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ASYMPTOTIC ANALYSIS OF TWO ELLIPTIC EQUATIONS WITH OSCILLATING TERMS (*)

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Abstract — *In a bounded, smooth open subset* Ω of R^N, *is disposed an* ε*-periodic distribution* $\bigcup_i T_{\varepsilon i}$ *of identical inclusions (fig 1) Then, the asymptotic behaviour of the solution* u_ε *of each of the two problems*

$$(H_\varepsilon)\left|\begin{array}{l} -\Delta u_\varepsilon + h_\varepsilon \chi_{\bigcup_i T_{\varepsilon i}} u_\varepsilon = f \quad in \quad \Omega\,, \\ u_\varepsilon \in H_0^1(\Omega)\,, \end{array}\right.$$

$$(M_\varepsilon)\left|\begin{array}{l} -\Delta u_\varepsilon = f \quad in \quad \Omega\Big\backslash\bigcup_i T_{\varepsilon i}\,, \\ \dfrac{\partial u_\varepsilon}{\partial n} + b_\varepsilon u_\varepsilon = 0 \quad on \quad \bigcup_i \partial T_{\varepsilon i}\,, \\ u_\varepsilon \in H_0^1(\Omega) \end{array}\right.$$

is studied, through epi-convergence methods

In this way, we simultaneously derive the asymptotic analysis of Neumann and Dirichlet boundary problems in open sets with holes Critical ratios combining the size r_ε *of the inclusions and the size of the highly oscillating parameters* h_ε *and* b_ε *are exhibited*

Résumé — *Soit* Ω *un ouvert borné et régulier de* R^N, *contenant une répartition* ε*-périodique* $\bigcup_i T_{\varepsilon i}$ *d'inclusions identiques (fig 1) Nous étudions, a l'aide des techniques d'épi-convergence,*

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le comportement asymptotique, lorsque ε *tend vers* 0, *des solutions* u_ε *pour chacun des deux problèmes :*

$$(H_\varepsilon)\quad \begin{cases} -\Delta u_\varepsilon + h_\varepsilon \chi_{\bigcup_i T_{\varepsilon i}} u_\varepsilon = f & \text{dans } \Omega, \\ u_\varepsilon \in H_0^1(\Omega), \end{cases}$$

$$(M_\varepsilon)\quad \begin{cases} -\Delta u_\varepsilon = f & \text{dans } \Omega \setminus \bigcup_i T_{\varepsilon i}, \\ \dfrac{\partial u_\varepsilon}{\partial n} + b_\varepsilon u_\varepsilon = 0 & \text{sur } \bigcup_i \partial T_{\varepsilon i}, \\ u_\varepsilon \in H_0^1(\Omega), \end{cases}$$

(où h_ε *et* b_ε *sont des réels positifs).*

Nous obtenons ainsi une approche unifiée des problèmes de Dirichlet et de Neumann dans « des ouverts à trous ». Nous montrons l'existence de rapports critiques liant la taille r_ε *des inclusions et l'amplitude* h_ε *ou* b_ε *des coefficients (termes fortement oscillants).*

I. INTRODUCTION

A. Two problems in an open set with holes

Let Ω be a bounded smooth open subset of R^N $(N \geq 2)$ and T be a smooth open subset of the unit ball $B(1)$ of R^N. Suppose that Ω is covered by a regular ε-mesh $\bigcup_{i=1}^{i=I(\varepsilon)} Y_{\varepsilon i}$ ($I(\varepsilon)$ is equivalent to $\frac{\text{Vol}(\Omega)}{\varepsilon^N}$). At the center $x_{\varepsilon i}$ of each ε-cell $Y_{\varepsilon i}$, a r_ε-homothetic $T_{\varepsilon i}$ of T $(r_\varepsilon < \varepsilon/2)$ is disposed, according to figure 1 below :

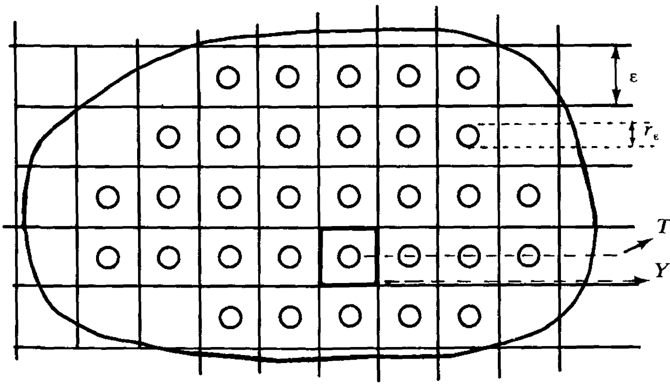

Figure 1.

Let us first recall the situation of the « crushed ice problem » [19], which will appear, indeed, as a particular case of the model problems (H_ε) and (M_ε). Let f be any fixed element of $L^2(\Omega)$ and u_ε the solution of the

Laplace equation in $\Omega_\varepsilon = \Omega \setminus \bigcup_i T_{\varepsilon i}$, with Dirichlet boundary conditions on $\partial\Omega_\varepsilon$:

$$(D_\varepsilon) \begin{cases} -\Delta u_\varepsilon = f & \text{in} \quad \Omega_\varepsilon , \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon = \partial\Omega \cup \left(\bigcup_i \partial T_{\varepsilon i} \right) . \end{cases}$$

The problem is to determine the behaviour of the solution u_ε of (D_ε), when the parameter ε goes to 0. Clearly, the limit of the sequence $(u_\varepsilon)_\varepsilon$ depends on the size r_ε of the inclusions.

When $\lim_\varepsilon \frac{r_\varepsilon}{\varepsilon} = 0$, the following result has been proved via different methods in [2], [9], [18], [19] :

THEOREM 1.1 : *The sequence $(P^\varepsilon u_\varepsilon)_\varepsilon$ of canonical extensions of u_ε, taking the value 0 on the inclusions, converges in the weak topology of $H_0^1(\Omega)$ to the solution u_0 of:*

$$(1) \qquad \begin{cases} -\Delta u_0 + C_D u_0 = f & \text{in} \quad \Omega , \\ u_0 = 0 & \text{on} \quad \partial\Omega , \end{cases}$$

where C_D is the constant given by

$$(2) \qquad C_D = \lim_\varepsilon \left\{ \frac{1}{\varepsilon^N} \operatorname*{Min}_{\substack{w = 0 \text{ on } \partial T_\varepsilon \\ w = 1 \text{ on } \partial B(\varepsilon/2)}} \int_{B(\varepsilon/2)\setminus T_\varepsilon} |\operatorname{grad} w|^2 \, dx \right\} .$$

Of course, C_D depends on the size r_ε of the inclusions and, for example, if N is greater or equal to 3, the change of variables $x = r_\varepsilon y$ in (2) shows the existence of a critical size $r_\varepsilon^c = \varepsilon^{N/(N-2)}$ such that :

1) if $\lim_\varepsilon \frac{r_\varepsilon}{r_\varepsilon^c} = 0$, then u_0 is the solution of :

$$\begin{cases} -\Delta u_0 = f & \text{in} \quad \Omega , \\ u_0 = 0 & \text{on} \quad \partial\Omega , \end{cases}$$

(the inclusions are too small to freeze Ω),

2) if $\lim_\varepsilon \frac{r_\varepsilon}{r_\varepsilon^c}$ belongs to $\mathbb{R}^{+*}$, then u_0 is the solution of (1), which contains a « strange term » $C_D u_0$ [9],

3) if $\lim_{\varepsilon} \frac{r_\varepsilon}{r_\varepsilon^c} = +\infty$, then u_0 is equal to 0. The inclusions are too large and Ω is frozen.

In [4], [20], a particular case of the third above situation is studied, by means of asymptotic expansions, that is the case : $r_\varepsilon = k\varepsilon$ $(0 < k < 1/2)$:

THEOREM 1.2 : *Suppose* $r_\varepsilon = k\varepsilon$ $(0 < k < 1/2)$, *then the sequence* $\left(\frac{1}{\varepsilon^2} P^\varepsilon u_\varepsilon\right)_\varepsilon$ *converges in the weak topology of* $L^2(\Omega)$ *to the function* u_1 *equal to*

$$u_1 = \tilde{Z} f , \tag{3}$$

where $\tilde{Z}$ *is the mean value* $\left(\tilde{Z} = \int_Y Z(y)\, dy\right)$ *of the solution* Z *of:*

$$\operatorname*{Min}_{\substack{z \text{ Y-periodic} \\ z = 0 \text{ on } \partial(kT)}} \left\{ \frac{1}{2} \int_{Y \setminus kT} |\operatorname{grad} z(y)|^2\, dx - \int_{Y \setminus kT} z(y)\, dy \right\}. \tag{4}$$

Let us now present the two model equations which will be considered here :

1) *Highly oscillating potentials*

u_ε is the solution of :

$$(H_\varepsilon) \begin{cases} -\Delta u_\varepsilon + a_\varepsilon u_\varepsilon = f & \text{in} \quad \Omega , \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega , \end{cases}$$

where a_ε takes the values h_ε on $\bigcup_i T_{\varepsilon i}$ $\left(h_\varepsilon \underset{\varepsilon \to 0}{\to} +\infty\right)$ and 0 elsewhere.

2) *Mixed problem*

u_ε is the solution of :

$$(M_\varepsilon) \begin{cases} -\Delta u_\varepsilon = f \quad \text{in} \quad \Omega_\varepsilon , \\ \dfrac{\partial u_\varepsilon}{\partial n} + b_\varepsilon u_\varepsilon = 0 \ \text{ on } \bigcup_i \partial T_{\varepsilon i} \ (n \text{ is the outer normal to } \partial T_{\varepsilon i},\ b_\varepsilon \in \mathbb{R}^+), \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega . \end{cases}$$

Clearly, when h_ε or b_ε are equal to $+\infty$, (H_ε) and (M_ε) coincide with (D_ε). When b_ε is equal to 0, (M_ε) is Laplace's problem in Ω_ε with Neumann boundary conditions on the boundary of the inclusions and Dirichlet boundary conditions on the fixed boundary $\partial\Omega$.

The asymptotic analysis of these two problems will be based on epi-convergence methods. Let us recall the main properties of this variational convergence well-fitted to the asymptotic analysis of minimization problems.

