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**NUMERICAL ANALYSIS  
OF THE TWO-DIMENSIONAL THERMO-DIFFUSIVE MODEL  
FOR FLAME PROPAGATION (\*)**

by F. BENKHALDOUN, B. LARROUTUROU (1)

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*Abstract. — We present the numerical analysis of the classical thermal-diffusional model describing a curved premixed flame propagating in a rectangular infinite tube. The adaptive moving mesh procedure used to numerically solve this problem leads to a system of non-linear integro-differential reaction-diffusion equations. We mathematically prove the existence and uniqueness of a solution to this problem and show the convergence of the numerical approximation. A particular feature of the analysis is that the estimates of the global numerical error not only depend on the time step and mesh spacing  $\Delta t$ ,  $\Delta x$ ,  $\Delta y$ , but also of the size of the computational domain.*

*Résumé. — Nous présentons l'analyse numérique du modèle thermo-diffusif employé en théorie de la combustion pour décrire la propagation d'une flamme plissée dans un tube rectangulaire infini. La méthode numérique utilise un maillage mobile adaptatif, ce qui conduit à résoudre un système d'équations intégral-différentielles non linéaires de type réaction-diffusion. Nous montrons l'existence et l'unicité d'une solution de ce problème, et prouvons la convergence de l'approximation numérique. Une particularité de cette analyse vient de ce que l'estimation de l'erreur numérique ne dépend pas seulement des pas en espace et en temps  $\Delta t$ ,  $\Delta x$ ,  $\Delta y$ , mais aussi de la dimension du domaine de calcul.*

## 1. INTRODUCTION

This paper deals with the numerical analysis of an adaptive explicit numerical method for the solution of the classical thermo-diffusive model describing the free propagation of a premixed wrinkled flame in an infinite gaseous medium.

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The main result of the paper is a convergence result for the numerical approximation : under adequate hypotheses on the computational grid, on the time step with which the method operates, *and on the size of the computational domain*, the numerical adaptive solution is shown to converge towards the continuous solution of the thermo-diffusive system of nonlinear governing equations, which is defined in the *unbounded* strip  $\mathbb{R} \times [0, L]$  (representing an infinite rectangular tube of width  $L$ ). This result is obtained using classical tools for the analysis of numerical schemes solving systems of parabolic equations, including in particular  $L^\infty$ -estimates of the numerical solution. Its main originality relies in the fact that the original continuous problem is posed in an unbounded domain. This leads to an estimate of the numerical error which not only depends on the mesh size and time step, but also on the size of the computational domain.

The paper is organized as follows : we begin by briefly presenting in Section 2 the thermo-diffusive model for wrinkled premixed flame propagation. This model consists of two coupled non linear reaction-diffusion equations in the strip  $\mathbb{R} \times [0, L]$ , associated to initial data and to homogeneous Neumann and non homogeneous Dirichlet conditions. Section 3 is devoted to a brief description of the explicit adaptive numerical method whose convergence forms the subject of this paper. The main feature of this method is that the adaptive moving mesh strategy transforms the original set of partial differential equations into a system of integro-differential equations, which will complicate the analysis of its convergence. In Section 4, we show that the thermo-diffusive model forms a well-posed mixed initial-boundary value problem and has a unique solution. The numerical analysis of this model is then presented in Section 5, and is divided into several classical steps : we first study the consistency of the scheme, we then derive sufficient conditions which insure that some  $L^\infty$ -estimates satisfied by the solution of the continuous problem also hold for the numerical solution, and then complete the convergence proof itself.

## 2. THE THERMO-DIFFUSIVE MODEL

The thermal diffusional model (equations (1) below) is a simplified system of non linear reaction-diffusion equations which was introduced in [1] to describe a premixed laminar flame propagating in a gaseous medium, in the framework of the well-known « *constant-density approximation* ». This model uncouples the flame propagation itself from the gas flow, but it retains many essential features of the phenomenon, including the cellular instabilities of the flame (see [3], [7], [10]) ; it is therefore physically relevant for qualitatively describing combustion phenomena in which the gas motion is almost uniform and plays a secondary role compared to the reactive and diffusive effects.

We are interested in the propagation of a wrinkled flame in a gaseous mixture confined in an infinite rectangular adiabatic channel  $S = \mathbb{R} \times [0, L]$ . Assuming that a single one-step chemical reaction  $A \rightarrow B$  takes place in the gaseous mixture, we describe the phenomenon with the following system of governing equations (see [1], [10]) :

$$\begin{aligned} T_t &= \Delta T + \Omega(T, Y), \\ Y_t &= \frac{1}{Le} \Delta Y - \Omega(T, Y). \end{aligned} \quad (1)$$

These equations are written using normalized variables :  $T$  is the reduced temperature of the mixture and  $Y$  is the normalized mass fraction of the reactant  $A$ . The positive constant parameter  $Le$  is the Lewis number of the reactant. From the law of mass action, the normalized reaction rate  $\Omega$  is of the form :

$$\Omega(T, Y) = Yf(T), \quad (2)$$

where  $f(T)$  is a nonlinear positive function of the temperature. The equations (1) are associated to the initial data :

$$T(x, y, t = 0) \equiv T_0(x, y), \quad Y(x, y, t = 0) \equiv Y_0(x, y), \quad (3)$$

and to the following boundary conditions :

$$T(-\infty, y, t) = 0, \quad Y(-\infty, y, t) = 1; \quad (4a)$$

$$T(+\infty, y, t) = 1, \quad Y(+\infty, y, t) = 0; \quad (4b)$$

$$\frac{\partial T}{\partial n} = \frac{\partial Y}{\partial n} = 0 \quad \text{on } W; \quad (4c)$$

where  $W$  denotes the channel walls :  $W = \{(x, y) \in \mathbb{R}^2, y = 0 \text{ or } y = L\}$ . The homogeneous Neumann condition (4c) for the temperature expresses that the tube walls are adiabatic.

*Remark 2.1 :* As mentioned above, the thermo-diffusive model (1) is a *simplified* system of governing equations. Such is not the case in a one-dimensional setting, where the analogous system

$$\begin{aligned} T_t &= T_{xx} + \Omega(T, Y), \\ Y_t &= \frac{1}{Le} Y_{xx} - \Omega(T, Y), \end{aligned} \quad (1')$$

appears to be the Lagrangian version of a much more complete system, which includes the mass and momentum balance equations and an isobaric equation of state :

$$\begin{aligned}
 \rho_\tau + (\rho u)_\xi &= 0, \\
 (\rho u)_\tau + (\rho u^2)_\xi &= -p_\xi, \\
 \rho c_p T_\tau + \rho u c_p T_\xi - (\lambda T_\xi)_\xi &= Q\omega(Y, T), \\
 \rho Y_\tau + \rho u Y_\xi - (\rho D Y_\xi)_\xi &= -m\omega(Y, T), \\
 \rho T &= \frac{mP_0}{R};
 \end{aligned} \tag{5}$$

(see [5], [8], [12]). The two systems (1') and (5) have been analysed in [8] from a rigorous mathematical point of view. It is of interest to notice that a numerical convergence result analogous to the one stated in Section 5 for system (1) also holds in the one-dimensional context. •

### 3. THE NUMERICAL METHOD

We now present the main features of the numerical method whose convergence properties will be investigated in Section 5. The reader is referred to [2] for more details about the method or for some results of numerical experiments.

In this method, the equations (1) are solved on a computational grid which continually moves towards the fresh mixture (i.e. towards negative  $x$ ) while the flame propagates in the same direction. To be more specific, the grid moves at each time  $t$  in the  $x$ -direction with a velocity  $V(t)$  equal to the instantaneous average flame speed. This procedure presents several advantages :

- \* The flame front may be deformed during the calculation, but it stays at the same place inside the moving computational domain (see the results in [2]). This allows one to rezone the grid much less often during the computation, and makes possible to reduce the size of the computational domain.

- \* In many cases (including some cellular unstable situations ; see [7]), the flame converges as  $t \rightarrow +\infty$  to a traveling wave propagating at constant speed (this asymptotic behaviour has not been proved mathematically, but is expected from a physical point of view and is actually observed numerically). In the moving grid reference frame, this traveling wave simply becomes a steady state. The investigation of the convergence of the time-dependent solution to steady state and the evaluation of the steady flame speed are then much easier.

