Y. Maday

A. Quarteroni

Error analysis for spectral approximation of the Korteweg-de Vries equation

<http://www.numdam.org/item?id=M2AN_1988__22_3_499_0>

$\mathcal{N}$umdam

ERROR ANALYSIS FOR SPECTRAL APPROXIMATION OF THE KORTEWEG-DE VRIES EQUATION (*)

by Y. MADAY (^{1}) & A. QUARTERONI (^{2})

Communicated by R. TEMAM

Abstract. — *The conservation and convergence properties of spectral Fourier methods for the numerical approximation of the Korteweg-de Vries equation are analyzed. It is proven that the (aliased) collocation pseudospectral method enjoys the same convergence properties as the spectral Galerkin method, which is less effective from the computational point of view. This result provides a precise mathematical answer to a question raised by several authors in the latest years.*

Résumé. — *Dans cet article on analyse les propriétés de conservation et de convergence de l'approximation des équations de Korteweg-de Vries par méthodes spectrales de type Fourier. On montre que la méthode de collocation pseudo-spectrale (sans correction du terme d'« aliasing ») possède les mêmes propriétés de convergence que la méthode spectrale de Galerkin qui est moins compétitive du point de vue numérique. Ces résultats donnent une réponse précise à un problème soulevé ces dernières années par de nombreux auteurs.*

1. INTRODUCTION

In this paper, we analyze the numerical approximation by Fourier spectral methods to the Korteweg-de Vries (briefly K.d.V.) equation with periodic solutions :

$$(I.1) \quad \begin{cases} \partial u/\partial t + u \, \partial u/\partial x + \alpha \, \partial^3 u/\partial x^3 = 0 \,, & x \in \mathbb{R} \,, t > 0 \,, \\ u(x + 2\,\pi, t) = u(x, t) \,, & x \in \mathbb{R} \,, t > 0 \,, \\ u(x, 0) = u^0(x) \,, & x \in \mathbb{R} \,, \end{cases}$$

where the initial condition u^0 is a real valued 2π-periodic function of α is a real non zero parameter.

The Korteweg-de Vries equation was formerly introduced in 1895 by Korteweg and de Vries [11] to model long, unidirectional, dispersive waves of small amplitude and, nowadays, it is used to describe phenomena arising from several different fields of the application of sciences. A survey of these numerous applications is given, e.g., in Bardos [1].

From a more theoretical point of view, after the early sixthies, the study of the K.d.V. equation has benefitted from the application of the scattering theory and from the discovery of many energy integrals. We refer to the pioneering work by Miura [14], Témam [22], and the more recent papers of Bona and Smith [5], Miura [15] and Bardos [1, 2]. Moreover, the analogies with the studies of Hamiltonian systems that have been focussed lately, have permitted to extend the applicability of the K.d.V. equation to new theoretical and physical models. These aspects, and their relevance for the interaction between pure and applied mathematics, are discussed in a very instructive review article of Bardos [1].

Numerical approximations of the K.d.V. equation based either on finite differences and finite elements methods are abundant in the literature. We refer, interalia, to the papers by Bona, Pritchard and Scott [4], Bona, Dougalis and Karakashian [3] and to the references quoted therein. Fourier spectral methods have been used also in many applications of the K.d.V. equations in the last decade. We refer, e.g., to the works by Tappert [21], Fornberg [8], Schamel and Elsässer [20], Fornberg and Whitham [9], He Ping and Ben Yu [13] and Pasciak [18].

The classical Fourier-Galerkin method has been used, as well as the more flexible collocation pseudospectral method in which the discrete Fourier transform is applied to deal with nonlinear terms (see e.g. Canuto, Hussaini, Quarteroni and Zang [6], chapters 2 and 4). A crucial question has been raised from several authors, and this is precisely whether the (aliased) collocation-pseudospectral method retains the same asymptotic accuracy as the pure Galerkin method. This is a sort of master question which is recurrent in the context of numerical approximations by spectral method. In the case of the K.d.V. equation, particularly, this doubt has induced several authors to introduce new pseudospectral methods with the aim of regaining the (presumably lost) exponential accuracy (see again Canuto, Hussaini, Quarteroni and Zang [6], sections 4.5 and 4.6). In this paper, we provide a precise mathematical answer to the above question. We prove that the genuine (non dealiased, non skew-symmetric) collocation-pseudospectral method enjoys the same convergence properties as the Galerkin method.

In section II we start proving that the Galerkin approximation conserves the three first energy integrals of the K.d.V. equation. Then, classical

energy methods allows us to prove that the Galerkin solution converges with spectral accuracy to the mathematical solution.

In section III, the collocation-pseudospectral approximation is considered. The related solution fails to conserve the second energy integral. However, convergence with spectral accuracy in any finite time interval $[0, T]$ can still be shown, by using a much more involved proof. In its essence, the new proof exhibits first that the pseudospectral solution cannot blow-up in a small subinterval $[0, t_1]$ of $[0, T]$. This property, joined to the property of consistence of the numerical method, allows us to initialize an inductive process which yields the desired result on the large time interval $[0, T]$.

In this paper, we will not be concerned with any time discretization of the K.d.V. equation. However, we recall that the semi-implicit time advancing schemes are customarily used for such a kind of equation. These schemes are computationally convenient since, at each time interval, they yield a diagonal system in terms of the unknown Fourier coefficients of the spectral solution. Moreover, for finite time intervals, they are stable without any restriction on the time and space discretization parameters. We refer the interested reader to Chan and Kerkhoven [7] where a linear stability analysis is presented for the K.d.V. equation, to Quarteroni [19] where a nonlinear stability analysis is carried out for a family of equations of the same kind and to Bona, Dougalis and Karakashian [3] and the references quoted therein where extended equations are also considered.

Working along with 2π-periodic functions, we introduce the periodic Sobolev spaces defined over $]0, 2\pi[$. We first recall the definition of classical Sobolev spaces. We set

$$L^2(0, 2\pi) = \left\{ f :]0, 2\pi[\to \mathbb{C}, \|f\| = \left[\int_0^{2\pi} |f(x)|^2 \, dx \right]^{1/2} < \infty \right\},$$

and we denote its scalar product by $(.\ ,.)$. Now, for any integer $r > 0$, we set

$$H^r(0, 2\pi) = \left\{ f \in L^2(0, 2\pi), \|f\|_r = \left[\sum_{j=0}^{r} \|\partial^j f / \partial x^j\|^2 \right]^{1/2} < \infty \right\},$$

and for any real $r > 0$, not in $\mathbb{N}$, the space $H^r(0, 2\pi)$ is defined by interpolation between $H^{E(r)}(0, 2\pi)$ and $H^{E(r)+1}(0, 2\pi)$ (we denote here by $E(r)$ the integral part of r). Next we consider the subspace $C_\#^\infty(0, 2\pi)$ of $C^\infty(0, 2\pi)$ of all functions that are 2π-periodic so as all their derivatives. Moreover, for any real $r \geq 0$, $H_\#^r(0, 2\pi)$ stands then for the closure of $C_\#^\infty(0, 2\pi)$ in $H^r(0, 2\pi)$. As pointed out in Lions and Magenes [12], if $r - 1/2$ is not an integer, then $H_\#^r(0, 2\pi)$ consists of all functions of

$\mathbf{H}^r(0, 2\pi)$ that are 2π-periodic so as their derivatives of order $\leqslant r - 1/2$. For any real number $r \leqslant 0$, we define $\mathbf{H}^r(\Omega)$ as the dual space of $\mathbf{H}^{-r}(\Omega)$. Its norm is again denoted by $\|.\|_r$.

Finally, if Λ is an interval of $\mathbb{R}$ and X is a Banach space, for any function f from $\mathbb{R}$ into $\mathbf{X}$, we will define

$$\|f\|_{\mathbf{L}^\infty(\Lambda, \mathbf{X})} = \sup_{t \in \Lambda} \|f(t)\|_{\mathbf{X}} .$$

With these notations it is known (see Témam [22]) that as soon as u^0 belongs to $\mathbf{H}^m_\#(0, 2\pi)$, with m in $\mathbb{N}$, then the solution to (1.1) satisfies

$$(\text{I.2}) \quad \begin{cases} \|u\|_{\mathbf{L}^\infty(0, T \,;\, \mathbf{H}^m_\#(0, 2\pi))} \leqslant \eta(m, T \,;\, \|u^0\|_m) , & m \geqslant 2 , \\ \|u\|_{\mathbf{L}^\infty(0, T \,;\, \mathbf{H}^1_\#(0, 2\pi))} \leqslant \eta(T \,;\, \|u^0\|_1) , \end{cases}$$

for any $T > 0$, where the constants η solely depend of the terms in bracket.

It is well known, that the family

$$(\text{I.3}) \qquad \varphi_k(x) = (2\pi)^{-1/2} \exp(ikx) , \quad k \in \mathbb{Z} ,$$

is orthonormal and complete in $L^2(0, 2\pi)$. Thus a natural approximation of $\mathbf{L}^2(0, 2\pi)$ by periodic functions will consist of the spaces defined by

$$(\text{I.4}) \qquad \forall N \in 2\mathbb{N} , \quad \mathbf{S}_N = \text{span} \{\varphi_k, -N/2 \leqslant k \leqslant N/2\} .$$

Let us denote by $\mathbf{P}_N$ the operator

$$\forall g \in L^2(0, 2\pi) , \quad \mathbf{P}_N g = \sum_{k = -N/2}^{N/2} \hat{g}_k \varphi_k ,$$

with

$$(\text{I.5}) \qquad \hat{g}_k = \int_0^{2\pi} g(x)\, \bar{\varphi}_k(x)\, dx , \quad k \in \mathbb{Z} .$$

Since $\mathbf{P}_N$ is in fact the orthogonal projection operator over $\mathbf{S}_N$ we have equivalently

$$(\text{I.6}) \qquad \forall \psi \in \mathbf{S}_N , \quad \int_0^{2\pi} (\mathbf{P}_N g - g)\, \psi\, dx = 0 .$$

For all g in $\mathbf{L}^2(0, 2\pi)$, $(\mathbf{P}_N g)_{N \in \mathbb{N}}$ converges to g. Moreover, for any $r \geqslant s$, $r \geqslant 0$ one has (see, e.g. Jackson [10], Pasciak [17])

$$(\text{I.7}) \qquad \forall g \in \mathbf{H}^r_\#(0, 2\pi), \quad \|g - \mathbf{P}_N g\|_s \leqslant cN^{s-r} \|g\|_r .$$

(Throughout this paper, c will denote a positive constant, independent of N, not necessarily the same in different contexts).

Besides, all the norms defined by the imbedding of $\mathbf{S}_N$ into $\mathbf{H}'_{\#}(0, 2\pi)$ are equivalent since $\mathbf{S}_N$ is a finite dimensional subspace of $\mathbf{H}'_{\#}(0, 2\pi)$; more precisely we can readily see that

(I.8) $\forall (r, s) \in (\mathbb{R}^+)^2$ with $r \leq s$, $\forall \varphi \in \mathbf{S}_N$,

$$\|\varphi\|_r \leq \|\varphi\|_s \leq \gamma(s) N^{s-r} \|\varphi\|_r .$$

The second inequality is known as « inverse inequality ».
   We finally notice that

(I.9) $\forall g \in \mathbf{S}_N$, $\quad \|g\|_1^2 = \displaystyle\sum_{k=-N/2}^{N/2} (1 + k^2)\, \hat{g}_k^2 \leq \hat{g}_0^2 + 2\|dg/dx\|^2 .$

This property will be frequently used in the sequel.