B. Epi-convergence [2], [13]

Let (X, τ) be a metrizable vector space and $(F^\varepsilon)_\varepsilon$ be a sequence of functionals defined on X. Then $(F^\varepsilon)_\varepsilon$ epi$_\tau$-converges to a functional F if :

$$(5) \qquad \forall x \in X \quad \left(\underset{\varepsilon}{\underline{\lim}_e} F^\varepsilon\right)(x) = \left(\underset{\varepsilon}{\overline{\lim}_e} F^\varepsilon\right)(x) = F(x)\,,$$

where

$$\left(\underset{\varepsilon}{\underline{\lim}_e} F^\varepsilon\right)(x) = \underset{x_\varepsilon \overset{\tau}{\to} x}{\mathrm{Min}} \ \underset{\varepsilon}{\underline{\lim}}\, F^\varepsilon(x_\varepsilon)\,, \quad \left(\underset{\varepsilon}{\overline{\lim}_e} F^\varepsilon\right)(x) = \underset{x_\varepsilon \overset{\tau}{\to} x}{\mathrm{Min}} \ \underset{\varepsilon}{\overline{\lim}}\, F^\varepsilon(x_\varepsilon)\,,$$

or, equivalently if :

$$(6) \qquad \forall x \in X\,, \quad \exists x_\varepsilon^0 \underset{\varepsilon}{\overset{\tau}{\to}} x \quad \overline{\lim_{\varepsilon \to 0}}\, F^\varepsilon(x_\varepsilon^0) \leq F(x)\,,$$

$$(7) \qquad \forall x \in X\,, \quad \forall x_\varepsilon \underset{\varepsilon}{\overset{\tau}{\to}} x \quad \underline{\lim_{\varepsilon \to 0}}\, F^\varepsilon(x_\varepsilon) \geq F(x)\,.$$

The main result about this convergence is :

THEOREM 1.3 : *Suppose that $(F^\varepsilon)_\varepsilon$ epi$_\tau$-converges to F and that x_ε is an o_ε-minimizer of $F^\varepsilon\left(o_\varepsilon \underset{\varepsilon \to 0}{\to} 0\right)$ that is :*

$$F^\varepsilon(x_\varepsilon) \leq \inf_{x \in X} F^\varepsilon(x) + o_\varepsilon\,.$$

Then every τ-converging subsequence $(x_{\varepsilon'})_{\varepsilon'}$ converges to a minimizer x of F and moreover $F(x) = \lim_{\varepsilon'} F^{\varepsilon'}(x_{\varepsilon'})$.

Notice that for any problem, the topology τ is choosen so that the sequence $(x_\varepsilon)_\varepsilon$ of minimizers of F^ε is τ-relatively compact.

Epi-convergence is related to the G-convergence of the linked operators in the sense of [21], [15] (see [2]). Consequently, the use of epi-convergence methods gives simultaneously, the limit problem, the convergence of total energy (see theorem 1.3) and the convergence of some mathematical objects linked to the problems such as eigenvalues of the operators [5] or solutions of the evolution problems [6].

The following result deals with the stability of epi-convergence under τ-continuous perturbations.

PROPOSITION 1.4 : *If* $(F^{\varepsilon})_{\varepsilon}$ *epi*$_{\tau}$*-converges to* F, *for every* τ*-continuous function* G, $(F^{\varepsilon}+G)_{\varepsilon}$ *epi*$_{\tau}$*-converges to* $F+G$.

For the asymptotic analysis of (H_{ε}), a direct method, consisting in the verification of (6) and (7) will be used, while for the study of (M_{ε}), a compacity method, using the results of [3], will be presented.

C. Notations

$L^2(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$ denote the classical function spaces.

$C_0^{\infty}(\Omega)$ denotes the space of functions which have partial derivatives of any order and with a compact support in Ω,

$\mathscr{B}$ is the family of the Borel subsets of Ω,

$\mathscr{O}$ is the family of the open Borel subsets of Ω,

I_A is the indicator function of the set A

$$I_A(x) = \begin{vmatrix} 0 & \text{if } x \text{ belongs to } A\,, \\ +\infty & \text{elsewhere}\,, \end{vmatrix}$$

χ_A is the characteristic function of the set A

$$\chi_A(x) = \begin{vmatrix} 1 & \text{if } x \text{ belongs to } A\,, \\ 0 & \text{elsewhere}\,, \end{vmatrix}$$

∂F is the subdifferential operator of the convex function F defined on a locally convex topological vector space V with dual V^*:

$$\partial F(u) = \{u^* \in V^* / \forall v \in V \quad F(v) \geqslant F(u) + \langle u^*, v-u \rangle\}\ .$$

Finally, let me express my thanks to H. Attouch and F. Murat for stimulating and very helpful discussions concerning the asymptotic analysis of (H_{ε}). The asymptotic analysis of (M_{ε}) was first considered in the Thesis [7].

II. ASYMPTOTIC ANALYSIS OF THE HIGHLY OSCILLATING POTENTIAL PROBLEM (H_{ε})

Throughout this paragraph, u_{ε} denotes the solution of (H_{ε}).

LEMMA 2.1 : a) u_ε *is the solution of the minimization problem :*

$$F_H^\varepsilon(u_\varepsilon) = \operatorname*{Min}_{u \in H_0^1(\Omega)} F_H^\varepsilon(u) ,$$

where F_H^ε *is the functional defined on* $H^1(\Omega)$ *by :*

$$F_H^\varepsilon(u) = \frac{1}{2} \int_\Omega |\operatorname{grad} u|^2 \, dx + \frac{1}{2} \int_\Omega a_\varepsilon u^2 \, dx - \int_\Omega fu \, dx . \tag{8}$$

b) *The sequence* $(u_\varepsilon)_\varepsilon$ *is bounded in* $H_0^1(\Omega)$.

Proof of Lemma 2.1 : a) Is an immediate consequence of (H_ε).

b) Notice that $F_H^\varepsilon(u_\varepsilon) \leq F_H^\varepsilon(0) = 0$. Then, use Poincaré's inequality [1] and the positivity of a_ε.

Our main result concerning the asymptotic analysis of (H_ε) is :

THEOREM 2.2 : a) *The sequence* $(F_H^\varepsilon)_\varepsilon$ $epi_{w - H_0^1(\Omega)}$*-converges to the functional* F_H *given by*

$$F_H(u) = \frac{1}{2} \int_\Omega |\operatorname{grad} u|^2 \, dx + \frac{1}{2} C_H \int_\Omega u^2 \, dx - \int_\Omega fu \, dx , \tag{9}$$

where C_H *is the positive constant given by*

$$C_H = \lim_\varepsilon \left\{ \frac{1}{\varepsilon^N} \operatorname*{Min}_{w = 1 \text{ on } \partial B(\varepsilon/2)} \int_{B(\varepsilon/2)} (|\operatorname{grad} w|^2(x) + a_\varepsilon(x) \, w^2(x)) \, dx \right\} . \tag{10}$$

b) *Consequently (see theorem* 1.3*) we have :*

b1) *the sequence* $(u_\varepsilon)_\varepsilon$ *of solutions of* (H_ε) *converges in the weak topology of* $H_0^1(\Omega)$ *to the solution* u_0 *of:*

$$(H_0) \begin{cases} -\Delta u_0 + C_H u_0 = f & in \quad \Omega \\ u_0 \in H_0^1(\Omega) , \end{cases}$$

b2) *the sequence* $\left(\int_\Omega (|\operatorname{grad} u_\varepsilon|^2 + a_\varepsilon u_\varepsilon^2) \, dx \right)_\varepsilon$ *of total energies converges to :*

$$\int_\Omega |\operatorname{grad} u_0|^2 \, dx + C_H \int_\Omega u_0^2 \, dx .$$

Before proving the theorem 2.2, let us give more precisely the value of C_H.

PROPOSITION 2.3 : *The constant C_H given by* (10) *has the following values* ($N \geqslant 3$) *which depend on the limit of the critical ratios :* $r_\varepsilon/\varepsilon^{N/(N-2)}$ *and* $h_\varepsilon r_\varepsilon^N/\varepsilon^N$:

	$\frac{h_\varepsilon r_\varepsilon^N}{\varepsilon^N} \underset{\varepsilon}{\to} 0$	$\frac{h_\varepsilon r_\varepsilon^N}{\varepsilon^N} = k_2$	$\frac{h_\varepsilon r_\varepsilon^N}{\varepsilon^N} \underset{\varepsilon}{\to} +\infty$
$\frac{r_\varepsilon^{N-2}}{\varepsilon^N} \underset{\varepsilon}{\to} 0$	$C_H = 0$	$C_H = 0$	$C_H = 0$
$\frac{r_\varepsilon^{N-2}}{\varepsilon^N} = k_1$	$C_H = 0$	$C_H = \underset{w \in H_0^1(R^N)}{\mathrm{Min}} \left\{ k_1 \int_{R^N} \lvert \mathrm{grad}\, w \rvert^2 dx + k_2 \int_T (w-1)^2 dx \right\}$	$C_H = k_1 \,\mathrm{Cap}_{R^N}(T) = k_1 \underset{\substack{w \in H_0^1(R^N) \\ w = 1 \text{ on } T}}{\mathrm{Min}} \int_{R^N \setminus T} \lvert \mathrm{grad}\, w \rvert^2 dx$
$\frac{r_\varepsilon^{N-2}}{\varepsilon^N} \underset{\varepsilon}{\to} +\infty$	$C_H = 0$	$C_H = k_2 \,\mathrm{meas}\,(T)$	$C_H = +\infty$

Proof of Proposition 2.3 : Write $x = r_\varepsilon y$ in (10). Then,

$$\text{(10bis)} \quad C_H = \lim_\varepsilon \underset{w = 1 \text{ on } \partial B(\varepsilon/2)}{\mathrm{Min}} \times \left(\frac{r_\varepsilon^{N-2}}{\varepsilon^N} \int_{B(\varepsilon/2 r_\varepsilon)} \lvert \mathrm{grad}\, w \rvert^2 dy + \frac{h_\varepsilon r_\varepsilon}{\varepsilon^N} \int_T w^2 dy \right).$$

Proof of Theorem 2.2 : Let w_ε be the solution of the local minimization problem occurring in (10) and denote $C(\varepsilon)$ the quantity given by :

$$\text{(11)} \quad C(\varepsilon) = \int_{B(\varepsilon/2)} (\lvert \mathrm{grad}\, w_\varepsilon \rvert^2 + a_\varepsilon w_\varepsilon^2)\, dx = \underset{\substack{w \in H^1(B(\varepsilon/2)) \\ w = 1 \text{ on } \partial B(\varepsilon/2)}}{\mathrm{Min}} \int_{B(\varepsilon/2)} (\lvert \mathrm{grad}\, w \rvert^2 + a_\varepsilon w^2)\, dx .$$

Then w_ε may be extended ε-periodically in a function still denoted by w_ε equal to 1 in $\bigcup_i Y_{\varepsilon i} \setminus B^i(\varepsilon/2)$. Let us admit for a moment the following properties of this function w_ε.