Since all the mesh points move at each time  $t$  with the same velocity  $V(t)$  in the  $x$ -direction, this moving grid procedure simply amounts to observing the flame propagation in a non galilean reference frame which moves with the velocity  $(V(t), 0)$  with respect to the original fixed reference frame. In other words, we solve in a *fixed* rectangular domain  $D = [a, b] \times [0, L]$  the flame propagation equations written in the moving reference frame, i.e. :

$$\begin{aligned} T_t &= \Delta T + \Omega(T, Y) + V(t) T_x, \\ Y_t &= \frac{1}{Le} \Delta Y - \Omega(T, Y) + V(t) Y_x, \end{aligned} \tag{6}$$

where the average instantaneous flame speed  $V(t)$  is given by (see [2]) :

$$V(t) = - \frac{1}{L} \int_S \Omega(T, Y). \tag{7}$$

The problem to be solved numerically therefore consists of the integro-differential equations (6)-(7) in  $D$ , with the initial data (3) and the following boundary conditions (in which  $\Gamma_a$  and  $\Gamma_b$  denote the boundaries  $\{x = a\}$  and  $\{x = b\}$  of  $D$ ) :

$$\frac{\partial T}{\partial n} = \frac{\partial Y}{\partial n} = 0 \quad \text{on } W \cap D; \tag{8}$$

$$T = 0, \quad Y = 1 \quad \text{on } \Gamma_a; \tag{9a}$$

$$T = 1, \quad Y = 0 \quad \text{on } \Gamma_b. \tag{9b}$$

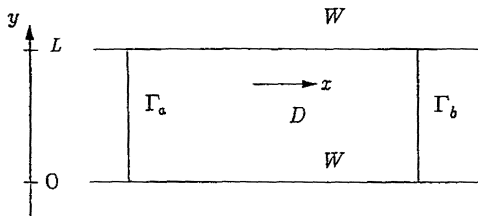


Figure 1. — The computational domain  $D$ .

The equations (6) are discretized using a non uniform mesh in the domain  $D$ . This grid is adaptively rezoned at some time levels during the calculation in order to maintain the accuracy of the spatial approximation (see [2]). It has the structure  $N_{j,k} = [x_{j,k}, y_k]$  (for  $1 \leq j \leq N_x$  and  $1 \leq k \leq N_y$ ), i.e. it is

divided into straight lines  $y = y_k$  parallel to the boundaries  $W \cap D$  (see Fig. 2).

In order to obtain a robust approximation of the spatial derivatives even on highly non uniform grids, we use a spatial discretization scheme which combines some features of the classical finite-difference and finite-element methodologies. We first define a triangulation of  $D$  by dividing each quadrangle  $[N_{j,k}, N_{j+1,k}, N_{j+1,k+1}, N_{j,k+1}]$  into two triangles  $[N_{j,k}, N_{j+1,k}, N_{j+1,k+1}]$  and  $[N_{j,k}, N_{j+1,k+1}, N_{j,k+1}]$  (see Fig. 2).

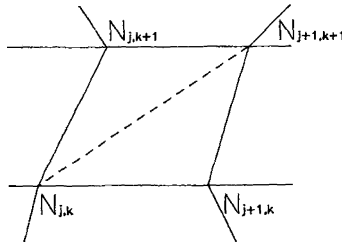


Figure 2. — The structure of the computational grid.

We then introduce the classical basis  $(\varphi_i)_{i \in I}$  of the P1 Lagrange triangular finite-element discretization space associated to this triangulation. In particular, we now number the mesh points in a different way :  $(S_i)_{i \in I}$  now denotes the set of all vertices in the triangulation, and Figure 2 becomes :

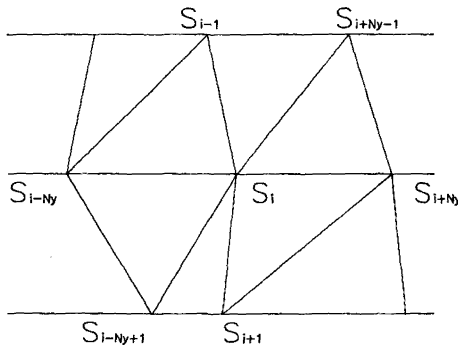


Figure 3. — The triangulation and the vertices  $(S_i)_{i \in I}$ .

The semi-discrete formulation of problem (6)-(9) is then written as :

$$\frac{dT_i}{dt} = - \frac{\int_D \left( \sum_{j \in I} T_j \nabla \varphi_j \right) \nabla \varphi_i}{\int_D \varphi_i} + \Omega_i + V(t) D_i(T_h),$$

$$\frac{dY_i}{dt} = - \frac{1}{Le} \frac{\int_D \left( \sum_{j \in I} Y_j \nabla \varphi_j \right) \nabla \varphi_i}{\int_D \varphi_i} - \Omega_i + V(t) D_i(Y_h), \quad \forall i \in I^0.$$
(10)

The unknowns are the values of  $T$  and  $Y$  at the vertices  $(S_i)_{i \in I^0}$  which are not located on the boundaries  $\Gamma_a$  and  $\Gamma_b$ , because of the Dirichlet conditions (9). In (10), the terms  $D_i(T_h)$  and  $D_i(Y_h)$  represent discrete expressions of the first derivatives  $T_x$  and  $Y_x$ . A finite-element formula can be used for these convective terms, but the particular structure of the grid also makes possible the use of a classical finite-difference formula.

Let us now end this section by writing down precisely the discrete scheme whose convergence will be analysed in Section 5. We will assume that a forward Euler scheme is used to integrate the differential system (10), that a centered finite-difference formula is employed for the first derivative terms  $D_i(T_h)$  and  $D_i(Y_h)$ , and that all segments  $[S_i, S_{i-1}]$  in Figure 3 are parallel to each other and of equal length (see Fig. 4). The aim of these hypotheses is only to simplify the algebra in the analysis presented below (on the other hand, considering that the grid is uniform and orthogonal would be a too drastic simplification with respect to the actual mesh used in the numerical

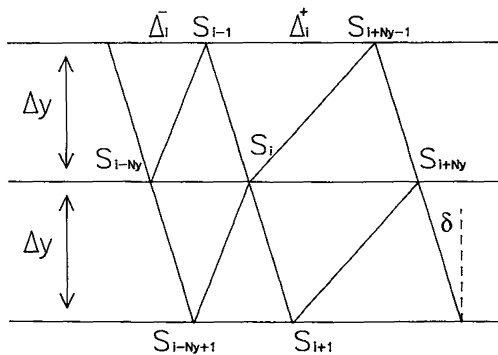


Figure 4. — The more regular triangulation used to analyse the method.



experiments ; Remark 5.4 below shows that it is interesting to take the non orthogonality of the grid into account in the analysis of the method). It can be checked that the convergence result of Section 5 remains true for the more general grid of Figure 3 or with another discretization of the convective terms (of course, some technical conditions such as (36) below need then to be modified).

