{3+} :

$$(4.14) \quad W(x_1, x_2, x_3) = \langle q_\xi, G_{ij}(x_1 - \cdot, x_2 - \cdot, x_3) \rangle_{H^{-1/2}(T^1) \times H^{1/2}(T^1)}$$

where $\xi = (\xi_1, \xi_2)$; $i, j = 1, 2, 3$. ■

The following theorem assures us that the weak solution of problem (4.6)-(4.8) coincides with the classic solution $G * \underline{q}$.

THEOREM 4.1 : Let $\underline{W}^k = (W_1^k, W_2^k, W_3^k)$ be the solution of problem (4.3), then

$$W_i^k = G_{ij} * \sigma_{jt}^k n_l |_{T^1} \quad i = 1, 2, 3 .$$

The proof of this theorem will be a consequence of the following propositions. Let $d(x, \bar{T}^1)$ denote $\inf_{y \in \bar{T}} |x - y|$,

$$|x - y| = ((x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2)^{1/2} .$$

PROPOSITION 4.1 : Let $q \in H^{-1/2}(T^1)$ and $W = q * G_{ij}$. Then $\forall x \in \mathbb{R}^{3+}$ the following estimates hold :

$$(4.15) \quad \begin{aligned} |W(x)| &\leq \text{Cte} \left(\frac{1}{d(x, \bar{T}^1)} + \frac{1}{d(x, \bar{T}^1)^2} \right) \\ \left| \frac{\partial W}{\partial x_p}(x) \right| &\leq \text{Cte} \left(\frac{1}{d(x, \bar{T}^1)^2} + \frac{1}{d(x, \bar{T}^1)^3} \right) \end{aligned}$$

where Cte indicates a positive constant ; $p = 1, 2, 3$.

Proof: Taking into account the fact that the injection of $H^1(T^1)$ in $H^{1/2}(T^1)$ is continuous, and relation (4.14) we have :

$$(4.16) \quad |W(x)| \leq \|q\|_{H^{-1/2}(T^1)} \|G_x\|_{H^{1/2}(T^1)} \leq C(q) \|G_x\|_{H^1(T^1)}$$

where $C(q)$ is a constant that depends on the distribution q and G_x is the function defined :

$$G_x(\xi) = G_{ij}(x_1 - \xi_1, x_2 - \xi_2, x_3), \quad \xi = (\xi_1, \xi_2).$$

As the norm

$$\|G_x\|_{H^1(T^1)}^2 = \int_{T^1} |G_x(\xi)|^2 d\xi + \sum_{p=1}^2 \int_{T^1} \left| \frac{\partial G_x}{\partial \xi_p}(\xi) \right|^2 d\xi$$

and $\frac{\partial G_x}{\partial \xi_p}(\xi) = -\frac{\partial G_x}{\partial x_p}(\xi)$, we can utilize estimates (4.13) with $r^2 = |x - \xi|^2$ and we have thus :

$$(4.17) \quad \|G_x\|_{H^1(T^1)}^2 \leq \text{Cte} \left(\frac{1}{d(x, \bar{T}^1)^2} + \frac{1}{d(x, \bar{T}^1)^4} \right).$$

From (4.17) and (4.16) we obtain the boundedness (4.15) for $W(x)$. The boundedness for the derivatives are found in analogous way, obtaining the relation :

$$(4.18) \quad \left| \frac{\partial W}{\partial x_p}(x) \right|^2 \leq C(q) \left\| \frac{\partial G_x}{\partial x_p} \right\|_{H^1(T^1)}^2.$$

Thus, the proposition is proved. ■

Before to second proposition we state a lemma (Deny [4]) that will be used in its proof.