PROPOSITION 2.4 : *Suppose that* $\overline{\lim_{\varepsilon}} \frac{C(\varepsilon)}{\varepsilon^N} < +\infty$, *then :*

a) $(w_\varepsilon)_\varepsilon$ *converges in the weak topology of* $H^1(\Omega)$ *to the constant function* 1.

b) $\left(\sum_i \frac{\partial w_\varepsilon}{\partial \nu} \bigg|_{\partial B^i(\varepsilon/2)} \right)_\varepsilon$ *converges in the strong topology of* $H^{-1}(\Omega)$ *to the constant* C_H *given by* (10),

and let us verify the two assertions (6) and (7) which become in this case

$$\text{(6)} \qquad \forall v \in H_0^1(\Omega)\,, \quad \exists v_\varepsilon^0 \xrightarrow[\varepsilon \to 0]{w - H_0^1(\Omega)} v \ \overline{\lim_{\varepsilon}}\, F_H^\varepsilon(v_\varepsilon^0) \leq F_H(v)\,,$$

$$\text{(7)} \qquad \forall v \in H_0^1(\Omega)\,, \quad \forall v_\varepsilon \xrightarrow[\varepsilon \to 0]{w - H_0^1(\Omega)} v \ \overline{\lim_{\varepsilon}}\, F_H^\varepsilon(v_\varepsilon) \geq F_H(v)\,.$$

1st *step :* verification of (6) when $\overline{\lim_{\varepsilon}} \frac{C(\varepsilon)}{\varepsilon^N} < +\infty$.

First consider a smooth function v in $C_0^\infty(\Omega)$ and let v_ε^0 be equal to $v w_\varepsilon$ where w_ε is the solution of (11).

Proposition 2.4. *a*) implies that $(v_\varepsilon^0)_\varepsilon$ converges to v in $w - H_0^1(\Omega)$.

$$F_H^\varepsilon(v_\varepsilon^0) = \frac{1}{2} \int_\Omega |\operatorname{grad} v|^2\, dx + \frac{1}{2} \sum_i v^2(x_{\varepsilon i}) \times \times \int_{B^i(\varepsilon/2)} (|\operatorname{grad} w_\varepsilon|^2 + a_\varepsilon w_\varepsilon^2)\, dx - \int_\Omega fv\, dx + o_\varepsilon\,,$$

From Proposition 2.4. *a*) and the regularity of v, one obtains :

$$F_H^\varepsilon(v_\varepsilon^0) = \frac{1}{2} \int_\Omega |\operatorname{grad} v|^2\, dx + \frac{1}{2} \sum_i v^2(x_{\varepsilon i}) \times \times \int_{B^i(\varepsilon/2)} (|\operatorname{grad} w_\varepsilon|^2 + a_\varepsilon w_\varepsilon^2)\, dx - \int_\Omega fv\, dx + o_\varepsilon\,,$$

where o_ε is a quantity which converges to 0 when ε goes to 0. Then, write

$$F_H^\varepsilon(v_\varepsilon^0) = \frac{1}{2}\int_\Omega |\operatorname{grad} v|^2\,dx + \frac{1}{2}\sum_i \varepsilon^N v^2(x_{\varepsilon i}) \times \\ \times \left(\frac{1}{\varepsilon^N}\int_{B'(\varepsilon/2)} (|\operatorname{grad} w_\varepsilon|^2 + a_\varepsilon w_\varepsilon^2)\,dx\right) - \int_\Omega fv\,dx + o_\varepsilon \\ = \frac{1}{2}\int_\Omega |\operatorname{grad} v|^2\,dx + \frac{1}{2}\sum_i \int_{Y_{\varepsilon i}} v^2(x)\,dx \times \\ \times \left(\frac{1}{\varepsilon^N}\int_{B'(\varepsilon/2)} (|\operatorname{grad} w_\varepsilon|^2 + a_\varepsilon w_\varepsilon^2)\,dx\right) - \int_\Omega fv\,dx + o_\varepsilon\,,$$

because v is smooth. The definition of $C_H(10)$ implies

$$\lim_\varepsilon F_H^\varepsilon(v_\varepsilon^0) = F_H(v)\,.$$

Then, for a general function v in $H_0^1(\Omega)$, let us apply a density argument. There exists a sequence $(v_n)_n$ of functions in $C_0^\infty(\Omega)$ which converges to v in the strong topology of $H_0^1(\Omega)$. By the previous argument, for each n :

$$\lim_\varepsilon F_H^\varepsilon(v_n w_\varepsilon) = F_H(v_n)\,,$$

hence

$$\overline{\lim_n}\,\overline{\lim_\varepsilon}\, F_H^\varepsilon(v_n w_\varepsilon) \leq F_H(v)$$

(F_H is continuous for the strong topology of $H_0^1(\Omega)$).

From the diagonalization argument of Corollary 1.16 [2], one derives the existence of a sequence $(n(\varepsilon))_\varepsilon$ growing to $+\infty$, such that : $(v_{n(\varepsilon)} w_\varepsilon)_\varepsilon$ converges to v in the weak topology of $H_0^1(\Omega)$, and

$$\overline{\lim_\varepsilon}\, F_H^\varepsilon(v_{n(\varepsilon)} w_\varepsilon) \leq F_H(v)\,. \quad \text{Then take} \quad v_\varepsilon^0 = v_{n(\varepsilon)} w_\varepsilon\,.$$

2nd *step* : verification of (7) when $\overline{\lim_\varepsilon}\, \dfrac{C(\varepsilon)}{\varepsilon^N} < +\infty$.

Let v be any element of $H_0^1(\Omega)$ and $(v_n)_n$ a sequence of functions in $C_0^\infty(\Omega)$ converging to v in the strong topology of $H_0^1(\Omega)$. Then, for every sequence $(v_\varepsilon)_\varepsilon$ converging to v in the weak topology of $H_0^1(\Omega)$, we write

$$F_H^\varepsilon(v_\varepsilon) \geq F_H^\varepsilon(v_n w_\varepsilon) + \langle \partial F_H^\varepsilon(v_n w_\varepsilon), v_\varepsilon - v_n w_\varepsilon \rangle$$

where

$$\langle \partial F_H^\varepsilon(v_n w_\varepsilon), v_\varepsilon - v_n w_\varepsilon \rangle = \int_\Omega \text{grad}\,(v_n w_\varepsilon) \,.\, \text{grad}\,(v_\varepsilon - v_n w_\varepsilon)\, dx +$$

$$+ \int_\Omega a_\varepsilon v_n w_\varepsilon (v_\varepsilon - v_n w_\varepsilon)\, dx - \int_\Omega f(v_\varepsilon - v_n w_\varepsilon)\, dx$$

$$= \int_\Omega \text{grad}\, v_n \,.\, \text{grad}\,(v_\varepsilon - v_n w_\varepsilon)\, w_\varepsilon\, dx$$

$$+ \sum_i \int_{B^i(\varepsilon/2)} (-\Delta w_\varepsilon + a_\varepsilon w_\varepsilon)\, v_n (v_\varepsilon - v_n w_\varepsilon)\, dx$$

$$+ \sum_i \int_{\partial B^i(\varepsilon/2)} \frac{\partial w_\varepsilon}{\partial v} v_n (v_\varepsilon - v_n w_\varepsilon)\, d\sigma_\varepsilon(x) + \int_\Omega f(v_\varepsilon - v_n w_\varepsilon)\, dx$$

$$- \int_\Omega \text{grad}\, w_\varepsilon . \,\text{grad}\, v_n (v_\varepsilon - v_n w_\varepsilon)\, dx\,.$$

From the definition of w_ε and Proposition 2.4, one deduces

$$\underline{\lim}_\varepsilon F_H^\varepsilon(v_\varepsilon) \geqslant F_H(v_n) + \int_\Omega \text{grad}\, v_n \,.\, \text{grad}\,(v - v_n)\, dx +$$

$$+ C_H \int_\Omega v_n (v - v_n)\, dx + \int_\Omega f(v - v_n)\, dx\,.$$

Then let n go to $+\infty$. The properties of $(v_n)_n$ give the conclusion.

3rd *step* : when $\overline{\lim}_\varepsilon \dfrac{C(\varepsilon)}{\varepsilon^N} = +\infty$.

In this case (see Proposition 2.3), r_ε is bigger than λr_ε^c for every λ in R^{+*}. Then, for every u in $H_0^1(\Omega)$:

$$F_H^\varepsilon(u) \geqslant F_{H,\lambda}^\varepsilon(u)\,,$$

where $F_{H,\lambda}^\varepsilon$ corresponds to the case $r_\varepsilon = \lambda r_\varepsilon^c$. Then, for every u in $H_0^1(\Omega)$:

$$\overline{\lim}_\varepsilon F_H^\varepsilon(u) \geqslant \underline{\lim}_\varepsilon F_H^\varepsilon(u) \geqslant \operatorname{Sup}_\lambda F_{H,\lambda}(u) = \begin{vmatrix} +\infty & \text{if } u \text{ is not } 0 \text{ (a.e. in } \Omega)\,, \\ 0 & \text{if } u \text{ is } 0 \text{ (a.e. in } \Omega)\,. \end{vmatrix}$$

The assertion (5) is verified with $F_H = I_{\{u = 0 \text{ in } \Omega\}}$.

Let us now prove the properties of the solution w_ε of the local problem, exposed in Proposition 2.4.

a) From the definition of w_ε (or its extension), one derives

$$\chi_{\bigcup_i Y_{\varepsilon i}\setminus B^i(\varepsilon/2)}(1 - w_\varepsilon) = 0\,. \tag{12}$$

The sequence $(\chi_{\bigcup_i Y_{\varepsilon i}\setminus B^i(\varepsilon/2)})_\varepsilon$ converges, in the weak topology of $L^2(\Omega)$ to the strictly positive constant Vol $(Y\setminus B(1/2))$ (see Lemma 4.1 of [20]). As soon as $(w_\varepsilon)_\varepsilon$ is bounded in $H^1(\Omega)$, and therefore strongly convergent in $L^2(\Omega)$, the assertion *a*) is a consequence of (12).