For an interior point  $S_i$  in  $D$ , we then get :

$$\begin{aligned} \frac{T_i^{n+1} - T_i^n}{\Delta t} = & - \left[ \frac{2(1+p^2)}{\Delta_i^+ \Delta_i^-} + \frac{2}{\Delta y^2} - \frac{2p}{\Delta_i \Delta y} \right] T_i^n \\ & + \left[ \frac{1}{\Delta y^2} - \frac{p}{\Delta_i \Delta y} \right] T_{i-1}^n + \left[ \frac{1}{\Delta y^2} - \frac{p}{\Delta_i \Delta y} \right] T_{i+1}^n \\ & + \left[ \frac{1+p^2}{\Delta_i \Delta_i^-} - \frac{p}{\Delta_i \Delta y} - \frac{V^n}{2 \Delta_i} \right] T_{i-N_y}^n + \left[ \frac{1+p^2}{\Delta_i \Delta_i^+} - \frac{p}{\Delta_i \Delta y} + \frac{V^n}{2 \Delta_i} \right] T_{i+N_y}^n \\ & + \left[ \frac{p}{\Delta_i \Delta y} \right] T_{i+1-N_y}^n + \left[ \frac{p}{\Delta_i \Delta y} \right] T_{i-1+N_y}^n + \Omega(T_i^n, Y_i^n). \end{aligned} \quad (11)$$

and :

$$\begin{aligned} \frac{Y_i^{n+1} - Y_i^n}{\Delta t} = & - \frac{1}{Le} \left[ \frac{2(1+p^2)}{\Delta_i^+ \Delta_i^-} + \frac{2}{\Delta y^2} - \frac{2p}{\Delta_i \Delta y} \right] Y_i^n \\ & + \frac{1}{Le} \left[ \frac{1}{\Delta y^2} - \frac{p}{\Delta_i \Delta y} \right] Y_{i-1}^n + \frac{1}{Le} \left[ \frac{1}{\Delta y^2} - \frac{p}{\Delta_i \Delta y} \right] Y_{i+1}^n \\ & + \frac{1}{Le} \left[ \frac{1+p^2}{\Delta_i \Delta_i^-} - \frac{p}{\Delta_i \Delta y} - \frac{V^n Le}{2 \Delta_i} \right] Y_{i-N_y}^n \\ & + \frac{1}{Le} \left[ \frac{1+p^2}{\Delta_i \Delta_i^+} - \frac{p}{\Delta_i \Delta y} + \frac{V^n Le}{2 \Delta_i} \right] Y_{i+N_y}^n \\ & + \frac{1}{Le} \left[ \frac{p}{\Delta_i \Delta y} \right] Y_{i+1-N_y}^n + \frac{1}{Le} \left[ \frac{p}{\Delta_i \Delta y} \right] Y_{i-1+N_y}^n - \Omega(T_i^n, Y_i^n). \end{aligned} \quad (12)$$

We have used the notations :

$$\begin{aligned} \Delta_i^+ &= x_{i+N_y} - x_i, \\ \Delta_i^- &= x_i - x_{i-N_y}, \\ \Delta_i &= \frac{1}{2} (\Delta_i^- + \Delta_i^+), \\ \Delta y &= y_{i-1} - y_i = y_i - y_{i+1}, \end{aligned}$$

and  $p = \text{tg } \delta$  (see Fig. 4).

In order to completely describe the numerical scheme, it remains to specify how the velocity  $V^n$  appearing in (11)-(12) is evaluated. A straightforward application of (7) would lead to perform a discrete integration of the reaction rate  $\Omega$  on the whole computational domain  $D$ . But this would create some difficulties in investigating the convergence of the numerical method : indeed, as already mentioned in the introduction, we will derive in Section 5 an upper bound for the numerical error in  $D \times [0, t_0]$ , which not only depends on the mesh sizes  $\Delta_x, \Delta_y, \Delta t$ , but also on the size on  $D$ . Thus the method will appear to converge in the limit  $\Delta_x \rightarrow 0, \Delta_y \rightarrow 0, \Delta t \rightarrow 0$ , and  $D \rightarrow S$ . In this limiting process, some difficulties would arise if we evaluate the velocity  $V^n$  with an integral on the whole domain  $D$ , which tends to become infinite.

For this reason, we define a fixed domain  $\hat{D} \subset D$  and compute  $V^n$  by numerically evaluating  $-\frac{1}{L} \int_{\hat{D}} \Omega$  instead of  $-\frac{1}{L} \int_D \Omega$ . Using a classical discrete integration rule, we then obtain an expression of the form :

$$V^n = -\frac{1}{L} \sum_{S_i \in \hat{D}} A_i \Omega(T_i^n, Y_i^n), \quad (13)$$

where the  $A_i$ 's are positive and satisfy :

$$\sum_{S_i \in \hat{D}} A_i = \hat{A} = \text{area}(\hat{D}). \quad (14)$$

The numerical method is then completely defined by the equations (11)-(12), by the analogous relations for those nodes  $S_i$  which are located on the walls  $W \cap D$ , the Dirichlet conditions (9) on  $\Gamma_a$  and  $\Gamma_b$ , the expression (13) of  $V^n$ , and the initial data (3) :

$$T_i^0 = T_0(S_i), \quad Y_i^0 = Y_0(S_i). \quad (15)$$

The method therefore relies on the knowledge of two domains, the computational domain  $D$  and the integration domain  $\hat{D}$ . Concerning the choice of  $\hat{D}$ , one has to keep in mind that, in a flame propagation problem, the reaction rate is negligibly small except in a very thin layer within the flame front. Therefore, a rather small domain  $\hat{D}$  is altogether adequate in practice.

#### 4. MATHEMATICAL ANALYSIS

In this section we are going to show the existence and uniqueness of a solution to the « continuous problem » (1)-(4). This mathematical study generalizes to a two-dimensional geometry the existence and uniqueness

results stated in [8] for the one-dimensional system (1'). Although many of the proofs are inspired from the one-dimensional case, we sketch most of the arguments for the sake of completeness.

Before stating the main result of this section, we need to introduce some notations and hypotheses. We keep the notations  $S = \mathbb{R} \times [0, L]$ , and  $W = \{(x, y) \in \mathbb{R}^2, y = 0 \text{ or } y = L\}$ . From now on, we will denote  $L^p = L^p(S)$ , for  $p \in [1, +\infty)$ , and  $\|\varphi\|_p = \|\varphi\|_{L^p}$  or  $\|(\varphi, \psi)\|_p = \max(\|\varphi\|_{L^p}, \|\psi\|_{L^p})$ . Furthermore, for  $m \in \mathbb{N}^*$ , we will use the notations  $H^m = H^m(S) = W^{m,2}(S)$ ,  $W^{m,\infty} = W^{m,\infty}(S)$  and  $\|\varphi\|_{m,\infty}, \|(\varphi, \psi)\|_{m,\infty}$ .

Let now  $\gamma$  and  $\gamma_1$  be two functions satisfying :

$$\begin{aligned} \gamma &\in C^\infty(\mathbb{R}, \mathbb{R}) ; \\ \gamma &= 0 \quad \text{on} \quad (-\infty, -1], \quad 0 \leq \gamma \leq 1 \quad \text{on} \quad [-1, 1], \\ &\hspace{20em} \gamma = 1 \quad \text{on} \quad [1, +\infty) ; \quad (16) \\ \gamma_1 &= 1 - \gamma . \end{aligned}$$

We will set :

$$\begin{aligned} \varphi_0(x, y) &= T_0(x, y) - \gamma(x) , \\ \psi_0(x, y) &= Y_0(x, y) - \gamma_1(x) . \end{aligned}$$

Finally, we define the functional spaces :

$$\begin{aligned} E^2 &= \left\{ \varphi \in H^2, \frac{\partial \varphi}{\partial y} = 0 \text{ on } W \right\} , \\ E^4 &= \left\{ \varphi \in H^4, \frac{\partial^3 \varphi}{\partial y^3} = \frac{\partial \varphi}{\partial y} = 0 \text{ on } W \right\} . \end{aligned}$$

The following assumptions will be used in the sequel :

$$\varphi_0, \psi_0 \in E^2 ; \tag{17}$$

$$T_0(x, y) \geq 0 \text{ a.e.}, \quad Y_0(x, y) \in [0, 1] \text{ a.e.} ; \tag{18}$$

$$f \in W_{\text{loc}}^{1,\infty}(\mathbb{R}_+, \mathbb{R}_+), \quad f(0) = 0 ; \tag{19}$$

$$\exists C_f > 0, \forall \theta \in \mathbb{R}_+, \quad f(\theta) \leq C_f \theta . \tag{20}$$

We can then state the following result :

**THEOREM 4.1 :** *Under the hypotheses (17)-(20), there exists a unique solution (T, Y) to problem (1)-(4) in  $S \times \mathbb{R}_+$ .* •

*Remark 4.2 :* The statement of Theorem 4.1 is not optimal in the sense that the assumption  $\varphi_0, \psi_0 \in L^2$  instead of (17) would be adequate to show the existence and uniqueness of a solution. Nevertheless, we will use the stronger hypothesis (17) in order to simplify the analysis below. •

To begin the proof, we first extend  $f$  on all of  $\mathbb{R}$  by setting  $f \equiv 0$  on  $\mathbb{R}_-$ , and define  $g$  by :

$$g(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0, \\ \xi & \text{if } \xi \geq 0. \end{cases}$$