II. ANALYSIS OF THE FOURIER-GALERKIN APPROXIMATION OF THE K.d.V. EQUATION

A spatial approximation (continuous in time) of problem (I.1) based on the Fourier-Galerkin method reads as follows :
   Find a mapping $u_N : [0, T] \to \mathbf{S}_N$ *such that*

(II.1)

$$\begin{cases} \forall \psi \in \mathbf{S}_N , \forall t, 0 \leq t \leq T , \quad (\partial u_N/\partial t + u_N\, \partial u_N/\partial x + \alpha\, \partial^3 u_N/\partial x^3, \psi) = 0 , \\ u_N(0) = \mathbf{P}_N u^0 . \end{cases}$$

This entails a nonlinear system of O.D.E.'s for the Fourier coefficients $(\hat{u}_N)_k(t)$ of the solution u_N. We present now the main properties enjoyed by the above Fourier-Galerkin approximation. They are concerned with the concepts of conservation, stability, uniqueness and convergence.

   LEMMA II.1 : *There exists a unique solution* u_N *to problem* (II.1). *Moreover this solution conserves the three first energy integrals of the K.d.V. equation, namely*

(II.2) $(\partial/\partial t)\left[\displaystyle\int_0^{2\pi} u_N(x, t)\, dx\right] = 0 ,$

(II.3) $(\partial/\partial t)\left[\displaystyle\int_0^{2\pi} |u_N(x, t)|^2\, dx\right] = 0 ,$

(II.4) $(\partial/\partial t)\left[\displaystyle\int_0^{2\pi} (\alpha(\partial u_N/\partial x)^2 - u_N^3/3)(x, t)\, dx\right] = 0 .$

Proof : The existence of a maximal time t_0, $0 < t_0 \leqslant T$ such that, for all $t < t_0$, there exists a unique solution $u_N(t)$ to problem (II.1) is a classical result of the theory of differential systems. The problem is to get the existence for an « arbitrary » time t_0, or equivalently to prove that one can take $t_0 = T$. This result will be achieved owing to (II.3), which ensures that the solution cannot blow-up.

Since the initial condition u^0 is real, then $u_N(0)$ is real too. We deduce that $u_N(t)$ is real for any $t < t_0$ from the uniqueness of the solution to problem (II.1).

In order to show (II.2), let us first choose $\psi = 1$ as a test function in (II.1). We get

$$(\partial/\partial t) \int_0^{2\pi} u_N(x, t)\, dx + (1/2) \int_0^{2\pi} (\partial u_N^2/\partial x)(x, t)\, dx +$$

$$+ \alpha \int_0^{2\pi} (\partial^3 u_N/\partial x^3)(x, t)\, dx = 0 ,$$

using the periodicity of u_N, we deduce then (II.2). Choosing now $\psi = u_N$ in (II.1), we obtain

$$(\text{II.5}) \quad (\partial/\partial t) \int_0^{2\pi} (u_N(x, t))^2\, dx + (1/3) \int_0^{2\pi} (\partial u_N^3/\partial x)(x, t)\, dx +$$

$$+ \alpha \int_0^{2\pi} (u_N\, \partial^3 u_N/\partial x^3)(x, t)\, dx = 0 .$$

Integrating by parts and using the periodicity of u_N yields

$$\int_0^{2\pi} (u_N\, \partial^3 u_N/\partial x^3)(x, t)\, dx = - (1/2) \int_0^{2\pi} \partial/\partial x (\partial u_N/\partial x)^2 (x, t)\, dx = 0 .$$

Similarly we have

$$\int_0^{2\pi} (\partial u_N^3/\partial x)(x, t)\, dx = 0 .$$

We derive now (II.3) from (II.5). Integrating (II.3) between 0 and t_0 proves that no blow-up occurs as time t_0. More precisely for any t, $0 \leqslant t < t_0$, we derive

$$(\text{II.6}) \qquad \| u_N(.\,, t) \| = \| u_N(.\,, 0) \| \leqslant \| u^0 \| .$$

whence t_0 is equal to T and we can state that the existence and uniqueness of the solution u_N holds for any time t, $0 \leqslant t \leqslant T$.

In order to prove now (II.4), let us take $\psi = P_N[u_N^2 + 2\,\alpha\,\partial^2 u_N/\partial x^2](.\,,t)$ in (II.1) and, for convenience of notation, let us drop for a while the explicit dependence on x and t. Then

(II.7) $\displaystyle\int_0^{2\pi} (\partial u_N/\partial t)\, P_N[u_N^2 + 2\,\alpha\,\partial^2 u_N/\partial x^2]\, dx$

$\displaystyle + (1/2) \int_0^{2\pi} (\partial/\partial x)[u_N^2 + 2\,\alpha\,\partial^2 u_N/\partial x^2]\, P_N[u_N^2 + 2\,\alpha\,\partial^2 u_N/\partial x^2]\, dx = 0\;.$

Since $\partial u_N/\partial t$ is in S_N we have from (I.6)

$\displaystyle\int_0^{2\pi} (\partial u_N/\partial t)\, P_N[u_N^2 + 2\,\alpha\,\partial^2 u_N/\partial x^2]\, dx$

$$= \int_0^{2\pi} \partial u_N/\partial t\, [u_N^2 + 2\,\alpha\,\partial^2 u_N/\partial x^2]\, dx$$

$$= (\partial/\partial t)\left[\int_0^{2\pi} (u_N^3/3 - \alpha\,(\partial u_N/\partial x)^2)\, dx\right]\;.$$

On the other hand using again (I.6) we derive

$\displaystyle\int_0^{2\pi} (\partial/\partial x)[u_N^2 + 2\,\alpha\,\partial^2 u_N/\partial x^2]\, P_N[u_N^2 + 2\,\alpha\,\partial^2 u_N/\partial x^2]\, dx$

$$= (1/2)\int_0^{2\pi} (\partial/\partial x)\{P_N[u_N^2 + 2\,\alpha\,\partial^2 u_N/\partial x^2]\}^2\, dx = 0\;.$$

From (II.7) we get now (II.4).

Remark II.1 : The estimates (II.2) to (II.4) are the discrete analogous of the conservation laws for the K.d.V. equation defined in, e.g., Miura, Gardner and Kruskal [16]. They hold for a large class of dispersive equations.

In the next two lemmas we state some a priori estimates for the Fourier-Galerkin solution in higher order norms.

LEMMA II.2 : *Assume that u^0 belongs to $H_\#^1(0, 2\pi)$. Then there exists a constant $c > 0$ independent of N such that for any t, $0 \leqslant t \leqslant T$:*

(II.8) $$\|u_N(.\,,t)\|_1 \leqslant c\;.$$

Proof: For any $t \leqslant T$, we derive from (II.4)

(II.9) $\displaystyle\int_0^{2\pi} (\alpha\,(\partial u_N/\partial x)^2 - u_N^3/3)(x, t)\, dx =$

$$= \int_0^{2\pi} (\alpha\,(\partial u_N/\partial x)^2 - u_N^3/3)(x, 0)\, dx\;.$$

Using now the continuous imbedding of $\mathbf{H}^1(0, 2\pi)$ into $\mathbf{L}^\infty(0, 2\pi)$ (see, e.g. [12]), we obtain first that

$$\int_0^{2\pi} (u_N^3/3)(x, 0)\, dx \leqslant (1/3)\|u_N(.\,, 0)\|_{\mathbf{L}^\infty}\|u_N(.\,, 0)\|^2$$

$$\leqslant c\|u_N(.\,, 0)\|_1 \|u_N(.\,, 0)\|^2 .$$

By the definition of $u_N(.\,, 0)$ and (I.7) it follows that

(II.10) $\quad \displaystyle\int_0^{2\pi} (u_N^3/3)(x, 0)\, dx \leqslant c\|u^0\|_1 \|u^0\|^2 \leqslant c(\|u^0\|_1^2 + \|u^0\|^4) .$

In a similar way, using (II.6), we get

$$\int_0^{2\pi} (u_N^3/3)(x, t)\, dx \leqslant c\|u_N(.\,, t)\|_1 \|u_N(.\,, t)\|^2 \leqslant c\|u_N(.\,, t)\|_1 \|u^0\|^2 ,$$

$$\leqslant (|\alpha|/2)\|u_N(.\,, t)\|_1^2 + c\|u^0\|^4 ,$$

The estimate (II.8) is then an easy consequence of (I.9), (II.9) and (II.10).

With this stability in the $\mathbf{H}_\#^1(0, 2\pi)$-norm we can prove now, as in the continuous case, the boundedness of another energy integral.

LEMMA II.3 : *Assume that u^0 belongs to $\mathbf{H}_\#^2(0, 2\pi)$. Then there exists a constant $c > 0$ independent of N such that for any t, $0 \leqslant t \leqslant T$:*

(II.11) $\qquad\qquad\qquad\qquad \|u_N(.\,, t)\|_2 \leqslant c .$

Proof : As in the proof of Lemma II.1 we have to choose properly a test function in (II.1). This time we take

$$\tilde{\psi} = \mathbf{P}_N[u_N^3 + 3\,\alpha\,(\partial u_N/\partial x)^2 + 6\,\alpha u_N\,\partial^2 u_N/\partial x^2 + (18/5)\,\alpha^2\,\partial^4 u_N/\partial x^4](.\,, t) .$$

This choice yields (here again we drop the dependence on x and t)

(II.12) $\displaystyle\int_0^{2\pi} [(\partial u_N/\partial t)\,\tilde{\psi} + u_N\,(\partial u_N/\partial x)\,\tilde{\psi} + \alpha\,(\partial^3 u_N/\partial x^3)\,\tilde{\psi}]\, dx = 0 .$

Let us examine now the first term in (II.12).
Since $\partial u_N/\partial t$ belongs to $\mathbf{S}_N$, by (I.6) we have

$$\int_0^{2\pi} \partial u_N/\partial t\,\tilde{\psi}\, dx = \int_0^{2\pi} \partial u_N/\partial t\,(u_N^3 + 3\,\alpha\,(\partial u_N/\partial x)^2 +$$

$$+ 6\,\alpha u_N\,\partial^2 u_N/\partial x^2 + (18/5)\,\alpha^2\,\partial^4 u_N/\partial x^4)\, dx ;$$

integrations by parts yield

$$(II.13) \qquad \int_0^{2\pi} \partial u_N/\partial t \; \tilde{\psi} \, dx = (\partial/\partial t) \int_0^{2\pi} [u_N^4/4 - 3 \, \alpha u_N (\partial u_N/\partial x)^2 +$$
$$+ (9/5) \, \alpha^2 (\partial^2 u_N/\partial x^2)^2] \, dx \, .$$