In order to prove that $(w_\varepsilon)_\varepsilon$ is bounded in $H^1(\Omega)$, we notice that (11) implies

$$C(\varepsilon) \leq \int_{B(\varepsilon/2)} |\text{grad } W_\varepsilon|^2\, dx\,,$$

where W_ε is the solution of the Dirichlet problem :

$$\begin{cases} -\Delta W_\varepsilon = 0 & \text{in} \quad B(\varepsilon/2)\setminus B(r_\varepsilon/2)\,, \\ W_\varepsilon = 1 & \text{on} \quad \partial B(\varepsilon/2)\,, \\ W_\varepsilon = 0 & \text{on} \quad \partial B(r_\varepsilon/2)\,. \end{cases} \tag{13}$$

W_ε is easily computable in terms of radial functions (see [9] p. 114). From the positivity of a_ε and Theorem 2.2 of [9], one deduces that $(w_\varepsilon)_\varepsilon$ is bounded in $H^1(\Omega)$.

b) The solution w_ε of (11) satisfies

$$\begin{cases} -\Delta w_\varepsilon + a_\varepsilon w_\varepsilon = 0 & \text{in} \quad B(\varepsilon/2)\,, \\ w_\varepsilon = 1 & \text{on} \quad \partial B(\varepsilon/2)\,. \end{cases} \tag{14}$$

We first deduce from (14) that w_ε is positive in $B(\varepsilon/2)$ (multiplying (14) par w_ε^-, the negative part of w_ε and integrating by parts [1]). With the same idea, we prove that

$$w_\varepsilon \geq W_\varepsilon \quad \text{in} \quad B(\varepsilon/2)\setminus B(r_\varepsilon/2) \tag{15}$$

where W_ε is the solution of (13)).

Since $w_\varepsilon = W_\varepsilon = 1$ on $\partial B(\varepsilon/2)$, (15) implies

$$\left.\frac{\partial W_\varepsilon}{\partial \nu}\right|_{\partial B(\varepsilon/2)} \geq \left.\frac{\partial w_\varepsilon}{\partial \nu}\right|_{\partial B(\varepsilon/2)} \geq 0\,.$$

Then, Lemma 2.3 and Lemma 2.8 of [9], imply that $\left(\sum_i \left.\frac{\partial w_\varepsilon}{\partial \nu}\right|_{\partial B^i(\varepsilon/2)}\right)_\varepsilon$ converges in the strong topology of $H^{-1}(\Omega)$.

This limit, whose computation is not necessary, for the proof of Theorem 2.2, may be calculated in the following way :

For every v in $C_0^\infty(\Omega)$:

$$(16) \quad \int_\Omega \operatorname{grad} w_\varepsilon \,.\, \operatorname{grad}\ (v w_\varepsilon)\, dx + \int_\Omega a_\varepsilon\, v w_\varepsilon^2\, dx =$$

$$= \sum_i \int_{B^i(\varepsilon/2)} (-\Delta w_\varepsilon + a_\varepsilon\, w_\varepsilon)\, v w_\varepsilon\, dx + \sum_i \int_{\partial B^i(\varepsilon/2)} \frac{\partial w_\varepsilon}{\partial \nu}\, v\, d\sigma_\varepsilon(x)$$

$$= \int_\Omega v(|\operatorname{grad} w_\varepsilon|^2 + a_\varepsilon\, w_\varepsilon^2)\, dx + o_\varepsilon\,,$$

because v is smooth. Using the same argument as in step 1 of the proof of Theorem 2.2, one derives

$$\langle \sum_i \frac{\partial w_\varepsilon}{\partial \nu}\bigg|_{\partial B^i(\varepsilon/2)}, v\rangle = C_H \int_\Omega v\, dx = o_\varepsilon$$

and therefore, for every v in $C_0^\infty(\Omega)$

$$\lim_\varepsilon \langle \sum_i \frac{\partial w_\varepsilon}{\partial \nu}\bigg|_{\partial B^i(\varepsilon/2)}, v\rangle = C_H \int_\Omega v\, dx\,.$$

One can improve the result of convergence exposed in Theorem 2.2*b*) in the following way.

PROPOSITION 2.5 :

a) *If* $C_H = 0$, *then* $(u_\varepsilon)_\varepsilon$ *converges to* u_0 *in the strong topology of* $H_0^1(\Omega)$.

b) *If* $C_H = +\infty$, *then* $(u_\varepsilon)_\varepsilon$ *converges to* 0 *in the strong topology of* $H_0^1(\Omega)$.

c) *If* C_H *is finite, but not* 0, *then* $(u_\varepsilon - w_\varepsilon\, u_0)_\varepsilon$ *converges to* 0 *in the strong topology of* $W_0^{1,1}(\Omega)$. *And, if* u_0 *is in* $C^1(\bar{\Omega})$, *that is, if* T *and* f *are sufficiently smooth so that* u_0 *is in* $C^1(\bar{\Omega})$, *then* $(u_\varepsilon - w_\varepsilon\, u_0)_\varepsilon$ *converges to* 0 *in the strong topology of* $H_0^1(\Omega)$.

Proof of Proposition 2.5 :

a) and *b*) are simple consequences of assertion *b*) in Theorem 2.2.

c) Take v in $C^1(\bar{\Omega}) \cap H_0^1(\Omega)$ and compute

$$(17) \quad \int_\Omega (|\operatorname{grad}(u_\varepsilon - w_\varepsilon v)|^2 + a_\varepsilon(u_\varepsilon - w_\varepsilon v)^2)\, dx =$$

$$= \int_\Omega (|\operatorname{grad} u_\varepsilon|^2 + a_\varepsilon(u_\varepsilon)^2)\, dx$$

$$+ \int_\Omega (|\operatorname{grad} w_\varepsilon|^2 + a_\varepsilon(w_\varepsilon)^2)\, v^2\, dx + 2 \int_\Omega \operatorname{grad} w_\varepsilon \,.\, \operatorname{grad} v w_\varepsilon v\, dx$$

$$+ \int_\Omega |\operatorname{grad} v|^2 (w_\varepsilon)^2\, dx - 2 \int_\Omega (\operatorname{grad} w_\varepsilon \,.\, \operatorname{grad} u_\varepsilon v + a_\varepsilon u_\varepsilon v w_\varepsilon)\, dx$$

$$- 2 \int_\Omega \operatorname{grad} u_\varepsilon \,.\, \operatorname{grad} v w_\varepsilon\, dx\, .$$

One can pass to the limit in (17) using Proposition 2.4 *a*) and *b*), Theorem 2.2 *b*) and the idea exposed in the computation of (16)

$$\int_\Omega (\operatorname{grad}(u_\varepsilon - w_\varepsilon)^2 +$$

$$+ a_\varepsilon(u_\varepsilon - w_\varepsilon v)^2)\, dx \underset{\varepsilon \to 0}{\longrightarrow} \int_\Omega (|\operatorname{grad}(u_0 - v)|^2 + C_H(u_0 - v)^2)\, dx\, .$$

From the inequality

$$\|u_\varepsilon - w_\varepsilon u_0\|_{W_0^{1,1}(\Omega)} \leq \|u_\varepsilon - w_\varepsilon v\|_{W_0^{1,1}(\Omega)} + \|w_\varepsilon(v - u_0)\|_{W_0^{1,1}(\Omega)}$$

and the density of smooth functions in $H_0^1(\Omega)$, we get the conclusion. If u_0 is in $C^1(\bar{\Omega})$, then we take in the above computation $v = u_0$.

Let us conclude this section giving some results of convergence concerning the mathematical objets linked to (H_ε). From [6], we deduce

THEOREM 2.6 : *Given f in $L^2((0, T)\,; \Omega)$ and g_0 in $H_0^1(\Omega)$, let u_ε be the solution of:*

$$\begin{cases} \dfrac{\partial u_\varepsilon(x,t)}{\partial t} - \Delta u_\varepsilon(x,t) + a_\varepsilon u_\varepsilon(x,t) = f(x,t) & \text{in } \Omega x]0, T[\,, \\ u_\varepsilon(.,0) = g_0(.) & \text{in } \Omega\,, \\ u_\varepsilon(x,t) = 0 & \text{for } t > 0, x \text{ on } \partial\Omega\,, \end{cases}$$

then $(u_\varepsilon)_\varepsilon$ *converges in the strong topology of* $L^2((0, T) ; \Omega)$ *to the solution* u_0 *of:*

$$\begin{cases} \dfrac{\partial u_0(x, t)}{\partial t} - \Delta u_0(x, t) + C_H u_0(x, t) = f(x, t) \text{ in } \Omega x]0, T[, \\ u_0(., 0) = g_0(.) , \\ u_0(x, t) = 0 \quad \text{for} \quad t > 0, x \text{ on } \partial\Omega. \end{cases}$$

And from [5], we deduce :

THEOREM 2.7 : *Let* $\mathscr{A}_\varepsilon$ *(resp.* $\mathscr{A}_0$*) be the operator associated to* (H_ε) *(resp. to* (H_0)*)*

$$D(\mathscr{A}_\varepsilon) = H^2(\Omega) \cap H_0^1(\Omega) ,$$
$$\mathscr{A}_\varepsilon u = - \Delta u + a_\varepsilon u$$

$$(\text{resp. } D(\mathscr{A}_0) = H^2 \cap H_0^1, \mathscr{A}_0 u = - \Delta u + C_H u) .$$

Then, the sequence of eigenvalues of $\mathscr{A}_\varepsilon$ *converges to the sequence of eigenvalues of* $\mathscr{A}_0$ *in the following sense. If* $(\lambda_{\varepsilon, k})_k$ *(resp.* $(\lambda_k)_k$*) is the nondecreasing sequence of eigenvalues of* $\mathscr{A}_\varepsilon$ *(resp.* $\mathscr{A}_0$*) and if* $u_{\varepsilon, k}$ *is an eigenvector associated to* $\lambda_{\varepsilon, k}$ *then :*

a) $(\lambda_{\varepsilon, k})_\varepsilon$ *converges to* λ_k ;

b) *the eigenspace* E_k *associated to* λ_k *is the limit in Kuratowski's sense* [17] *and for the strong topology of* $L^2(\Omega)$ *of the subspace generated by* $(u_{\varepsilon k}, u_{\varepsilon, k+1}, \dots, u_{\varepsilon, k+m})$ *if m is the multiplicity order of* λ_k.

