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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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UNIFORM IN ε DISCRETIZATION ERROR ESTIMATES FOR CONVECTION DOMINATED CONVECTION-DIFFUSION PROBLEMS (*)

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Abstract. — *Asymptotically fitted variants of the standard Galerkin finite element method and of the streamlined diffusion method of Hughes and Brooks for solving a mixed boundary value problem for convection dominated convection-diffusion flow problems are considered*

We discuss the existence of global and local L_2-discretization error estimates (and sometimes in energy norm) which are uniformly valid with respect to the diffusion parameter ε

Résumé — *On considère des variantes « asymptotiquement adaptées » de la méthode des éléments finis habituelle de Galerkin, ainsi que de la méthode dite de « streamlined diffusion » de Hughes et Brooks pour résoudre un problème aux limites mixtes pour des problemes de convection-diffusion avec convection dominante*

On discute l'existence d'estimation L_2 de l'erreur, globales et locales (et aussi en norme de l'énergie), qui sont valables uniformément par rapport au paramètre de diffusion ε.

1. INTRODUCTION

In this paper, we consider convection-reaction-diffusion problems of the form

$$(\mathscr{L}_\varepsilon) \qquad \begin{cases} -\varepsilon\, \Delta u_\varepsilon + b \,.\, \nabla u_\varepsilon + c u_\varepsilon = f & \text{in} \quad \Omega \subset \mathbb{R}^N \\ u_\varepsilon = g_1 \quad \text{on} \quad \Gamma_1\,, \quad \varepsilon \dfrac{\partial u_\varepsilon}{\partial \nu} + \sigma(u_\varepsilon - g_2) = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

where Ω is bounded domain with boundary $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. ε is the diffusion parameter with $0 \leqq \varepsilon \leqq 1$. In the singularly perturbed case $0 < \varepsilon \ll 1$, the solution of $(\mathscr{L}_\varepsilon)$ is characterized by sharp boundary layers.

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In general, the numerical solution of standard finite element methods (f.e.m.) applied to $(\mathscr{L}_\varepsilon)$ has undesired oscillations unless the mesh is very fine. In the case of the Galerkin f.e.m. (with piecewise polynomials of degree k), and for essential boundary conditions on the downstream layer part of $\partial\Omega$, we have the L_2-discretization error estimate

$$\|u_\varepsilon - u_h\|_{0,2} \leqq C\left(\frac{h}{\varepsilon}\right)^{k+1}.$$

Well-known modifications of standard f.e.m. are mesh-refinements, upwind-schemes and exponentially fitted schemes. In this paper, we derive for adapted finite element schemes global and local discretization error estimates in the L_2-norm and sometimes in the energy norm which are uniformly valid with respect to the full range of ε. More precisely, a family of approximations, u_h, $0 < h \leqq h_0$ converges uniformly (with respect to $0 \leqq \varepsilon \leqq \varepsilon_0$) to u_ε in the norm $\|\,.\,\|$ of order p if with a constant C independent of ε and h

$$\sup_{0 \leqq \varepsilon \leqq \varepsilon_0} \|u_\varepsilon - u_h\| \leqq Ch^p.$$

An important consequence of such an uniform in ε discretization error estimate is that the given f.e.m. well approximates the solution u_0 of the limit problem

$$(\mathscr{L}_0) \qquad \begin{cases} b\,.\,\nabla u_0 + cu_0 = f & \text{in} \quad \bar{\Omega}\backslash\bar{\Gamma}_- \\ u_0 = g_1 & \text{on} \quad \bar{\Gamma}_- \end{cases}$$

if ε, $h \to 0$ (independent on the relation between ε and h). (Compare the discussion concerning finite difference schemes in Giles/Rose [6].) $\Gamma_- = \{x \in \partial\Omega \mid \exists \nu, b\,.\,\nu < 0\}$ denotes the inflow part of $\partial\Omega$. For simplicity we assume $\bar{\Gamma}_- \subseteq \Gamma_1$. In case of $b \equiv 0$, uniformly in ε convergent finite difference schemes are considered by several authors (Miller [12], Niijima [15], Shishkin-Titov [21]). Jemeljanov [9], Kellogg [11] and Shishkin [20] considered special one- and two-dimensional cases with $|b| > 0$. Concerning uniformly in ε convergent finite element schemes in case of $b \equiv 0$, we refer to Schatz-Wahlbin [18]. In case of $|b| > 0$ Nävert [14] derived uniformly in ε valid local energy and L_2-norm estimates for the streamlined diffusion f.e.m. The question of local L_∞-estimates for this situation is studied in Risch [17] for a hybrid f.e.m. preserving the maximum principle. The problem of uniformly in ε convergent (with respect to global norms) f.e.m. in case of $b \not\equiv 0$ seems to be open. In the present paper, we propose "switching" algorithms of the following kind : in case of $\varepsilon \geqq h^\kappa$ (with some $\kappa > 0$), the f.e.m. is exactly the standard Galerkin f.e.m. (SG-FEM). The first variant in the case $0 \leqq \varepsilon < h^{\kappa}$ (modified Galerkin f.e.m.-MG-FEM) consists on replacing the given boundary conditions on $\partial\Omega \backslash \bar{\Gamma}_{-}$ by homogeneous conditions on Neumann type (*cf.* Axelsson [1] with no rigorous asymptotic analysis). A second variant is to approximate the limit problem $(\mathscr{L}_0)$ instead of $(\mathscr{L}_{\varepsilon})$ by the streamlined diffusion f.e.m. of Hughes/Brooks [8] (scheme MSD.1) or to solve $(\mathscr{L}_{\varepsilon})$ with homogeneous conditions of Neumann type on $\partial\Omega \backslash \bar{\Gamma}_{-}$ approximately by the streamlined diffusion method (scheme MSD.2).

This paper is organized as follows : Section 2 is concerned with notations and with estimates for problem $(\mathscr{L}_{\varepsilon})$. In Section 3 we consider the Galerkin f.e.m. and the modified Galerkin scheme MG-FEM. Section 4 contains error estimates for the streamlined diffusion f.e.m. and its modifications MSD.1 and MSD.2. In Section 5 sufficient conditions on the limit problem $(\mathscr{L}_0)$ which guarantee uniform in ε error estimates are given. Concluding remarks are contained in Section 6.

2. ESTIMATES FOR THE CONTINUOUS PROBLEM

We use the conventional Sobolev spaces $W_p^k(G)$ on $G \subseteq \Omega$ and their norms (or seminorms) $\| \cdot \|_{k,p,G}$ (or $| \cdot |_{k,p,G}$). $\overset{\circ}{W}{}_p^k(G)$ denotes the closure of $C_0^{\infty}(G)$ in the norm of W_p^k. Further let $| \cdot |_{k,p,\Gamma'}$ be the norm on $W_p^k(\Gamma')$, $\Gamma' \subseteq \partial\Omega$. $(\cdot , \cdot)_G$ denotes the inner product in $L_2(G) = W_2^0(G)$. If there is no doubt we omit the index Ω.

We shall consider the convection-reaction-diffusion problem

$$(\mathscr{L}_{\varepsilon}) \qquad \begin{cases} L_{\varepsilon} u_{\varepsilon} = - \varepsilon \, \Delta u_{\varepsilon} + b \cdot \nabla u_{\varepsilon} + c u_{\varepsilon} = f & \text{in} \quad \Omega \subset \mathbb{R}^N \\ u = g_1 \quad \text{on} \quad \Gamma_1, \quad \varepsilon \dfrac{\partial u_{\varepsilon}}{\partial \nu} + \sigma (u_{\varepsilon} - g_2) = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

where some scalar field u_{ε} (concentration of heat or a chemical) is driven by a velocity field b. ε is the diffusion parameter with $0 \leqq \varepsilon \leqq 1$, ν is the outward pointing normal vector on $\partial\Omega$ and

$$\Gamma_{\substack{- \\ (+)}} = \left\{ x \in \partial\Omega \mid \exists \nu(x) \, , \, (b \cdot \nu)(x) \underset{(>)}{<} 0 \right\} ,$$

$$\Gamma_0 = \{ x \in \partial\Omega \mid \exists \nu(x) \, , \, (b \cdot \nu)(x) = 0 \}$$

are the inflow (outflow) and characteristic parts, respectively, of $\partial\Omega$.
For the given data we assume :

(H.1) $\Omega \subset \mathbb{R}^N$ bounded domain with the boundary $\partial\Omega \subset C^2$ or a polyhedron

$$\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2\,, \quad \Gamma_1 \cap \Gamma_2 = \varnothing\,.$$

(H.2) $b \in [C^1(\bar{\Omega})]^N$, $c \in L_\infty(\Omega)$, $f \in L_2(\Omega)$; $g_i \in L_\infty(\Gamma_i)$, $i = 1, 2$, $\sigma \in L_\infty(\Gamma_2)$ with $0 \leqq \sigma(x, \varepsilon) \leqq C\varepsilon^\tau$, $\tau \geqq 0$.