Now we notice that

$$\tilde{\psi} = \mathbf{P}_N [u_N^3 + 3 \, \alpha (\partial u_N/\partial x)^2 + 6 \, \alpha u_N \, \partial^2 u_N/\partial x^2] + (18/5) \, \alpha^2 \, \partial^4 u_N/\partial x^4 \, ,$$

hence we obtain

$$(II.14) \qquad \int_0^{2\pi} [u_N (\partial u_N/\partial x) + \alpha \, \partial^3 u_N/\partial x^3] \, \tilde{\psi} \, dx = A + B + C + D + E$$

with

$$A := (18/5) \, \alpha^3 \int_0^{2\pi} \partial^3 u_N/\partial x^3 \, \partial^4 u_N/\partial x^4 \, dx = 0 \, ,$$

$$B := \int_0^{2\pi} u_N (\partial u_N/\partial x) \, \mathbf{P}_N (u_N^3) \, dx \, ,$$

$$C := \alpha \int_0^{2\pi} [\partial^3 u_N/\partial x^3 \, u_N^3 + 3 \, u_N \, \partial u_N/\partial x \, \mathbf{P}_N (u_N \, \partial^2 u_N/\partial x^2)] \, dx \, ,$$

$$D := 3 \, \alpha \int_0^{2\pi} u_N \, \partial u_N/\partial x \, \mathbf{P}_N ((\partial u_N/\partial x)^2 + u_N \, \partial^2 u_N/\partial x^2) \, dx \, ,$$

$$E := \alpha^2 \int_0^{2\pi} [(18/5) \, u_N \, \partial u_N/\partial x \, \partial^4 u_N/\partial x^4 +$$
$$+ 3 \, \partial^3 u_N/\partial x^3 ((\partial u_N/\partial x)^2 + 2 \, u_N \, \partial^2 u_N/\partial x^2)] \, dx \, .$$

In order to bound B we use the continuous imbedding of $\mathbf{H}_\#^1 (0, 2\pi)$ into $\mathbf{L}^\infty (0, 2\pi)$

$$B \leqslant c \, \|u_N\|_1 \, \|\partial u_N/\partial x\|_0 \, \|\mathbf{P}_N (u_N^3)\| \leqslant c \, \|u_N\|_1^2 \, \|u_N^3\| \leqslant c \, \|u_N\|_1^2 \, \|u_N^3\|_1 \, .$$

Since $\mathbf{H}_\#^1 (0, 2\pi)$ is an algebra, $\|u_N^3\|_1 \leqslant c \, \|u_N\|_1^3$ and therefore we deduce from (II.8) that

$$(II.15) \qquad\qquad\qquad |B| \leqslant c \, .$$

Let us consider now the term C ; integrating by parts the first addendum we obtain

$$C = - \alpha \int_0^{2\pi} [\partial^2 u_N/\partial x^2 (3 u_N^2 \, \partial u_N/\partial x) -$$

$$- 3 u_N \, \partial u_N/\partial x \, \mathbf{P}_N (u_N \, \partial^2 u_N/\partial x^2)] \, dx$$

$$= 3 \alpha \int_0^{2\pi} u_N \, \partial u_N/\partial x \, [\mathbf{P}_N (u_N \, \partial^2 u_N/\partial x^2) - (u_N \, \partial^2 u_N/\partial x^2)] \, dx \, ,$$

so that

$$C = 3 \alpha \|u_N\|_{\mathbf{L}^\infty} \|u_N\|_1 \|\mathbf{P}_N (u_N \, \partial^2 u_N/\partial x^2) - (u_N \, \partial^2 u_N/\partial x^2)\| \, .$$

From (I.7) (using again the imbedding of $\mathbf{H}_\#^1 (0, 2\pi)$ into $\mathbf{L}^\infty (0, 2\pi)$), we derive

$$|C| \leq c \|u_N\|_{\mathbf{L}^\infty} \|u_N\|_1 \|u_N \, \partial^2 u_N/\partial x^2\|$$

$$\leq c \|u_N\|_{\mathbf{L}^\infty} \|u_N\|_1 \|u_N\|_{\mathbf{L}^\infty} \|\partial^2 u_N/\partial x^2\|$$

$$\leq c \|u_N\|_1^3 \|u_N\|_2 \, ,$$

and by (II.8) we conclude that

(II.16) $$|C| \leq c \|u_N\|_2 \, .$$

We now have

$$D = 3 \alpha \int_0^{2\pi} u_N \, \partial u_N/\partial x \, \mathbf{P}_N ((\partial/\partial x)(u_N \, \partial u_N/\partial x)) \, dx \, ,$$

$$= 3 \alpha \int_0^{2\pi} \mathbf{P}_N (u_N \, \partial u_N/\partial x) \, \mathbf{P}_N ((\partial/\partial x) (u_N \, \partial u_N/\partial x)) \, dx$$

$$= 3 \alpha \int_0^{2\pi} \mathbf{P}_N (u_N \, \partial u_N/\partial x)(\partial/\partial x) \, \mathbf{P}_N (u_N \, \partial u_N/\partial x) \, dx = 0 \, .$$

Similarly we obtain

$$E = \alpha^2 \int_0^{2\pi} [(- 18/5) ((\partial u_N/\partial x)^2 + u_N \, \partial^2 u_N/\partial x^2)(\partial^3 u_N/\partial x^3))$$

$$+ 3 (\partial^3 u_N/\partial x^3)((\partial u_N/\partial x)^2 + 2 u_N \, \partial^2 u_N/\partial x^2)] \, dx$$

$$= - \alpha^2 \int_0^{2\pi} [(3/5)(\partial u_N/\partial x)^2 \, \partial^3 u_N/\partial x^3 - (6/5) u_N \, \partial/\partial x (\partial^2 u_N/\partial x^2)^2] \, dx$$

$$= \alpha^2 \int_0^{2\pi} [(3/5)(\partial/\partial x)((\partial u_N/\partial x)^2) \, \partial^2 u_N/\partial x^2 +$$

$$+ (6/5) u_N (\partial/\partial x)(\partial^2 u_N/\partial x^2)^2] \, dx \, .$$

After integration by parts, we deduce that $E = 0$.

From (II.14), (II.15), (II.16) and the fact that $A = D = E = 0$ we derive

$$\left| \int_0^{2\pi} [(1/2) \, \partial u_N^2/\partial x + \alpha \, \partial^3 u_N/\partial x^3] \, \tilde{\psi} \, dx \right| \leqslant c \, (\|u_N\|_2 + 1) .$$

Due to (II.12) and (II.13) we obtain

$$(\partial/\partial t) \int_0^{2\pi} [u_N^4/4 - 3 \, \alpha u_N \, (\partial u_N/\partial x)^2 +$$

$$+ (9/5) \, \alpha^2 (\partial^2 u_N/\partial x^2)^2] \, dx \leqslant c \, (\|u_N\|_2 + 1) .$$

After integrating between 0 and t, $0 \leqslant t \leqslant T$ and using (II.8) to bound the two first resulting terms under the integral, we have

$$\int_0^{2\pi} (9/5) \, \alpha^2 (\partial^2 u_N(., t)/\partial x^2)^2 \, dx \leqslant c \left(1 + \int_0^t \|u_N(., s)\|_2^2 \, ds \right)$$

where c depends on the $\mathbf{H}_\#^2(0, 2\pi)$-norm of the initial condition u^0 and on T. Using now the Gronwall lemma yields (II.11).

We turn now to the convergence estimate for the Galerkin approximation of the K.d.V. equation.

THEOREM II.1 : *Assume that u^0 belongs to $\mathbf{H}_\#^m(0, 2\pi)$, for some integer $m \geqslant 2$. Then there exists a constant $c > 0$ independent of N such that for any t, $0 \leqslant t \leqslant T$:*

(II.17)
$$\|u(., t) - u_N(., t)\| \leqslant cN^{1-m} .$$

Proof: For any time t, $0 \leqslant t \leqslant T$, we set $e(t) = \mathbf{P}_N \, u(t) - u_N(t)$. From (I.1) and (II.1), setting $E[f, g] = f \, \partial f/\partial x - g \, \partial g/\partial x$ we derive for any ψ in $\mathbf{S}_N$

(II.18) $(\partial e/\partial t + \alpha \, \partial^3 e/\partial x^3, \psi) = (E[\mathbf{P}_N \, u, u] - E[\mathbf{P}_N \, u, u_N], \psi) .$

Let us choose $\psi = e$ as test function in (II.18) and bound each term on the right-hand side. We obtain first, using (I.2) and (I.7)

(II.19) $|(E[\mathbf{P}_N \, u, u], e)| \leqslant (\|u\|_1 + \|\mathbf{P}_N \, u\|_1) \|u - \mathbf{P}_N \, u\|_1 \|e\| \leqslant$

$$\leqslant c \|u - \mathbf{P}_N \, u\|_1 \|e\| .$$

Moreover, a repeated use of integration by parts yields

$$\left| (E[\mathbf{P}_N u, u_N], e) \right| = \left| (1/2) \int_0^{2\pi} (\partial/\partial x)[(\mathbf{P}_N u)^2 - u_N^2] e \, dx \right|$$

$$= \left| (1/2) \int_0^{2\pi} (\mathbf{P}_N u + u_N) e \, \partial e/\partial x \, dx \right|$$

$$= \left| (1/4) \int_0^{2\pi} (\partial/\partial x)[\mathbf{P}_N u + u_N] e^2 \, dx \right| .$$

From (I.2), (II.11) and the imbedding of $\mathbf{H}_\#^1(0, 2\pi)$ into $\mathbf{L}^\infty(0, 2\pi)$ we deduce that

$$(\text{II}.20) \quad \left| (E[\mathbf{P}_N u, u_N], e) \right| \leqslant c \left\| (\partial/\partial x)[\mathbf{P}_N u + u_N] \right\|_{\mathbf{L}^\infty} \|e\|^2 \leqslant c \|e\|^2 .$$

Let us note now that $e(0) = 0$; by (II.18) — (II.20) and the Gronwall lemma we obtain

$$\|e(t)\| \leqslant c \exp(ct) \left[\int_0^t \|u(s) - \mathbf{P}_N u(s)\|_1^2 \, ds \right]^{1/2} .$$

The estimate (II.17) is then an easy consequence of (I.2), (I.7), and the triangle inequality :

$$\|u - u_N\| \leqslant \|u - \mathbf{P}_N u\| + \|e\| .$$

The above result yields the following error estimate in the $\mathbf{H}_\#^1(0, 2\pi)$-norm.