We also use the functions  $\gamma$  and  $\gamma_1$  defined in (16) and set :

$$\begin{aligned} \varphi(x, y, t) &= T(x, y, t) - \gamma(x), \\ \psi(x, y, t) &= Y(x, y, t) - \gamma_1(x). \end{aligned}$$

We can then rewrite the problem (1)-(4) as :

$$\varphi_t - \Delta\varphi = f(\varphi + \gamma)g(\psi + \gamma_1) + \gamma_{xx}, \tag{21}$$

$$\psi_t - \frac{\Delta\psi}{Le} = -f(\varphi + \gamma)g(\psi + \gamma_1) - \frac{\gamma_{xx}}{Le};$$

$$\varphi(x, y, 0) = \varphi_0(x, y), \quad \psi(x, y, 0) = \psi_0(x, y); \tag{22}$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial y} = 0 \quad \text{on } W; \tag{23a}$$

$$\varphi(-\infty, y, t) = \varphi(+\infty, y, t) = \psi(-\infty, y, t) = \psi(+\infty, y, t) = 0. \tag{23b}$$

Defining the linear operator :

$$A : \begin{cases} D(A) = E^2 \times E^2 \rightarrow L^2 \times L^2, \\ (\varphi, \psi) \rightarrow \left( -\Delta\varphi, -\frac{\Delta\psi}{Le} \right), \end{cases} \tag{24}$$

we can state :

LEMMA 4.3 :  $A$  is a maximal monotone operator. •

*Proof:* It suffices to show that the linear operator

$$A_1 : \begin{cases} E^2 \rightarrow L^2, \\ \varphi \rightarrow -\Delta\varphi, \end{cases}$$

is maximal monotone, i.e. that  $(A_1 u, u)_2 \geq 0$  for all  $u \in E^2$ , and that  $Id + A_1$  is surjective on  $L^2$ .

For  $u \in E^2$ , we have :  $(A_1 u, u)_2 = - \int_S \Delta u u = \int_S \nabla u^2 \geq 0$  and  $A_1$  is monotone.

It remains to check that, for any  $v \in L^2$ , there exists  $u \in E^2$  such that  $u - \Delta u = v$ . The variational formulation of this problem reads :

$$u \in H^1, \int_S u w + \int_S \nabla u \nabla w = \int_S v w, \quad \forall w \in H^1,$$

which has a solution from Riesz representation theorem. It is obvious to check that this solution satisfies  $u - \Delta u = v$  in the sense of distributions, whence  $\Delta u = u - v$  a.e. This shows that  $\Delta u \in L^2$ , from which one classically deduces that  $u \in H^2$ . •

Thus, as in the one-dimensional study [8], the proof of Theorem 4.1 relies on the application of the *linear semigroups theory* to partial differential equations. We will use the following classical result from functional analysis (see [4], [9]) :

**THEOREM 4.4 :** *Let  $H$  be a real Hilbert space and  $B$  be a linear maximal monotone operator defined on the subspace  $D(B) \subset H$ . Assume that  $F$  is a Lipschitz-continuous mapping from  $H$  into itself. Then for any  $u_0 \in D(B)$ , there exists a unique solution  $u \in C(\mathbb{R}, D(B)) \cap C^1(\mathbb{R}, H)$  of :*

$$\frac{du}{dt} + Bu = F(u) \quad \text{for } t \geq 0, \\ u(0) = u_0. \quad \bullet$$

In order to apply this general result, we define, for any functions  $\bar{f}$  and  $\bar{g}$  of  $C(\mathbb{R}, \mathbb{R})$ , the mapping :

$$F(\bar{f}, \bar{g}) : \begin{cases} L^2 \times L^2 \rightarrow L^2 \times L^2, \\ (\varphi, \psi) \rightarrow \left[ \begin{aligned} & \bar{f}(\varphi + \gamma) \bar{g}(\psi + \gamma_1) + \gamma_{xx}, \\ & - \bar{f}(\varphi + \gamma) \bar{g}(\psi + \gamma_1) - \frac{\gamma_{xx}}{Le} \end{aligned} \right]. \end{cases} \quad (25)$$

The problem (21)-(23) now takes the form :

$$\text{Find } U = (\varphi, \psi) \in D(A), U_t + AU = F(f, g)(U), \\ U(t = 0) = U_0 = (\varphi_0, \psi_0). \quad (26)$$

Applying Theorem 4.4, we have :

**LEMMA 4.5 :** *Assume that  $\varphi_0, \psi_0 \in E^2$  and that  $f \in W_{loc}^{1,\infty}(\mathbb{R})$ ,  $f(0) = 0$ . Then there exists  $t_{\max} \in \mathbb{R}_+^* \cup \{+\infty\}$  such that a unique solution  $(\varphi, \psi)$  of*

problem (21)-(23) exists on  $S \times [0, t_{\max})$ . Moreover, the following properties hold :

$$\varphi, \psi \in C([0, t_{\max}), E^2) \cap C^1([0, t_{\max}), L^2); \tag{27}$$

$$\text{If } t_{\max} < +\infty, \text{ then } \lim_{t \rightarrow t_{\max}} \|(\varphi, \psi)(t)\|_{\infty} = +\infty. \bullet \tag{28}$$

*Proof:* Let  $K \geq \|(\varphi_0, \psi_0)\|_{\infty} + 1$  and consider two functions  $f_K$  and  $g_K$  which are Lipschitz-continuous on  $\mathbb{R}$  and coincide with  $f$  and  $g$  respectively in  $[-K, K]$ . Since  $f_K(0) = g_K(0) = 0$ , it is easy to check that the mapping  $F(f_K, g_K)$  defined by (25) is Lipschitz-continuous from  $L^2 \times L^2$  into itself. Theorem 4.4 then shows the existence of a solution  $U = (\varphi, \psi)$  of :

$$U_t + AU = F(f_K, g_K)(U), U(t = 0) = (\varphi_0, \psi_0).$$

For small positive  $t$ , we still have  $\|(\varphi(t), \psi(t))\|_{\infty} < K$ , which shows that  $(\varphi, \psi)$  is a solution of (26). Since  $\varphi, \psi \in H^2$ , it is classical to check that the conditions (23b) are satisfied :  $\varphi(\pm \infty, y, t) = \psi(\pm \infty, y, t) = 0$ . Then (21)-(23) has a solution on a small interval  $[0, t_0]$ , and the property (28) follows from classical arguments of ordinary differential equations theory (the details are left to the reader).  $\bullet$

*Remark 4.6 :* In fact, since  $\varphi, \psi \in C([0, t_{\max}), H^2)$ , it can easily be shown that, for all  $t_0 < t_{\max}$ ,  $\varphi$  and  $\psi$  tend to 0 uniformly with respect to  $y \in [0, L]$  and  $t \in [0, t_0]$  as  $|x| \rightarrow +\infty$  :

$$\forall \varepsilon > 0, \exists R > 0, \sup_{\substack{|x| > R \\ 0 \leq y \leq L \\ 0 \leq t \leq t_0}} [|\varphi(x, y, t)| + |\psi(x, y, t)|] < \varepsilon. \bullet$$

The two next results show that the solution of (21)-(23) defined in Lemma 4.5 is a physically acceptable solution of (1)-(4) (i.e. satisfies  $0 \leq Y \leq 1$  and  $0 \leq T$  a.e.), and exists for all time :

**LEMMA 4.7 :** *In addition to the hypotheses of Lemma 4.5, suppose that the assumptions (18) hold. Let  $(\varphi, \psi)$  be the solution of (21)-(23) defined in Lemma 4.5. For  $(x, y, t) \in S \times [0, t_{\max})$ , define :*

$$\begin{aligned} T(x, y, t) &= \varphi(x, y, t) + \gamma(x), \\ Y(x, y, t) &= \psi(x, y, t) + \gamma_1(x). \end{aligned}$$

Then the following inequalities hold :

$$T(x, y, t) \geq 0, 0 \leq Y(x, y, t) \leq 1 \text{ a.e. on } S \times [0, t_{\max}). \bullet \tag{29}$$