(H.3)$^+$ $\exists \tilde{g} \in W_2^2(\Omega)$ with $\tilde{g} = g_1$ on Γ_1, $\varepsilon \dfrac{\partial \tilde{g}}{\partial \nu} + \sigma(\tilde{g} - g_2) = 0$ on Γ_2.

(H.4)$_0$ $\inf\limits_{\Omega} \left\{c(x) - \dfrac{1}{2}(\nabla \cdot b)(x)\right\} \geqq \alpha_0 > 0$.

(H.5) $\bar{\Gamma}_- \subseteq \Gamma_1$.

Let $W^\varepsilon = W_2^1(\Omega)$ if $\varepsilon > 0$, $W^0 = \{w \in L_2(\Omega) \mid L_0 w \in L_2(\Omega)\}$, $\Sigma_\varepsilon = \Gamma_1$ if $\varepsilon > 0$ and $\Sigma_0 = \Gamma_-$. Further let $V^\varepsilon = \{w \in W^\varepsilon \mid w = g_1 \text{ on } \Sigma_\varepsilon\}$ and $\mathring{V}^\varepsilon = \{w \in W^\varepsilon \mid w = 0 \text{ on } \Sigma_\varepsilon\}$.

The variational formulation of $(\mathscr{L}_\varepsilon)$ is

$$u_\varepsilon \in V^\varepsilon : \quad B_{\varepsilon,\sigma}(u_\varepsilon, v) = l_\sigma(v) \quad \forall v \in \mathring{V}^\varepsilon \tag{2.1}$$

where

$$B_{\varepsilon,\sigma}(u, v) = \varepsilon(\nabla u, \nabla v) + (b \cdot \nabla u + cu, v) + \int_{\Gamma_2} \sigma uv\, ds \tag{2.2}$$

$$l_\sigma(v) = (f, v) + \int_{\Gamma_2} \sigma g_2 v\, ds\,. \tag{2.3}$$

With respect to the weighted norm $|||\,.\,|||_{\varepsilon,0}$ defined by

$$|||u|||_{\varepsilon,0} = \left(\varepsilon \|\nabla u\|_{0,2}^2 + \alpha_0 \|u\|_{0,2}^2 + \int_{\Gamma_2} \sigma u^2\, ds + \frac{1}{2}\int_{\Gamma_+} (b \cdot \nu) u^2\, ds\right)^{1/2} \tag{2.4}$$

we obtain using Green's formula and standard inequalities.

LEMMA 2.1 : *Under the hypotheses* (H.1)-(H.5) *it holds for any* $u, v \in W_2^1(\Omega)$ *and* $0 \leqq \varepsilon \leqq 1$.

(i) $$B_{\varepsilon,\sigma}(u, u) \geqq |||u|||_{\varepsilon,0}^2 - \frac{1}{2}\int_{\Gamma_-} |b \cdot \nu|\, u^2\, ds\,.$$

(ii) $$|B_{\varepsilon,\sigma}(u, v)| \leqq C\left(|||u|||_{\varepsilon,0}\, |||v|||_{\varepsilon,0} + \|u\|_{0,2}\, \|b \cdot \nabla v\|_{0,2} + \left|\int_{\Gamma_-} |b \cdot \nu|\, uv\, ds\right|\right)\,. \quad \blacksquare$$

As a consequence of Lemma 2.1 (i) there exists an unique solution u_ε of (2.1), $0 \leqq \varepsilon \leqq 1$.

Let us now assume a sharper hypothesis as $(\mathrm{H}.3)^+$.
(H.3) The solution of (2.1) belongs to $W_2^2(\Omega)$ and satisfies

$$\varepsilon |u_\varepsilon|_{2,2} \leqq C\left(\|L_\varepsilon u_\varepsilon\|_{0,2} + \|u_\varepsilon\|_{1,2}\right).$$

Remark 2.1 : Consider the following imbedded form of (2.1) with $0 \leqq \delta \leqq 1$

$$(2.1)_\delta \qquad u \in V^\varepsilon : \quad B^\delta_{\varepsilon,\sigma}(u, v) = l^\delta_\sigma(v) \quad \forall v \in \mathring{V}^\varepsilon$$

with

$$(2.2)_\delta \quad B^\delta_{\varepsilon,\sigma}(u, v) = \varepsilon(\nabla u, \nabla v) - \varepsilon\delta \sum_{i=1}^{I} (\Delta u, b \cdot \nabla v)_{\Omega_i} + $$
$$+ (L_0 u, v + \delta b \cdot \nabla v) + \int_{\Gamma_2} \sigma u v \, ds .$$

$$(2.3)_\delta \qquad l^\delta_\sigma(v) = (f, v + \delta b \cdot \nabla v) + \int_{\Gamma_2} \sigma g_2 v \, ds$$

where $\bar{\Omega} = \bigcup_{i=1}^{I} \bar{\Omega}_i$, $\Omega_i \cap \Omega_j = \varnothing$ if $i \neq j$. Under our hypotheses the solutions of (2.1) and $(2.1)_\delta$ are identical. ∎

Further we have the following a priori estimates.

LEMMA 2.2 : *Under the hypotheses* (H.1)-(H.5) *it holds for the solution* u_ε *of* (2.1) *with a constant* $K(f, \tilde{g})$ *independent on* ε.

$$\text{(i)} \qquad \varepsilon^{3/2} |u_\varepsilon|_{2,2} + |\!|\!| u_\varepsilon |\!|\!|_{\varepsilon,0} \leqq K(f, \tilde{g}) .$$

In case of $u_\varepsilon \in W_2^{l+1}(\Omega)$, *it is*

$$\text{(ii)} \qquad \varepsilon^{l+1/2} |u_\varepsilon|_{l+1,2} \leqq K(f, \tilde{g}), \quad l \geqq 1 . \ ∎$$

Proof: The estimate $|\!|\!| u_\varepsilon |\!|\!|_{\varepsilon,0} \leqq K(f, \tilde{g})$ follows from Lemma 2.1 with foregoing homogenization of $(\mathscr{L}_\varepsilon)$ by the aid of $(\mathrm{H}.3)^+$. Hence the assertion (i) follows by (H.3).

The second assertion follows by induction. ∎

Remark 2.2 : The estimates of Lemma 2.2 are sharp in the case $\Gamma_+ \cap \Gamma_1 \neq \varnothing$. ∎

In the singularly perturbed case $0 \leqq \varepsilon \leqq \varepsilon_0 \ll 1$, the properties of the solution u_ε of $(\mathscr{L}_\varepsilon)$ are depending on the properties of the solution u_0 of the limit problem $(\mathscr{L}_0)$. It holds

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u_0\|_{0,2} = 0$$

if $\partial\Omega = \Gamma_1$ (Bardos-Rauch [3], Th. 1). Note that we cannot ensure in general that $u_0 \in W_2^1(\Omega)$. In regard of uniform in ε discretization error estimates we need the following sharpened regularity and convergence hypotheses on the solutions of $(\mathscr{L}_\varepsilon)$ and $(\mathscr{L}_0)$, respectively.

(H.6) $$\exists r \in N_0 : \quad u_0 \in W_p^{r+1}(\Omega) , \quad 2 \leqq p \leqq +\infty .$$

(H.7) $$\exists \beta > 0 : \quad \|u_\varepsilon - u_0\|_{0,2} \leqq K(f, \tilde{g})\, \varepsilon^\beta$$

with a constant $K(f, \tilde{g})$ independent on ε.

Sometimes it holds even

(H.8) $$\exists \tilde{\beta} > 0 : \quad |\!|\!| u_\varepsilon - u_0 |\!|\!|_{\varepsilon,0} \leqq K(f, \tilde{g})\, \varepsilon^{\tilde{\beta}}$$

with a constant $K(f, \tilde{g})$ independent on ε.