COROLLARY II.1 : *Assume that u^0 belongs to $\mathbf{H}_\#^m(0, 2\pi)$, for some $m \geqslant 2$. Then there exists a constant $c > 0$ independent of N such that for any t, $0 \leqslant t \leqslant T$:*

$$(\text{II}.21) \qquad \|u(\,.\,, t) - u_N(\,.\,, t)\|_1 \leqslant cN^{2-m} .$$

Proof : This result is a consequence of the inverse inequality in (I.8). Indeed, we have

$$\|u_N(\,.\,, t) - \mathbf{P}_N u(\,.\,, t)\|_1 \leqslant N \|u_N(\,.\,, t) - \mathbf{P}_N u(\,.\,, t)\|$$

$$\leqslant N \left[\|u(\,.\,, t) - u_N(\,.\,, t)\| + \|u(\,.\,, t) - \mathbf{P}_N u(\,.\,, t)\| \right] ,$$

and from (I.7) and (II.17) we derive

$$\|u_N(\,.\,, t) - \mathbf{P}_N u(\,.\,, t)\|_1 \leqslant N^{2-m} .$$

Now (II.21) follows using again (I.7).

III. ANALYSIS OF THE COLLOCATION METHOD FOR THE APPROXIMATION OF THE K.d.V. EQUATION

Despite its mathematical interest, the Fourier-Galerkin method is generally abandoned in the applications in favor of the Fourier collocation method. The latter method allows a very efficient treatment of the nonlinear term $u(\partial u/\partial x)$ by transform techniques at the expense, however, of introducing an extra error due to the aliasing. This has induced many authors to dealiase the Fourier collocation solution by resorting to different kinds of techniques. For a more involved discussion about these arguments, the reader is referred to [6] (see e.g. chapter 3 and section 4.4.2).

We will show in this chapter that the aliased Fourier collocation method is stable and convergent, and that its asymptotic rate of convergence is the same as that of the Fourier Galerkin method.

III.1. Position of the problem

Let us introduce the collocation points $\zeta_j = 2\,\pi j/(N+1)$, for $j = 0, ..., N$. Then we associate with this set the interpolation operator $I_N : \mathbf{C}^0(0, 2\,\pi) \to \mathbf{S}_N$ defined by :

$$(\text{III.1}) \quad \forall f \in \mathbf{C}^0(0, 2\,\pi), \quad I_N f \in \mathbf{S}_N \quad \text{and} \quad I_N f(\zeta_j) = f(\zeta_j),$$
$$j = 0, ..., N.$$

Now we define the pseudo-spectral derivative operator ∂_N as

$$(\text{III.2}) \quad \forall f \in \mathbf{C}^1(0, 2\,\pi), \quad \partial_N f = \partial \circ I_N f = \partial (I_N f)/\partial x.$$

Remark III.1 : The calculation of the nodal values of $\partial_N f$ in term of those of f is accomplished by two F.F.T.s plus N complex multiplications (see, e.g., [6] chapter 2). This requires $5\,N \log_2 N$ operations if N is a power of 2.

Let us introduce the following « discrete » scalar product in $\mathbf{C}^0(0, 2\,\pi)^2$

$$(\text{III.3}) \quad \forall \varphi, \psi \in \mathbf{C}^0(0, 2\,\pi), \quad (\varphi, \psi)_N = ((2\,\pi)/(N+1)) \sum_{j=0}^{N} \varphi(\zeta_j)\,\bar{\psi}(\zeta_j).$$

It is well-known that it coincides with the $\mathbf{L}^2$-scalar product when the product $\varphi\psi$ belongs to $\mathbf{S}_{2N}$, hence in particular

$$(\text{III.4}) \quad \forall \varphi, \psi \in \mathbf{S}_N, \quad (\varphi, \psi)_N = (\varphi, \psi).$$

Then the operator I_N is precisely the orthogonal projection operator onto $\mathbf{S}_N$ with respect to $(.\,,.)_N$. Moreover, it has been proved in [17] that this

operator satisfies the following inequality for any real numbers r and s, $0 \leqslant r \leqslant s$,

(III.5) $\forall \varphi \in \mathbf{H}_{\#}^{s}(0, 2\pi)$, $\quad \| \varphi - I_N \varphi \|_r \leqslant cN^{r-s} \| \varphi \|_s$.

With these notations we can now introduce the formulation of the approximate problem obtained by a collocation pseudo-spectral method :
Find a mapping $u_N : [0, T] \to \mathbf{S}_N$ such that

(III.6)
$$
\begin{cases}
\forall t, 0 \leqslant t \leqslant T, \quad \forall j, 0 \leqslant j \leqslant N, \\
[\partial u_N / \partial t + (1/2) \partial_N (u_N^2) + \alpha \, \partial^3 u_N / \partial x^3](\zeta_j) = 0, \\
\forall j, 0 \leqslant j \leqslant N, \quad u_N(0, \zeta_j) = u^0(\zeta_j).
\end{cases}
$$

or, equivalently, since these equalities entail equalities between polynomials of $\mathbf{S}_N$

(III.7)
$$
\begin{cases}
\forall \psi \in \mathbf{S}_N, \quad \forall t, 0 \leqslant t \leqslant T, \\
(\partial u_N / \partial t + (1/2) \partial_N (u_N^2) + \alpha \, \partial^3 u_N / \partial x^3, \psi)_N = 0, \\
u_N(\, . \, , 0) = I_N u^0.
\end{cases}
$$

The proof of the existence of the solution to this scheme and of the convergence of u_N to u will be more technical than the one for the Galerkin method.

Let us introduce two extended problems that will be useful in the analysis of (III.6). The first one is a standard K.d.V. problem with initial condition v^0 (that will be equal to $u(\, . \, , t)$ for various times t) :

(III.8)
$$
\begin{cases}
\partial v / \partial t + v \, \partial v / \partial x + \alpha \, \partial^3 v / \partial x^3 = 0, \quad x \in \mathbb{R}, t > 0, \\
v(x + 2\pi, t) = v(x, t), \quad x \in \mathbb{R}, t > 0, \\
v(x, 0) = v^0(x), \quad x \in \mathbb{R}.
\end{cases}
$$

The second problem is a collocation pseudo-spectral problem whose initial condition $v_N^0 \in \mathbf{S}_N$ is a suitable approximation of v^0, which may differ from $I_N v^0$ (the one that was considered in (III.7)) :
Find a mapping $v_N : [0, T] \to \mathbf{S}_N$ such that

$$
\begin{cases}
\forall t, 0 \leqslant t \leqslant T, \quad \forall j, 0 \leqslant j \leqslant N, \\
[\partial v_N / \partial t + (1/2) \partial_N (v_N^2) + \alpha \, \partial^3 v_N / \partial x^3](\zeta_j) = 0, \\
\forall j, 0 \leqslant j \leqslant N, \quad v_N(0, \zeta_j) = v_N^0(\zeta_j),
\end{cases}
$$

or equivalently

(III.9)
$$
\begin{cases}
\forall \psi \in \mathbf{S}_N, \quad \forall t, 0 \leqslant t \leqslant T, \\
(\partial v_N / \partial t + (1/2) \partial_N (v_N^2) + \alpha \, \partial^3 v_N / \partial x^3, \psi)_N = 0, \\
v_N(\, . \, , 0) = v_N^0.
\end{cases}
$$

We first exhibit a time interval $[0, t_1^*]$ $(t_1^* < T)$ in which there exists a solution to the collocation problem (III.9). Then we prove that this solution, together with its three first derivatives, can be bounded in $[0, t_1^*]$ by some constants depending only on the initial data. This is accomplished in Lemma III.2 and in Lemma III.3. We then prove an estimate of the error between $v - v_N$ in terms of $v^0 - v_N^0$. Finally, in section III.4, we show by an iteration argument that the above convergence result, applied to $u - u_N$, can be in fact extended to cover the whole time interval $[0, T]$.

III.2. Three lemmas about the boundedness of the solution of the collocation problem

It is readily seen by the classical theory of differential systems, that problem (III.9) admits a local solution. This means that there exists $t_0 > 0$, such that for all $t \leq t_0$, the solution of (III.9) exists and is unique (note that t_0 may depend on N).

However, no information about the boundedness of the solution in any norm independently of N is provided from this result. For this we shall take now well suited test functions in (III.9) as we did in the previous section.

The first choice $\psi = 1$ in (III.9) yields

$$(\text{III.10}) \qquad (\partial / \partial t) \left[\int_0^{2\pi} v_N(x, t)\, dx \right] = 0 \,,$$

as in (II.2), or again

(III.11) $\forall t, 0 \leq t \leq t_0 \,,$

$$\int_0^{2\pi} v_N(x, t)\, dx = \int_0^{2\pi} v_N(x, 0)\, dx = \int_0^{2\pi} v_N^0(x)\, dx \,,$$

which expresses the conservation of the Fourier coefficient $(\hat{v}_N)_0 (t)$.

We turn now to the proof of

LEMMA III.1 : *For any real number R, there exist three positive constants $t_0^* \leq t_0$, β_0, γ_0 depending only on R such that for any initial value v_N^0 verifying*

$$(\text{III.12}) \qquad\qquad \left\| v_N^0 \right\|_1 \leq R$$

and any t, $0 \leq t < t_0^*$, we have

$$(\text{III.13}) \qquad\qquad \left\| v_N(\,\cdot\,, t) \right\| \leq \beta_0 (t_0^* - t)^{-1/2} \,,$$
$$(\text{III.14}) \qquad\qquad \left\| v_N(\,\cdot\,, t) \right\|_1 \leq \gamma_0 (1 + (t_0^* - t)^{-5/6}) \,.$$

Proof: Let us take $\psi = v_N$ in (III.9) ; this gives (we drop again the dependence on x and t)

$$(1/2)(\partial/\partial t)(v_N, v_N)_N + (1/2)((\partial/\partial x) I_N(v_N^2), v_N)_N +$$
$$+ \alpha(\partial^3 v_N/\partial x^3, v_N)_N = 0 .$$

From property (III.4) the third term vanishes by integration by parts, thus

$$(\partial/\partial t)\|v_N\|^2 + ((\partial/\partial x) I_N(v_N^2), v_N) = 0 .$$

Using now the Cauchy-Schwarz inequality gives

$$(\partial/\partial t)\|v_N\|^2 \leq |(I_N(v_N^2), \partial v_N/\partial x)| \leq \|I_N(v_N^2)\| \|\partial v_N/\partial x\|$$
$$\leq (I_N(v_N^2), I_N(v_N^2))_N^{1/2} \|v_N\|_1 \leq (v_N^2, v_N^2)_N^{1/2} \|v_N\|_1$$
$$\leq \|v_N\|_{L^\infty} \|v_N\| \|v_N\|_1 .$$

Then by the following Gagliardo-Nirenberg inequality

(III.15) $\forall \varphi \in \mathbf{H}^1(0, 2\pi) , \quad \|\varphi\|_{L^\infty} \leq C_1 \|\varphi\|^{1/2} \|\varphi\|_1^{1/2} ,$

we have

(III.16) $(\partial/\partial t)\|v_N\|^2 \leq C_1(\|v_N\| \|v_N\|_1)^{3/2} .$

Using now $\psi = I_N(v_N^2 + 2\alpha \partial^2 v_N/\partial x^2)$ in (III.9) yields

$$(\partial v_N/\partial t, v_N^2 + 2\alpha \partial^2 v_N/\partial x^2)_N + (1/2)(\partial\psi/\partial x, \psi)_N = 0 .$$