*Proof:* a) Let us first show that  $Y \geq 0$ . This is essentially the maximum principle. For any function  $Z$  of  $L^2_{loc}(\mathbb{R})$  we define as usual:  $Z^- = \max(0, -Z)$ ,  $Z^+ = \max(0, Z)$ . For  $t \in [0, t_{\max})$ , it is known that  $\psi^-(t) \in H^1$ ,  $(\psi(t) + \gamma_1)^- \in H^1_{loc}(\mathbb{R})$  (see [11]). It follows easily from the properties (16) of  $\gamma$  and  $\gamma_1$  that  $(\psi(t) + \gamma_1)^- = Y^- \in H^1$ . Since  $(\varphi, \psi)$  satisfies (27), we can multiply the equation  $Y_t - \frac{\Delta Y}{Le} = -f(T)g(Y)$  by  $Y^-$  and integrate in  $S$  to get :

$$\frac{d}{dt} \left[ \frac{1}{2} \int_S (Y^-)^2 \right] + \frac{1}{Le} \int_S (\nabla Y^-)^2 = 0 \tag{30}$$

(we have used the identity  $g(Y)Y^- \equiv 0$ ). On the other hand, it can be checked easily that the mapping  $\psi \rightarrow (\psi + \gamma_1)^-$  is continuous from  $L^2$  into itself. Thus  $Y^- \in C([0, t_{\max}), L^2)$ . Since  $\int_S (Y^-)^2$  is decreasing on  $[0, t_{\max})$  from (30) and  $\int_S [Y^-(t=0)]^2 = 0$  from (18), we obtain :

$$Y^-(t) \equiv 0 \quad \text{for } t \in [0, t_{\max}),$$

or equivalently  $Y(t) \geq 0$  for  $t \in [0, t_{\max})$ .

b) Using  $(Y - 1)^+$  and  $T^-$  in place of  $Y^-$  in the preceding proof gives the other inequalities (29). •

**LEMMA 4.8 :** *In addition to the hypotheses of Lemmas 4.5 and 4.7, suppose that the assumption (20) holds. Then the solution  $(T, Y)$  of (1)-(4) defined in Lemma 4.7 exists on  $S \times \mathbb{R}_+$  :*

$$t_{\max} = +\infty . \quad \bullet$$

*Proof:* For  $p \in [2, +\infty)$  and  $t \in [0, t_{\max})$ , we multiply the equation  $T_t - \Delta T = f(T)g(Y)$  by  $pT^{p-1}$  and integrate in  $S$  as in the proof of Lemma 4.7. Using the inequalities (20) and (29), we can write :

$$\frac{d}{dt} \left( \int_S T^p \right) \leq C_f p \int_S T^p .$$

Since  $T_0 \in L^p$ , we can apply Gronwall's lemma to get :

$$\int_S T(t)^p \leq e^{pC_f t} \int_S (T_0)^p ,$$

or  $\|T(t)\|_p \leq e^{C_f t} \|T_0\|_p$ . Taking the limit of this inequality as  $p \rightarrow \infty$ , we obtain  $\|T(t)\|_\infty \leq e^{C_f t} \|T_0\|_\infty$ , which together with (28) and (29) implies  $t_{\max} = +\infty$ . •

*Remark 4.9:* Under additional hypotheses on the non linear function  $f$  and on the initial data  $\varphi_0, \psi_0$ , it could be checked that the solution  $(T, Y)$  defined in Lemma 4.7 is a classical solution of (1)-(4):  $T, Y \in C(\mathbb{R}_+, C^2(S)) \cap C^1(\mathbb{R}_+, C(S))$ . We omit the proof of this fact for the sake of simplicity ; the reader is referred to [8] for the analogous regularity results in a one-dimensional context. •

We end this section by investigating a slightly different problem, inspired from the adaptive numerical method described in the preceding section. We are now interested in finding  $(\bar{T}, \bar{Y})$  satisfying the initial and boundary conditions (3)-(4) and the integro-differential equations (6)-(7). The solutions of this problem are of course related to the solutions of (1)-(4), as we now show :

**PROPOSITION 4.10 :** *In addition to (17)-(20), assume that  $\varphi_0, \psi_0 \in L^1$ . Then there exists a unique solution  $(\bar{T}, \bar{Y})$  of problem (3)-(4)-(6)-(7).* •

*Proof:* a) Let  $F$  be the mapping  $F(f, g)$  defined by (25) ; let  $(\varphi, \psi)$  be the solution of (21)-(23) defined in Lemma 4.5, and  $t_0 \in \mathbb{R}_+^*$ . Using (16) and considering separately the domains  $(-\infty, -1) \times (0, L)$ ,  $(-1, 1) \times (0, L)$  and  $(1, +\infty) \times (0, L)$ , it is easy to show that there exists a constant  $K$  such that :

$$\forall t \in [0, t_0], \|F(\varphi, \psi)(t)\|_1 \leq \|(\varphi, \psi)(t)\|_1 .$$

Let now  $R(t)$  be the linear semigroup generated by  $-A$ , where  $A$  is the maximal monotone operator (24) ; for  $\hat{\varphi}_0, \hat{\psi}_0 \in E^2$ ,  $R(t)(\hat{\varphi}_0, \hat{\psi}_0) = (\hat{\varphi}(t), \hat{\psi}(t))$  is the solution of :

$$\hat{\varphi}_t - \Delta \hat{\varphi} = 0, \quad \hat{\psi}_t - \frac{\Delta \hat{\psi}}{Le} = 0 \quad \text{in } S ,$$

$$\hat{\varphi}_y = \hat{\psi}_y = 0 \quad \text{on } W ,$$

$$\hat{\varphi}(x, y, 0) = \hat{\varphi}_0(x, y), \quad \hat{\psi}(x, y, 0) = \hat{\psi}_0(x, y) \quad \text{in } S .$$

It is then classical to show that the solution  $(\varphi, \psi)$  of (21)-(23) satisfies the relation :

$$(\varphi, \psi)(t) = R(t)(\varphi_0, \psi_0) + \int_0^t R(t-s) F[(\varphi, \psi)(s)] ds .$$



On the other hand, the semigroup  $R(t)$  satisfies the inequality :

$$\|R(t)(\varphi, \psi)\|_1 \leq \|(\varphi, \psi)\|_1 .$$

If  $\varphi_0, \psi_0 \in L^1$ , we get :

$$\|(\varphi, \psi)(t)\|_1 \leq K + K \int_0^t [\|(\varphi, \psi)(s)\|_1 + 1] ds ,$$

and Gronwall's lemma yields :

$$\varphi(t), \psi(t) \in L^1 . \quad (31)$$

b) Let  $(T, Y)$  be the solution of (1)-(4) defined in Theorem 4.1. It is easy to check that (31) implies  $\Omega(T, Y) \in C(\mathbb{R}_+, L^1)$  (see [8]). Then (7) defines a continuous function  $V(t)$  on  $\mathbb{R}_+$ . For  $(x, y, t) \in S \times \mathbb{R}_+$ , we set :

$$\begin{aligned} \bar{T}(x, y, t) &= T\left(x + \int_0^t V(t') dt', y, t\right) , \\ \bar{Y}(x, y, t) &= Y\left(x + \int_0^t V(t') dt', y, t\right) . \end{aligned}$$

Then  $(\bar{T}, \bar{Y})$  satisfies (6) and the proof is complete. •

*Remark 4.11 :* For  $R > 0$ , let  $S_R = \{(x, y) \in S, |x| > R\}$ . Since  $\Omega(t) \in L^1$ , the integral  $\int_{S_R} \Omega(t)$  tends to 0 as  $R \rightarrow +\infty$ . Moreover, one can easily check that this limit is uniform on any compact set of  $\mathbb{R}_+$  :

$$\forall \varepsilon > 0, \exists R > 0, \sup_{0 \leq t \leq t_0} \int_{S_R} \Omega(t) < \varepsilon . \quad \bullet$$

Of course, the reason for considering the integro-differential system (6) clearly appears : we will investigate in the next section the convergence of the numerical solution defined in Section 3 towards the solution of (6) defined in Proposition 4.10. But, as mentioned in Section 3, the equations (6) have their own interest, since we conjecture (and this conjecture is strongly supported by numerical evidence (see [2], [7]) and by the conclusions of several analytical studies developed by physicists) that the time-dependent solution of (6) converges as  $t \rightarrow +\infty$  to a steady state (while the solution of (1)-(4) converges to a traveling wave).

