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A VARIATIONAL METHOD FOR PARAMETER IDENTIFICATION (*)

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Abstract. — *We study a parameter identification problem associated with the two dimensional diffusion equation. We consider the steady state situation, where the equation is elliptic and of divergence form. For Neumann boundary conditions, a variational method is proposed for the reconstruction of the unknown scalar conductivity. The variational method is based on the minimization of a convex functional, and the reconstructed conductivity is a continuous, piecewise linear function on a triangulation of the two dimensional domain. Stability results are proved and numerical examples are considered to test the performance. A method is then proposed for the reconstruction of a matrix conductivity.*

Résumé. — *On étudie un problème d'identification de paramètres associé à des équations de diffusion en dimension 2. On considère l'état stationnaire lorsque l'équation elliptique est sous forme divergentielle ; pour les fonctions aux limites de Neumann une méthode variationnelle est proposée pour la reconstruction de la conductivité scalaire inconnue ; la méthode variationnelle est basée sur la minimisation d'une fonctionnelle convexe et la conductivité reconstruite est continue linéaire par morceaux sur une triangulation du domaine bidimensionnel. Des résultats de stabilité sont démontrés et des exemples numériques sont considérés pour tester les performances. Une méthode est ensuite proposée pour la reconstruction d'une conductivité matricielle.*

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1. INTRODUCTION

This paper is concerned with the following parameter identification problem : given measurements u^m, f^m, and g^m of u^*, f^*, and g^* corresponding to an elliptic equation on a two dimensional domain Ω,

$$- \operatorname{div} (a^*(x) \nabla u^*) = f^* \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \tag{1.1}$$

$$a^*(x) \frac{\partial u^*}{\partial n} = g^* \quad \text{on} \quad \partial\Omega , \tag{1.1a}$$

with the compatibility condition $\int_\Omega f^* \, dx + \int_{\partial\Omega} g^* \, ds = 0$, find the spatially varying conductivity $a^*(x)$.

Equation (1.1) corresponds to a steady state solution of

$$\frac{\partial u^*}{\partial t} = \frac{\partial}{\partial x} \left(a^* \frac{\partial u^*}{\partial x} \right) + \frac{\partial}{\partial y} \left(a^* \frac{\partial u^*}{\partial y} \right) + f^* . \tag{1.2}$$

This equation is important in hydrology, where it describes the flow of water through an aquifer. Equation (1.2) is derived under the assumptions that the flow is independent of depth and that the flow rate in porous rock is proportional to the pressure gradient. The latter assumption is known as Darcy's law. In equation (1.2), u^* is the pressure or piezometric head and a^* is the transmissivity, the function of proportionality in Darcy's law which measures the ability of the water to move in the aquifer. The quantity f^* represents a source term. A detailed discussion of the various terms and a derivation of equation (1.2) can be found in Bear [4]. An important inverse problem is to determine the transmissivity a^* under steady state conditions by measuring u^* and f^* at well sites.

In this paper, our attention is focused on u^* in $H_1(\Omega)$ and f^* and g^* in $L_2(\Omega)$ and $L_2(\partial\Omega)$ respectively. The " measurement " u^m is an approximation of u^* in $H_1(\Omega)$; in practice u^* would be measured at well sites, and u^m would be the piecewise linear interpolate of these measurements on a triangulation of Ω, with the well sites as the vertices of the triangles. Because f^* represents a source term, an analysis of equation (1.1) with f^* a delta function is also relevant. The reader is referred to Alessandrini [2] for a study of that situation.

Equation (1.1) can be viewed as a first order partial differential equation in the unknown conductivity $a^*(x)$. The equation is

$$\nabla u^* \, . \, \nabla a^* + a^* \, \Delta u^* = - f^* \tag{1.3}$$

which can be solved by integrating along the lines of steepest ascent of u^*. Evidently, some assumption on ∇u^* is required, as (1.3) becomes singular when $\nabla u^* = 0$. If ∇u^* should vanish on some open set, then (1.3) provides no information about the behavior of $a^*(x)$ on this set. Although this situation is possible when $f^* \neq 0$, Alessandrini [1] has shown that if $f^* = 0$ and if $u^*|_{\partial\Omega}$ has a finite number of relative maxima and minima, then ∇u^* vanishes only at a finite number of points in Ω, with finite multiplicity.

If $a^*(x)$ is prescribed along that portion of the boundary where $\frac{\partial u^*}{\partial n} < 0$, the so-called inflow boundary, then Richter [20] proved that (1.3) can be solved uniquely under the condition

$$\inf_{\Omega} \max \{ |\nabla u^*|, \Delta u^* \} > 0 . \tag{1.4}$$

Evidently, the problem with which this paper deals is overdetermined, since we assume the boundary data to be supplied along the entire boundary. In practice the inflow boundary is not known, so our hypothesis is a reasonable one.