The second term vanishes using (III.4) and integration by parts, hence

(III.17) $(\partial/\partial t)[\alpha\|\partial v_N/\partial x\|^2 - (1/3)(v_N^3, 1)_N] = 0 .$

Let us integrate this equality between 0 and t, with $t \leq t_0$ and use (III.4) to get

$$\|\partial v_N(.\,, t)/\partial x\|^2 - \|\partial v_N(.\,, 0)/\partial x\|^2$$
$$= (1/3\alpha)[(v_N^3(.\,, t), 1)_N - (v_N^3(.\,, 0), 1)_N]$$
$$\leq (1/3\alpha)(\|v_N(.\,, t)\|_{L^\infty} \|v_N(.\,, t)\|^2 + \|v_N(.\,, 0)\|_{L^\infty} \|v_N(.\,, 0)\|^2) .$$

Using again inequality (III.15) gives the following result

$$\|\partial v_N(.\,, t)/\partial x\|^2 \leq K^0 + C_2(\|v_N\|_1^{1/2} \|v_N\|^{5/2}) ,$$

where $C_2 = (1/3 \, \alpha) \, C_1$ and

$$K^0 = \|\partial v_N(\,.\,,0)/\partial x\|^2 + C_2 \|v_N(\,.\,,0)\|_{L^\infty} \|v_N(\,.\,,0)\|^2 \leqslant R^2(1 + C_2 \, R) \,.$$

Now a bound for the $\mathbf{H}^1(0, 2 \, \pi)$-norm of $v_N(\,.\,,t)$ is easily recovered by means of (I.9). We deduce

$$(III.18) \qquad \|v_N(\,.\,,t)\|_1^2 \leqslant K' + C_2(\|v_N(\,.\,,t)\|_1^{1/2} \|v_N(\,.\,,t)\|^{5/2}) \,,$$

where $K' = 2 \, K^0 + \left(\displaystyle\int_0^{2\pi} v_N(x, 0) \, dx \right)^2 \leqslant 2 \, R^2(1 + C_2 \, R) + 2 \, \pi R^2.$

For some technical reasons which will be clarified at the end of our proof, if necessary we take in (III.18) a possibly larger K' in order to satisfy the following inequality

$$(III.19) \qquad\qquad K' > \|v_N^0\|^3 \,.$$

This is achieved for instance for $K' = \max \, \{R^3, 2 \, R^2(\pi + 1 + C_1 \, R)\}$.
   Let us set

$$t_0^* = \min \, (T, 1/((1 + C_2) \, C_2 \, K'^{2/3})) \quad \text{and} \quad \beta_0^2 = 1/[(1 + C_2) \, C_2] \,.$$

We show that (III.13) holds with these constants.
   Assume by contradiction that there exists $t^* < t_0^*$ such that (III.13) is not verified for $t = t^*$, i.e.

$$(III.20) \qquad \|v_N(\,.\,,t^*)\|^2 > [(1/K')^{2/3} - (1 + C_2) \, C_2 \, t^*]^{-1} \,.$$

We shall prove now that the derivative of the mapping $t \to \|v_N(\,.\,,t)\|$ is bounded, then we will derive a lower bound for $\|v_N^0\|$ which contradicts (III.19). First, we note that, for any time $t \leqslant t_0$ such that

$$(III.21) \qquad\qquad \|v_N(\,.\,,t)\| \geqslant K'^{1/3} \,,$$

we get

$$K' < \|v_N(\,.\,,t)\|^3 \leqslant \|v_N(\,.\,,t)\|_1^{1/2} \|v_N(\,.\,,t)\|^{5/2} \,,$$

since

$$\|v_N(\,.\,,t)\|_1 \geqslant \|v_N(\,.\,,t)\| \,.$$

From (III.18) it follows that for any t satisfying (III.21), we have

$$(III.22) \qquad \|v_N(\,.\,,t)\|_1^2 \leqslant (1 + C_2)(\|v_N(\,.\,,t)\|_1^{1/2} \|v_N(\,.\,,t)\|^{5/2}) \,;$$

we deduce then

(III.23) $$\|v_N(.\,,t)\|_1 \leq (1 + C_2)^{2/3} \|v_N(.\,,t)\|^{5/3} \,.$$

Noting that

$$[(1/K')^{2/3} - (1 + C_2)\,C_2\,t^*]^{-1} \geq K'^{2/3} \,,$$

we deduce from (III.20) that (III.21) holds for $t = t^*$, hence (III.23) is satisfied for $t = t^*$. Let us introduce now (III.23) in (III.16). For any t, $0 \leq t \leq t^*$ such that $\|v_N(.\,,t)\| \geq K'^{1/3}$ we have

$$(\partial/\partial t)\|v_N(.\,,t)\|^2 \leq (1 + C_2)\,C_2\|v_N(.\,,t)\|^4 \,,$$

which can be written as

(III.24) $$(\partial/\partial t)[-1/\|v_N\|^2] \leq (1 + C_2)\,C_2 \,.$$

Let us consider now the set $A \subset [0, t^*]$ defined by

$$A = \{s, 0 \leq s \leq t^* : \text{for any } t \text{ in } [s, t^*] \text{ we have } \|v_N(.\,,t)\| \geq K'^{1/3}\}$$

It is an easy matter to check that, by virtue of the continuity of the function $t \to \|v_N(.\,,t)\|$, A is a closed subinterval $[\sigma^*, t^*]$ of $[0, t^*]$. Besides, from (III.24) we obtain for any s in A that

$$\|v_N(.\,,s)\|^2 \geq [(1/\|v_N(.\,,t^*)\|^2) + (1 + C_2)\,C_2(t^* - s)]^{-1} \,,$$

and by (III.20)

(III.25) $\forall s \in A$, $$\|v_N(.\,,s)\|^2 > [(1/K')^{2/3} - (1 + C_2)\,C_2\,s]^{-1} > K'^{2/3} \,.$$

Applying (III.25) to σ^* shows that $\sigma^* = 0$ hence A turns out to be equal to $[0, t^*]$. We arrive now to the contradiction between (III.25) with $s = 0$ and (III.19). We deduce first that the maximal time t_0 is independent of N since no explosion occurs before t_0^*, and that (III.20) cannot hold. Whence, we have (III.13) for all $t < t_0^*$. Now (III.14) follows from (III.13), (III.18) and Cauchy-Schwarz inequality.

Remark III.2 : We note that equation (III.17) is the discrete analog of the equation (II.4) of conservation of the third energy integral for the solution of the collocation problem. Unfortunately, the conservation of the second integral does not hold anymore in the current case. This failure does not allow us to deduce directly the conservation of the $H_{\#}^1 (0, 2\pi)$-norm of

v_N. On the other hand, this would still be possible if the following skew symmetric decomposition of the nonlinear term

$$(1/3)\, \partial_N(v_N^2) + (1/3)\, v_N\, \partial v_N/\partial x$$

was considered in (III.9) instead of $(1/2)\, \partial_N(v_N^2)$. However, this choice would slow down the efficiency of the numerical scheme since an extra nonlinear term should be computed. Besides, we will prove also for the genuine pseudospectral scheme (III.9) a uniform bound of the $\mathbf{H}_\#^1$ $(0, 2\,\pi)$-norm of v_N on the large time interval $[0, T]$. To this end, we start by proving a bound on a « small » time interval in the next corollary.

COROLLARY III.1 : *For any real number R, there exist three positive constants $\tilde{t}_1$, β_1, γ_1 depending only on R such that for any initial value v_N^0 verifying*

$$(III.26) \qquad\qquad \left\| v_N^0 \right\|_1 \leqslant R$$

and any t, $0 \leqslant t \leqslant \tilde{t}_1$, we have

$$(III.27) \qquad\qquad \left\| v_N(\,.\,,t) \right\| \leqslant \beta_1\,,$$

$$(III.28) \qquad\qquad \left\| v_N(\,.\,,t) \right\|_1 \leqslant \gamma_1\,.$$

Proof: It is a simple consequence of the previous lemma. For instance one can take $\tilde{t}_1 = t_0^*/2$, in which case (III.27) and (III.28) hold with $\beta_1 = \beta_0[2/t_0^*]^{1/2}$ and $\gamma_1 = \gamma_0(1 + (t_0^*/2)^{-5/6})$.

The same kind of bounds we obtained for v_N will now be proved for the derivative $\partial v_N/\partial t$. Let us first recall that in view of (III.10) the following equality holds

$$(III.29) \qquad \forall t, 0 \leqslant t \leqslant t_0\,, \qquad \int_0^{2\pi} \partial v_N/\partial t\,(x, t)\, dx = 0\,.$$

Further information concerning the boundedness of the norms of $\dfrac{\partial v_N}{\partial t}$ in $\mathbf{L}^2(0, 2\,\pi)$ and $\mathbf{H}_\#^1$ $(0, 2\,\pi)$ are obtained by differentiating equation (III.9) with respect to the time variable. We get

$$(III.30) \quad \forall \psi \in \mathbf{S}_N, \forall t, 0 \leqslant t \leqslant T\,, \quad (\partial^2 v_N/\partial t^2 + \partial_N(v_N\, \partial v_N/\partial t)$$
$$+ \alpha(\partial^3/\partial x^3)(\partial v_N/\partial t), \psi)_N = 0\,.$$

We can now prove the following result.

LEMMA III.2 : *For any real number R, there exist three positive constants* $t_1^* \leq \tilde{t}_1$, β_1^* *and* γ_1^*, *depending only on R, such that for any initial value* v_N^0 *verifying*

(III.31) $$\|v_N^0\|_4 \leq R$$

and any t, $0 \leq t < t_1^*$, *we have*

(III.32) $$\|(\partial v_N/\partial t)(.\,,t)\| \leq \beta_1^*,$$

(III.33) $$\|(\partial v_N/\partial t)(.\,,t)\|_1 \leq \gamma_1^*.$$

Proof: Let us take $\psi = \partial v_N/\partial t$ in (III.30). We obtain, using (III.4) and noting that $\partial v_N/\partial t$ is periodic

$$\forall t, 0 \leq t \leq \tilde{t}_1, \quad (1/2)(\partial/\partial t)\|\partial v_N/\partial t\|^2 + (\partial_N(v_N\,\partial v_N/\partial t), \partial v_N/\partial t)_N = 0.$$

By the definition (III.2) and the property (III.4), integrating by parts and using the Cauchy-Schwarz inequality, it follows

$$\forall t, 0 \leq t \leq \tilde{t}_1 \quad (\partial/\partial t)\|\partial v_N/\partial t\|^2 \leq 2\,|\,(I_N(v_N\,\partial v_N/\partial t), (\partial/\partial x)\,\partial v_N/\partial t)\,|$$

$$\leq 2\,\|I_N(v_N\,\partial v_N/\partial t)\|\,\|\partial v_N/\partial t\|_1,$$

$$\leq 2\,\|v_N\|_{L^\infty}\,\|\partial v_N/\partial t\|\,\|\partial v_N/\partial t\|_1.$$

Using the Gagliardo-Nirenberg inequality (III.15) and Corollary III.1, we can find a constant $\tilde{K}''$ depending only on the $H^1_\#(0, 2\pi)$-norm of the initial data such that