LEMMA 4.1 : *Let μ be a positive measure on \mathbb{R}^3 with total finite mass, and let the potential be defined as*

$$(4.19) \quad U_\alpha^\mu(x) = \int \frac{1}{|x - y|^{3-\alpha}} d\mu(y)$$

where α is a number such that $0 < \alpha < 3$. Then, the following relation holds :

$$(4.20) \quad \int |U_{\alpha/2}^\mu(x)|^2 dx = C(\alpha) \iint \frac{1}{|x - y|^{3-\alpha}} d\mu(x) d\mu(y)$$

where $C(\alpha)$ is a constant dependant on α and μ .

PROPOSITION 4.2 : Let $\underline{q} \in (\mathcal{D}(T^1))^3$. Then the function $\underline{W} = G * \underline{q} \in \mathcal{V}$.

Proof: 1) First, we shall demonstrate that $e_{ij}(\underline{W}) \in L^2(\mathbb{R}^{3+})$. In effect, let k, l, r, s be any of the subindices 1, 2, 3. We verify that the function $D_l(q_k * G_{rs}) = \frac{\partial}{\partial x_l}(q_k * G_{rs})$ belongs to $L^2(\mathbb{R}^{3+})$. Taking into account estimates (4.13) we obtain the boundedness : $\forall x \in \mathbb{R}^{3+}$

$$(4.21) \quad |D_l(q_k * G_{rs})(x)| = |q_k * D_l G_{rs}(x)| \leq \\ \leq \text{Cte} \int_{T^1} \frac{1}{|x-y|^2} |q_k(y)| dy = \text{Cte} U_{\alpha/2}^\mu(x)$$

where $\alpha = 2$ and the measure $\mu = |q_k| dT^1$.

On the other hand, the application of Lemma 4.1, the regularity of the function q_k and relation (4.21) leads to the inequalities :

$$\int_{\mathbb{R}^{3+}} |D_l W_i|^2 dx \leq \text{Cte} \iint_{T^1} \frac{1}{|x-y|} |q_k(x)| |q_k(y)| dx dy \leq \\ \leq \text{Cte} \iint_{T^1} \frac{1}{|x-y|} dx dy$$

where the constant, Cte, depends on q_k ; and as this last integral is finite we have the stated result for each $e_{ij}(\underline{W})$, $i, j = 1, 2, 3$.

2) Now we shall demonstrate that $\underline{W} \in \mathcal{V}$. We take a function $\chi \in \mathcal{D}(\mathbb{R}^3)$ defined by :

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2 \end{cases}.$$

Then, $\forall R > 0$ the function $\chi_R(x) = \chi\left(\frac{x}{R}\right)$ satisfies :

$$(4.22) \quad 0 \leq \chi_R(x) \leq 1 ; \quad \left| \frac{\partial \chi_R}{\partial x_i}(x) \right| \leq \text{Cte} \frac{1}{R}.$$

Taking into account the first part of this proof, $e_{ij}(\underline{W})$ is a function whose square is integrable outside of B_{2R}^+ and besides $W_i \in L^2(B_{2R}^+)$, where $B_{2R}^+ = B(0, 2R) \cap \mathbb{R}^{3+}$. This allows us to affirm :

$$(4.23) \quad e_{ij}(\underline{W}\chi_R) \in L^2(\mathbb{R}^{3+}).$$

On the other hand, by the definition of function χ_R , we consider :

$$\|e_{ij}(\underline{W}\chi_R - \underline{W})\|_{L^2(\mathbb{R}^{3+})}^2 = \|e_{ij}(\underline{W}(\chi_R - 1))\|_{L^2(B_{2R}^+ - \bar{B}_R^+)}^2 + \|e_{ij}(\underline{W})\|_{L^2(CB_{2R}^+)}^2$$

by the first part of the proposition, the second term of the summation converges to zero when $R \rightarrow \infty$; and by the estimates of Proposition 4.1 and

relation (4.22) we can demonstrate that the first term of the summation also converges to zero ; therefore, we have :

$$(4.24) \quad e_{ij}(\underline{W}\chi_R) \underset{R \rightarrow \infty}{\rightarrow} e_{ij}(\underline{W}) \text{ in } L^2(\mathbb{R}^{3+}).$$