Our goal is to design a numerical method which is easy to implement, and which gives a reconstructed conductivity that has predictive value when " good " measurements are available. More precisely, the reconstructed conductivity must give " good " approximations to the associated elliptic forward problems. If a_p is the reconstructed conductivity, assumed positive, let

$$- \nabla . (a_p \nabla v) = - \nabla . (a^* \nabla w) \quad \text{in} \quad \Omega$$

$$v = w \quad \text{on } \partial\Omega \quad \text{or} \quad a_p \frac{\partial v}{\partial n} = a^* \frac{\partial w}{\partial n} \quad \text{on} \quad \partial\Omega .$$

An integration by parts against $v - w$ gives

$$\|\nabla(w - v)\|_{L_2} \leq C \|a_p - a^*\|_{L_2} , \tag{1.5}$$

where $C = C\left(\inf_{\Omega} a_p, \|w\|_{1,\infty}\right)$, and a^* and $\partial\Omega$ are assumed sufficiently smooth so that $\|w\|_{1,\infty}$ is finite. In view of (1.5), we shall think of a " good " reconstructed conductivity as one that is close to a^* in L_2.

If the goal were an accurate reconstruction of a^* in the L^∞ norm, then the problem would be ill-posed. This is due to the smoothing property of elliptic operators, on account of which rapid oscillations in the conductivity are suppressed in the forward solution. The following one-dimensional example demonstrates the phenomenon : let $a^*(x) = \frac{1}{2}$, $u^*(x) = x^2$, and consider

$$a_N(x) = (2 + \cos Nx)^{-1}, \quad u_N(x) = x^2 + \left(\frac{x}{N}\right) \sin Nx + \left(\frac{1}{N^2}\right) \cos Nx.$$

One verifies that for every N

$$(a^*(u^*)')' = (a_N(u_N)')' \quad \text{on} \quad (0, \pi)$$

$$(a^*(u^*)')(0) = (a_N(u_N)')(0), \quad (a^*(u^*)')(\pi) = (a_N(u_N)')(\pi)$$

and $|u_N - u^*|_{L^\infty} \leq \frac{c}{N} \to 0$, while $|a_N - a^*|_{L^\infty} = \frac{1}{2}$ for every N. More sophisticated examples of a similar type can be found in Murat [17] and Alessandrini [2].

If, on the other hand, the goal were an accurate reconstruction of a^* in the H^{-1} norm, then the problem would be well-posed. Indeed, one can show (with some reasonable hypotheses) that if v solves

$$\begin{aligned} \nabla \cdot (a^* \nabla u^*) &= \nabla \cdot (b \nabla v) \quad \text{in} \quad \Omega \\ a^* \frac{\partial u^*}{\partial n} &= b \frac{\partial v}{\partial n} \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1.6}$$

then

$$\|a^* - b\|_{H^{-1}} \leq C \|u^* - v\|_{H_1}, \tag{1.6a}$$

where H^{-1} is the dual of $H_1(\Omega)$. One proof of (1.6a) is as follows. Assume that $u^* \in C^2(\bar{\Omega})$ and $|\nabla u^*| \neq 0$ in $\bar{\Omega}$. Lemma 5 of section 3, which is based on a theorem of da Veiga [8], shows that for each $\phi \in H_1(\Omega)$, there is a ψ_ϕ with

$$\begin{aligned} \nabla u^* \cdot \nabla \psi_\phi &= \phi \\ \|\psi_\phi\|_{H_1} &\leq C \|\phi\|_{H_1} \end{aligned} \tag{1.7}$$

where C is independent of ϕ. Using (1.6) and integrating by parts against ψ_ϕ gives

$$\int_\Omega (a^* - b) \nabla u^* \cdot \nabla \psi_\phi = \int_\Omega b \nabla (v - u^*) \cdot \nabla \psi_\phi .$$

Using (1.7), this gives

$$\begin{aligned} \left| \int_\Omega (a^* - b) \phi \right| &= \left| \int_\Omega b \nabla (u^* - v) \cdot \nabla \psi_\phi \right| \\ &\leq |b|_{L^\infty} \|\nabla (u^* - v)\|_{L_2} \|\nabla \psi_\phi\|_{L_2} \\ &\leq C \|u^* - v\|_{H_1} \|\phi\|_{H_1} . \end{aligned} \tag{1.8}$$

Since ϕ was an arbitrary H_1 function, this gives (1.6a). Unfortunately, a good approximation in H^{-1} of the coefficient doesn't have predictive value.

Our goal of estimating a^* in L_2 lies somewhere between the ill-posed and well-posed problems just discussed. In particular, if a^* is smooth and b is an element of some finite element space, perhaps associated with a triangulation of Ω of mesh h, then $\|a^* - b\|_{H_1} \leq Ch^{-1}\|a^* - b\|_{L_2} + O(h)$ by standard finite element results. Choosing $\phi = a^* - b$ in (1.8) gives

$$\|a^* - b\|_{L_2} \leq Ch^{-1}\|u^* - v\|_{H_1} + O(h^2)\,. \tag{1.9}$$

This inequality suggests that an algorithm producing a reconstructed coefficient a_p which lies in a *finite element space* should satisfy

$$\|a^* - a_p\|_{L_2} \sim h^{-1}\|u^* - u^m\|_{H_1},$$

at least if $f^m = f^*$, $g^m = g^*$. In other words, errors of numerical approximation may amplify errors of measurement — a well-known phenomenon in the numerical solution of ill-posed problems, see e.g. Natterer [18].