(III.34) $$\forall t, 0 \leq t \leq \tilde{t}_1, \quad \|v_N\|_{L^\infty} \leq \tilde{K}''.$$

We can therefore obtain the inequality

(III.35) $$\forall t, 0 \leq t \leq \tilde{t}_1, \quad (\partial/\partial t)\|\partial v_N/\partial t\|^2 \leq 2\,\tilde{K}''\|\partial v_N/\partial t\|\,\|\partial v_N/\partial t\|_1.$$

Our goal is now to provide a bound for $\|\partial v_N/\partial t\|_1$. To this end, let us take

$$\psi = I_N(v_N\,\partial v_N/\partial t + \alpha\,(\partial^2/\partial x^2)\,\partial v_N/\partial t)$$

as test function in (III.30). This choice yields the equality

$$-(\partial^2 v_N/\partial t^2, \alpha\,(\partial^2/\partial x^2)\,\partial v_N/\partial t) = (\partial^2 v_N/\partial t^2, v_N\,\partial v_N/\partial t)_N.$$

After integration between 0 and t, $0 \leqslant t \leqslant \tilde{t}_1$, we obtain

$$(\text{III.36}) \quad \| (\partial/\partial x)(\partial v_N/\partial t)(.\,,t) \|^2 - \| (\partial/\partial x)(\partial v_N/\partial t)(.\,,0) \|^2$$
$$= 2\,\alpha^{-1} \int_0^t (\partial^2 v_N/\partial t^2, v_N\,\partial v_N/\partial t)_N\,ds\,.$$

Let us now focus on the right-hand side of the previous equality. Integrating by parts with respect to the time variable gives

$$\int_0^t (\partial^2 v_N/\partial t^2, v_N\,\partial v_N/\partial t)_N\,ds = [(\partial v_N/\partial t, v_N\,\partial v_N/\partial t)_N]_0^t$$

$$- \int_0^t (\partial v_N/\partial t, (\partial/\partial t)(v_N\,\partial v_N/\partial t))_N\,ds$$

$$= [(\partial v_N/\partial t, v_N\,\partial v_N/\partial t)]_0^t - \int_0^t (\partial v_N/\partial t, (\partial v_N/\partial t)^2)_N\,ds$$

$$- \int_0^t (\partial v_N/\partial t, v_N\,\partial^2 v_N/\partial t^2)_N\,ds\,.$$

Noting that the last term on the righ-hand side is the opposite of the left-hand term, we get

$$2 \int_0^t (\partial^2 v_N/\partial t^2, v_N\,\partial v_N/\partial t)_N\,ds = ((\partial v_N/\partial t)^2\,(.\,,t), v_N(.\,,t))_N -$$

$$- ((\partial v_N/\partial t)^2\,(.\,,0), v_N(.\,,0))_N$$

$$- \int_0^t (1, (\partial v_N/\partial t)^3)_N\,ds\,.$$

By virtue of (III.34) and the Gagliardo-Nirenberg inequality (III.15), we deduce that

$$2 \left| \int_0^t (\partial^2 v_N/\partial t^2, v_N\,\partial v_N/\partial t)_N\,ds \right| \leqslant \tilde{K}'' \| (\partial v_N/\partial t)(.\,,t) \|^2 +$$

$$+ \| (\partial v_N/\partial t)(.\,,0) \|^2 \| v_N^0 \|_{L^\infty}, + \int_0^t \| \partial v_N/\partial t \|_{L^\infty} \| \partial v_N/\partial t \|^2\,ds\,,$$

$$\leqslant \tilde{K}'' \| (\partial v_N/\partial t)(.\,,t) \|^2 + \| (\partial v_N/\partial t)(.\,,0) \|^2 \| v_N^0 \|_{L^\infty}$$

$$+ C_1 \int_0^t \| \partial v_N/\partial t \|_1^{1/2} \| \partial v_N/\partial t \|^{5/2}\,ds\,.$$

Injecting this inequality in (III.36) and using (I.9) and (III.29) gives

$$(\alpha/2)\|(\partial v_N/\partial t)(.\,,t)\|_1^2 \leqslant \alpha\|(\partial v_N/\partial t)(.\,,0)\|_1^2$$
$$+ \|(\partial v_N/\partial t)(.\,,0)\|^2 \|v_N^0\|_{L^\infty}$$
$$+ C_1 \int_0^t \|\partial v_N/\partial t\|_1^{1/2} \|\partial v_N/\partial t\|^{5/2} ds$$
$$+ \tilde{K}''\|(\partial v_N/\partial t)(.\,,t)\|^2 .$$

We can rewrite (III.9) as follows

$$\partial v_N/\partial t = (1/2)\,\partial_N(v_N^2) + \alpha\,\partial^3 v_N/\partial x^3 .$$

Then using (III.31), it is an easy matter to deduce from this equality that

$$\|(\partial v_N/\partial t)(.\,,0)\| \leqslant C\,[\|\partial/\partial x((v_N^2) - I_N(v_N^2))\|\,]$$
$$+ \|\partial/\partial x(v_N^2)\| + \|\partial^3 v_N/\partial x^3\|$$

and using (III.5) with $r = s = 1$, we deduce that $\|(\partial v_N/\partial t)(.\,,0)\|$ is bounded by $R^2(1 + \alpha R)$. On the other hand, by taking the derivative in the x-direction of the previous equality, we derive that the term $\|(\partial v_N/\partial t)(.\,,0)\|_1$ can be bounded by $R^2(1 + \alpha R)$. Finally, with a new constant K'' depending only on R, we obtain the inequality

(III.37) $\forall t, 0 \leqslant t \leqslant \tilde{t}_1 ,$

$$\|(\partial v_N/\partial t)(.\,,t)\|_1^2 \leqslant K''(1 + \|(\partial v_N/\partial t)(.\,,t)\|^2)$$
$$+ C_1 \int_0^t \|\partial v_N/\partial t\|_1^{1/2}\|\partial v_N/\partial t\|^{5/2} ds .$$

Let us now consider the set B defined by

$$B = \left\{ s \in]0, \tilde{t}^1] : \forall t \in [0, s] : C_1 \int_0^t \|\partial v_N/\partial t\|_1^{1/2}\|\partial v_N/\partial t\|^{5/2} ds \leqslant K'' \right\} .$$

It is an easy matter to check that B is not empty ; indeed, we derive from (I.8) and (III.35) that

$\forall \tau, 0 \leqslant \tau \leqslant \tilde{t}^1 ,$

$$(\partial/\partial t)\|\partial v_N/\partial t(.\,,\tau)\|^2 \leqslant 2\,cN\,\tilde{K}''\|\partial v_N/\partial t(.\,,\tau)\|^2 .$$

Then, we get, using again (I.8)

$$\forall \tau, 0 \leqslant \tau \leqslant \tilde{t}^1 ,$$

$$\| \partial v_N / \partial t (. , \tau) \|^2 + cN^{-1} \| \partial v_N / \partial t (. , \tau) \|_1^2 \leqslant$$

$$\leqslant 2 \| \partial v_N / \partial t (. , 0) \|^2 \exp (2 \, cN \tilde{K}'' \tau) .$$

This proves the existence of a time $t_i > 0$ (obviously depending on N) such that $]0, t_i] \subset B$. Next, we derive from (III.37) that

$$(\text{III.38}) \qquad \forall t \in B , \quad \| (\partial v_N / \partial t)(. , t) \|_1^2 \leqslant K'' (2 + \| (\partial v_N / \partial t) (. , t) \|^2) ,$$

and due to (III.35)

$$\forall t \in B , \quad (\partial / \partial t) \| \partial v_N / \partial t \|^2 \leqslant \tilde{K}'' \sqrt{K''} (2 \| \partial v_N / \partial t \| + \| \partial v_N / \partial t \|^2)$$

$$\leqslant 4 \, \tilde{K}'' \sqrt{K''} (1 + \| \partial v_N / \partial t \|^2) .$$

After integration on time, we obtain

$$(\text{III.39}) \quad \forall t \in B , \quad (2 + \| (\partial v_N / \partial t)(. , t) \|^2) \leqslant$$

$$\leqslant (2 + \| (\partial v_N / \partial t)(. , 0) \|^2) \exp (4 \, \tilde{K}'' \sqrt{K''} \, t) .$$

From (III.38) we then state

$$(\text{III.40}) \quad \forall t \in B , \quad \| (\partial v_N / \partial t)(. , t) \|_1^2 \leqslant$$

$$\leqslant K'' (2 + \| (\partial v_N / \partial t)(. , 0) \|^2) \exp (4 \, \tilde{K}'' \sqrt{K''} \, t) .$$

We deduce that there exists a constant τ_0, that depends only on the $H_\#^4 (0, 2 \pi)$-norm of the initial function v_N^0 and on T such that

$$\forall t \in B , \quad \| (\partial v_N / \partial t)(. , t) \|_1^2 \leqslant \tau_0 .$$

From the definition of B we can state that this set contains an interval $[0, t_1^*]$ with $t_1^* = \min (\tilde{t}_1, K'' / (C_1 \tau_0^{3/2}))$, which is therefore independent of N. Thus the desired results follow from (III.39) and (III.40).

In order to prove the convergence of the discretization we need a further stability result

LEMMA III.3 : *For any real number R, there exists a constant γ_3 depending only on R such that for any initial value v_N^0 verifying*

$$(\text{III.41}) \qquad\qquad \| v_N^0 \|_4 \leqslant R$$

and any t, $0 \leqslant t \leqslant t_1^*$, we have

(III.42) $\|v_N(\,.\,,t)\|_3 \leqslant \gamma_3$.

Proof: Let us choose $\psi = \partial^3 v_N/\partial x^3$ in the equation (III.9). We obtain

$$|\alpha|\,\|\partial^3 v_N/\partial x^3\|^2 = -\,(\partial v_N/\partial t,\, \partial^3 v_N/\partial x^3) -$$
$$-\,(1/2)(\partial/\partial x(I_N(v_N^2)),\, \partial^3 v_N/\partial x^3)$$
$$\leqslant \|\partial v_N/\partial t\|\,\|\partial^3 v_N/\partial x^3\| + [\,\|\partial/\partial x((v_N^2) - I_N(v_N^2))\| +$$
$$+\, c\,\|v_N\|_1^2]\|\partial^3 v_N/\partial x^3\| \ .$$

Using (III.5) with $r = s = 1$ we have

(III.43) $|\alpha|\,\|\partial^3 v_N/\partial x^3\|^2 \leqslant \|\partial v_N/\partial t\|\,\|\partial^3 v_N/\partial x^3\| + c\,\|v_N\|_1^2\,\|\partial^3 v_N/\partial x^3\|$.