Let us precise the connections between the two problems (H_ε) and (D_ε). Let $a_{\varepsilon, h}$ be the oscillating potential defined on Ω by :

$$a_{\varepsilon, h} = \begin{cases} h & \text{on } \bigcup_i T_{\varepsilon i} , \quad h \in \mathbb{R}^{+ *} , \\ 0 & \text{elsewhere} , \end{cases}$$

and let F_h^ε be the functional associated to this oscillating potential $a_{\varepsilon, h}$:

$$F_h^\varepsilon(u) = \frac{1}{2} \int_\Omega |\text{grad } u|^2 \, dx + \frac{1}{2} \int_\Omega a_{\varepsilon, h} u^2 \, dx - \int_\Omega fu \, dx (u \in H_0^1(\Omega)) .$$

From Theorem 2.40 of [2], one deduces that when h goes to $+ \infty$, the sequence $(F_h^\varepsilon)_h$ $\text{epi}_{w - H_0^1(\Omega)}$-converges to the functional F^ε defined on $H_0^1(\Omega)$ by :

$$F^\varepsilon(u) = \frac{1}{2} \int_\Omega |\text{grad } u|^2 \, dx + I_{H_0^1(\Omega_\varepsilon)}(u) - \int_\Omega fu \, dx .$$

(Notice that F^{ε} is the functional associated to the problem (D_{ε}).)

Hence, a diagonalization argument [2] implies the existence of a sequence $(h(\varepsilon))_{\varepsilon}$ going to $+\infty$, such that the sequence $(F^{\varepsilon}_{h(\varepsilon)})_{\varepsilon}$ $\text{epi}_{w-H_0^1(\Omega)}$-converges to the epi-limit of $(F^{\varepsilon})_{\varepsilon}$ i.e. to the functional F defined on $H_0^1(\Omega)$ by :

$$F(u)=\frac{1}{2}\int_{\Omega}|\operatorname{grad} u|^2\,dx+\frac{1}{2}C_D\int_{\Omega}u^2\,dx-\int_{\Omega}fu\,dx$$

(where C_D is defined in (2). Notice that F is the functional associated to (1)).

The Proposition 2.3 gives some information about the sequence $(h(\varepsilon))_{\varepsilon}$: if N is greater or equal to 3, then the last column in the array shows that the sequence $(h(\varepsilon))_{\varepsilon}$ has to be choosen so that :

$$\lim_{\varepsilon}\frac{h(\varepsilon)\,r_{\varepsilon}^N}{\varepsilon^N}=+\infty\,,$$

and since the critical size r_{ε}^c is equal to $C\varepsilon^{N/(N-2)}$:

$$\lim_{\varepsilon} h_{\varepsilon}\,\varepsilon^{2N/(N-2)}=+\infty\,.$$

III. ASYMPTOTIC ANALYSIS OF THE MIXED PROBLEM (M_{ε})

In this section u_{ε} denotes the solution of

$$(M_{\varepsilon})\begin{cases}-\Delta u_{\varepsilon}=f & \text{in } \Omega_{\varepsilon}\\ \dfrac{\partial u_{\varepsilon}}{\partial n}+b_{\varepsilon}\,u_{\varepsilon}=0 & \text{on } \bigcup_i \partial T_{\varepsilon i}\ \ (b_{\varepsilon}>0)\\ u_{\varepsilon}=0 & \text{on } \partial\Omega\,.\end{cases}$$

LEMMA 3.1 :

a) *There exists a linear continuous operator P^{ε} from $H^1_{\partial\Omega}(\Omega_{\varepsilon})$ $(H^1_{\partial\Omega}(\Omega_{\varepsilon})=\{u\in H^1(\Omega_{\varepsilon})/u=0 \text{ on } \partial\Omega\})$, into $H_0^1(\Omega)$, satisfying :*

$$\sup_{\varepsilon}\|P^{\varepsilon}\|_{(H^1_{\partial\Omega}(\Omega_{\varepsilon}),\,H_0^1(\Omega))}<+\infty$$

b) *u_{ε} is the solution of the minimization problem :*

$$F^{\varepsilon}_M(u_{\varepsilon})=\operatorname*{Min}_{H^1_{\partial\Omega}(\Omega_{\varepsilon})}F^{\varepsilon}_M(u)$$

where F_M^ε is the functional defined by :

$$(18)\quad F_M^\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |\operatorname{grad} u|^2\, dx + \frac{1}{2} b_\varepsilon \sum_i \int_{\partial T_{\varepsilon i}} u^2\, dH_{N-1}^\varepsilon(x) - \int_{\Omega_\varepsilon} fu\, dx\,,$$

$dH_{N-1}^\varepsilon(.)$ denoting the Hausdorff measure of the regular $N-1$ dimensional manifold $\partial T_{\varepsilon i}$.

c) *$(P^\varepsilon u_\varepsilon)_\varepsilon$ is bounded in $H_0^1(\Omega)$.*

Proof of Lemma 3.1 :

a) See [22], [10], Note that such an operator is not unique. However, in the sequel we will choose one among them. The results obtained below do not depend on such a choice.

b) Is an immediate consequence of (M_ε).

c) Is a consequence of *a*) and *b*), and the positivity of b_ε.

A) The case $\lim_\varepsilon \frac{r_\varepsilon}{\varepsilon} = 0$: a low concentration of inclusions.

Our main result of convergence is :

THEOREM 3.2 :

a) *The sequence $(F_M^\varepsilon)_\varepsilon$ defined by* (18) *$epi_{w-H_0^1(\Omega)}$-converges to the functional F_M*

$$F_M(u) = \frac{1}{2}\int_\Omega |\operatorname{grad} u|^2\, dx + \frac{1}{2} C_M \int_\Omega u^2\, dx - \int_\Omega fu\, dx\,,$$

where C_M is the constant

$$(19)\quad C_M = \lim_\varepsilon \left\{ \frac{1}{\varepsilon^N} \operatorname*{Min}_{w=1 \text{ on } \partial B(\varepsilon/2)} \times \left(\left(\int_{B(\varepsilon/2)\setminus T_\varepsilon} |\operatorname{grad} w|^2\, dx + b_\varepsilon \int_{\partial T_\varepsilon} w^2(x)\, dH_{N-1}^\varepsilon(x) \right) \right) \right\}.$$

b) *The sequence $(P^\varepsilon u_\varepsilon)_\varepsilon$ converges in the weak topology of $H_0^1(\Omega)$ to the solution u_0 of:*

$$(M_0)\begin{cases} -\Delta u_0 + C_M u_0 = f & \text{in } \Omega\,, \\ u_0 = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Moreover the sequence $\left(\int_{\Omega_\varepsilon} |\operatorname{grad} u_\varepsilon|^2\,dx + b_\varepsilon \sum_i \int_{\partial T_{\varepsilon i}} u_\varepsilon^2\,dH_{N-1}^\varepsilon(x)\right)_\varepsilon$ *converges to*

$$\int_\Omega |\operatorname{grad} u_0|^2\,dx + C_M \int_\Omega u_0^2\,dx\,.$$

As in Proposition 2.3, depending on the limit of the critical ratios : $r_\varepsilon^{N-2}/\varepsilon^N$ and $b_\varepsilon r_\varepsilon^{N-1}/\varepsilon^{N-2}$, the values of C_M are :

PROPOSITION 3.3 : *If* $N \geq 3$, *then*

	$\dfrac{b_\varepsilon r_\varepsilon^{N-1}}{\varepsilon^N} \underset{\varepsilon}{\to} 0$	$\dfrac{b_\varepsilon r_\varepsilon^{N-1}}{\varepsilon^N} = k_2$	$\dfrac{b_\varepsilon r_\varepsilon^{N-1}}{\varepsilon^N} \underset{\varepsilon}{\to} +\infty$
$\dfrac{r_\varepsilon}{\varepsilon^{N/(N-2)}} \underset{\varepsilon}{\to} 0$	$C_M = 0$	$C_M = 0$	$C_M = 0$
$\dfrac{r_\varepsilon}{\varepsilon^{N/(N-2)}} = k_1$	$C_M = 0$	$C_M = \underset{u=1 \text{ à l'infini}}{\operatorname{Min}} \left\{k_1^{N-2} \int_{R^N \setminus T} \lvert\operatorname{grad} u\rvert^2\,dx + k_2 \int_{\partial T} u^2\,dH_{N-1}(x)\right\}$	$C_M = k_1^{N-2} \operatorname{cap}(T)$
$\dfrac{r_\varepsilon}{\varepsilon^{N/(N-2)}} \underset{\varepsilon}{\to} +\infty$	$C_M = 0$	$C_M = k_2 H_{N-1}(\partial T)$	$C_M = +\infty$

Proof of Theorem 3.2 : One can apply the same « direct method » as the one given in the Proof of Theorem 2.3. One has now to take w_ε the solution of the following minimization problem

$$(20)\quad \int_{B(\varepsilon/2)} |\operatorname{grad} w_\varepsilon|^2\,dx + b_\varepsilon \int_{\partial T_\varepsilon} w_\varepsilon^2\,dH_{N-1}^\varepsilon(x) = \\ = \underset{w=1 \text{ on } \partial B(\varepsilon/2)}{\operatorname{Min}} \left(\int_{B(\varepsilon/2)} |\operatorname{grad} w|^2\,dx + b_\varepsilon \int_{\partial T_\varepsilon} w^2\,dH_{N-1}^\varepsilon(x)\right)$$

and to derive, for this function w_ε, properties similar to the ones given in Proposition 2.4 (see [7]). However, we present here a different method based on a « compacity argument » and a decomposition of the functional F_M^ε into a quadratic term $\frac{1}{2}\int_{\Omega_\varepsilon} |\operatorname{grad} u|^2 \, dx$ and the « constraints » $\frac{1}{2}\sum_i b_\varepsilon \int_{\partial T_{\varepsilon i}} u^2 \, dH_{N-1}^\varepsilon(x)$. We will point out that the limit constraints still keep the same expression in the two cases: $\lim_\varepsilon r_\varepsilon/\varepsilon = 0$ and $r_\varepsilon = k\varepsilon$ $(0 < k < 1/2)$. Indeed, only the limit of the quadratic term is changed.