At least four methods have been proposed in the literature for the solution of our parameter identification problem. As already noted, one idea is to view (1.1) as a first order partial differential equation in $a^*(x)$ and to integrate along characteristics [21]. Other methods include a least-squares approach [10], the large time asymptotics of an associated dynamical system [11], and a singular perturbation technique [1]. A more detailed description of these approaches is given in the next section.

This paper develops a new, variational approach, which appears in some respects preferable to the existing ones. It is motivated by the simple observation that for any positive weights γ_1 and γ_2,

$$\int_\Omega |\sigma - a\,\nabla u^*|^2\,dx + \gamma_1 \int_\Omega |\operatorname{div}\sigma + f^*|^2\,dx + \\ + \gamma_2 \int_{\partial\Omega} |\sigma\,.\,n - g^*|^2\,ds \geq 0 \tag{1.10}$$

for any choice of $a(x)$ and any vector field σ, the minimum being achieved only when $\sigma = a\,\nabla u^*$ with $a(x)$ a solution of (1.1, 1.1a). Our method for reconstructing the unknown coefficient involves minimizing (1.10) numerically over suitable finite-dimensional spaces of coefficients and vector fields, using the measured data u^m, f^m, and g^m in (1.10). The weights γ_1 and γ_2 are chosen so that each term of the sum has the same magnitude for the choice $a = a_I^*$, $\sigma = (a^*\,\nabla u^*)_I$, where a_I^* and $(a^*\,\nabla u^*)_I$ are the interpolates of a^* and $a^*\,\nabla u^*$ respectively on the chosen spaces. This technique is similar to the method of equation error, see e.g. [19], but it takes advantage of the structure of the underlying equation.

Variations of (1.10) are possible. For example, one might consider using $\sigma . n = g^m$ and $\operatorname{div} \sigma = - f^m$ as constraints and minimizing $\|\sigma - a \nabla u^m\|_{L_2}^2$ or perhaps $\|\sigma - a \nabla u^m\|_{L_2}^2 + \varepsilon \|a\|_{H_1}^2$ for some small positive ε. Other possibilities include minimizing

$$\frac{1}{2} \int_\Omega |a^{-1/2} \sigma - a^{1/2} \nabla u^*|^2 \, dx \tag{1.10a}$$

with respect to $a(x) \geq 0$ (by hand), then with respect to σ satisfying $\operatorname{div} \sigma = - f^m$ and $\sigma . n = g^m$ (numerically). Minimization over $a(x)$ leads to the choice

$$a(x) = \frac{|\sigma|}{|\nabla u^*|}, \tag{1.11}$$

which on substitution into (1.10*a*) gives the convex functional

$$\int_\Omega (|\sigma| |\nabla u^*| - \langle \sigma, \nabla u^* \rangle) \, dx . \tag{1.12}$$

This modified approach will be useful in section 5, where we develop a method for reconstructing a *matrix* conductivity.

Choosing finite element spaces of continuous, piecewise polynomial a's and σ's, (1.10) becomes a quadratic minimization problem in $\mathbb{R}^q$, for some q. From a numerical standpoint, one could not hope for an easier minimization problem. A wide variety of numerical methods are available, including steepest descent, relaxation, and conjugate gradient.

The method proposed is a flexible one, by no means restricted to the specific problem treated here. For example, if N different solutions u_i^* $(1 \leq i \leq N)$ of (1.1, 1.1*a*) are known, corresponding to different choices f_i^*, g_i^*, we easily make use of them all by adding the N inequalities (1.10). More generally, the method seems suited to situations where the equation of state is the Euler-Lagrange equation of a convex variational problem.

We shall prove several stability theorems for the variational approach. The arguments draw heavily upon ideas introduced by Falk [9] and Alessandrini [1]. Let $\{\Delta_h\}$ be a family of regular, quasi-uniform triangulations of Ω, with each Δ_h composed of triangles of diameter $\leq h < 1$. Using piecewise linear, bounded a's, piecewise linear σ's, and piecewise polynomial u^m, the following results will be proved for sufficiently smooth a^* and u^*:

(1) *Assume condition* (1.4), *and let* $a_{p,h}$, $\sigma_{p,h}$ *minimize*

$$\|\sigma - a \nabla u^m\|_{L_2}^2 + h^2 \|\operatorname{div} \sigma + f^m\|_{L_2}^2 + h \|\sigma . n - g^m\|_{L_2(\partial\Omega)}^2 .$$

Then

$$\|a_{p,h} - a^*\|_{L_2} \leq C\left\{h + h^{-1}\|u^* - u^m\|_{H_1} + \|f^* - f^m\|_{L_2} + h^{-1/2}\|g^* - g^m\|_{L_2(\partial\Omega)}\right\} \tag{1.13}$$

In particular, if u^m is the piecewise quadratic interpolate of u^* on Δ_h, $f^m = f^*$, and $g^m = g^*$, then

$$\|a_{p,h} - a^*\|_{L_2} \leq Ch . \tag{1.13a}$$

This is Theorem 1 (stated more fully in section 3). It is similar to Falk's least-squares stability result ; however, Falk's argument required that $\nabla u^* . \tau > 0$ for some constant vector τ, whereas Theorem 1 assumes only the far weaker hypothesis (1.4). The second result uses a regularizing term $\nu\|\nabla a\|_{L^2}^2$ as part of the functional :