5. NUMERICAL ANALYSIS

We are now going to study the convergence of the numerical solution defined in Section 3 to the solution of the continuous problem (3)-(4)-(6)-(7). Although the numerical method has been defined using some finite-element concepts, we will analyse it as a finite-difference scheme, using its developed expression (11)-(12).

Throughout this section, it will be assumed that the hypotheses (17)-(20) hold, and that  $\varphi_0, \psi_0 \in L^1$ ;  $(T, Y)$  and  $(\bar{T}, \bar{Y})$  will denote the solutions of (1)-(4) and (3)-(4)-(6)-(7) respectively.

As for many other similar results, the proof of our convergence result requires the solution of the continuous problem to be smooth enough ; we will therefore assume that :

$$\forall t_0 > 0, T, Y \in W^{2,\infty}([0, t_0], L^\infty) \cap L^\infty([0, t_0], W^{3,\infty}), \tag{32}$$

i.e. that the solution has bounded second-order time derivatives and third-order spatial derivatives. Furthermore, we will assume that the non linear function  $f$  is Lipschitz-continuous on all of  $\mathbb{R}_+$  :

$$f \in W^{1,\infty}(\mathbb{R}_+). \tag{33}$$

We keep the notations of Section 3, and set :

$$\Delta x = \max_{i \in I} [\max(\Delta_i, \Delta_i^+, \Delta_i^-)]$$

( $\Delta x$  is an upper bound of all mesh sizes in the  $x$ -direction). We also rewrite the scheme (11)-(12) under the form

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \sum_j a_j^i T_j^n + V^n \sum_j b_j^i T_j^n - \Omega(T_i^n, Y_i^n), \tag{34a}$$

$$\frac{Y_i^{n+1} - Y_i^n}{\Delta t} = \frac{1}{Le} \sum_j a_j^i Y_j^n + V^n \sum_j b_j^i Y_j^n - \Omega(T_i^n, Y_i^n), \tag{34b}$$

where the terms  $\sum_j a_j^i T_j^n$  and  $\sum_j b_j^i T_j^n$  are the discrete approximations of  $\Delta T(S_i)$  and  $T_x(S_i)$  respectively. The next Lemma shows that these approximations are consistent :

LEMMA 5.1 : *Let  $v \in W^{2,\infty}(\mathbb{R}_+, L^\infty) \cap L^\infty(\mathbb{R}_+, W^{3,\infty})$ . There exists a positive constant  $K$  such that, for all  $i \in I$  and  $t \geq 0$  :*

$$\left| \sum_j a_j' v(S_j, t) - \Delta v(S_i, t) \right| < K(\Delta x + \Delta y), \tag{35a}$$

$$\left| \sum_j b_j' v(S_j, t) - v_x(S_i, t) \right| < K \Delta x, \tag{35b}$$

$$\left| \frac{v(S_i, t + \Delta t) - v(S_i, t)}{\Delta t} - v_t(S_i, t) \right| < K \Delta t. \bullet \tag{35c}$$

*Proof:* These properties are obtained in a classical way using truncated Taylor expansions. For instance, for the inequality (35b), we have :

$$v(S_{i+N_y}, t) = v(S_i, t) + \Delta_i^+ v_x(S_i, t) + \frac{(\Delta_i^+)^2}{2} v_{xx}(S^+, t),$$

where  $S^+$  is a point in the interval  $[S_i, S_{i+N_y}]$ . We also have :

$$v(S_{i-N_y}, t) = v(S_i, t) - \Delta_i^- v_x(S_i, t) + \frac{(\Delta_i^-)^2}{2} v_{xx}(S^-, t),$$

whence :

$$\begin{aligned} \frac{v(S_{i+N_y}, t) - v(S_{i-N_y}, t)}{\Delta_i^+ + \Delta_i^-} - v_x(S_i, t) &= \\ &= \frac{(\Delta_i^+)^2 v_{xx}(S^+, t) - (\Delta_i^-)^2 v_{xx}(S^-, t)}{2(\Delta_i^+ + \Delta_i^-)}, \end{aligned}$$

from which (35b) easily follows with  $K = \|v_{xx}\|_\infty$ . •

The next lemma gives the technical sufficient conditions on the grid and on the time step which insure that the  $L^\infty$  estimates (29) are satisfied by the numerical solution. It uses the notations  $\sigma_0 = \min(1, Le^{-1})$ ,  $\sigma_1 = \max(1, Le^{-1})$  and  $\Delta_i^\pm = \max(\Delta_i^+, \Delta_i^-)$ .

LEMMA 5.2 : *Let  $N \in \mathbb{N}^*$ . Assume that the grid defined in Figure 4 satisfies the conditions :*

$$p \geq 0, p \Delta y \leq \Delta_i \text{ for all } i \in I, \tag{36a}$$

*and that the grid and the numerical solution satisfy the condition :*

$$\frac{1+p^2}{\Delta_i^\pm} - \frac{p}{\Delta y} \geq \frac{1}{2\sigma_0} |V^n| \text{ for all } n < N \text{ and } i \in I. \tag{36b}$$

Assure moreover that the time step satisfies :

$$\Delta t \leq \left[ 2 \sigma_1 \left( \frac{1+p^2}{\Delta_i^+ \Delta_i^-} + \frac{1}{\Delta y^2} - \frac{p}{\Delta_i \Delta y} \right) + f(T_i^n) \right]^{-1}$$

for all  $n < N$  and  $i \in I$ . (36c)

Then, the following inequalities hold :

$$0 \leq Y_i^n \leq 1, 0 \leq T_i^n, \text{ for all } n \leq N \text{ and } i \in I. \bullet$$

*Proof:* Setting  $Z_i^n = 1 - Y_i^n$ , we have :

$$Y_i^{n+1} = \sum_j \left[ \delta_{ij} + \frac{\Delta t}{Le} a_j' + \Delta t V^n b_j' - \Delta t \delta_{ij} f(T_i^n) \right] Y_j^n, \quad (37a)$$

$$T_i^{n+1} = \sum_j \left[ \delta_{ij} + \Delta t a_j' + \Delta t V^n b_j' \right] T_j^n + \Delta t Y_i^n f(T_i^n), \quad (37b)$$

$$Z_i^{n+1} = \sum_j \left[ \delta_{ij} + \frac{\Delta t}{Le} a_j' + \Delta t V^n b_j' \right] Z_j^n + \Delta t Y_i^n f(T_i^n), \quad (37c)$$

(where  $\delta_{ij}$  is the Kronecker delta), and we want  $Y_i^{n+1}, T_i^{n+1}, Z_i^{n+1}$  to be positive as soon as the  $Y_j^n, T_j^n, Z_j^n$  are positive. This property is true if all the terms in brackets in (37) are positive, which gives exactly (36). •

*Remark 5.3 :* To rigorously complete the proof of Lemma 5.2, it would be necessary to examine the discrete scheme for the nodes  $S_i$  located on the boundaries  $W \cap D$ . We leave it to the reader to check that the conditions (36) remain sufficient to insure the positivity of  $Y_i^{n+1}, T_i^{n+1}, Z_i^{n+1}$  when the boundary scheme is taken into account. •

*Remark 5.4 :* The conditions (36) imply in particular :

$$p \geq 0, p \Delta y \leq \Delta_i, \frac{1+p^2}{\Delta_i^\pm} \geq \frac{p}{\Delta y}.$$

These conditions exactly amount to saying that there is no obtuse angle in the triangulation of Figure 4. This is a classical condition for insuring the positivity of a triangular finite-element approximation of the Laplace operator (see [6]). •

*Remark 5.5 :* If the grid is uniform ( $\Delta_i^+ = \Delta_i^- = \Delta x$ ) and orthogonal ( $p = 0$ ), the sufficient conditions (36) become :

$$\frac{|V^n| \Delta x}{\sigma_0} \leq 2,$$

which is the classical « cell Reynolds number » restriction on the mesh size, and :

$$\Delta t \leq \left[ 2 \sigma_1 (\Delta x^{-2} + \Delta y^{-2}) + \max_{i \in I} f(T_i^n) \right]^{-1},$$

a classical stability condition. •

We can now turn to the convergence of the numerical approximation. We will state two results : Theorem 5.6 shows the convergence of the adaptive numerical solution defined in Section 3 to the solution of (6), while Theorem 5.7 deals with the convergence of a non adaptive numerical solution [defined by setting  $V^n \equiv 0$  in (11)-(12)] to the solution of the original system (1). The proof of Theorem 5.6 is more lengthy and technical, essentially because of the integro-differential term in the equations (6).