On the other hand, using the definition of $\mathbf{H}_{\#}^2(0, 2\pi)$-norm, the inequality (III.15) and the inverse inequality (I.8) we deduce

$$\|v_N^2\|_2 \leqslant c[\,\|v_N\|_2\,\|v_N\|_1 + \|v_N\|_1^2] \leqslant cN\,\|v_N\|_1^2 \ .$$

By virtue of (III.43) we obtain

$$|\alpha|\,\|\partial^3 v_N/\partial x^3\|^2 \leqslant \|\partial v_N/\partial t\|\,\|\partial^3 v_N/\partial x^3\| + c\,\|v_N\|_1^2\,\|\partial^3 v_N/\partial x^3\| \ .$$

The two previous lemmas yield now that the term $\|\partial^3 v_N/\partial x^3\|$ is bounded and we derive the desired result (III.42) by noting that

$$\|\partial^2 v_N/\partial x^2\| \leqslant C\,\|\partial^3 v_N/\partial x^3\| \ .$$

III.3. Analysis of the convergence in the local interval $[0, t_1^*]$

We are now in the position of stating a first local convergence result of the solution of the collocation problem (III.9) to the one of the K.d.V. equation (III.8) in the time interval $[0, t_1^*]$. Precisely we have :

PROPOSITION III.1 : *Assume that v^0 belongs to $\mathbf{H}_{\#}^m(0, 2\pi)$, for some $m \geqslant 4$, and that v_N^0 is bounded in $\mathbf{H}_{\#}^4(0, 2\pi)$ independently of N. Then there exists a constant $\Lambda_m > 0$, depending continuously on $\|v_N^0\|_m$ but independent of N, such that for any t, $0 \leqslant t \leqslant t_1^*$:*

(III.44) $\|v_N(\,.\,,t) - v(\,.\,,t)\|_1 \leqslant \Lambda_m N^{2-m} + \|v_N^0 - v^0\|_1$.

Proof: From (III.8) it is easy to check that (remind that $E(x)$ denotes the integral part of x)

$\forall t, 0 \leqslant t \leqslant T$,

$$\mathbf{P}_{E(N/2)}(\partial v/\partial t + v\, \partial v/\partial x + \alpha\, \partial^3 v/\partial x^3) = 0 \ .$$

so that

$\forall \psi \in \mathbf{L}^2(0, 2\pi)$, $\forall t, 0 \leqslant t \leqslant T$,

$$(\mathbf{P}_{E(N/2)}(\partial v/\partial t + v\, \partial v/\partial x + \alpha\, \partial^3 v/\partial x^3), \psi) = 0 \ .$$

For a fixed time t, let us substract this equation from the collocation equation (III.9) at the same time. If we set $\tilde{v}(t) = \mathbf{P}_{E(N/2)}v(t)$ and $e(t) = \tilde{v}(t) - v_N(t)$, for any ψ in $\mathbf{S}_N$ we derive the identity

(III.45) $\quad (\partial e/\partial t + \alpha\, \partial^3 e/\partial x^3, \psi) = (1/2)(\partial/\partial x(I_N(v_N^2) - \mathbf{P}_{E(N/2)}(v^2)), \psi) \ .$

Taking $\psi = e(t)$ for all t and noting that $\tilde{v}^2(t) \in \mathbf{S}_N$ we obtain

(III.46) $\quad (\partial/\partial t)\|e\|^2 = (\partial/\partial x(\tilde{v}^2 - v^2), e)$
$$+ (\partial/\partial x(v^2 - \mathbf{P}_{E(N/2)}(v^2)), e) + (\partial/\partial x(I_N(v_N^2 - \tilde{v}^2)), e) \ .$$

The last term on the right can be bounded as follows

$(\partial/\partial x(I_N(v_N^2 - \tilde{v}^2)), e) = ((v_N^2 - \tilde{v}^2), \partial e/\partial x)_N$
$$= ((2\,\tilde{v} - e)\, e, \partial e/\partial x)_N$$
$$\leqslant 2(\tilde{v}e, \tilde{v}e)_N^{1/2} \|\partial e/\partial x\| + \|\partial e/\partial x\|_{\mathbf{L}^\infty} \|e\|^2$$
$$\leqslant 2 \|\tilde{v}\|_{\mathbf{L}^\infty} \|\partial e/\partial x\| \|e\| + \|\partial e/\partial x\|_{\mathbf{L}^\infty} \|e\|^2 \ .$$

Then from (III.46), we deduce

(III.47) $\quad (\partial/\partial t)\|e\|^2 \leqslant \|e\| \left\{ [\|\partial\tilde{v}/\partial x\|_{\mathbf{L}^\infty} \right.$
$$+ \|\partial v/\partial x\|_{\mathbf{L}^\infty}]\|\tilde{v} - v\|_1 + \|v^2 - \mathbf{P}_{E(N/2)}v^2\|_1$$
$$\left. + 2\|\tilde{v}\|_{\mathbf{L}^\infty} \|\partial e/\partial x\| + \|\partial e/\partial x\|_{\mathbf{L}^\infty} \|e\| \right\} \ .$$

From (1.2) and (1.7) we get, as soon as v^0 belongs to $\mathbf{H}^m(0, 2\pi)$ with $m \geqslant 3$

(III.48) $[\|\partial\tilde{v}/\partial x\|_{\mathbf{L}^\infty} + \|\partial v/\partial x\|_{\mathbf{L}^\infty}]\|\tilde{v} - v\|_1$
$$+ \|v^2 - \mathbf{P}_{E(N/2)}v^2\|_1 \leqslant C(m)\, N^{1-m} \ .$$

Here $C(m)$ is a constant that depends only on the norm $\eta_m \|v^0\|_m$. Moreover, from (III.42) we have (we enlarge, if needed, the value of $C(m)$ keeping although its dependence only on $\eta_m \|v^0\|_m$)

(III.49) $2\|\tilde{v}\|_{L^\infty} + \|\partial e/\partial x\|_{L^\infty} \leq C(m) + \gamma_3$.

Taking into account (III.47), (III.48) and (III.49) we obtain

(III.50)

$$(\partial/\partial t)\|e\|^2 \leq (C(m) + \gamma_3)(N^{1-m}\|e\| + \|\partial e/\partial x\|\,\|e\| + \|e\|^2) ,$$

where the constant depends on the initial data only.

Our next goal is to obtain an estimate for $\|\partial e/\partial x\|$ to be used in (III.50). For this, we take $\psi = (I_N v_N^2 - \mathbf{P}_{E(N/2)} v^2) - 2\alpha\, \partial^2 e/\partial x^2$ in (III.45), so that we find the identity

$$-2\alpha(\partial e/\partial t, \partial^2 e/\partial x^2) = (\partial e/\partial t, \mathbf{P}_{E(N/2)} v^2 - I_N v_N^2) .$$

Therefore

(III.51) $\alpha(\partial/\partial t)\|\partial e/\partial x\|^2$
$$= (\partial e/\partial t, \mathbf{P}_{E(N/2)} v^2 - \tilde{v}^2) + (\partial e/\partial t, I_N(\tilde{v}^2 - v_N^2)) ,$$

or again, if we set $w = \mathbf{P}_{E(N/2)} v^2 - \tilde{v}^2$,

(III.52) $\alpha(\partial/\partial t)\|\partial e/\partial x\|^2 = (\partial e/\partial t, w) + (\partial e/\partial t, I_N[e(\tilde{v} + v_N)])$.

Besides we remark that

$(\partial e/\partial t, I_N[e(\tilde{v} + v_N)]) = (\partial e/\partial t, e(\tilde{v} + v_N))_N = (1/2)(\partial e^2/\partial t, \tilde{v} + v_N)_N$.

Then integrating (III.52) between 0 and t for any $t \leq t_1^*$ we derive

(III.53) $\alpha\|(\partial e/\partial x)(.\,,t)\|^2 - \alpha\|(\partial e/\partial x)(.\,,0)\|^2$
$$= \int_0^t (\partial e/\partial t, w)\, ds + (1/2)\int_0^t (\partial e^2/\partial t, \tilde{v} + v_N)_N\, ds .$$

Setting $z = \tilde{v} + v_N$ and integrating by parts with respect to t gives

(III.54) $\left| \int_0^t (\partial e^2/\partial t, z)_N\, ds \right| = \left| -[(e^2, z)_N]_0^t + \int_0^t (e^2, \partial z/\partial t)_N \right|$

$$\leq |(e^2(.\,,t), z(.\,,t))_N| + |(e^2(.\,,0), z(.\,,0))_N|$$

$$+ \max_{0 \leq \tau \leq t} [\|\partial z/\partial t(.\,,\tau)\|_{L^\infty}] \int_0^t \|e\|^2\, ds$$

Furthermore, we have

(III.55)

$$\left| \int_0^t (\partial e/\partial t, w) \, ds \right| = \left| - (e(t), w(t)) + (e(0), w(0)) + \int_0^t (e, \partial w/\partial t) \right|$$

$$\leq \|e(t)\| \, \|w(t)\| + \|e(0)\| \, \|w(0)\| + t \max_{0 \leq \tau \leq t} \|e(\tau)\|_1 \|\partial w/\partial t (\tau)\|_{-1} .$$

Finally, we can deal in a similar way the last term of the right-hand side of (III.53).

Let us set now

$$K_0 = \|\partial e/\partial x (0)\|^2 + |(e^2(0), z(0))_N| + \|e(0)\| \, \|w(0)\| \, ;$$

then, from (III.53) to (III.55) we deduce that

$$\forall t, 0 \leq t \leq t_1^* ,$$

$$\|\partial e/\partial x (t)\|^2 \leq K_0 + c \max_{0 \leq \tau \leq t} \left\{ [\|z(\tau)\|_{\mathbf{L}^\infty} + \|\partial z/\partial t (\tau)\|_{\mathbf{L}^\infty}] \int_0^t \|e(s)\|^2 \, ds \right.$$

$$\left. + \|\partial w/\partial t (\tau)\|_{-1}^2 + (1/2 \, c) \|e(\tau)\|_1^2 \right\}$$

$$+ \|e(t)\|^2 + \|w(t)\|^2 .$$

It is readily seen from (I.2) and (III.8) that if v^0 belongs to $\mathbf{H}_\#^m(0, 2\pi)$ with $m \geq 3$, then v belongs to $\mathbf{L}^\infty(0, T ; \mathbf{H}_\#^m(0, 2\pi))$ and $\partial v/\partial t$ belongs to $\mathbf{L}^\infty(0, T ; \mathbf{H}_\#^{m-3}(0, 2\pi))$. Thus, by a straightforward application of (I.7), we deduce

$$K_0 + C \max_{0 \leq \tau \leq t_1^*} [\|\partial w/\partial t (\tau)\|_{-1}^2 + \|w(\tau)\|^2] \leq C(m) \{N^{2(2-m)} + \|e(0)\|_1^2\} .$$

Hence we derive from (III.28) and (III.33) that

(III.56) $$\max_{0 \leq \tau \leq t} \|\partial e/\partial x (\tau)\|^2 \leq$$

$$\leq C(m) \left[N^{2(2-m)} + \int_0^t \|e(s)\|^2 \, ds + \|e(0)\|_1^2 \right] .$$

Let us now introduce this last inequality into (III.50) ; we obtain

$$(\partial/\partial t) \|e(t)\|^2 \leq$$

$$\leq (C(m) + \gamma_3) \left(N^{2(2-m)} + \|e(t)\|^2 + \int_0^t \|e(s)\|^2 \, ds + \|e(0)\|_1^2 \right) ,$$

whence, after integration with respect to the time variable

$$\forall t, 0 \leq t \leq t_1^*, \quad \|e(t)\|^2$$

$$\leq \|e(0)\|^2 + (C(m) + \gamma_3)\left(t(N^{2(2-m)} + \|e(0)\|_1^2) + \int_0^t \|e(s)\|^2 ds\right)$$

$$\leq (C(m) + \gamma_3)\left(t(N^{2(2-m)} + \|e(0)\|_1^2) + \int_0^t \|e(s)\|^2 ds\right).$$

The Gronwall lemma yields now

$$(III.57) \quad \forall t, 0 \leq t \leq t_1^*, \quad \|e(t)\|^2 \leq (C(m) + \gamma_3)(N^{2(2-m)} + \|e(0)\|_1^2).$$

We recall that the constant $(C(m) + \gamma_3)$ that appears here depends only on the $\mathbf{H}^m(0, 2\pi)$-norm of the initial value v^0 and on γ_3, which in turn, from Lemma III.3, provides a bound for the $\mathbf{H}^3(0, 2\pi)$-norm of v_N. Hence c is a constant independent of N and t.