(2) *Let $a_{p,h}$, $\sigma_{p,h}$ minimize*

$$\|\sigma - a\,\nabla u^m\|_{L_2}^2 + h^2\|\operatorname{div}\sigma + f^m\|_{L_2}^2 + h\|\sigma . n - g^m\|_{L_2(\partial\Omega)}^2 + \nu\|\nabla a\|_{L_2}^2 ,$$

where $\quad \nu \sim (h^2 + \|u^* - u^m\|_{H_1} + h\|f^* - f^m\|_{L_2} + h^{1/2}\|g^* - g^m\|_{L_2(\partial\Omega)})^2.$

Then

$$\int_\Omega |a^* - a_{p,h}|\,|\nabla u^*|^2\,dx \leq C\left\{h + h^{-1}\|u^* - u^m\|_{H_1} + \|f^* - f^m\|_{L_2} + h^{-1/2}\|g^* - g^m\|_{L_2(\partial\Omega)}\right\} . \tag{1.14}$$

In particular, if u^m is the piecewise quadratic interpolate of u^* on Δ_h, $f^m = f^*$, $g^m = g^*$, and $\nu = h^4$, then

$$\int_\Omega |a^* - a_{p,h}|\,|\nabla u^*|^2\,dx \leq Ch .$$

This is Theorem 2 (see section 3). An analogous, weighted estimate for an output least-squares method is given in [15].

If condition (1.4) is satisfied, then Theorem 2 is weaker than Theorem 1, as Theorem 1 gives the same bound in L_2. Note however that the two results are based on minimization of slightly different functionals. If $\|u^m - u_I^*\|_{H_1} = O(\varepsilon)$, where u_I^* is the piecewise quadratic interpolate of u^* on Δ_h and $\varepsilon > 0$ due to random measurement error, then both (1.13) and (1.14) exhibit the εh^{-1} behavior suggested by (1.9).

These two results give convergence only if $\|u^* - u^m\|_{H_1} = O(h^\alpha)$ with

$\alpha > 1$. Piecewise linear measurements u^m — the simplest case numerically — are excluded, because they give $\|u^* - u^m\|_{H_1} = O(h)$ in general. However, a result that includes such u^m can be obtained by making a different choice of the weights :

(3) *Let* $a_{p,h}$, $\sigma_{p,h}$ *minimize*

$$\|\sigma - a\,\nabla u^m\|_{L_2}^2 + \|\operatorname{div}\sigma + f^m\|_{L_2}^2 + h^{-1}\|\sigma \,.\, n - g^m\|_{L_2(\partial\Omega)}^2 + \nu\|\nabla a\|_{L_2}^2$$

where $\nu \sim (h + \|u^* - u^m\|_{H_1} + \|f^* - f^m\|_{L_2} + h^{-1/2}\|g^* - g^m\|_{L_2(\partial\Omega)})^2$.
Then

$$\int_\Omega |a_{p,h} - a^*|\,|\nabla u^*|^2\,dx \leqslant C\left\{h^{1/2} + h^{-1/2}\|u^* - u^m\|_{H_1}\right.$$
$$\left. + h^{-1/2}\|f^* - f^m\|_{L_2} + h^{-1}\|g^* - g^m\|_{L_2(\partial\Omega)}\right\}.$$

In particular, if u^m *is the piecewise linear interpolate of* u^* *on* Δ_h, $f^m = f^*$, $g^m = g^*$, *and* $\nu = h^2$, *then*

$$\int_\Omega |a_{p,h} - a^*|\,|\nabla u^*|^2\,dx \leqslant Ch^{1/2}.$$

This is Theorem 3 (see section 3). If $|\nabla u^*| > 0$, then using the a priori L^∞ bound on $a_{p,h}$, we have

$$\|a_{p,h} - a^*\|_{L_2} \leqslant C\left\{\int_\Omega |a_{p,h} - a^*|\,|\nabla u^*|^2\,dx\right\}^{1/2} \leqslant Ch^{1/4}.$$

Under certain conditions (see Theorem 4 in section 3), this inequality can be improved to

$$\|a_{p,h} - a^*\|_{L_2} \leqslant Ch^{1/2}.$$

Numerical implementation has shown the method to work even better than predicted. A typical example is as follows. For $\Omega = (0,1)\times(0,1)$, let Δ_h be composed of triangles $(\frac{i-1}{N}, \frac{j-1}{N})$, $(\frac{i}{N}, \frac{j-1}{N})$, $(\frac{i}{N}, \frac{j}{N})$ and $(\frac{i-1}{N}, \frac{j-1}{N})$, $(\frac{i-1}{N}, \frac{j}{N})$, $(\frac{i}{N}, \frac{j}{N})$, $1 \leqslant i, j \leqslant N$, $h = \frac{1}{N}$. Take $a^* = 1 + y^2$, $u^* = x + y + \frac{1}{3}(x^3 + y^3)$ and let u^m, f^m, and g^m be the piecewise linear interpolates of u^*, f^*, and g^* on Δ_h. Minimization of the functional

$$\|\sigma - a\,\nabla u^m\|_{L_2}^2 + \|\operatorname{div}\sigma + f^m\|_{L_2}^2 + h^2\|\nabla a\|_{L_2}^2$$

over piecewise linear σ with $\sigma . n = g^m$ on $\partial\Omega$ and piecewise linear and bounded a appears to give

$$\| a_{p,h} - a^* \|_{L_2} \sim C h^{1.6} ,$$

where $a_{p,h}$ is the discrete solution. This is the observed behavior using different N in the range $10 \leq N \leq 50$. By contrast, the best estimate gives $\| a_{p,h} - a^* \|_{L_2} \leq C h^{1/2}$ — in fact, the observed convergence is even better than that predicted by (1.13a) for the piecewise quadratic interpolate of u^* ! More extensive information on numerical tests will be found in section 4.