The question of sufficient conditions for (H.6)-(H.8) will be discussed in Section 5.

For subdomains $G \subseteq \Omega$ we denote by $(\partial G)_-$, $(\partial G)_+$ and $(\partial G)_0$, respectively, the inflow, outflow and characteristic parts of ∂G, respectively. In preparation of interior discretization error estimates, we give the following result for subdomains of special type. First of all, we need some notations. Let $\xi_x(\tau)$ be the solution of

$$\dot{\xi}(\tau) = b(\xi(\tau)) , \quad \xi(0) = x \in \bar{\Omega}$$

(the "streamline" passing through $x \in \bar{\Omega}$). For any point $x \in \Omega \cup \bar{\Gamma}_-$

$$\tau_+(x) = \inf \{\tau > 0 \,|\, \xi_x(\tau) \notin \Omega\}$$

denotes the first exit time of $\xi_x(\tau)$ from Ω. Let for $\Gamma' \subseteq \bar{\Gamma}_-$ be $E(\Gamma') = \{\xi_x(\tau) \,|\, x \in \Gamma', \ 0 \leqq \tau \leqq \tau_+(x)\}$.

DEFINITION : *A domain Ω is of "channel type" if $\bar{\Omega} = E(\bar{\Gamma}_-)$. In particular, this property implies that all streamlines $\xi_x(\tau)$, $x \in \bar{\Omega}$ leave Ω in finite time.*

The result announced above is :

LEMMA 2.3 : *Let Ω_i, $i = 1, 2$ be simply connected domains of "channel type" with $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$. Further let*

$$\operatorname{dist}\left((\partial\Omega_1)_+, (\partial\Omega_2)_+\right) \geqq C_1(s)\, q\, |\ln q|\,,$$
$$\operatorname{dist}\left((\partial\Omega_1)_0, (\partial\Omega_2)_0\right) \geqq C_2(s)\, \sqrt{q}\, |\ln \sqrt{q}\,|$$

and $b \cdot \nu \Big|_{\underset{(-)}{(\partial\Omega_1)_+}} \underset{(\leqq)}{\geqq} b_0 \underset{(<)}{>} 0$. *Then it holds with $q = \varepsilon$ and constants $K(f, \tilde{g})$ independent on ε*

(i) $$\|u_\varepsilon\|_{l+1, 2, \Omega_1} \leqq K(f, \tilde{g}) \quad if \quad u_\varepsilon \in W_2^{l+1}(\Omega_2)\,.$$

(ii) $$\|u_\varepsilon\|_{0, 2, \Omega_1} \leqq K(f, \tilde{g})\varepsilon^s \quad if \quad L_\varepsilon u_\varepsilon = 0 \quad in \quad \Omega_2\,.$$

(iii) $$\|u_\varepsilon - u_0\|_{0, p, \Omega_1} \leqq K(f, \tilde{g})\, \varepsilon \quad with \quad 2 \leqq p \leqq +\infty$$

if $u_\varepsilon, u_0 \in W_\infty^2(\Omega_2)$. ■

Proof: Assertion (i) holds in case of $\partial\Omega = \Gamma_1$ (*cf.* Nävert [14], Th. 2.3). We can generalize the proof given there to our case since another structure of the boundary values on $\partial\Omega \setminus \bar{\Gamma}_-$ is not essential.

(ii) We use a modification of the proof of Lemma 2.2 in Schatz-Wahlbin [18] in the case $b \equiv 0$. The proof is somewhat technical and therefore omitted.

(iii) We can generalize the proof given in Goering *et al.* [7], Theorem 4.3 using a generalized maximum principle (*cf.* Lemma 5.3). ■

3. AN ASYMPTOTICALLY FITTED GALERKIN FINITE ELEMENT METHOD

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of $\bar{\Omega}$ with (for simplicity) $\bar{\Omega} = \bigcup_{i=1}^{I(h)} \bar{\tau}_i$. Let $S_h \subset W_\infty^1(\Omega)$ be a finite element space of piecewise polynomials of degree k satisfying

(A.1) (inverse property)

$$\forall v \in S_h, \forall \tau_i \in \mathcal{T}_h: \quad \|v\|_{m, 2, \tau_i} \leqq C h^{-1} \|v\|_{m-1, 2, \tau_i}\,, \quad m \geqq 1\,.$$

(A.2) (approximation property)

$$\forall v \in W_2^{l+1}(\Omega)\,, \quad 1 \leqq l \leqq k \quad \text{with} \quad v = g \quad \text{on} \quad \Gamma' \subseteq \partial\Omega$$
$$\exists \Pi_h v \in S_h\,, \quad \Pi_h v = g_h \quad \text{on} \quad \Gamma' \quad \text{with} \quad |g_h|_{0, \infty, \Gamma'} \leqq |g|_{0, \infty, \Gamma'}$$

such that

$$h^2\|v-\Pi_h v\|_{2,2,h}+h\|v-\Pi_h v\|_{1,2}+\|v-\Pi_h v\|_{0,2}+ \\ +\sqrt{h}|v-\Pi_h v|_{0,2,\Gamma'}\leqq Ch^{l+1}|v|_{l+1,2}$$

where

$$\|\cdot\|_{2,2,h}=\left(\sum_{i=1}^{I(h)}\|\cdot\|_{2,2,\tau_i}^2\right)^{1/2}.$$

Let $V_h^{\varepsilon}=\{w\in S_h\,|\,w=g_{1,h}\text{ on }\Sigma_\varepsilon\}$ with $g_{1,h}=\Pi_h g_1$ and $\mathring{V}_h^{\varepsilon}=S_h\cap\mathring{V}^{\varepsilon}$. The standard Galerkin finite element method (SG-FEM) is

$$\text{(3.1)}\qquad u_{\varepsilon,h}\in V_h^{\varepsilon}:\quad B_{\varepsilon,\sigma}(u_\varepsilon-u_{\varepsilon,h},v)=0\quad\forall v\in\mathring{V}_h^{\varepsilon}$$

and it holds

LEMMA 3.1 : *Suppose that the hypotheses* (H.1)-(H.5) *and* (A.1), (A.2) *are valid. Then we have the bounds for the scheme* SG-FEM

(i) $$|||u_\varepsilon-u_{\varepsilon,h}|||_{\varepsilon,0}\leqq C\inf_{w\in V_h^{\varepsilon}}\left\{|||u_\varepsilon-w|||_{\varepsilon,0}+\frac{1}{\sqrt{\varepsilon}}\|u_\varepsilon-w\|_{0,2}\right\}.$$

(ii) $$|||u_\varepsilon-u_{\varepsilon,h}|||_{0,2}\leqq C\frac{h}{\varepsilon}\left(1+\frac{h}{\varepsilon}\right)|||u_\varepsilon-u_{\varepsilon,h}|||_{\varepsilon,0}.$$

Moreover if $u_\varepsilon\in W_2^{l+1}(\Omega)$, $1\leqq l\leqq k$ *and* $\varepsilon\geqq h$, *we find*

$$\|u_\varepsilon-u_{\varepsilon,h}\|_{0,2}+\frac{h}{\varepsilon}|||u_\varepsilon-u_{\varepsilon,h}|||_{\varepsilon,0}\leqq C\frac{h^{l+1}}{\sqrt{\varepsilon}}|u_\varepsilon|_{l+1,2}.\quad\blacksquare$$

Proof: It is a straightforward generalization of the result given in Nävert [14] for the special case $\partial\Omega=\Gamma_1$ using Lemma 2.1 and Aubin's duality trick. ■

Remark 3.1 : Lemma 2.2 implies that the estimates in Lemma 3.1 are uniformly valid on $h^\kappa\leqq\varepsilon\leqq 1$, $0\leqq\kappa<1$

$$\sup_{h^\kappa\leqq\varepsilon\leqq 1}\left(\|u_\varepsilon-u_\varepsilon,h\|_{0,2}+\frac{h}{\varepsilon}|||u_\varepsilon-u_{\varepsilon,h}|||_{\varepsilon,0}\right)\leqq K(f,\tilde{g})\,h^{(l+1)(1-\kappa)}.\quad\blacksquare$$