The result (III.44) follows now using (I.7) oncemore, together with (III.56) and (III.57).

III.4. Convergence results for the approximation

We can state now the main result of this section.

THEOREM III.1 : *Assume that u^0 belongs to $\mathbf{H}^m(0, 2\pi)$, for some $m > 4$. Then for any t, $0 \leq t \leq T$ and any N large enough the following estimate holds*

$$(III.58) \qquad \|u(.\,, t) - u_N(.\,, t)\|_1 \leq cN^{2-m}.$$

Proof: Let us choose a class of initial conditions for (III.9) such that

$$(III.59) \qquad \|v_N^0\|_4 \leq 2\,\eta_4 \|u^0\|_4,$$

where η_4 is the constant which appears in (I.2). From Lemma III.2 (applied with $R = 2\,\eta_4 \|u^0\|_4$), we deduce the existence of a time t^* independent of N such that, for any initial condition verifying (III.59), the solution of (III.9) exists for any t, $0 \leq t \leq t^*$. Under the current hypotheses, by virtue of Proposition III.1 there exists a constant $\Lambda_m = \Lambda(\|u^0\|_m)$ such that, for any time t, $0 \leq t \leq t^*$, we have

$$(III.60) \qquad \|v_N(.\,, t) - v(.\,, t)\|_1 \leq \Lambda_m N^{2-m} + \|v_N^0 - v^0\|_1.$$

Finally, we denote by N^* the integral part of

$$1 + ([[(4K + 13)\Lambda_m]^{-1}\eta_4\|u^0\|_4)^{1/(5-m)},$$

where we have set $K = T/t^*$.

From (III.5), it can be deduced that there exists a constant $N_1 \geq N^*$ such that for any $N \geq N_1$ the estimate (III.59) is true if we take $v_N^0 = u_N^0$, and that

$$(\text{III.61}) \quad \|u^0 - u_N^0\|_1 + \|u(.,t) - \mathbf{P}_N u(.,t)\|_1 +$$
$$+ N^{-3}\|u(.,t) - \mathbf{P}_N u(.,t)\|_4 \leq \Lambda_m N^{1-m}$$

(if needed, we have enlarged the value of the constant Λ_m).

We are going to prove by induction on $k \leq K$ that

$$(H_k) \quad \begin{cases} \text{For any } t : kt^* \leq t \leq (k+1)t^*, u_N \text{ exists and satisfies} \\ \|u_N(.,kt^*)\|_4 \leq 2\eta_4\|u^0\|_4 \text{ and} \\ \forall t : kt^* \leq t \leq (k+1)t^*, \quad \|u(.,t) - u_N(.,t)\|_1 \\ \qquad\qquad\qquad\qquad \leq (k+2)\Lambda_m N^{2-m}. \end{cases}$$

It is an easy matter to check that (H_0) is a simple consequence of (III.60) and (III.61). Let us assume that H_k is true for k and let us prove it for $k + 1$. It is readily seen that the solution of (III.8) with $v^0 = u(., (k+1)t^*)$ is the solution of (I.1) and that the solution of (III.9) with $v_N^0 = u_N(., (k+1)t^*)$ is the solution of (III.7) for $t \geq (k+1)t^*$. First, we have to prove a bound for $\|u_N(., (k+1)t^*)\|_4$. Using (I.7) and the inverse inequality (I.8) we deduce from (III.61) and the previous estimate that

$$\forall t, 0 \leq t \leq (k+1)t^*, \quad \|u_N(.,t) - \mathbf{P}_N u(.,t)\|_4$$
$$\leq 4 N^3 \|u_N(.,t) - \mathbf{P}_N u(.,t)\|_1$$
$$\leq 4 N^3 [\|u_N(.,t) - u(,t)\|_1 + \|u(.,t) - \mathbf{P}_N(.,t)\|_1]$$
$$\leq 4(k+3)\Lambda_m N^{5-m}.$$

Therefore

$$\forall t, 0 \leq t \leq (k+1)t^*, \quad \|u_N(.,t)\|_4$$
$$\leq \|u_N(.,t) - \mathbf{P}_N u(.,t)\|_4 + \|u(.,t) - \mathbf{P}_N(.,t)\|_4 + \|u(.,t)\|_4$$
$$\leq (4K + 13)\Lambda_m N^{5-m} + \|u(.,t)\|_4.$$

If N is chosen greater than N^* we derive from (I.2) that $\|u_N(.,(k+1)t^*)\|_4 \leq 2\eta_4\|u^0\|_4$. Moreover, from Lemma III.1, we de-

duce that u_N exists for any t, $(k + 1)\, t^* \le t \le (k + 2)\, t^*$ and from (III.60)

$\forall t,\, 0 \le t \le t^*$,

$$\| u_N(.\,,\, (k + 1)\, t^* + t) - u(.\,,\, (k + 1)\, t^* + t)\|_1 \le \Lambda_m\, N^{2-m} +$$

$$+ \| u_N(.\,,\, (k + 1)\, t^*) - u(.\,,\, (k + 1)\, t^*)\|_1 \,.$$

Using now the induction hypothesis we obtain

$\forall t,\, 0 \le t \le t^*$,

$$\| u_N(.\,,\, (k + 1)\, t^* + t) - u(.\,,\, (k + 1)\, t^* + t)\|_1 \le (k + 3)\, \Lambda_m\, N^{2-m} \,.$$

This proves (H_{k+1}). Thus the induction procedure applies successfully and the desired result (III.58) follows.

Remark III.3 : The stability result in the $\mathbf{H}^4(0, 2\,\pi)$-norm stated in (H_k) plays a fundamental role in the proof of the global convergence result (III.58). As we have seen, stability has been obtained by exploiting the spectral decay of the error on each local interval $[kt^*,\, (k + 1)\, t^*]$.

BIBLIOGRAPHY

[1] C. BARDOS, *Historique sommaire de l'équation de Korteweg-de Vries. Un exemple de l'interaction entre les mathématiques pures et appliques ;* Publ. n° **45** (1983), Université de Paris Nord.

[2] C. BARDOS, *Ondes solitaires et solitons ;* Boll. U.M.I. 5, **16-A** (1979), pp. 21-47.

[3] J. L. BONA, V. A. DOUGALIS & O. A. KARAKASHIAN, *Fully discrete Galerkin methods for the Korteweg-de Vries equation,* to appear in Comput. and Math. with applications **12 A.**

[4] J. L. BONA, W. G. PRITCHARD & L. R. SCOTT, *Numerical schemes for a model nonlinear, dispersive wave ;* to appear in J. Comput. Phys.

[5] J. L. BONA & R. SMITH, *The initial value problem for the Korteweg-de Vries equations ;* Phil. Roy. Soc. London, **278** (1975), pp. 555-604.

[6] C. CANUTO, M. Y. HUSSAINI, A. QUARTERONI & T. A. ZANG, *Spectral Methods in Fluid Dynamics ;* Springer-Verlag, Berlin, 1988.

[7] T. F. CHAN & T. KERKHOVEN, *Fourier methods with extended stability intervals for the Korteweg-de Vries equation ;* S.I.A.M. J. Numer. Anal. **22** (1985), pp. 441-454.

[8] B. FORNBERG, *Numerical computation of nonlinear waves ;* Technical Report Plenum, Nonlinear Phenomena in Physics and Biology (1981).

[9] B. FORNBERG & F. R. S. WHITHAM, *A numerical and theoretical study of certain nonlinear phenomena ;* Phil. Trans. Roy. Soc. **289** (1978), pp. 373-404.

[10] D. JACKSON, *The Theory of Approximation,* A.M.S. Colloquium publications, vol. XI (1930), New York.

[11] D. J. KORTEWEG & G. DE VRIES, *On the change of form of long waves advancing in rectangular canal, and on a new type of long stationary waves ;* Philos. Mag. (1895), pp. 422-443.

[12] J. L. LIONS & E. MAGENES, *Honhomogeneous Boundary Value Problems and Applications,* Vol. I, Springer Verlag (1972), Berlin, Heildelberg and New-York.

[13] M. HE PING & G. BEN YU, *The Fourier pseudospectral method with a restrain operator for the Korteweg-de Vries equation,* J. Comp. Phys. **65** (1986), pp. 120-137.

[14] R. M. MIURA, *Korteweg-de Vries equation and generalization. I. A remarkable explicit nonlinear transformation ;* J. Math. Phys. **9** (1968), pp. 1202-1204.

[15] R. M. MIURA, *The Korteweg-de Vries equation : A survey of results ;* S.I.A.M. Review **18** (1976), pp. 412-459.

[16] R. M. MIURA, C. S. GARDNER & M. D. KRUSKAL ; *Korteweg-de Vries equation and generalization. II. Existence of conservation laws and constants of motion ;* J. Math. Phys. **9** (1968), pp. 1204-1209.

[17] J. E. PASCIAK, *Spectral and pseudo-spectral methods for advection equation ;* Math. Comput. **35** (1980), pp. 1081-1092.

[18] J. E. PASCIAK, *Spectral method for a nonlinear initial value problem involving pseudo-differential operators ;* S.I.A.M. J. Numer. Maths., **19,** 1982, pp. 142-154.

[19] A. QUARTERONI, *Fourier spectral methods for pseudo-parabolic equations ;* S.I.A.M. J. Numer. Anal., 24 (1987), pp. 323-335.

[20] H. SCHAMEL & K. ELSÄSSER, *The application of spectral method to nonlinear wave propagation,* J. Comp. Phys. **22** (1976), pp. 501-516.

[21] F. TAPPERT, *Numerical solution of the Korteweg-de Vries equation and its generalizations by the split-step Fourier method, in Lecture in Nonlinear wave Motion,* A.C. Newell, ed., Applied Mathematics, vol. **15,** A.M.S., Providence, Rhode Island (1974), pp. 215-217.

[22] R. TÉMAM, *Sur un problème non linéaire ;* J. Math. Pures Appl. **48** (1969), pp. 159-172.