Let us first recall the following « compacity theorem » :

THEOREM 3.4 [3] : *Let E_q be the family of quadratic energy functionals on $H_0^1(\Omega)$:*

$$E_q = \{\Phi/\Phi(u) = \int_\Omega \sum_{i,j} a_{ij} D_j u D_i u \, dx \; ; \quad a_{ij} = a_{ji} \text{ and } \forall \xi \in \mathbb{R}^N$$
$$\lambda_0 |\xi|^2 \leq \sum_{ij} a_{ij} \xi_i \xi_j \leq \Lambda_0 |\xi|^2 , \quad 0 < \lambda_0 \leq \Lambda_0\} .$$

Let $\mathcal{F}$ be the family of unilateral constraints :

$$\mathcal{F} = \Big\{F : H_0^1(\Omega) \times \mathcal{B} \to \bar{\mathbb{R}}^+ / \text{ i)} \forall v \in H_0^1 : B \to F(v, B) \text{ is a positive outer regular Borel measure,}$$
$$\text{ii)} \forall \omega \in \mathcal{O}, v \to F(v, \omega) \text{ is lower semicontinuous on } H_0^1(\Omega) \text{ and proper, convex,}$$
$$\text{iii)} \forall u, v \in H_0^1(\Omega), \forall \omega \in \mathcal{O} : u|_\omega = v|_\omega \Rightarrow F(u, \omega) = F(v, \omega),$$
$$\text{iv)} \forall u, v \in H_0^1(\Omega), \forall \omega \in \mathcal{O} :$$
$$F(\inf(u, v), \omega) + F(\sup(u, v), \omega) \leq F(u, \omega) + F(v, \omega)\Big\} .$$

Let $(\Phi_n)_n$ be a sequence in E_q. Let $(F_n^1)_n$ and $(F_n^2)_n$ be two sequences in $\mathcal{F}$ such that F_n^1 is decreasing and F_n^2 is increasing (with respect to v in $H_0^1(\Omega)$).

Suppose there exist z and $(z_n)_n$ converging to z in the strong topology of $L^2(\Omega)$, such that

$$\Phi_n(z_n) \underset{n}{\to} \Phi(z) , \quad F_n^1(z_n, B) = F_n^2(z_n, B) = 0 \text{ for every } B \text{ in } \mathcal{B} .$$

Then, there exist a subsequence still denoted n, Φ in E_q, F^1 and F^2 in $\mathcal{F}$,

F^1 decreasing and F^2 increasing and a rich family $\mathscr{R}$ of Borel subsets of Ω in $\mathcal{O}$ [14] *such that :*

$(\Phi_n)_n$ *épi*$_{s-L^2(\Omega)}$*-converges to* Φ ,

$(\Phi_n + F_n^i(.,B))_n$ *épi*$_{s-L^2(\Omega)}$*-converges to* $\Phi + F^i(.,B)$ *for* B *in* $\mathscr{R}$,

$(\Phi_n + F_n^1(.,B) + F_n^2(.,B))_n$ *épi*$_{s-L^2(\Omega)}$*-converges to* $\Phi +$

$$+ F^1(\inf(.,z), B) + F^2(\sup(.,z), B) .$$

We have the following representation theorem for the functionals of $\mathscr{F}$.

THEOREM 3.5 [12], [3] : *For every F in $\mathscr{F}$, there exist μ, ν Radon measure (positive), and $f\colon \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, Borel measurable with respect to the first variable and convex, lower semicontinuous with respect to the second one, such that :*

$$F(v,B) = \int_B f(x, \tilde{\tilde{v}}(s))\, d\mu(x) + \nu(B)$$

($\tilde{\tilde{v}}$ is the quasi-continuous representant of v) and if F is decreasing (resp. increasing) then f is decreasing (resp. increasing) with respect to v.

Let us come back to the proof of Theorem 3.2*a*).

1st *step :* Decomposition of F_M^ε (18) and use of Theorems 3.4 and 3.5. We write :

$$F_M^\varepsilon(u) = \Phi^\varepsilon(u) + F_\varepsilon^1(u,\Omega) + F_\varepsilon^2(u,\Omega) \quad \text{with}$$

$$(21) \qquad \begin{cases} \Phi^\varepsilon(u) &= \dfrac{1}{2}\displaystyle\int_{\Omega_\varepsilon} |\operatorname{grad} u|^2\, dx , \\ F_\varepsilon^1(u,B) &= b_\varepsilon \displaystyle\sum_i \int_{B \cap \partial T_{\varepsilon i}} u^{-2}\, dH_{N-1}^\varepsilon(x) \quad (B \in \mathscr{B}) , \\ F_\varepsilon^2(u,B) &= b_\varepsilon \displaystyle\sum_i \int_{B \cap \partial T_{\varepsilon i}} u^{+2}\, dH_{N-1}^\varepsilon(x) . \end{cases}$$

One difficulty is that $(\Phi^\varepsilon)_\varepsilon$ is not uniformly coercive on $H_0^1(\Omega)$. But this is not really a problem, because

— the existence of the extension operator P^ε (Lemma 3.1 *a*)) guarantees that the epi-limit of $(F_M^\varepsilon)_\varepsilon$ is $+\infty$ outside $H_0^1(\Omega)$ (see [2], p. 163),

— we may add a small perturbation

$$\frac{1}{n}\sum_i \int_{T_{\varepsilon i}} |\operatorname{grad} u|^2\, dx \quad (n \in N^*) ,$$

and then let n go to $+\infty$, after the study of the limit constraints. So, let us apply Theorem 3.4 with Φ^ε defined in (21).

Take $z_\varepsilon = z = 0$. $(\chi_{\Omega_\varepsilon})_\varepsilon$ converges in the strong topology of $L^2(\Omega)$ to 1, therefore $(\Phi^\varepsilon)_\varepsilon$ epi$_{w-H_0^1(\Omega)}$-converges to Φ :

$$\Phi(u) = \frac{1}{2}\int_\Omega |\operatorname{grad} u|^2\, dx .$$

2nd *step :* Determination of the limit constraints F^1 and F^2.

We first show, using the ε-periodic distribution of the inclusions in (M_ε) that the measures μ_1 and μ_2 appearing in the limit functionals F^1 and F^2 (Theorem 3.5) are « invariant under translations » [7]. As F^1_ε, F^2_ε, Φ^ε are positively homogeneous of degree 2, F^1 and F^2 are also positively homogeneous of degree 2. In fact from these properties and the more specific formulas (21), one derives :

$$F^1(u, B) = C'_M \int_B u^{-2}\, dx ,$$

$$F^2(u, B) = C'_M \int_B u^{+2}\, dx \quad (B \in \mathscr{R}) ,$$

for a constant C'_M in $\bar{R}^+$.

3rd *step :* Computation of C'_M.

From Theorem 3.4, one deduces that

$$(22)\quad C'_M = \frac{1}{\operatorname{meas}(\Omega)} \underset{y_\varepsilon \xrightarrow{s-L^2(\Omega)} 1}{\operatorname{Min}} \ \underline{\lim}_{\varepsilon} \left(\Phi^\varepsilon(y_\varepsilon) + F^1_\varepsilon(y_\varepsilon, \Omega) + F^2_\varepsilon(y_\varepsilon, \Omega)\right) .$$

In order to prove the equality between C'_M and C_M, we have to use the solution w_ε of the local problem (19) [7].

Remark 3.6 : One advantage to use such a compacity argument is to determine the epi-limit of $(\Phi^\varepsilon)_\varepsilon$. In the case just considered, we saw that $(\Phi^\varepsilon)_\varepsilon$ epi-converge to Φ : $\Phi(u) = \frac{1}{2}\int_\Omega |\operatorname{grad} u|^2\, dx$, thanks to the convergence of $(\chi_{\Omega_\varepsilon})_\varepsilon$ to 1 in the strong topology of $L^2(\Omega)$. As we will see in the next subsection, when $r_\varepsilon = k\varepsilon$, $(\chi_{\Omega_\varepsilon})_\varepsilon$ converges only in the weak topology of $L^2(\Omega)$ to a constant. Hence, the epi-convergence of $(\Phi^\varepsilon)_\varepsilon$ is modified, while the limit unilateral constraints F^1 and F^2 are the same.

Let us finally conclude this subsection by pointing out that similar results as those exposed in Proposition 2.5, Theorem 2.6 and Theorem 2.7 are available in this case [7].

B) The case : $r_\varepsilon = k\varepsilon\,(0 < k < 1/2)$: a high density of inclusions.

THEOREM 3.7 : *The sequence* $(P^\varepsilon u_\varepsilon)_\varepsilon$ *converges in the weak topology of* $H_0^1(\Omega)$ *to the solution* u_{hom} *of the « homogenized problem »*

$$\operatorname*{Min}_{H_0^1(\Omega)} \left\{ \int_\Omega j_{\text{hom}}(\text{grad}\, u)\, dx + \frac{C_M}{2} \int_\Omega u^2\, dx - \int_\Omega fu\, dx \left(\int_{Y\setminus T} dy \right) \right\}$$

(23) *where*

$$j_{\text{hom}}(x) = \operatorname*{Min}_{\substack{w \in H^1(Y\setminus T) \\ w\ Y\text{-periodic}}} \int_{Y\setminus T} |\text{grad}\, w(y) + x|^2\, dy \quad (x \in \mathbb{R}^N)$$

C_M *is given in* (19) :

$$C_M = \lim_\varepsilon \frac{b_\varepsilon}{\varepsilon} H_{N-1}(\partial T)\,.$$

Moreover the energy

$$\left(\int_{\Omega_\varepsilon} |\text{grad}\, u_\varepsilon|^2\, dx + b_\varepsilon \sum_i \int_{\partial T_{\varepsilon i}} (u_\varepsilon)^2\, dH_{N-1}^\varepsilon(x) \right)_\varepsilon$$

converges to

$$\int_\Omega j_{\text{hom}}(\text{grad}\, u_{\text{hom}})\, dx + C_M \int_\Omega (u_{\text{hom}})^2\, dx\,.$$

Proof of Theorem 3.7 : As we already pointed out in Remark 3.6, we have only to determine the epi-limit of $(\Phi^\varepsilon)_\varepsilon$. But in this case

$$\chi_{\Omega_\varepsilon}(x) = \chi\left(\frac{x}{\varepsilon}\right),$$

where χ is the characteristic function of $Y\setminus T$. In order to determine the $\text{epi}_{w-L^2(\Omega)}$-limit of $(\Phi^\varepsilon)_\varepsilon$, we use the same idea as the one exposed in the proof of Theorem 1.20 of [2], based on the density of piecewise linear functions in $H^1(\Omega)$ [16].

Let us conclude this subsection by giving a proof, by epi-convergence methods, of Theorem 1.2 concerning the case « $r_\varepsilon = k\varepsilon$, $b_\varepsilon = +\infty$ », obtained by means of asymptotic expansions in [4] and [20].