Remark 3.2 : For the scheme SG-FEM we obtain with respect to the adapted norm $||||\,.\,||||$ (*cf.* Schieweck [19]) defined by

$$||||u||||=|||u|||_{\varepsilon,0}+\sup_{0\neq w\in V_h^{\varepsilon}}\frac{B_{\varepsilon,\sigma}(u,w)}{|||w|||_{\varepsilon,0}}$$

the following error estimate

$$(3.2) \qquad |||\, u_\varepsilon - u_{\varepsilon,h} \,|||_{\varepsilon,0} \leqq C \inf_{w \in V_h^\varepsilon} ||||\, u_\varepsilon - w \,|||| \,. \quad \blacksquare$$

Exponentially fitted schemes with adding boundary layer-like functions to the trial space are one way to obtain useful error estimates in case of $0 \leqq \varepsilon \ll 1$ (*cf.* Schieweck [19] with $N \leqq 2$). Because of the difficulties arising for such schemes in case of a more complex geometry, we propose another modified Galerkin scheme. Let $\bar{u}_\varepsilon$ be the solution of $(\mathscr{L}_\varepsilon)$ in the special case $\Gamma_- = \Gamma_1,\ \Gamma_2 = \Gamma_+ \cup \Gamma_0,\ \sigma \equiv g_2 \equiv 0$:

$$(3.3) \quad \bar{u}_\varepsilon \in W_2^1(\Omega) \cap V^0 : \quad B_{\varepsilon,0}(\bar{u}_\varepsilon, v) = (f, v) \quad \forall v \in W_2^1(\Omega) \cap \mathring{V}^0$$

and let $\bar{u}_{\varepsilon,h}$ be the projection corresponding to SG-FEM

$$(3.4) \qquad \bar{u}_{\varepsilon,h} \in V_h^0 : \quad B_{\varepsilon,0}(\bar{u}_\varepsilon - \bar{u}_{\varepsilon,h}, v) = 0 \quad \forall v \in \mathring{V}_h^0 \,.$$

Then we define the modified Galerkin scheme MG-FEM

$$(\text{MG-FEM}) \qquad u_h^{(M)} = \begin{cases} u_{\varepsilon,h} \text{ defined by (3.1) if } h^\kappa \leqq \varepsilon \leqq 1 \\ \bar{u}_{\varepsilon,h} \text{ defined by (3.4) if } 0 \leqq \varepsilon \leqq h^\kappa \,. \end{cases}$$

We obtain the following result.

LEMMA 3.2 : *Under the hypotheses* (H.1)-(H.5) *and* (A.1), (A.2) *it holds for the scheme* MG-FEM *with* $0 \leqq \varepsilon < h^\kappa,\ \kappa > 0$

$$(3.5) \quad \left\{ \begin{array}{l} \| u_\varepsilon - u_h^{(M)} \|_{0,2} \\ \\ |||\, u_\varepsilon - u_h^{(M)} \,|||_{\varepsilon,0} \end{array} \right. \leqq \left\{ \begin{array}{l} \| u_\varepsilon - \bar{u}_\varepsilon \|_{0,2} \\ \\ |||\, u_\varepsilon - \bar{u}_\varepsilon \,|||_{\varepsilon,0} \end{array} \right. + $$

$$+ C \left(||||\, \bar{u}_\varepsilon - u_0 \,|||| + \inf_{w \in V_h^0} ||||\, u_0 - w \,|||| \right) .$$

Moreover, if (H.6), (H.7) *with* $r \geqq 1$ *and* (H.6), (H.8) *with* $r \geqq 1$, *respectively, are satisfied, we have the bounds*

$$\text{(i)} \qquad \| u_\varepsilon - u_h^{(M)} \|_{0,2} \leqq K(f, \tilde{g})\, (h^r + \varepsilon^{\min\{1/2\,;\,\beta\}})$$

and

$$\text{(ii)} \qquad |||\, u_\varepsilon - u_h^{(M)} \,|||_{\varepsilon,0} \leqq K(f, \tilde{g})\, (h^r + \varepsilon^{\min\{1/2\,;\,\tilde{\beta}\}}) \,,$$

respectively. ■

Proof: (3.5) follows from triangle inequality and (3.2). By Lemma 5.2 we have

$$(3.6) \qquad ||||\, \bar{u}_\varepsilon - u_0 \,|||| \leqq C\,\sqrt{\varepsilon}\,|u_0|_{1,2}\,.$$

Further it is

$$||||\, u_0 - w \,|||| \leqq |||\, u_0 - w \,|||_{\varepsilon,0} + C\,\|u_0 - w\|_{1,2}$$

and with the approximation property (A.2)

$$(3.7) \qquad \inf_{w \in V_h^0} ||||\, u_0 - w \,|||| \leqq C h^r |u_0|_{r+1,2}\,.$$

The assertions (i), (ii) are obtained combining (3.5)-(3.7) and (H.6), (H.7) or (H.6), (H.8). ∎

Regarding uniform in ε discretization error estimates we obtain now from Remark 3.1 and Lemma 3.2.

THEOREM 3.1 : *Let the hypotheses of Lemma* 3.2 *be satisfied and let* $u_\varepsilon \in W_2^{l+1}(\Omega)$ *and* $u_0 \in W_2^{r+1}(\Omega)$, $1 \leqq l \leqq k$, $1 \leqq r \leqq k$ *be the solutions of* $(\mathscr{L}_\varepsilon)$ *and* $(\mathscr{L}_0)$, *respectively. Then it holds for the scheme* MG-FEM

$$\text{(i)} \quad \|u_\varepsilon - u_h^{(M)}\|_{0,2} \leqq K(f, \tilde{g}) \min \left\{ \left(\frac{h}{\varepsilon}\right)^{l+1} ; h^r + \varepsilon^{\min\{1/2\,;\,\beta\}} \right\}$$

with $\kappa = \kappa_1 = \dfrac{l+1}{l+1+\min\{1/2\,;\,\beta\}}$ *if* (H.6), (H.7) *are valid.*

$$\text{(ii)} \quad |||\, u_\varepsilon - u_h^{(M)} \,|||_{\varepsilon,0} \leqq K(f, \tilde{g}) \min \left\{ \left(\frac{h}{\varepsilon}\right)^{l} ; h^r + \varepsilon^{\min\{1/2\,;\,\tilde{\beta}\}} \right\}$$

with $\kappa = \kappa_2 = \dfrac{l}{l+\min\{1/2\,;\,\tilde{\beta}\}}$ *if* (H.6), (H.8) *are valid.* ∎

Conclusion 3.1 : Theorem 3.1 implies the uniform in ε estimates with the corresponding assumptions of Theorem 3.1 (i), (ii)

$$\text{(i)} \qquad \sup_{0 \leqslant \varepsilon \leqslant 1} \|u_\varepsilon - u_h^{(M)}\|_{0,2} \leqq K(f, \tilde{g})\, h^{\kappa_1 \min\{1/2\,;\,\beta\}}$$

$$\text{(ii)} \qquad \sup_{0 \leqslant \varepsilon \leqslant 1} |||\, u_\varepsilon - u_h^{(M)} \,|||_{\varepsilon,0} \leqq K(f, \tilde{g})\, h^{\kappa_2 \min\{1/2\,;\,\tilde{\beta}\}}\,. \quad ∎$$

4. ASYMPTOTICALLY FITTED STREAMLINED DIFFUSION FINITE ELEMENT SCHEMES

We consider now the streamlined diffusion scheme (SD-FEM) of Hughes-Brooks [8] starting from the imbedded form $(2.1)_\delta$ with $0 \leqq \varepsilon,\ \delta \leqq 1$.

$$(4.1) \qquad u^\delta_{\varepsilon,h} \in V^\varepsilon_h : \quad B^\delta_{\varepsilon,\sigma}(u_\varepsilon - u^\delta_{\varepsilon,h}, v) = 0 \quad \forall v \in \overset{\circ}{V}{}^\varepsilon_h .$$

Additionally we assume

$$(\mathrm{H}.4)_\delta \qquad \forall \delta \in [0\,;1] : \quad \varepsilon\delta \leqq Ch^2 , \quad \inf_{\partial\Omega} \{1 + \delta c(x)\} \geqq c_\delta \geqq 0$$

$$\inf_{\Omega} \left\{ c(x) - \frac{1}{2}(\nabla \cdot b)(x) - \frac{\delta}{2}(\nabla \cdot bc)(x) \right\} \geqq \alpha_\delta > 0 .$$

Further let $|||\,.\,|||_{\varepsilon,\delta}$ be the norm defined by

$$(4.2) \quad |||\,u\,|||_{\varepsilon,\delta} = \Big(\varepsilon \|\nabla u\|^2_{0,2} + \delta \|b \cdot \nabla u\|^2_{0,2} + \alpha_\delta \|u\|^2_{0,2} + + \int_{\Gamma_2} \sigma u^2 \, ds + \int_{\Gamma_+} (b \cdot \nu)(1 + \delta c)\, u^2 \, ds \Big)^{1/2} .$$

Remark 4.1 : The term $\sqrt{\delta}\|b \cdot \nabla u\|_{0,2}$ in (4.2) represents an essential effect of the scheme SD-FEM with $\delta > 0$. There is additionally control on the derivative $b \cdot \nabla u$. ■

First of all, we prove S_h-ellipticity and continuity of the bilinearform $B^\delta_{\varepsilon,\sigma}(\cdot,\cdot)$.