**THEOREM 5.6 :** *Let  $(\bar{T}, \bar{Y})$  be the solution of (3)-(4)-(6)-(7) defined in Proposition 4.10, and let  $(T_i^n, Y_i^n)$  be the numerical solution defined in Section 3,  $D$  and  $\hat{D}$  denoting respectively the computational and integration domains. Assume that the hypotheses (32)-(33) hold, and that the conditions (36) are satisfied for all  $n \geq 0$  and  $i \in I$ .*

*For any  $t_0 > 0$  and  $\varepsilon > 0$ , there exists a bounded domain  $\hat{D}_0 \subset S$  such that, for any domain  $\hat{D}$  satisfying  $\hat{D}_0 \subset \hat{D}$ , then :*

*There exist a bounded domain  $D_0$  with  $\hat{D} \subset D_0 \subset S$ , and positive numbers  $\Delta x_0, \Delta y_0, \Delta t_0$  such that :*

*If  $D_0 \subset D$ ,  $\Delta x \leq \Delta x_0$ ,  $\Delta y \leq \Delta y_0$ ,  $\Delta t \leq \Delta t_0$ , then :*

$$\max_{\substack{0 \leq n \Delta t \leq t_0 \\ i \in I}} \left| \bar{T}(S_i, n \Delta t) - T_i^n \right| < \varepsilon,$$

$$\max_{\substack{0 \leq n \Delta t \leq t_0 \\ i \in I}} \left| \bar{Y}(S_i, n \Delta t) - Y_i^n \right| < \varepsilon. \quad \bullet$$

The similar result for the system (1) with no integro-differential term is somewhat simpler :

**THEOREM 5.7 :** *Let  $(T, Y)$  be the solution of (1)-(4) defined in Theorem 4.1, and let  $(\tilde{T}_i^n, \tilde{Y}_i^n)$  be the non adaptive numerical solution defined on the computational domain  $D$  by setting  $V^n \equiv 0$  in (11)-(12). Assume that the hypotheses (32)-(33) hold, and that the conditions (36) (with  $V^n \equiv 0$ ) are satisfied for all  $n \geq 0$  and  $i \in I$ .*

For any  $t_0 > 0$  and  $\varepsilon > 0$ , there exist a bounded domain  $D_0$  and positive numbers  $\Delta x_0, \Delta y_0, \Delta t_0$  such that :

If  $D_0 \subset D, \Delta x \leq \Delta x_0, \Delta y \leq \Delta y_0, \Delta t \leq \Delta t_0$ , then

$$\max_{\substack{0 \leq n \Delta t \leq t_0 \\ i \in I}} |T(S_i, n \Delta t) - \tilde{T}_i^n| < \varepsilon ,$$

$$\max_{\substack{0 \leq n \Delta t \leq t_0 \\ i \in I}} |Y(S_i, n \Delta t) - \tilde{Y}_i^n| < \varepsilon . \bullet$$

*Proof of Theorem 5.6 :* a) To simplify the algebra, we now assume that the tube width  $L$  is equal to unity :  $L = 1$ .

Let  $t_0 > 0$  and  $\varepsilon > 0$ , and let  $\hat{D}_0$  be a bounded domain in  $S$ . We look for  $(T^0, Y^0)$  satisfying the initial data (3), the boundary conditions (4) and the equations :

$$T_t^0 = \Delta T^0 + \Omega(T^0, Y^0) + V_0(t) T_x^0 ,$$

$$Y_t^0 = \frac{1}{Le} \Delta Y^0 - \Omega(T^0, Y^0) + V_0(t) Y_x^0 ,$$

with  $V_0(t) = - \int_{\hat{D}_0} \Omega(T^0, Y^0)(t)$ . Arguing as in the proof of Proposition

4.10, it is easy to show that this problem has a unique solution given by :

$$T^0(x, y, t) = \bar{T}(x + X_0(t), y, t) ,$$

$$Y^0(x, y, t) = \bar{Y}(x + X_0(t), y, t) ,$$

where the function  $X_0(t)$  satisfies the (well-posed) ordinary differential problem :

$$X_0(0) = 0 ,$$

$$X_0'(t) = - \left[ \int_{\hat{D}_0} \Omega(\bar{T}(x + X_0(t), y, t), \bar{Y}(x + X_0(t), y, t)) - \int_S \Omega(\bar{T}(x, y, t), \bar{Y}(x, y, t)) \right] .$$

Let  $M = \sup_{S \times [0, t_0]} [|\bar{T}_x| + |\bar{Y}_x|]$ , and let  $\varepsilon_1 > 0$  be such that  $2 M t_0 \varepsilon_1 < \varepsilon$ .

For any domain  $\hat{D}_0$  and any  $t \leq t_0$ , we have  $|X_0'(t)| \leq \|\Omega\|_{L^\infty([0, t_0], L^1)}$ .

Then, using Remark 4.11, we can choose  $\hat{D}_0$  such that :

$$\forall t \in [0, t_0], |X'_0(t)| < \varepsilon_1. \quad (38)$$

We then have, for  $t \leq t_0$  :

$$|\bar{T}(x, y, t) - T^0(x, y, t)| \leq M |X_0(t)| \leq M t_0 \varepsilon_1$$

which yields :

$$\|\bar{T} - T^0\|_{L^\infty(S \times [0, t_0])} < \frac{\varepsilon}{2}, \quad \|\bar{Y} - Y^0\|_{L^\infty(S \times [0, t_0])} < \frac{\varepsilon}{2}.$$

b) From now on, we assume that the integration domain  $\hat{D}$  which appears in the definition of the numerical method is fixed and satisfies  $\hat{D}_0 \subset \hat{D}$ , where  $\hat{D}_0$  is chosen such that (38) holds. Let  $(\hat{T}, \hat{Y})$  be defined by (3)-(4) and

$$\begin{aligned} \hat{T}_t &= \Delta \hat{T} + \Omega(\hat{T}, \hat{Y}) + \hat{V}(t) \hat{T}_x, \\ \hat{Y}_t &= \frac{1}{Le} \Delta \hat{Y} - \Omega(\hat{T}, \hat{Y}) + \hat{V}(t) \hat{Y}_x, \end{aligned}$$

with  $\hat{V}(t) = - \int_{\hat{D}} \Omega(\hat{T}, \hat{Y})(t)$ . It follows from a) above that :

$$\|\bar{T} - \hat{T}\|_{L^\infty(S \times [0, t_0])} < \frac{\varepsilon}{2}, \quad \|\bar{Y} - \hat{Y}\|_{L^\infty(S \times [0, t_0])} < \frac{\varepsilon}{2}.$$

To conclude the proof, we now want to prove that, if the computational domain  $D$  is large enough and if  $\Delta x$ ,  $\Delta y$ ,  $\Delta t$  are small enough, then :

$$\begin{aligned} \max_{\substack{0 \leq n \Delta t \leq t_0 \\ i \in I}} |\hat{T}(S_i, n \Delta t) - T_i^n| &< \frac{\varepsilon}{2}, \\ \max_{\substack{0 \leq n \Delta t \leq t_0 \\ i \in I}} |\hat{Y}(S_i, n \Delta t) - Y_i^n| &< \frac{\varepsilon}{2}. \end{aligned}$$

From Lemma 5.1, we can write :

$$\begin{aligned} \frac{\hat{T}(S_i, t^{n+1}) - \hat{T}(S_i, t^n)}{\Delta t} &= \sum_j a'_j \hat{T}(S_j, t^n) + \hat{V}(t^n) \sum_j b'_j \hat{T}(S_j, t^n) + \\ &\quad + \Omega(\hat{T}(S_i, t^n), \hat{Y}(S_i, t^n)) + O(\Delta t, \Delta x, \Delta y), \quad (39) \end{aligned}$$

where  $O(\Delta t, \Delta x, \Delta y)$  is a term which satisfies an upper bound of the form :

$$\exists R > 0, \quad O(\Delta t, \Delta x, \Delta y) < R \cdot (\Delta t + \Delta x + \Delta y).$$