Now, as a consequence of relation (4.23) we have : $\underline{W}\chi_R \in (H^1(\Omega_R))^3$, where Ω_R is a bounded domain of \mathbb{R}^{3+} with a Lipschitz boundary that contains B_{2R}^+ . Therefore $\underline{W}\chi_R \in \mathcal{V}$; and taking into account the convergence (4.24), we have the result of the proposition. ■

LEMMA 4.2 : *For each $i, j = 1, 2, 3$ the application $q \rightarrow q * G_{ij}$ is continuous from $H_{\text{comp}}^{-1/2}(\mathbb{R}^2)$ into $H_{\text{Loc}}^{1/2}(\mathbb{R}^2)$.*

Proof: Each term of tensor G is reduced on plane $\{x_3 = 0\}$ to a summation of constants by terms of the type $|x|^{-1}, x_l|x|^{-2}, x_l x_m|x|^{-3}$ where $|x| = (x_1^2 + x_2^2)^{1/2}$, $x = (x_1, x_2, 0)$, and $l, m = 1, 2$. The Fourier transformations of these functions are, respectively, constants by terms of the type $|\xi|^{-1}, \xi_j|\xi|^{-2}, \xi_l \xi_m|\xi|^{-3}$ where $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2}$, $\xi = (\xi_1, \xi_2)$ (See Sneddon [16]).

Then, the result of this lemma follows, once it is demonstrated for $G_{ij} = 1/r$. See Mac Camy & Stephan [11] for this proof. ■

PROPOSITION 4.3 : *Let $\underline{q} \in (H^{-1/2}(T^1))^3$ then the function $\underline{W} = G * \underline{q} \in \mathcal{V}$.*

Proof: In the first place consider $\underline{q} \in (\mathcal{D}(T^1))^3$. Taking into account the result of Proposition 4.2, the properties of tensor G and relation (4.23), we can apply the Generalized Green's formula to obtain :

$$\int_{\mathbb{R}^{3+}} \sigma_{ij} e_{ij}(\underline{W}\chi_R) dy = \langle q_i, W_i \chi_R \rangle_{H^{-1/2}(T^1) \times H^{1/2}(T^1)}, \quad \sigma_{ij} = a_{ijkl} e_{khy}(\underline{W}).$$

Taking the limits when $R \rightarrow \infty$, by relation (4.24) we have :

$$(4.25) \quad \int_{\mathbb{R}^{3+}} \sigma_{ij} e_{ij}(\underline{W}) dy = \langle q_i, W_i \rangle_{H^{-1/2}(T^1) \times H^{1/2}(T^1)}, \quad \sigma_{ij} = a_{ijkl} e_{khy}(\underline{W}).$$

Lemma 4.2 assures the continuity of the application $\underline{q} \rightarrow G * \underline{q}$ from $(H^{-1/2}(T^1))^3$ into $(H^{1/2}(T^1))^3$ (see Lions & Magenes [9]). By the coercivity of the elasticity operator, we can deduce from relation (4.25) the inequality :

$$(4.26) \quad \|\underline{W}\|_{\mathcal{V}}^2 \leq \text{Cte} \|\underline{q}\|_{(H^{-1/2}(T^1))^3}^2$$

and, therefore, the continuity of the application $\underline{q} \rightarrow G * \underline{q}$ from $(\mathcal{D}(T^1))^3$ into \mathcal{V} , $(\mathcal{D}(T^1))^3$ with the norm of $(H^{-1/2}(T^1))^3$. As $(\mathcal{D}(T^1))^3$ is dense in

$(H^{-1/2}(T^1))^3$ (Lions & Magenes [9]), we can extend this application to a continuous application from $(H^{-1/2}(T^1))^3$ into \mathcal{V} , thus obtaining the result of the proposition. ■