LEMMA 4.1 : *Under the assumptions* (H.1)-(H.5) *and* (A.1), (A.2) *it holds form the scheme* SD-FEM *with* $0 \leqq \delta \leqq 1$

(i) $\forall u_h \in S_h : \quad B^\delta_{\varepsilon,\sigma}(u_h, u_h) \geqq \frac{1}{2} |||\,u_h\,|||^2_{\varepsilon,\delta} + \frac{c_\delta}{2} \int_{\Gamma_-} (b \cdot \nu)\, u_h^2 \, ds$

(ii) $\forall u, v \in W^1_2(\Omega)$:

$$|B^\delta_{\varepsilon,\sigma}(u, v)| \leqq C_1 |||\,u\,|||_{\varepsilon,\delta}\, |||\,v\,|||_{\varepsilon,\delta} + C_2(\varepsilon\delta \|u\|_{2,2,h} + \|u\|_{0,2}) \times \|b \cdot \nabla v\|_{0,2} + \left(\int_{\Gamma_-} |b \cdot \nu|\, u^2 \, ds \right)^{1/2} \left(\int_{\Gamma_-} |b \cdot \nu|\, v^2 \, ds \right)^{1/2} . \quad ■$$

Proof: It is a straightforward generalization of the result in Johnson *et al.* [10] for the special case $\partial\Omega = \Gamma_1$. ■

A first consequence is the stability result.

LEMMA 4.2 : *Under the hypotheses of Lemma* 4.1 *there is a unique solution* $u^{\delta}_{\varepsilon, h} \in S_h$ *of the scheme* SD-FEM, $0 \leqq \delta \leqq 1$ *satisfying*

$$|||u^{\delta}_{\varepsilon, h}|||_{\varepsilon, \delta} \leqq C(\|f\|_{0,2} + \|\Pi_h \tilde{g}\|_{1,2} + \\ + |\sqrt{\sigma} g_2|_{0,2,\Gamma_2} + |\sqrt{|b \cdot \nu|}\, g_{1,h}|_{0,2,\Gamma_-}) . \quad \blacksquare$$

The following error estimates generalize the results given in Johnson et al. [10] for $\partial\Omega = \Gamma_1$ and Axelsson [2] for $\Gamma_1 = \Gamma_-$, $\Gamma_2 = \Gamma_0 \cup \Gamma_+$, $\sigma \equiv 0$.

LEMMA 4.3 : *Under the assumptions of Lemma* 4.1 *the solutions* $u^{\delta}_{\varepsilon, h}$ *of* SD-FEM *with* $\delta > 0$ *and* u_ε *of* $(\mathscr{L}_\varepsilon)$, *respectively, satisfy*

$$(4.3) \qquad |||u_\varepsilon - u^{\delta}_{\varepsilon, h}|||_{\varepsilon, \delta} \leqq C \inf_{w \in V^{\varepsilon}_h} \left\{ |||u_\varepsilon - w|||_{\varepsilon, \delta} + \varepsilon\sqrt{\delta}\,\|u_\varepsilon - w\|_{2,2,h} + \\ + \frac{1}{\sqrt{\delta}}\|u_\varepsilon - w\|_{0,2} \right\} . \quad \blacksquare$$

Together with approximation property

$$|||u - \Pi_h u|||_{\varepsilon, \delta} \leqq Ch^l(\sqrt{\varepsilon} + \sqrt{\delta} + \sqrt{h})\,|u|_{l+1,2}$$

Lemma 4.3 implies :

Conclusion 4.1 : Let the assumptions of Lemma 4.3 be valid. Moreover, if $u_\varepsilon \in W^{l+1}_2(\Omega)$, $1 \leqq l \leqq k$ holds with $\delta = Ch \geqq \varepsilon \geqq 0$, we have the bound

$$(4.4) \qquad |||u_\varepsilon - u^{\delta}_{\varepsilon, h}|||_{\varepsilon, h} \leqq Ch^{l+1/2}|u_\varepsilon|_{l+1,2} . \quad \blacksquare$$

Remark 4.2 : According to Lemma 2.2 the estimates of Conclusion 4.1 are not uniformly valid on $0 \leqq \varepsilon \leqq 1$ in the general case setting $\delta = 0$ if $\varepsilon \geqq Ch$. ■

Regarding discretization error estimates, uniformly on $0 \leqq \varepsilon \leqq 1$, we propose asymptotically fitted schemes of the following kind. For $\varepsilon \geqq h^{\kappa}$ we solve $(\mathscr{L}_\varepsilon)$ by the scheme SG-FEM. In case of $0 \leqq \varepsilon < h^{\kappa}$, we solve in a first variant the limit problem $(\mathscr{L}_0)$ instead of $(\mathscr{L}_\varepsilon)$ by SD-FEM

$$(4.5) \qquad u^{\delta}_{0,h} \in V^0_h : \quad B^{\delta}_{0,0}(u - u^{\delta}_{0,h}, v) = 0 \quad \forall v \in \mathring{V}^0_h .$$

A second variant is analogously to scheme MG-FEM. We solve problem (3.3) instead of $(\mathscr{L}_\varepsilon)$ by SD-FEM

$$(4.6) \qquad \bar{u}^{\delta}_h \in V^0_h : \quad B^{\delta}_{\varepsilon,0}(\bar{u}_\varepsilon - \bar{u}^{\delta}_h, v) = 0 \quad \forall v \in \mathring{V}^0_h .$$

Thus, we define the modified streamlined diffusion schemes

$$\text{(MSD.1)} \qquad u_h^{(S1)} = \begin{cases} u_{\varepsilon,h} & \text{defined by (3.1) if } h^{\kappa} \leqq \varepsilon \leqq 1 \\ u_{0,h}^{\delta} & \text{defined by (4.5) if } 0 \leqq \varepsilon < h^{\kappa} \end{cases}$$

and

$$\text{(MSD.2)} \qquad u_h^{(S2)} = \begin{cases} u_{\varepsilon,h} & \text{defined by (3.1) if } h^{\kappa} \leqq \varepsilon \leqq 1 \\ \bar{u}_h^{\delta} & \text{defined by (4.6) if } 0 \leqq \varepsilon < h^{\kappa} . \end{cases}$$

Because of definition of the adapted schemes, we have to perform the analysis only in the case $0 \leqq \varepsilon < h^{\kappa}$. First of all consider scheme (MSD.1).