THEOREM 3.8 : *When* $r_\varepsilon = k\varepsilon$ $(0 < k < 1/2)$, $b_\varepsilon = +\infty$ (*Dirichlet boundary on the inclusions*).

a) $\left(\frac{1}{\varepsilon^2} P^\varepsilon u_\varepsilon\right)_\varepsilon$ *is bounded in* $L^2(\Omega)$ (P^ε *is now the canonical extension operator from* $H_0^1(\Omega_\varepsilon)$ *into* $H_0^1(\Omega)$ *by* 0 *on the inclusions*).

b) $\left(\frac{1}{\varepsilon^2} P^\varepsilon u_\varepsilon\right)_\varepsilon$ *converges in the weak topology of* $L^2(\Omega)$ *to* $\tilde{Z} f$, *where* $\tilde{Z}$ *is the mean value of the solution* Z *of:*

$$\operatorname*{Min}_{\substack{z \text{ } Y\text{-periodic} \\ z = 0 \text{ on } \partial(kT)}} \left\{ \frac{1}{2} \int_{Y \setminus kT} |\operatorname{grad} z|^2 (y)\, dy - \int_{Y \setminus kT} z(y)\, dy \right\}.$$

Moreover, $\left(\frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} |\operatorname{grad} u_\varepsilon|^2 (x)\, dx\right)_\varepsilon$ *converges to* $\tilde{Z} \int_\Omega f^2(x)\, dx$.

c) *The sequence* $(G^\varepsilon)_\varepsilon$ *defined on* $H_0^1(\Omega)$ *by*

$$G^\varepsilon(u) = \frac{\varepsilon^2}{2} \int_{\Omega_\varepsilon} |\operatorname{grad} u|^2 (x)\, dx + I_{H_0^1(\Omega_\varepsilon)}(u) - \int_\Omega f u\, dx$$

$epi_{w - L^2(\Omega)}$*-converges to the functional* G *defined on* $L^2(\Omega)$ *by*

$$G(u) = \frac{1}{2\tilde{Z}} \int_\Omega u^2\, dx - \int_\Omega f u\, dx .$$

Proof of Theorem 3.8 :

a) This is a consequence of the following Lemma, whose proof is obtained by means of changes of variables.

LEMMA 3.9 : *There exists a positive constant* $C(Y)$ *such that for every* u *in* $H^1(Y)$, $u = 0$ *on* kT *:*

$$\|u\|_{L^2(Y)} \leq C(Y) \|\operatorname{grad} u\|_{(L^2(Y))^{N \times N}} .$$

For every u *in* $H^1(Y_\varepsilon)$, $u = 0$ *on* T_ε :

$$\|u\|_{L^2(Y_\varepsilon)} \leq C(Y)\, \varepsilon\, \|\operatorname{grad} u\|_{(L^2(Y_\varepsilon))^{N \times N}} .$$

From Lemma 3.9 and Lemma 3.1 *b*), with « $b_\varepsilon = +\infty$ », one deduces :

$$(24) \qquad \int_{\Omega_\varepsilon} |\operatorname{grad} u_\varepsilon|^2\, dx \leq \|f\|_{L^2(\Omega_\varepsilon)} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$
$$\leq \|f\|_{L^2(\Omega)}\, C(Y)\, \varepsilon\, \|\operatorname{grad} u_\varepsilon\|_{(L^2(\Omega_\varepsilon))^{N \times N}} .$$

From (24), we infer

$$\int_{\Omega_\varepsilon} |\text{grad } u_\varepsilon|^2 \, dx \leq C(f, Y)\, \varepsilon^2 ,$$

and Lemma 3.9 implies

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C(f, Y)\, \varepsilon^2 .$$

c) The two assertions (5) and (6) become in this case :

$$\forall v \in L^2(\Omega) , \quad \exists v_{\varepsilon,0} \xrightarrow[\varepsilon]{w - L^2(\Omega)} v : \ \overline{\lim_{\varepsilon}}\, G^\varepsilon(v_{\varepsilon,0}) \leq G(v) , \tag{25}$$

$$\forall v \in L^2(\Omega) , \quad \forall v_\varepsilon \xrightarrow[\varepsilon]{w - L^2(\Omega)} v : \ \underline{\lim_{\varepsilon}}\, G^\varepsilon(v_\varepsilon) \geq G(v) . \tag{26}$$

Let Z be the solution of the minimization problem (4), and Z_ε be the Y_ε-periodic function defined in the ε-cell Y_ε by $Z_\varepsilon(x) = Z\left(\frac{x}{\varepsilon}\right)$. Z_ε satisfies

$$\begin{cases} -\Delta Z_\varepsilon = \dfrac{1}{\varepsilon^2} & \text{in} \quad Y_\varepsilon \backslash T_\varepsilon , \\ Z_\varepsilon = 0 & \text{on} \quad \partial T_\varepsilon . \end{cases}$$

$(Z_\varepsilon)_\varepsilon$ converges in the weak topology of $L^2(\Omega)$ to $\tilde{Z} = \int_{Y \backslash kT} Z(y)\, dy$. Now, for every v in $C_0^\infty(\Omega)$, let $v_{\varepsilon,0}$ be defined by :

$$v_{\varepsilon,0} = v Z_\varepsilon \frac{1}{\tilde{Z}} . \tag{27}$$

Then, the sequence $(v_{\varepsilon,0})_\varepsilon$ converges to v, in the weak topology of $L^2(\Omega)$. The computation of $G^\varepsilon(P^\varepsilon v_{\varepsilon,0})$ gives

$$\begin{aligned} G^\varepsilon(P^\varepsilon v_{\varepsilon,0}) = \frac{\varepsilon^2}{2\tilde{Z}^2} \Bigg\{ & \int_{\Omega_\varepsilon} |\text{grad } v|^2 Z_\varepsilon^2 \, dx + \\ & + 2 \int_{\Omega_\varepsilon} \text{grad } v \,.\, \text{grad } Z_\varepsilon \, v Z_\varepsilon \, dx + \\ & + \int_{\Omega_\varepsilon} |\text{grad } Z_\varepsilon|^2 \, v^2 \, dx \Bigg\} - \frac{1}{\tilde{Z}} \int_{\Omega_\varepsilon} f v Z_\varepsilon \, dx . \end{aligned}$$

An integration by parts and the properties of Z_ε give

$$\lim_{\varepsilon} G^\varepsilon(P^\varepsilon v_{\varepsilon,0}) = G(v) .$$

For v in $L^2(\Omega)$, we choose a sequence $(v_n)_n$ of smooth functions converging to v in the strong topology of $L^2(\Omega)$, and we use a diagonalization argument similar to the one used in the proof of theorem 2.2. So (25) is proved. Take any sequence $(v_\varepsilon)_\varepsilon$ converging to v in the weak topology of $L^2(\Omega)$. One may suppose

$$\overline{\lim_{\varepsilon}}\ \varepsilon^2 \int_{\Omega_\varepsilon} |\operatorname{grad} v_\varepsilon|^2\, dx < +\infty \tag{28}$$

otherwise, (26) is automatically satisfied. We choose a sequence $(v_n)_n$ of smooth functions converging to v in the strong topology of $L^2(\Omega)$ and use a subdifferential inequality :

$$G^\varepsilon(v_\varepsilon) \geqslant G^\varepsilon((v_n)_{\varepsilon,0}) + \langle \partial G^\varepsilon((v_n)_{\varepsilon,0}), v_\varepsilon - (v_n)_{\varepsilon,0}\rangle$$

where

$$\begin{aligned} \langle \partial G^\varepsilon((v_n)_{\varepsilon,0}), v_\varepsilon - (v_n)_{\varepsilon,0}\rangle &= \\ &= \frac{\varepsilon^2}{\tilde{Z}} \int_{\Omega_\varepsilon} \operatorname{grad} v_n \cdot \operatorname{grad}\,(v_\varepsilon - (v_n)_{\varepsilon,0})\, Z_\varepsilon\, dx \\ &+ \frac{\varepsilon^2}{\tilde{Z}} \int_{\Omega_\varepsilon} \operatorname{grad} Z_\varepsilon \cdot \operatorname{grad}\,(v_\varepsilon - (v_n)_{\varepsilon,0})\, v_n\, dx \\ &- \int_{\Omega_\varepsilon} f(v_\varepsilon - (v_n)_{\varepsilon,0})\, dx\,. \end{aligned} \tag{29}$$

From the properties of Z_ε and (27), (28), the first term of (29) converges to 0. An integration by parts of the second term of (29) gives :

$$\lim_{\varepsilon}\ \langle \partial G^\varepsilon((v_n)_{\varepsilon,0}), v_\varepsilon - (v_n)_{\varepsilon,0}\rangle \geqslant \frac{1}{\tilde{Z}} \int_\Omega (v - v_n)\, v_n\, dx - \int_\Omega f(v - v_n)\, dx\,.$$

Let n go to $+\infty$: (26) is proved.

IV. FURTHER RESULTS

First, notice that all the preceding results are still true when the laplacian operator is replaced by a second-order elliptic operator with constant coefficients, that is a linear operator from $H_0^1(\Omega)$ in to $H^{-1}(\Omega)$ defined by :

$$u \mapsto -\sum_{ij} D_j(a_{ij}\, D_i\, u)\,,$$

where the matrix $(a_{ij})_{ij}$ is constant symmetric and positively definite [9].

Let us present some results connected to the Stokes system and which may be obtained by similar technics.

In [8], [11], was studied the asymptotic behaviour of an incompressible fluid slowly flowing in the porous medium Ω_ε with Fourier conditions on the boundary of the inclusions $T_{\varepsilon i}$.