Proof of Theorem 4.1: Proposition 4.3 assures that the function $\underline{u} = (\underline{u}_1, \underline{u}_2, \underline{u}_3)$, $\underline{u}_i = G_{ij} * \sigma_{jl}^k n_l |_{T^1}$ is an element of space \mathcal{V} . Applying the generalized Green's formula we deduce that \underline{u} is also a solution of equation (4.10) and, therefore, $\underline{u} = \underline{W}^k$; thus we have the result of the theorem. ■

Remark 4.5: The function \underline{W}^k , weak solution of local problem (3.14)-(3.17), is going to play an important part in the boundary condition satisfied by \underline{u}^0 on the manifold Σ . In order to determine this condition we must define a matrix \mathcal{C} that plays a role analogous to that of the capacity in the elliptical problems related to an equation (see Sanchez-Palencia [14], Murat [12], Picard [13]). ■

DEFINITION 4.1: Let us define the matrix $\mathcal{C} = (C_{kh})_{k,h=1,2,3}$

$$(4.27) \quad C_{kh} = \int_{\mathbb{R}^{3+}} \sigma_{ij}^k e_{ij}(\underline{W}^h) dy, \quad \sigma_{ij}^k = a_{i,jlh} e_{lhy}(\underline{W}^k).$$

Applying the Generalized Green's formula we obtain:

$$(4.28) \quad C_{kh} = - \langle \sigma_{h3}^k, 1 \rangle_{H^{-1/2}(T^1) \times H^{1/2}(T^1)}.$$

The properties of symmetry and ellipticity of the elasticity operator allows us to demonstrate that the matrix \mathcal{C} is symmetric and positive definite.

5. LIMIT PROBLEM

According to the relations obtained (3.2)-(3.6) \underline{u}^0 is the solution of an elasticity problem posed in Ω , whose boundary condition on Σ we do not know.

This condition is obtained as a consequence of supposing that there is an overlap between the extended domains of validity of outer stress expansion (3.1) and the local stress expansion, valid near Σ :

$$(5.1) \quad \sigma_{i3}^e = \sum_{\tilde{x}} \frac{1}{\varepsilon} \sigma_{i3y}^{-1} \left(\frac{x - \tilde{x}}{\varepsilon} \right) + \ll \text{terms} \gg$$

so, taking into account relation (3.13) we obtain:

$$(5.2) \quad \sigma_{i3}^0 |_{x_3=0} = - \lim_{\varepsilon \rightarrow 0} \sum_{\tilde{x}} \frac{1}{\varepsilon} u_k^0(\tilde{x}) \sigma_{i3}^k \left(\frac{x - \tilde{x}}{\varepsilon} \right) \Big|_{x_3=0}$$

where the summation of the right side of relation (5.1) and (5.2) is extended to all the centers \tilde{x} of zones T^ε contained in Σ (i.e. the number of terms is $n(\varepsilon) = O(\eta^{-2})$), and where $\sigma_{i3}^k \left(\frac{x - \tilde{x}}{\varepsilon} \right) \Big|_{x_3=0}$ is the change to the variable x of the distribution $\sigma_{i3}^k|_{y_3=0}$.

Remark 5.1 : $\sigma_{i3}^k \left(\frac{x - \tilde{x}}{\varepsilon} \right) \Big|_{x_3=0}$ is a distribution with compact support contained in \bar{T}^ε , defined as : $\forall \phi \in \mathcal{D}(\mathbb{R}^2)$:

$$(5.3) \quad \left\langle \sigma_{i3}^k \left(\frac{x - \tilde{x}}{\varepsilon} \right) \Big|_{x_3=0}, \phi \right\rangle_{\mathcal{D}'(\mathbb{R}^2) \times \mathcal{D}(\mathbb{R}^2)} = \left\langle \sigma_{i3}^k|_{y_3=0}, \varepsilon^2 \phi(\tilde{x} + \varepsilon y) \right\rangle_{\mathcal{D}'(\mathbb{R}^2) \times \mathcal{D}(\mathbb{R}^2)}. \quad \blacksquare$$

The condition that \underline{u}^0 satisfies on Σ is now obtained from the following proposition.