Wexler and his coworkers [23, 24] recently and independently proposed a similar variational technique for a different but related problem. Their work does not, however, include stability or convergence estimates.

2. REVIEW OF METHODS

In this section we discuss some of the methods that have been used for solving our parameter identification problem.

A. Integrating along characteristics

It is natural to view (1.1) as a partial differential equation in $a^*(x)$ and to integrate along its characteristics. As already mentioned, Richter [20] proved that (1.3) can be solved uniquely under condition (1.4), if $a^*(x)$ is prescribed along the inflow boundary. In a companion paper, Richter [21] proposes a finite difference scheme for the solution of (1.3) under condition (1.4). The method is developed on the unit square $(0, 1) \times (0, 1)$, with an easy extension to more general domains in $\mathbb{R}^2$. He chooses the uniform grid

$$(x_i, y_j) = (ih, jh) , \quad 0 \leq i, j \leq n + 1 , \quad h = \frac{1}{n+1} ,$$

denoting by Ω_h the interior grid points and Γ_1^h the discrete inflow boundary (a boundary grid point is in Γ_1^h if its nearest neighboring grid point in Ω_h has a larger value of u^*). Equation (1.3) is discretized as

$$\frac{a_{i,j}^* - a_{k,j}^*}{h} \cdot \frac{u_{i,j}^* - u_{k,j}^*}{h} + \frac{a_{i,j}^* - a_{i,\ell}^*}{h} \cdot \frac{u_{i,j}^* - u_{i,\ell}^*}{h} + a_{i,j}^* \, Hu_{i,j}^* = - f_{i,j}^*$$

$$Hu_{i,j}^* = \frac{u_{i+1,j}^* + u_{i-1,j}^* + u_{i,j+1}^* + u_{i,j-1}^* - 4\, u_{i,j}^*}{h^2} \tag{2.1}$$

where k is the first index of the minimum of $\{ u^*_{i-1,j}, u^*_{i,j}, u^*_{i+1,j} \}$ and ℓ is the second index of the minimum of $\{ u^*_{i,j-1}, u^*_{i,j}, u^*_{i,j+1} \}$. Solving for $a^*_{i,j}$ gives

$$a^*_{i,j} = \frac{a^*_{k,j}\left(\dfrac{u^*_{i,j} - u^*_{k,j}}{h}\right) + a^*_{i,\ell}\left(\dfrac{u^*_{i,j} - u^*_{i,\ell}}{h}\right) - hf^*_{i,j}}{\left(\dfrac{u^*_{i,j} - u^*_{k,j}}{h}\right) + \left(\dfrac{u^*_{i,j} - u^*_{i,\ell}}{h}\right) + hHu^*_{i,j}} .$$

The reader will note that the scheme is explicit in the direction of increasing $u^*_{i,j}$. Hence, " the discrete solution is developed in a manner consistent with the characteristics of the continuous problem, which are curves of steepest ascent in u^*. "

Richter showed that under condition (1.4), the discrete problem (2.1) has a unique solution $a^*_{i,j}$ assuming prescribed values on Γ^h_1. Moreover, if u^* and a^* are sufficiently regular in $\bar{\Omega}$, $u^* \in C^3(\bar{\Omega})$ and $a^* \in C^2(\bar{\Omega})$, then

$$\max_{0 \leq i,j \leq n+1} \left| a^*_{i,j} - a^*(x_i, y_j) \right| = 0(h) \quad \text{as} \quad h \to 0$$

assuming $a^*_{i,j} = a^*(x_i, y_j)$ on Γ^h_1. Although this convergence theorem requires more regularity than our stability results (see section 3), it gives $O(h)$ convergence under perfect nodal measurements. In contrast, Theorem 1, with $f^m = f^*$ and $g^m = g^*$ gives $O(h)$ convergence in $L_2(\Omega)$ under piecewise quadratic interpolation of perfect nodal measurements of u^* on a triangulation Δ_h of mesh h. However, the situation where approximate measurements are given was not addressed by Richter. We note that the method involves approximation of second derivatives from discrete data, a source of potential instability.

B. Output Least-Squares Minimization

A second approach is the output least-squares method. To our knowledge, it was first applied to (1.1, 1.1*a*) by Frind & Pinder [10]. The least-squares philosophy says that if u^*_b is the solution of (1.1, 1.1*a*) with the coefficient a^* replaced with b, then b is a good approximation of a^* if the difference of the forward solutions, $u^* - u^*_b$, is small in $L_2(\Omega)$. For practical purposes, a finite dimensional implementation of this idea is necessary.