LEMMA 4.4 : *Under the hypotheses of Lemma* 4.1 *it holds for the scheme* MSD.1 *with* $0 \leqq \varepsilon < h^{\kappa}$, $0 < \kappa < 1$, $\delta = Ch$

$$\text{(4.7)} \qquad \left\{ \begin{array}{l} \| u_\varepsilon - u_h^{(S1)} \|_{0,2} \\ ||| u_\varepsilon - u_h^{(S1)} |||_{0,\delta} \end{array} \right. \leqq \left\{ \begin{array}{l} \| u_\varepsilon - u_0 \|_{0,2} \\ ||| u_\varepsilon - u_0 |||_{0,\delta} \end{array} \right. + \\ + C \inf_{w \in V_h^0} \left\{ ||| u_0 - w |||_{0,h} + \frac{1}{\sqrt{h}} \| u_0 - w \|_{0,2} \right\} .$$

Moreover, if (H.6), (H.7) *and* (H.6), (H.8), *respectively, are valid, it holds*

$$\text{(i)} \qquad \| u_\varepsilon - u_h^{(S1)} \|_{0,2} \leqq K(f, \tilde{g})(h^{r+1/2} + \varepsilon^{\beta})$$

and

$$\text{(ii)} \qquad ||| u_\varepsilon - u_h^{(S1)} |||_{0,\varepsilon^{1/\kappa}} \leqq K(f, \tilde{g})(h^{r+1/2} + \varepsilon^{\tilde{\beta}}) ,$$

respectively. ■

Proof: (4.7) follows from triangle inequality and Lemma 4.3 with $\varepsilon = 0$. Assertion (i) is now a consequence of (4.7), (4.3) with $\varepsilon = 0$ and (H.6), (H.7). Further it holds for $0 \leqq \varepsilon < h^{\kappa}$ with $0 < \kappa < 1$

$$\text{(4.8)} \qquad ||| u_\varepsilon - u_0 |||_{0,\varepsilon^{1/\kappa}} \leqq ||| u_\varepsilon - u_0 |||_{0,\varepsilon} \leqq C ||| u_\varepsilon - u_0 |||_{\varepsilon,0} \\ \leqq K(f, \tilde{g})\, \varepsilon^{\tilde{\beta}}$$

and

$$\text{(4.9)} \qquad ||| u_0 - u_{0,h}^{\delta} |||_{0,\varepsilon^{1/\kappa}} \leqq ||| u_0 - u_{0,h}^{\delta} |||_{0,h} \leqq C h^{r+1/2} | u_0 |_{r+1,2} .$$

Thus we obtain (ii) combining (4.7)-(4.9). ■

Summarizing Lemma 4.4 and Lemma 3.1, we obtain :

THEOREM 4.1 : *Let the hypotheses of Lemma* 3.2 *be satisfied and let* $u_\varepsilon \in W_2^{l+1}(\Omega)$ *and* $u_0 \in W_2^{r+1}(\Omega)$, $1 \leqq l, r \leqq k$ *be the solutions of* $(\mathscr{L}_\varepsilon)$ *and* $(\mathscr{L}_0)$, *respectively. Then it holds for the streamlined diffusion scheme* MSD.1

$$\text{(i)} \qquad \left\|u_\varepsilon - u_h^{(S1)}\right\|_{0,2} \leqq K(f, \tilde{g}) \min \left\{ \left(\frac{h}{\varepsilon}\right)^{l+1} ; h^{r+1/2} + \varepsilon^\beta \right\}$$

with $\kappa = \kappa_3 = \dfrac{l+1}{l+\beta+1}$ *if* (H.6), (H.7) *are valid and*

$$\text{(ii)} \qquad \left|\!\left|\!\left| u_\varepsilon - u_h^{(S1)} \right|\!\right|\!\right|_{0, \varepsilon^{1/\kappa}} \leqq K(f, \tilde{g}) \min \left\{ \left(\frac{h}{\varepsilon}\right)^{l} ; h^{r+1/2} + \varepsilon^{\tilde{\beta}} \right\}$$

with $\kappa = \kappa_4 = \dfrac{l}{l+\tilde{\beta}}$ *if* (H.6), (H.8) *are valid.* ∎

Conclusion 4.2 : Theorem 4.1 implies the uniform in ε estimates (with the corresponding assumptions of Theorem 4.1 (i) (ii))

$$\text{(i)} \qquad \sup_{0 \leqq \varepsilon \leqq 1} \left\|u_\varepsilon - u_h^{(S1)}\right\|_{0,2} \leqq K(f, \tilde{g})\, h^{\kappa_3 \beta}$$

$$\text{(ii)} \qquad \sup_{0 \leqq \varepsilon \leqq 1} \left\|u_\varepsilon - u_h^{(S1)}\right\|_{0, \varepsilon^{1/\kappa}} \leqq K(f, \tilde{g})\, h^{\kappa_4 \tilde{\beta}} . \quad ∎$$

Considering scheme MSD.2, we obtain the following global error estimate for sufficiently small ε.

THEOREM 4.2 : *Under the hypotheses of Lemma* 4.1 *it holds for the modified streamlined diffusion scheme* MSD.2 *with*

$$0 \leqq \varepsilon \leqq h^\kappa, \quad \kappa > 2 \quad \text{and} \quad \delta = Ch$$

$$\text{(i)} \qquad \left\|u_\varepsilon - u_h^{(S2)}\right\|_{0,2} \leqq K(f, \tilde{g}) \left(h^{r+1/2} + \varepsilon^\beta + \frac{\sqrt{\varepsilon}}{h} \right)$$

if (H.6), (H.7) *are valid and*

$$\text{(ii)} \qquad \left|\!\left|\!\left| u_\varepsilon - u_h^{(S2)} \right|\!\right|\!\right|_{0,\varepsilon} \leqq K(f, \tilde{g}) \left(h^{r+1/2} + \varepsilon^{\tilde{\beta}} + \frac{\sqrt{\varepsilon}}{h} \right)$$

if (H.6), (H.8) *are valid.* ∎

Proof: We have to modify the proof of Lemma 4.4.

$$(4.10) \qquad \left|u_\varepsilon - u_h^{(S2)}\right| \leqq \left|u_\varepsilon - u_0\right| + \left|u_0 - u_{0,h}^\delta\right| + \left|u_{0,h}^\delta - \bar{u}_h^\delta\right| .$$

The first right hand side term is bounded by (H.7) or (H.8). For the second term we obtain from Conclusion 4.1 with $\varepsilon = 0$

$$\text{(4.11)} \qquad |\!|\!| u_0 - u_{0,h}^{\delta} |\!|\!|_{0,h} \leqq C h^{r+1/2} |u_0|_{r+1,2} \,.$$

For the third term we estimate with $v = \bar{u}_h^{\delta} - u_{0,h}^{\delta} \in \mathring{V}_h^0$ and using Lemma 4.1 and inverse properties (A.1)

$$\begin{aligned}
\frac{1}{2} |\!|\!| v |\!|\!|_{0,\delta}^2 \leqq B_{0,0}^{\delta}(v, v) &= B_{\varepsilon,0}^{\delta}(\bar{u}_h^{\delta}, v) - \varepsilon(\nabla \bar{u}_h^{\delta}, \nabla v) + \\
&\quad + \varepsilon\delta \sum_{i=1}^{I(h)} (\Delta \bar{u}_h^{\delta}, b \cdot \nabla v)_{\tau_i} - B_{0,0}^{\delta}(u_{0,h}^{\delta}, v) \\
&= -\varepsilon(\nabla \bar{u}_h^{\delta}, v) + \varepsilon\delta \sum_{i=1}^{I(h)} (\Delta \bar{u}_h^{\delta}, b \cdot \nabla v)_{\tau_i} \\
&\leqq \varepsilon \left\| \nabla \bar{u}_h^{\delta} \right\|_{0,2} \cdot \| \nabla v \|_{0,2} + \varepsilon\delta \left\| \bar{u}_h^{\delta} \right\|_{2,2,h} \| b \cdot \nabla v \|_{0,2} \\
&\leqq \frac{C_\varepsilon}{h} \left\| \bar{u}_h^{\delta} \right\|_{1,2} \| v \|_{0,2} + \frac{C\varepsilon\delta}{h} \left\| \bar{u}_h^{\delta} \right\|_{1,2} \| b \cdot \nabla v \|_{0,2} \\
&\leqq \frac{\alpha_0}{4} \| v \|_{0,2}^2 + \frac{C\varepsilon^2}{h^2} \left\| \bar{u}_h^{\delta} \right\|_{1,2}^2 + \frac{\delta}{4} \| b \cdot \nabla v \|_{0,2}^2 \\
&\quad + \frac{C\varepsilon^2 \delta}{h^2} \left\| \bar{u}_h^{\delta} \right\|_{1,2}^2 \,.
\end{aligned}$$