PROPOSITION 5.1 : *Let \underline{u}^0 be a regular function, if $\lim_{\varepsilon \rightarrow 0} (\varepsilon/\eta^2) = \mathcal{K}$, $\mathcal{K} \geq 0$, then*

$$(5.4) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sum_{\tilde{x}} u_k^0(\tilde{x}) \sigma_{ij}^k \left(\frac{x - \tilde{x}}{\varepsilon} \right) n_j \Big|_{x_3=0} = \mathcal{K} C_{ik} u_k^0|_{x_3=0} \quad \text{in } \mathcal{D}'(\Sigma)$$

if $\lim_{\varepsilon \rightarrow 0} (\varepsilon/\eta^2) = +\infty$, then

$$(5.5) \quad \lim_{\varepsilon \rightarrow 0} \frac{\eta^2}{\varepsilon} \sum_{\tilde{x}} u_k^0(\tilde{x}) \sigma_{ij}^k \left(\frac{x - \tilde{x}}{\varepsilon} \right) n_j \Big|_{x_3=0} = C_{ik} u_k^0|_{x_3=0} \quad \text{in } \mathcal{D}'(\Sigma).$$

Proof : Consider $\phi \in \mathcal{D}(\Sigma)$, taking into account relations (4.28) and (5.3) :

$$(5.6) \quad \begin{aligned} & \left\langle \frac{1}{\varepsilon} \sum_{\tilde{x}} u_k^0(\tilde{x}) \sigma_{i3}^k \left(\frac{x - \tilde{x}}{\varepsilon} \right) \Big|_{x_3=0}, \phi \right\rangle_{\mathcal{D}'(\mathbb{R}^2) \times \mathcal{D}(\mathbb{R}^2)} + \\ & \quad + \left\langle \mathcal{K} C_{ik} u_k|_{x_3=0}, \phi \right\rangle_{\mathcal{D}'(\mathbb{R}^2) \times \mathcal{D}(\mathbb{R}^2)} \\ & = \left\langle \sigma_{i3}^k|_{y_3=0}, \sum_{\tilde{x}} u_k^0(\tilde{x}) \phi(\tilde{x} + \varepsilon y) \varepsilon - \mathcal{K} \int_{\Sigma} \phi u_k^0 d\Sigma \right\rangle_{\mathcal{D}'(\mathbb{R}^2) \times \mathcal{D}(\mathbb{R}^2)} \end{aligned}$$

taking limits in relation (5.6) and bearing in mind that :

$$\sum_{\bar{x}} u_k^0(\bar{x}) \phi(\bar{x} + \varepsilon y) \varepsilon - \mathcal{K} \int_{\Sigma} \phi u_k^0 \Big|_{x_3=0} dx_1 dx_2 \rightarrow 0 \text{ in } \mathcal{E}' \rightarrow 0(\mathbb{R}^2)$$

we have the result (5.4). Analogously, in the case where ε/η^2 tends to ∞ we can demonstrate relation (5.5) and therefore the proposition. ■

As a consequence of Proposition 5.1 and relation (5.2), it can be deduced that the limit problem fulfills the following boundary condition on Σ :

$$(5.7) \quad \sigma_{ij}^0 n_j \Big|_{\Sigma} + \mathcal{K} C_{ij} u_j^0 \Big|_{\Sigma} = 0 \quad \text{if} \quad \lim_{\varepsilon \rightarrow 0} (\varepsilon/\eta^2) = \mathcal{K}, \mathcal{K} \geq 0$$

$$(5.8) \quad u_i^0 \Big|_{\Sigma} = 0 \quad \text{if} \quad \lim_{\varepsilon \rightarrow 0} (\varepsilon/\eta^2) = +\infty.$$

Consequently, $\varepsilon = O(\eta^2)$ is the critical dimension of the stuck zones that gives a Fourier-type limit problem that is intermediate between the unstuck case, which we obtain for $\mathcal{K} = 0$, and the totally stuck case, for $\mathcal{K} = \infty$.