The method begins with a triangulation Δ_h of Ω of mesh h. Given an L_2 measurement u^m of u^*, select a finite dimensional class of coefficients $a_h \in A_h$. To each coefficient a_h, we associate a $u^*_{a_h} \in V_h$, where $u^*_{a_h}$ solves (1.1, 1.1*a*) in a Galerkin approximation :

$$\int_\Omega a_h \nabla u_{a_h}^* . \nabla v \, dx = \int_\Omega f^* v \, dx + \int_{\partial\Omega} g^* v \, ds \quad \forall v \in V_h$$

$$\int_\Omega u_{a_h}^* \, dx = \int_\Omega u^m \, dx$$

where V_h is a finite dimensional space. To find an approximation of a^*, we select that $a_h \in A_h$ which minimizes $\left\|u_{a_h}^* - u^m\right\|_{L_2}^2$ over A_h. Versions of this method have recently been studied by Kunisch and White [15, 16], and (using regularization) Kravaris and Seinfeld [14].

As already noted, a stability result for this technique was given by Falk [9] under the hypothesis that $\nabla u^* . \nu > 0$ for some constant vector ν. If A_h are continuous, piecewise polynomials of degree r on a triangulation Δ_h with $\alpha \leqslant a_h \leqslant \beta$, and V_h are continuous, piecewise polynomials of degree $r+1$ on Δ_h, then Falk's theorem is

$$\|a_h - a^*\|_{L_2} \leqslant C\,(h^r + h^{-2}\|u^* - u^m\|_{L_2}) ,$$

where
$$\left\|u_{a_h}^* - u^m\right\|_{L_2}^2 = \min_{b \in A_h} \|u_b^* - u^m\|_{L_2}^2 .$$

Our stability result is analogous, although we restrict our attention to piecewise linear a_h for numerical ease. In this case $r = 1$, and Falk's result gives

$$\|a^* - a_h\|_{L_2} \leqslant C\,(h + h^{-2}\|u^* - u^m\|_{L_2}) .$$

Moreover, instead of a general measurement u^m of u^* in L_2, our u^m is a piecewise polynomial interpolate of the measured u^* at nodes. Our attention was restricted to such u^m because this is what would normally be used in applications. Now Theorem 1, with $f^m = f^*$, and $g^m = g^*$, and standard finite element results give

$$\|a_{p,h} - a^*\|_{L_2} \leqslant C\left\{h + h^{-1}\|u^* - u^m\|_{H_1}\right\} \leqslant C\left\{h + h^{-2}\|u^* - u^m\|_{L_2}\right\} ,$$

where $a_{p,h}$ is the reconstructed conductivity defined in Theorem 1. Hence, the estimates are the same, although Theorem 1 holds under the more general condition

$$\inf_\Omega \max \{|\nabla u^*|, \Delta u^*\} > 0 .$$

Although the variational method might appear more unstable than the least-squares method, as it requires differentiation of the measurement u^m, the estimates give no such indication.

Output least-squares minimization is a commonly used method suited to many parameter identification problems, see e.g. Chavent [5] and Banks and Kunisch [3]. However, the above least-squares Galerkin method appears computationally onerous. Calculating the gradient of $G(a_h) = \left\|u^*_{a_h} - u^m\right\|^2_{L_2}$ requires solving an elliptic equation, and a steepest descent method would have to do this repeatedly. A further difficulty arises from the potential nonconvexity of G, and the consequent danger of getting trapped in a local minimum. Because the variational approach is based on the minimization of a convex functional, it has no such difficulty.

C. Singular Perturbation

Alessandrini [1] proposed a singular perturbation technique to determine the spatially varying coefficient in the *special case* $f^* = 0$. He solves the (stiff) elliptic problem

$$\begin{aligned} \varepsilon \, \Delta a_\varepsilon + \nabla \, . \, (a_\varepsilon \nabla u^*) = 0 \quad & \text{in} \quad \Omega \\ a_\varepsilon = a^* \quad & \text{on} \quad \partial\Omega \end{aligned} \tag{2.2}$$

with $\varepsilon > 0$ small, where it is now assumed that a^* is known on the entire boundary. Equation (2.2) is elliptic, and it can be written in divergence form as

$$\begin{aligned} \nabla \, . \, (e^{-u^*/\varepsilon} \nabla v) = 0 \quad & \text{in} \quad \Omega \\ v = e^{u^*/\varepsilon} a^* \quad & \text{on} \quad \partial\Omega \end{aligned} \tag{2.2a}$$

with $v = e^{u^*/\varepsilon} a_\varepsilon$. Using a very elegant argument, Alessandrini showed that

$$\int_\Omega |a_\varepsilon - a^*| \; |\nabla u^*|^2 \, dx \leq C \varepsilon^{1/2} \, .$$

He also proved a stability result with u^* in (2.2) replaced with an approximate measurement u^m. Since our Theorem 2 with perfect measurements gives the estimate

$$\int_\Omega |a_{p,h} - a^*| \; |\nabla u^*|^2 \, dx \leq Ch \, ,$$

(where $a_{p,h}$ is the reconstructed conductivity), his result is comparable to ours when $\varepsilon = h^2$.

D. Long-time behavior of an associated dynamical system

Hoffman and Sprekels [11] have proposed a new and ingenious technique to reconstruct coefficients in elliptic equations. It is not based on a minimization algorithm at all, but instead on the long time behavior of an associated dynamical system. The specific equation they considered is

$$-\nabla . (A^* \nabla u^*) = f^* \tag{2.3}$$

where $u^* \in H_1^0$. The algorithm seeks to determine a matrix A^* which solves (2.3).