Hence, we obtain together with Lemma 4.2

$$\frac{1}{4} |\!|\!| v |\!|\!|_{0,\delta}^2 \leqq C \frac{\varepsilon^2}{h^2} (1 + \delta) \left\| \bar{u}_h^{\delta} \right\|_{1,2}^2 \leqq K(f, \tilde{g}) \frac{\varepsilon}{h^2} (1 + \delta)$$

and thus

$$\text{(4.12)} \qquad |\!|\!| \bar{u}_h^{\delta} - u_{0,h}^{\delta} |\!|\!|_{0,\delta} \leqq K(f, \tilde{g}) \frac{\sqrt{\varepsilon}}{h} \,.$$

Summarizing (4.10)-(4.12), we have with (H.6), (H.7) or (H.6), (H.8)

$$\left\| u_\varepsilon - \bar{u}_h^{\delta} \right\|_{0,2} \leqq K(f, \tilde{g}) \left(\varepsilon^{\beta} + h^{r+1/2} + \frac{\sqrt{\varepsilon}}{h} \right)$$

if (H.6), (H.7) are valid
and

$$\begin{aligned}
|\!|\!| u_\varepsilon - \bar{u}_h^{\delta} |\!|\!|_{0,\varepsilon} &\leqq C |\!|\!| u_\varepsilon - u_0 |\!|\!|_{\varepsilon,0} + |\!|\!| u_0 - \bar{u}_h^{\delta} |\!|\!|_{0,h} \\
&\leqq K(f, \tilde{g}) \left(\varepsilon^{\tilde{\beta}} + h^{r+1/2} + \frac{\sqrt{\varepsilon}}{h} \right)
\end{aligned}$$

if (H.6), (H.8) are valid. ■

Conclusion 4.3 : Theorem 4.2 implies in case of $0 \leqq \varepsilon < h^{\kappa}$, $\kappa > 2$ the uniform in ε estimates

$$\text{(i)} \qquad \sup_{0 \leqq \varepsilon \leqq h^{\kappa}} \left\| u_{\varepsilon} - u_h^{(S2)} \right\|_{0,2} \leqq K(f, \tilde{g})\, h^{\min\{r+1/2,\, \kappa/2-1,\, \beta\kappa\}}$$

in case of (H.6), (H.7)

$$\text{(ii)} \qquad \sup_{0 \leqq \varepsilon \leqq h^{\kappa}} |\!|\!| u_{\varepsilon} - u_h^{(S2)} |\!|\!|_{0,\varepsilon} \leqq K(f, \tilde{g})\, h^{\min\{r+1/2,\, \kappa/2-1,\, \tilde{\beta}\kappa\}}$$

in case of (H.6), (H.8). ■

Remark 4.3 : The question of global error estimates, uniformly on $0 \leqq \varepsilon \leqq 1$, for the modified streamlined diffusion scheme MSD.2 is open. ■

Lastly, we consider local discretization error estimates in case of sufficiently small ε, say $0 \leqq \varepsilon < h$. Let additionally be the local interpolation property (A.3) of Nävert [14] be satisfied which is valid in case of Lagrangian triangular elements or tensor products of one-dimensional Lagrangian elements. Nävert [14] proved in case of $\partial\Omega = \Gamma_1$ for subdomains of "channel type" :

LEMMA 4.5 : *Let* $\partial\Omega = \Gamma_1$ *and let the hypotheses of Lemma* 4.1 *and* (A.3) *be valid. Let* Ω_i, $i = 1, 2$ *be subdomains of* Ω *satisfying the hypotheses of Lemma* 2.3 *with* $q = h$. *Then it holds for the streamlined diffusion scheme* SD-FEM, $0 \leqq \varepsilon < h$, $\delta = Ch$ *with* $u_{\varepsilon} \in W_2^{l+1}(\Omega_2)$, $1 \leqq l \leqq k$

$$|\!|\!| u_{\varepsilon} - u_{\varepsilon,h}^{\delta} |\!|\!|_{\varepsilon, h, \Omega_1} \leqq K(f, \tilde{g})\, h^{\min\{s, l+1/2\}} \,. \quad ■$$

By the aid of Lemma 4.5 which is valid also for the problem $(\mathscr{L}_{\varepsilon})$, we obtain for the modified streamlined diffusion schemes MSD.1 and MSD.2.

THEOREM 4.3 : *Under the assumption of Lemma* 4.5 *(without the restriction* $\partial\Omega = \Gamma_1$*) it holds for the solutions* $u_h^{(S1)}$ *and* $u_h^{(S2)}$, *respectively, of scheme* MSD.1 *and* MSD.2, *respectively, and with* $s \geqq l + 1/2$.

$$\text{(i)} \qquad \left\| u_{\varepsilon} - u_h^{(S1)} \right\|_{0,2,\Omega_1} \leqq K(f, \tilde{g})\, (h^{l+1/2} + \varepsilon)$$

$$\text{(ii)} \qquad \left\| u_{\varepsilon} - u_h^{(S2)} \right\|_{0,2,\Omega_1} \leqq K(f, \tilde{g})\, (h^{l+1/2}) \,. \quad ■$$

Proof: Assertion (i) follows from Lemma 2.3 (iii), Lemma 4.5 with $\varepsilon = 0$ and triangle inequality. Assertion (ii) is a consequence of the estimate

$$\left\| u_{\varepsilon} - u_h^{(S2)} \right\|_{0,2,\Omega_1} \leqq \left\| u_{\varepsilon} - \bar{u}_{\varepsilon} \right\|_{0,2,\Omega_1} + \left\| \bar{u}_{\varepsilon} - \bar{u}_h^{\delta} \right\|_{0,2,\Omega_1}$$

where the first right hand side term is bounded by Lemma 2.3 (ii) (because of $L_\varepsilon(u_\varepsilon - \bar{u}_\varepsilon) = 0$) and the second term by Lemma 4.5 with $s \geqq l + 1/2$. ∎

Remark 4.4 : Asume that the inverse property

$$\|v_h\|_{0,\infty,\tau_i} \leqq C h^{-N/2} \|v_h\|_{0,2,\tau_i}, \quad \forall v_h \in S_h, \quad \forall \tau_i \in \mathcal{T}_h$$

holds. Then we obtain the (non-optimal) L_∞-estimate

$$\begin{aligned} \|u_\varepsilon - u_h^{(S2)}\|_{0,\infty,\Omega_1} &\leqq C \left\{ \inf_{w \in V_h^\varepsilon} \|u_\varepsilon - w\|_{0,\infty,\Omega_1} + \right. \\ &\qquad \left. + h^{-N/2} \|u_\varepsilon - u_h^{(S2)}\|_{0,2,\Omega_1} \right\} \\ &\leqq K(f, \tilde{g})\, h^{l + 1/2 - N/2} \end{aligned}$$

if $u_\varepsilon \in C(\bar{\Omega}_1) \cap W_\infty^{l+1}(\Omega_1)$. The boundedness of $|u_\varepsilon|_{l+1,\infty,\Omega_1}$ follows from Lemma 2.3 (i) and Sobolev's imbedding theorem. ∎

5. SOME ASYMPTOTIC ERROR ESTIMATES

It remains to give conditions which are sufficient for the hypotheses (H.6)-(H.8). Clearly, such conditions depend essentially on the asymptotic behaviour of the solution u_ε of $(\mathscr{L}_\varepsilon)$ and on the limit problem $(\mathscr{L}_0)$.

We restrict the considerations to domains of "channel type" or to situations where Γ_- is closed in $\partial\Omega$. More precisely, let one of the following conditions be satisfied :

(C.1) Γ_- closed in $\partial\Omega$; $c(x) \geqq c_0(r)$ with sufficiently large c_0 (estimates of c_0 are given in [7], Th. 3.1).

(C.2) Γ_- closed in $\partial\Omega$; $\exists \eta(x) :\ b \,.\, \nabla\eta \geqq b_0 > 0$ in $\bar{\Omega}$, $c(x) \geqq c_0$ with c_0 arbitrary.

(C.3) Γ_- simply connected ; $\bar{\Omega} = \mathrm{E}(\bar{\Gamma}_-)$, $b \,.\, \nu \neq 0$ on $\bar{\Gamma}_- \cup \bar{\Gamma}_+$, $c(x) \geqslant c_0$ with c_0 arbitrary.

Lemma 5.1 yields sufficient conditions for the regularity assumption (H.6) (*cf.* [7], Sect. 3).