Subtracting (39) from (34a) and defining the numerical error  $e_i^n$  as  $e_i^n = T_i^n - \hat{T}(S_i, t^n)$ , we get :

$$\frac{e_i^{n+1} - e_i^n}{\Delta t} = \sum_j a_j^n e_j^n + V^n \sum_j b_j^n e_j^n + (V^n - \hat{V}(t^n)) \sum_j b_j^n \hat{T}(S_j, t^n) + \Omega(T_i^n, Y_i^n) - \Omega(\hat{T}(S_i, t^n), \hat{Y}(S_i, t^n)) + O(\Delta t, \Delta x, \Delta y),$$

which can be rewritten as :

$$e_i^{n+1} = \sum_j [\delta_{ij} + \Delta t a_j^n + \Delta t V^n b_j^n] e_j^n + \Delta t (V^n - \hat{V}(t^n)) \sum_j b_j^n \hat{T}(S_j, t^n) + \Delta t [\Omega(T_i^n, Y_i^n) - \Omega(\hat{T}(S_i, t^n), \hat{Y}(S_i, t^n))] + \Delta t O(\Delta t, \Delta x, \Delta y). \quad (40)$$

Let  $F = \|f\|_{W^1 \infty(\mathbb{R}_+)}$ . One easily checks that :

$$|\Omega(T_1, Y_1) - \Omega(T_2, Y_2)| \leq F |T_1 - T_2| + F |Y_1 - Y_2|. \quad (41)$$

On the other hand :

$$\left| \sum_j b_j^n \hat{T}(S_j, t^n) \right| = \left| \frac{\hat{T}(S_{i+N_y}, t^n) - \hat{T}(S_{i-N_y}, t^n)}{\Delta_i^+ + \Delta_i^-} \right| \leq M.$$

Lastly, all the terms in brackets in the first line of (40) are positive and sum to unity from Lemmas 5.1 and 5.2. Therefore, setting

$$E^n = \max_{i \in I} [ |e_i^n| + |\hat{Y}(S_i, t^n) - Y_i^n| ], \quad (42)$$

we get :

$$E^{n+1} \leq E^n + \Delta t M |\hat{V}(t^n) - V^n| + \Delta t F E^n + \Delta t O(\Delta t, \Delta x, \Delta y). \quad (43)$$

Let us now consider the term  $|\hat{V}(t^n) - V^n|$ . We can write :

$$\begin{aligned} |\hat{V}(t^n) - V^n| &= \left| \int_{\hat{D}} \Omega(\hat{T}, \hat{Y})(t^n) - \sum_{S_i \in \hat{D}} A_i \Omega(T_i^n, Y_i^n) \right| \leq \\ &\leq \left| \int_{\hat{D}} \Omega(\hat{T}, \hat{Y})(t^n) - \sum_{S_i \in \hat{D}} A_i \Omega(\hat{T}(S_i, t^n), \hat{Y}(S_i, t^n)) \right| \\ &+ \sum_{S_i \in \hat{D}} A_i |\Omega(\hat{T}(S_i, t^n), \hat{Y}(S_i, t^n)) - \Omega(T_i^n, Y_i^n)|. \end{aligned}$$



The first term in the right-hand side of this inequality is a numerical integration error, which can be classically estimated to give :

$$|\hat{V}(t^n) - V^n| \leq \hat{A}Q(\Delta x + \Delta y) + \sum_{S_i \in \hat{D}} A_i |\Omega(\hat{T}(S_i, t^n), \hat{Y}(S_i, t^n)) - \Omega(T_i^n, Y_i^n)|$$

where  $\hat{A} = \text{area}(\hat{D})$ ,  $Q = \|\Omega\|_{L^\infty([0, T], W^{1, \infty})}$ . Using now (41), (42) and (14), we get :

$$|\hat{V}(t^n) - V^n| \leq \hat{A}Q(\Delta x + \Delta y) + F\hat{A}E^n.$$

(43) then yields :

$$E^{n+1} \leq (1 + \Delta t F + \Delta t MF\hat{A})E^n + \Delta t O(\Delta t, \Delta x, \Delta y). \tag{44}$$

We leave it to the reader to check that this recurrence relation, which we have derived by considering only interior nodes  $S_i$ , still holds when the nodes located on the boundaries  $W \cap D$  are taken into account. But we still have to consider the nodes located on the boundaries  $\Gamma_a$  and  $\Gamma_b$ . If for instance  $S_i \in \Gamma_a$ , where an homogeneous Dirichlet condition is used for the temperature, we have :

$$|e_i^n| = |T_i^n - \hat{T}(S_i, t^n)| = |\hat{T}(S_i, t^n)|,$$

and some error is introduced in the numerical solution since  $\hat{T}$  is not exactly zero on  $\Gamma_a$ .

This boundary error can be made small if the domain  $D$  is large enough. Indeed, from Remark 4.6, if  $\varepsilon_2$  is a positive number (which will be adequately chosen hereafter), we can choose  $D_0 = [a_0, b_0] \times [0, L]$  such that :

$$\sup_{\substack{x < a_0 \\ 0 \leq y \leq L \\ 0 \leq t \leq t_0}} |\hat{T}(x, y, t)| + |1 - \hat{Y}(x, y, t)| < \varepsilon_2,$$

$$\sup_{\substack{x > b_0 \\ 0 \leq y \leq L \\ 0 \leq t \leq t_0}} |1 - \hat{T}(x, y, t)| + |\hat{Y}(x, y, t)| < \varepsilon_2.$$

Thus, if the computational domain  $D = [a, b] \times [0, L]$  satisfies  $D_0 \subset D$ , we will have :

$$|T_i^n - \hat{T}(S_i, t^n)| + |Y_i^n - \hat{Y}(S_i, t^n)| < \varepsilon_2 \quad \text{on} \quad \Gamma_a \cup \Gamma_b.$$

The recurrence relation (44) is then to be rewritten as :

$$E^{n+1} \leq \max [\varepsilon_2, (1 + P \Delta t) E^n + \Delta t O(\Delta t, \Delta x, \Delta y)]$$

(we have set  $P = F + MF\hat{A}$ ). Since the initial data (3) are used for the numerical solution (see (15)), we have  $E^0 = 0$  and the sequence  $(E^n)$  is bounded by the sequence  $(F^n)$  satisfying :

$$\begin{aligned} F^{n+1} &= (1 + P \Delta t) F^n + \Delta t O(\Delta t, \Delta x, \Delta y), \\ F^0 &= \varepsilon_2. \end{aligned}$$

We easily get :

$$F^n = (1 + P \Delta t)^n \varepsilon_2 + \Delta t O(\Delta t, \Delta x, \Delta y) \frac{(1 + P \Delta t)^n - 1}{P \Delta t}.$$

Since  $(1 + P \Delta t)^n \leq e^{nP \Delta t} \leq e^{Pt_0}$  when  $n \Delta t \leq t_0$ , we finally obtain :

$$E^n \leq F^n \leq e^{Pt_0} \varepsilon_2 + \frac{e^{Pt_0} - 1}{P} O(\Delta t, \Delta x, \Delta y),$$

and  $\varepsilon_2, \Delta t, \Delta x, \Delta y$  can be chosen small enough to give :

$$E^n \leq \frac{\varepsilon}{2},$$

which ends the proof. •

*Remark 5.8:* The reason for using an integration domain  $\hat{D}$  strictly imbedded in the computational domain  $D$  is now clear. If  $\hat{D} = D$ , the constant  $P$  depends on  $A = \text{area}(D)$ . Then, when  $D$  is taken large enough to lower the value of  $\varepsilon_2$ ,  $e^{Pt_0}$  tends to  $+\infty$  and  $\varepsilon_2 e^{Pt_0}$  does not tend to 0 any longer. •

*Remark 5.9:* The interpolation errors which are introduced in the numerical solution when the grid is changed to a better adapted grid at some time levels could also be taken into account without modifying the statements of Theorems 5.6 and 5.7, provided that the number of these grid adaptations performed in the interval  $[0, t_0]$  is bounded independently of the number of time steps. •

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