For fixed $\varepsilon > 0$, they consider the dynamical system

$$\begin{gathered} -\varepsilon \frac{\partial}{\partial t} \Delta u(t) - \nabla . (A(t) \nabla u(t)) = f^* \\ \dot{A}(t) = \nabla u(t) \otimes \nabla (u(t) - u^*) \\ u(0) = u^0 \in H_1^0, \quad A(0) = A^0 \in L^\infty(\Omega) \text{ symmetric} \end{gathered} \tag{2.4}$$

where u^0 and A^0 are arbitrary initial conditions. Clearly, (2.4) has (2.3) as a steady state. Now, (2.4) has a unique solution $(u(t), A(t))$ for all t. They show that $u(t_m) \to u^*$ and $A(t_m) \to A_\infty$ for some subsequence $t_m \to \infty$, where A_∞ solves (2.3), under the hypothesis that (2.3) has a positive definite solution A^* ; the key tool is the energy estimate

$$\sup_{t \geq 0} \left\{ \|\nabla u(t) - \nabla u^*\|_{L_2}^2 + \|A(t) - A^*\|_{L_2}^2 \right\} + \\ + \int_0^\infty \|\nabla u(t) - \nabla u^*\|_{L_2}^2 \, dt \leq C < \infty$$

where $C = C(u^0, A^0, A^*)$. With (2.4) replaced with

$$\begin{gathered} -\varepsilon \frac{\partial}{\partial t} \Delta u(t) - \nabla . (A(t) \nabla u(t)) = f^* \\ \left(\varepsilon \frac{\partial}{\partial t} \nabla u(t) + A(t) \nabla u(t) \right) . n = g^* \\ \dot{A}(t) = \nabla u(t) \otimes \nabla (u(t) - u^*) \\ u(0) = u^0 \in H_1, \quad A(0) = A^0 \in L^\infty(\Omega) \quad \text{symmetric} \end{gathered}$$

the same techniques will give a matrix A_∞ satisfying

$$\begin{aligned} -\nabla . (A_\infty \nabla u^*) &= f^* \quad \text{in} \quad \Omega \\ A_\infty \nabla u^* . n &= g^* \quad \text{on} \quad \partial\Omega . \end{aligned}$$

For the purpose of calculation, a finite dimensional analogue of this technique must of course be used. Hoffman & Sprekels propose a semidiscrete Galerkin method. Under the same assumptions as before, energy estimates analogous to the continuous version are proved. In this case, the estimate is used to show that $A_\infty^n = \lim_{t \to \infty} A^n(t)$ and $A_\infty^n \xrightarrow{L_2(\Omega)} A_\infty$, where A_∞ solves (2.3), and $A^n(t)$ is the n-dimensional Galerkin solution of (2.4).

One problem with this method is that it gives a matrix coefficient, not a scalar one. Moreover, one can specify the Dirichlet data or the Neumann data, but not both simultaneously. In this context the solution of (2.3) is not unique. The method of [11] presumably chooses a particular solution, but it is not clear which one.

E. Further remarks

Our discussion has been limited to methods whose convergence has been proved at least in some cases. A variety of other methods could be considered — see for example the survey articles of Chavent [5], and Polis & Goodson [19].

The variational method is as good as (or better than) all of the methods just discussed, as far as stability and convergence theorems are concerned. Being a quadratic minimization problem, it is also extremely easy to implement. The method's principal disadvantage is the large number of variables it uses : if σ and a are piecewise linear on a triangulation with N^2 nodes, then the functional to be minimized depends on $3N^2$ variables.

3. CONVERGENCE THEOREMS FOR THE VARIATIONAL METHOD

In this section, we prove stability results for several forms of the variational method, under finite element approximation. Once again, the underlying elliptic equation is

$$\begin{aligned} -\nabla \cdot (a^*(x)\nabla u^*) &= f^* \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \\ a^*(x)\frac{\partial u^*}{\partial n} &= g^* \quad \text{on} \quad \partial\Omega . \end{aligned} \tag{3.1}$$

It will be assumed that $a^*(x)$ is a scalar with a priori bounds $0 < \alpha \leqslant a^*(x) \leqslant \beta < \infty$. In addition, the following is assumed,

(i) $u^* \in H_3(\Omega)$

(ii) $\Delta u^* \in C^0(\bar{\Omega})$

(iii) $a^* \in H_2(\Omega)$.

Let $\{\Delta_h\}$, $0 < h < 1$, be a family of regular, quasi-uniform triangulations of Ω, a Lipschitz continuous domain. That is, for each triangle $T \in \Delta_h$,

$$\frac{h_T}{\rho_T} \leqslant \sigma \quad \text{and} \quad \frac{h}{h_T} \leqslant \nu$$

where $h_T =$ diameter of $T \leqslant h$, and $\rho_T = \sup \{\text{diam}(S) ; S$ is a ball contained in $T\}$. Should the boundary be curved, then we use boundary triangles with one edge replaced by a segment of the boundary.