Let $\vec{u}_\varepsilon$ be the solution of :

$$(H'_\varepsilon) \quad \left| \begin{array}{ll} -\Delta \vec{u}_\varepsilon + a_\varepsilon \vec{u}_\varepsilon = -\operatorname{grad} p_\varepsilon + \vec{f} & \text{in } \Omega \,, \\ \operatorname{div} \vec{u}_\varepsilon = 0 & \text{in } \Omega \,, \\ \vec{u}_\varepsilon \in (H_0^1(\Omega))^N \,, & \end{array} \right.$$

where the potential a_ε takes the two values

$$\left| \begin{array}{ll} h_\varepsilon & \text{on the inclusions} \,, \\ 0 & \text{elsewhere} \,. \end{array} \right.$$

Our main result concerning the asymptotic behaviour of $\vec{u}_\varepsilon$ is :

THEOREM 4.1 :

$(\vec{u}_\varepsilon)_\varepsilon$ converges in the weak topology of $(H_0^1(\Omega))^N$ to the solution $\vec{u}_0$ of

$$\left| \begin{array}{ll} -\Delta \vec{u}_0 + C_S \vec{u}_0 = \operatorname{grad} p_0 + \vec{f} & in\ \Omega \,, \\ \operatorname{div} \vec{u}_0 = 0 & in\ \Omega \,, \\ \vec{u}_0 \in (H_0^1(\Omega))^N \,, & \end{array} \right.$$

where C_S is the symmetric matrix given by

$$(C_S)_{k\ell} = \lim_\varepsilon \left\{ \frac{1}{\varepsilon^N} \int_{B(\varepsilon/2)} \operatorname{grad} \vec{w}_\varepsilon^k \cdot \operatorname{grad} \vec{w}_\varepsilon^\ell \, dx + \right.$$

$$\left. + \frac{h_\varepsilon}{\varepsilon^N} \int_{T_\varepsilon} \vec{w}_\varepsilon^k \cdot \vec{w}_\varepsilon^\ell \, dx \right\}, \quad (k, \ell \in \{1, \ldots, N\}) \,,$$

$\vec{w}_\varepsilon^k$ being the solution of the local minimization problem

$$(30) \qquad \underset{\substack{\vec{w} \in (H^1(B(\varepsilon/4)))^N, \\ \operatorname{div} \vec{w} = 0 \ in\ B(\varepsilon/4), \\ \vec{w} = \vec{e}_k\ on\ \partial B(\varepsilon/4)}}{\operatorname{Min}} \int_{B(\varepsilon/4)} (|\operatorname{grad} \vec{w}|^2 + a_\varepsilon |\vec{w}|^2) \, dx \,,$$

($\vec{e}_k$ is the k-th canonical vector of R^N).

$\left(\int_\Omega (|\text{grad } \vec{u}_\varepsilon|^2 + a_\varepsilon |\vec{u}_\varepsilon|^2)\, dx\right)_\varepsilon$ *converges to*

$$\int_\Omega |\text{grad } \vec{u}_0|^2\, dx + \sum_{k\ell} \int_\Omega (C_S)_{k\ell}\, (\vec{u}_0)_k\, (\vec{u}_0)_\ell\, dx\,,$$

$(p_\varepsilon)_\varepsilon$ *converges in the weak topology of* $L^2(\Omega)/R$ *to the limit pressure* p_0.

Sketch of the proof: When T is equal to $B(1)$, the solution $\vec{w}_\varepsilon^k$ of the local problem (30) is computable in terms of radial functions (see [8] for similar computations). When T is a subset of $B(1)$, we conjecture some pointwise estimates similar to those obtained by Marchenko and Hruslov in [18].

1st step : For a smooth function $\vec{v}$ in $(C_0^\infty(\Omega))^N$ and divergence-free in Ω, we set

$$(31) \qquad \vec{v}_\varepsilon^0(x) = \begin{cases} \vec{v}(x) - \sum_k (\vec{e}_k - \vec{w}_\varepsilon^k(x))\, v_k(x_{\varepsilon\iota}) & \text{in } B^\iota(\varepsilon/4)\,, \\ \vec{v}(x) & \text{in } Y_{\varepsilon\iota} \backslash B^\iota(\varepsilon/4)\,, \end{cases}$$

and suppose that for every k

$$(32) \qquad (C_S)_{kk} < +\infty\,.$$

Then, (32) implies

$$\vec{w}_\varepsilon^k \xrightarrow[\varepsilon]{w - (H^1(\Omega))^N} \vec{e}_k\,, \quad \vec{v}_\varepsilon^0 \xrightarrow[\varepsilon]{w - (H_0^1(\Omega))^N} \vec{v}\,.$$

As in the proof of Theorem 2.2, one obtains :

$$\lim_\varepsilon \left(\int_\Omega (|\text{grad } \vec{v}_\varepsilon^0|^2 + a_\varepsilon |\vec{v}_\varepsilon^0|^2)\, dx\right) =$$

$$= \int_\Omega |\text{grad } \vec{v}\,|^2\, dx + \sum_{k\ell} \int_\Omega (C_S)_{k\ell}\, v_k\, v_\ell\, dx\,.$$

For a function $\vec{v}$ in $(H_0^1(\Omega))^N$ and divergence-free in Ω, use a diagonalization argument (see proof of Theorem 2.2).

Take $\vec{v}$ in $(H_0^1(\Omega))^N$ and divergence-free in Ω, then two sequences $(\vec{v}_\varepsilon)_\varepsilon$ and $(\vec{v}_n)_n$ converging to $\vec{v}$ in the weak or strong topology of

$(H_0^1(\Omega))^N$, with $\vec{v}_n$ smooth. As in the proof of Theorem 2.2 (second step), we have to pass to the limit on the term :

$$\int_\Omega \{\operatorname{grad} (\vec{v}_n)^0_\varepsilon \cdot \operatorname{grad} (\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon) + a_\varepsilon (\vec{v}_n)^0_\varepsilon (\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon)\} \, dx =$$

$$= \int_\Omega \operatorname{grad} \vec{v}_n \cdot \operatorname{grad} (\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon) \, dx + \sum_i \sum_k (\vec{v}_n)_k \, (x_{\varepsilon i}) \times$$

$$\times \int_{B^i(\varepsilon/4)} \operatorname{grad} \vec{w}^k_\varepsilon \cdot \operatorname{grad} (\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon) \, dx$$

$$+ \sum_i h_\varepsilon \int_{T_{\varepsilon i}} (\vec{v}_n(x) - \vec{v}_n(x_{\varepsilon i}))(\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon) \, dx + \sum_i \sum_k (\vec{v}_n)_k \, (x_{\varepsilon i}) \times$$

$$\times \int_{B^i(\varepsilon/4)} a_\varepsilon \, \vec{w}^k_\varepsilon \cdot (\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon) \, dx =$$

$$= \int_\Omega \operatorname{grad} \vec{v}_n \cdot \operatorname{grad} (\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon) \, dx +$$

$$+ \sum_i h_\varepsilon \int_{T_{\varepsilon i}} (\vec{v}_n(x) - (\vec{v}_n)(x_{\varepsilon i})) \cdot (\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon) \, dx +$$

$$+ \sum_i \sum_k (\vec{v}_n)_k \, (x_{\varepsilon i}) \int_{\partial B^i(\varepsilon/4)} \left(\frac{\partial \vec{w}^k_\varepsilon}{\partial \nu} - q^k_\varepsilon \, \vec{v} \right) \cdot (\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon) \, d\sigma_\varepsilon(x) , \tag{33}$$

using the Euler equation associated to (30).

The first term of (33) converges to $\int_\Omega \operatorname{grad} \vec{v}_n \cdot \operatorname{grad} (\vec{v} - \vec{v}_n) \, dx$, when ε goes to 0. The second term of (33) converges to 0, thanks to (32) and the smoothness of $\vec{v}_n$. In order to pass to the limit on the third term of (33), we first need the following interpolation lemma (see [8]).

LEMMA 4.2 : *For every v in $C^1(\bar{\Omega})$, there exists v^*_ε in $C^1(\bar{\Omega})$ such that*

$v^*_\varepsilon(x) = v(x_{\varepsilon i})$ in $B^i(\varepsilon/4)$ for $i = 1$ to $I(\varepsilon)$,

$\|\operatorname{grad} v^*_\varepsilon\|_{(L^\infty(\Omega))^N} \leq C \|\operatorname{grad} v\|_{(L^\infty(\Omega))^N}$, for a constant C independant of v and ε.

$(v^*_\varepsilon)_\varepsilon$ converges to v in the strong topology of $L^2(\Omega)$.

The third term of (33) may be written as :

$$\sum_k \sum_i \int_{\partial B^i(\varepsilon/4)} \left(\frac{\partial \vec{w}^k_\varepsilon}{\partial \nu} - q^k_\varepsilon \, \vec{v} \right) (\vec{v}_\varepsilon - (\vec{v}_n)^0_\varepsilon)((\vec{v}_n)_k)^*_\varepsilon \, d\sigma_\varepsilon(x) .$$

This is the reason why appears $B^i(\varepsilon/4)$ instead of $B^i(\varepsilon/2)$ in (30) and (31). Under the assumption (32) :

$$\sum_i \left(\frac{\partial \vec{w}_\varepsilon^k}{\partial \nu} - q_\varepsilon^k \vec{v} \right) \Bigg|_{\partial B^i(\varepsilon/4)} \xrightarrow[\varepsilon]{s - H^{-1}(\Omega)} (C_S)_k \text{ (the } k\text{-th column of the matrix } C_S),$$

thanks to the explicit computations (compare to Lemma 2.3 [9]).

Then, the third term of (33) converges to

$$\int_\Omega \sum_{k\ell} (C_S)_{k\ell} \, (\vec{v}_n)_k \, (\vec{v} - \vec{v}_n)_\ell \, dx \, .$$

Finally, let n go to $+\infty$. Theorem 4.1 *a*), *b*) are proved.

In order to study the behaviour of the pressure, take $\vec{\varphi}$ in $(C_0^\infty(\Omega))^N$ and compute

$$\begin{aligned} \langle \operatorname{grad} p_\varepsilon, \vec{\varphi}_\varepsilon^0 \rangle &= \int_\Omega p_\varepsilon \operatorname{div} \vec{\varphi} \, dx \\ &= \langle - \Delta \vec{u}_\varepsilon + a_\varepsilon \vec{u}_\varepsilon - \vec{f}, \vec{\varphi}_\varepsilon^0 \rangle \\ &= \int_\Omega (\operatorname{grad} \vec{u}_\varepsilon \, . \operatorname{grad} \vec{\varphi}_\varepsilon^0 + a_\varepsilon \, \vec{u}_\varepsilon \, \vec{\varphi}_\varepsilon^0) \, dx + \int_\Omega \vec{f} \vec{\varphi}_\varepsilon^0 \, dx \, , \end{aligned}$$

where $\vec{\varphi}_\varepsilon^0$ is associated to $\vec{\varphi}$ by (31).

Then, one obtains :

$$\begin{aligned} \int_\Omega \left(\lim_\varepsilon p_\varepsilon \right) \operatorname{div} \vec{\varphi} \, dx &= \int_\Omega \operatorname{grad} \vec{u}_0 \, . \operatorname{grad} \vec{\varphi} \, dx \\ &\quad + \sum_{k\ell} \int_\Omega (C_S)_{k\ell} \, (\vec{u}_0)_k \, \varphi_\ell \, dx + \int_\Omega \vec{f} \vec{\varphi} \, dx \\ &= \int_\Omega p_0 \operatorname{div} \vec{\varphi} \, dx \, . \end{aligned}$$

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