LEMMA 5.1 : *Let the datas of* $(\mathscr{L}_0)$ *and* $\bar{\Gamma}_-$, $\bar{\Gamma}_+$, Γ_0 *be sufficiently smooth. Then any of the conditions* (C.1)-(C.3) *is sufficient for* (H.6). ∎

Remark 5.1 : Condition on η in (C.2) is equivalent to the fact that all streamlines $\xi_x(\tau)$, $x \in \Omega$ leave $\bar{\Omega}$ in finite time (Devinatz-Ellis-Friedman [4]). ■

Remark 5.2 : We find local regularity statements of type (H.6) applying Lemma 5.1 on subdomains $G \subseteq \Omega$, especially in case of $G \subseteq E(\bar{\Gamma}_-)$. ■

Remark 5.3 : Sometimes we need instead of (H.6) the weaker hypothesis. $(\text{H.6})^+$ u_0 is semiregular in Ω (there exists a constant $M > 0$ such that $\Delta u_0 \leq u$ in the sense of $W_2^{-1}(\Omega)$.

Felgenhauer ([5], Th. 5) gives sufficient conditions for $(\text{H.6})^+$ in case of (C.3) and piecewise smooth $\bar{\Gamma}_-$. ■

Consider now the assumptions (H.7) and (H.8) of asymptotic convergence. In case of the special mixed boundary value problem

$$(5.1) \qquad \begin{cases} L_\varepsilon u_\varepsilon = f \quad \text{in} \quad \Omega \subset \mathbb{R}^N \\ u_\varepsilon = g \text{ on } \Gamma_1 = \Gamma_-, \ \varepsilon \dfrac{\partial u_\varepsilon}{\partial \nu} + \sigma u = 0 \text{ on } \Gamma_2 = \Gamma_+ \cup \Gamma_0 \\ \text{with} \quad 0 \leqq \sigma(x, \varepsilon) \leqq C\varepsilon^\tau, \quad \tau > 0 \end{cases}$$

it holds

LEMMA 5.2 : *Under the hypotheses* (H.1)-(H.5) *and* (H.6) *with* $r = 0$, $p = 2$ *the solution of* (5.1) *satisfies the hypotheses* (H.7), (H.8) *with* $\beta = \tilde{\beta} = 1/2 \min \{1 ; \tau\}$. ■

Verification of (H.7), (H.8) is more complicated if essential boundary conditions are given on $\Gamma' \subseteq \Gamma_0 \cup \Gamma_+$. Let us consider for simplicity Dirichlet's problem

$$(5.2) \qquad \begin{cases} L_\varepsilon u_\varepsilon = f & \text{in} \quad \Omega \subset \mathbb{R}^N \\ u_\varepsilon = g & \text{on} \quad \partial\Omega = \Gamma_1 . \end{cases}$$

As a rule, we need some more information about the asymptotic behaviour of u_ε for $0 < \varepsilon \leqq \varepsilon_0 \ll 1$. We discuss two variants.

In case of sufficient smooth datas of $(\mathscr{L}_\varepsilon)$ and of (H.6) with sufficiently large r, there exists an asymptotic expansion

$$u^{as}(x, \varepsilon) = u_0(x) + \sum_{j=1}^{J} v_j(x, \varepsilon)$$

where u_0 denotes the solution of $(\mathscr{L}_0)$ and the v_j are boundary layer terms. If asymptotic error estimates and estimates of the v_j in integrals norms are available we find

$$\|u_\varepsilon - u_0\|_{0,2} \leqq \|u_\varepsilon - u^{as}\|_{0,2} + \sum_{j=1}^{J} \|v_j\|_{0,2}$$

or

$$|||u_\varepsilon - u_0|||_{\varepsilon,0} \leqq |||u_\varepsilon - u^{as}|||_{\varepsilon,0} + \sum_{j=1}^{J} |||v_j|||_{\varepsilon,0} .$$

Remark 5.4 : Let $\Gamma_1 \cap \Gamma_+ \neq \varnothing$. Then there exists an index j such that $|||v_j|||_{\varepsilon,0} \nrightarrow 0$ if $\varepsilon \to 0$. Hence (H.8) cannot be valid. ■

The disadvantages of the first method are the smoothness assumptions and the explicit construction of boundary layer terms. Another way is to use a generalized maximum principle and boundary layer-like barrier functions.

LEMMA 5.3 (*cf.* Felgenhauer [5], Th. 2) : *Under the hypotheses* (H.1)-(H.5) *let be* $v \leqq s$ *on* ∂G. *Further let* $L_\varepsilon v \leqq L_\varepsilon s$ *in the sense of* $W_2^{-1}(G)$, $G \subseteq \Omega$. *Then it holds* $v \leqq s$ a.e. *in* G. ■

Hence we prove (H.7) by the aid of a pointwise valid estimate

$$|(u_\varepsilon - u_0)(x)| \leqq s(x, \varepsilon) \quad \text{a.e.} \quad \text{in } \Omega \tag{5.3}$$

via $\|u_\varepsilon - u_0\|_{0,2} \leqq \|s\|_{0,2.}$ Pointwise estimates of type (5.3) are given in [7], Section 4.1 by the aid of classical maximum principle (*cf.* Protter-Weinberger [16]). Using Lemma 5.3 we can weaken the smoothness assumptions. We omit the somewhat technical details refering to [7] and give only results.

LEMMA 5.4 : *Let* Γ_- *be closed in* $\partial\Omega$ (*cf.* (C.1), (C.2)) *and let* $\bar{\Gamma}_-$, $\bar{\Gamma}_+$, Γ_0 *be sufficiently smooth. Let* $u_0 \in W_\infty^2(\Omega)$ *or let* u_0 *be semiregular in* Ω. *Then one of the following conditions is sufficient for* (H.7) :

(i) $\partial\Omega = \Gamma_-$ (*hence* $\Gamma_+ = \Gamma_0 = \varnothing$) *with* $\beta = 1$.

(ii) $\partial\Omega = \Gamma_- \cup \Gamma_+$ (*hence* $\Gamma_0 = \varnothing$) *with* $\beta = 1/2$.

(iii) *Let for* $F(x) = -\operatorname{dist}(x, \bar{\Gamma}_+ \cup \Gamma_0)$ *in* $U_\gamma(\bar{\Gamma}_+ \cup \Gamma_0) = \{x \in \Omega \mid -F(x) \leqq \gamma\}$ *be* $b \,.\nabla F \geqq 0$. *Then it is* $\beta = 1/4$. ■

LEMMA 5.5 : *Let* (C.3) *be valid and let* $\bar{\Gamma}_+$, Γ_0 *be sufficiently smooth. Let* $u_0 \in W_\infty^2(\Omega)$ *or let* u_0 *be semiregular in* Ω. *Then it holds* (H.7) *with* $\beta = 1/2$ *if* $\Gamma_0 = \varnothing$ *and* $\beta = 1/4$ *else.* ■

Remark 5.5 : In case of a convex polyhedron Ω (or convex Ω with piecewise smooth $\bar{\Gamma}_+ \cup \Gamma_0$), the result of Lemma 5.5 remains valid. ■

Remark 5.6 : In case of interior boundary layers (« shocks ») we can prove estimates of type (H.7), (H.8) in subdomains away from the shock. ■

6. CONCLUDING REMARKS.

Under certain assumptions concerning the asymptotic behaviour of the solution u_ε of $(\mathscr{L}_\varepsilon)$ and the limit solution u_0 of $(\mathscr{L}_0)$ we proved global L_2-error estimates which are uniformly valid on $0 \leqq \varepsilon \leqq \varepsilon_0$ for the modified Galerkin f.e.m. MG-FEM and for the modified streamlined diffusion f.e.m.'s MSD.1 and MSD.2. Sometimes error estimates in the weighted norm $|||\,.\,|||_{\varepsilon,\delta}$ are possible. Obviously, the modified streamlined diffusion schemes MSD.1 and MSD.2 seem to be favourable because of the improved global L_2-error estimate $O(h^{r+1/2})$ and the high order and uniform in ε local error estimates in case of small ε.

The question of quasioptimal interior L_∞-estimates which are uniformly in ε valid has been discussed in Schatz-Wahlbin [18] for $b \equiv 0$, $N \leqq 2$. In case of $b \not\equiv 0$ this seems to be an open problem. One way to overcome this situation is perhaps given by the scheme proposed in Mizukami-Hughes [13], where a discrete maximum principle holds in the special case $N = 2$, $k = 1$.

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