The limit problems (3.2)-(3.5), (5.7) with $\mathcal{K} = 0$ and (3.2)-(3.5), (5.8) correspond to mixed homogeneous Neumann-Dirichlet problems and, consequently, they are well posed. The problem (3.2)-(3.5), (5.7) with $\mathcal{K} > 0$ also is well posed since matrix \mathcal{C} is symmetric and positive definitive.

Remark 5.2 : For the study of local stresses, it should be pointed out that in a neighbourhood of Σ , they are given by (3.7), which gives stresses on the order of ε^{-1} due to the presence of the boundary layer. ■

Remark 5.3 : The study of the convergence of the solutions towards the limit problem solution (see Brillard & Lobo & Perez [2]) is performed within the framework of epi-convergence (see Attouch [1]). For the study of convergence in other boundary homogenization problems see Attouch [1], Picard [13], Murat [12], Cioranescu & Murat [3]. ■

REFERENCES

- [1] H. ATTOUCH, *Variational convergence for functions and operators*, Pitman, London (1984).
- [2] A. BRILLARD, M. LOBO, E. PEREZ, *Homogénéisation de frontières par épi-convergence en élasticité linéaire*. (A paraître).
- [3] D. CIORANESCU, F. MURAT, *Un terme étrange venu d'ailleurs*. Collège de France Seminar, Research Notes in Mathematics, Pitman, London, (1982) n° 60, p. 98-138, n° 70, pp. 154-178.

- [4] J. DENY, *Sur la convergence de certaines intégrales de la théorie du potentiel*. Arch. Math. Vol. V, (1954), p. 367-371.
- [5] G. DUVAUT, J. L. LIONS, *Les inéquations en mécanique et en physique*. Dunod. Paris (1972).
- [6] W. ECKHAUS, *Asymptotic Analysis and Singular Perturbations*, North-Holland, Amsterdam (1979).
- [7] O. A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach. London (1969).
- [8] L. LANDAU, E. LIFCHITZ, *Théorie de l'Elasticité*. Mir. Moscou (1967).
- [9] J. L. LIONS, E. MAGENES, *Problèmes aux limites non homogènes et applications*. Vol. I. Dunod, Paris (1968).
- [10] M. LOBO, E. PEREZ, *Comportement asymptotique d'un corps élastique dont une surface présente de petites zones de collage*. C.R. Acad. Sc. Paris, t. 304, Série II n° 5, 1987.
- [11] R. C. Mac CAMY, E. STEPHAN, *Solution Procedures for Three-Dimensional Eddy Current Problems*. J. Math. Anal. Appl. 101 (1984), 348-379.
- [12] F. MURAT, Neumann's Sieve. *Proceedings of the meeting on variational methods in nonlinear analysis*, Isle of Elba 1983. *Research Notes in Mathematics*, N° 127 Pitman, London, 1985.
- [13] C. PICARD, *Analyse limite d'équations variationnelles dans un domaine contenant une grille*. Thèse d'Etat. Université de Paris-Sud. Orsay (1984).
- [14] E. SANCHEZ-PALENCIA, *Boundary value problems in domains containing perforated walls*. In *Nonlinear Differential Equations*, Collège de France Seminar, Vol. III, *Research Notes in Mathematics*, 70, p. 309-325, Pitman, London (1982).
- [15] J. SANCHEZ-HUBERT, E. SANCHEZ-PALENCIA, *Acoustic fluid flow through holes and permeability of perforated walls*. Jour. Math. Anal. Appl., 87, (1982) p. 427-453.
- [16] I. N. SNEDDON, *Fourier transforms*, McGraw-Hill. London (1951).
- [17] R. TEMAM, *Problèmes mathématiques en plasticité*. Gauthier-Villars. Paris (1983).