For a triangle $T \in \Delta_h$, we say

$$T \in \partial\Delta_h \quad \text{if} \quad \bar{T} \cap \partial\Omega \neq \varnothing .$$

Let us define the spaces

$$\begin{aligned} Q_h^{(k)} &= \left\{ w \in C^0(\bar{\Omega}) : w|_T \text{ is a polynomial of degree} \leqslant k, \forall T \in \Delta_h \right\} \\ A_h &= \left\{ w \in Q_h^{(1)} : 0 < \alpha \leqslant w \leqslant \beta \right\} \\ K_h &= Q_h^{(1)} \times Q_h^{(1)} . \end{aligned}$$

We now give a precise statement of Theorem 1.

THEOREM 1 : *Let* u^m, f^m, *and* g^m *be measurements of* u^*, f^*, *and* g^* *corresponding to equation* (3.1). *Assume that* $u^m \in Q_h^{(k)}$ *for some fixed* k *and* $\|u^* - u^m\|_{H_1} < \varepsilon$, $\|f^* - f^m\|_{L_2} < \lambda_1$, *and* $\|g^* - g^m\|_{L_2(\partial\Omega)} < \lambda_2$. *Let* $\sigma_{p,h} \in K_h$, $a_{p,h} \in A_h$ *be such that*

$$\begin{aligned} J(\sigma_{p,h}, a_{p,h}) &= \min_{\substack{\sigma \in K_h \\ a \in A_h}} J(\sigma, a) \\ &= \min_{\substack{\sigma \in K_h \\ a \in A_h}} \left\{ \|\sigma - a \nabla u^m\|_{L_2}^2 + h^2 \|\text{div}\, \sigma + f^m\|_{L_2}^2 \right. \\ &\quad \left. + h \|\sigma \cdot n - g^m\|_{L_2(\partial\Omega)}^2 \right\} . \end{aligned}$$

If $\inf_{\Omega} \max \{|\nabla u^*|, \Delta u^*\} > 0$, *then*

$$\|a_{p,h} - a^*\|_{L_2} \leqslant C \left\{ h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2} \lambda_2 \right\} ,$$

where C *is independent of* h, ε, λ_1, *and* λ_2. *If* $u^m = u_{h,I}^{*(2)}$, *the piecewise quadratic interpolate of* u^* *on* Δ_h, *then*

$$\|a_{p,h} - a^*\|_{L_2} \leqslant C \left\{ h + \lambda_1 + h^{-1/2} \lambda_2 \right\} .$$

Our proof will control the quantity $\|a_{p,h} - a^*\|_{L_2}$ by the magnitude of $J(\sigma_{p,h}, a_{p,h})$. Hence although Theorem 1 is stated for minimizers of J, an analogous result holds for any $\sigma \in K_h$, $a \in A_h$ which make J small enough.

Before proceeding with the proof of Theorem 1, some preliminary results are necessary.

LEMMA 1 :

(1*a*) *If* $\psi \in H_1(\Omega)$, *then for any* $T \in \Delta_h$,

$$\|\psi\|_{L_2(\partial T)} \leq C\left\{h^{-1/2}\|\psi\|_{L_2(T)} + h^{1/2}\|\nabla\psi\|_{L_2(T)}\right\}$$

where C is independent of h and T. Additionally,

$$\|\psi\|_{L_2(\partial\Omega)} \leq C\left\{h^{-1/2}\|\psi\|_{L_2(\Omega)} + h^{1/2}\|\nabla\psi\|_{L_2(\Omega)}\right\}.$$

(1*b*) *If* $\psi|_T$ *is a polynomial of degree* $\leq k$ *on each* $T \in \Delta_h$, *then*

$$\|\nabla\psi\|_{L_2(T)} \leq Ch^{-1}\|\psi\|_{L_2(T)}$$

where C is independent of h and T.

(1*c*) *If* $\phi_{h,I}^{(\ell)}$ *is the* ℓ-th *degree interpolate of* ϕ *on* Δ_h, *then for any* $T \in \Delta_h$

$$\|\phi - \phi_{h,I}^{(\ell)}\|_{H_m(T)} \leq Ch^{\ell+1-m}\|\phi\|_{H_{\ell+1}(T)}, \quad 0 \leq m \leq \ell\,,$$

where C is independent of h and T.

(1*d*) *For* $\phi = a_{p,h} - a^*$,

$$\|\nabla\phi\|_{L_2(\Omega)} \leq C\left\{h^{-1}\|\phi\|_{L_2(\Omega)} + h\|a^*\|_{H_2(\Omega)}\right\}$$

and

$$\|\phi\|_{L_2(\partial\Omega)} \leq C\left\{h^{-1/2}\|\phi\|_{L_2(\Omega)} + h^{3/2}\|a^*\|_{H_2(\Omega)}\right\}.$$

Proof: Statements (1*a*-*c*) are standard results from the theory of finite elements ; they can be found in Ciarlet [7] for polygonal domains and Scott [22] for domains with curved boundaries. Assertion (1*d*) is easily proved by using (1*b*, *c*) on each triangle, then adding and using (1*a*).

At this point, we adopt the convention $L_2 = L_2(\Omega)$ and $H_\ell = H_\ell(\Omega)$. In the following analysis, the constant C denotes a generic constant.

LEMMA 2 :

$$(J(\sigma_{p,h}, a_{p,h}))^{1/2} \leq C(h^2 + \varepsilon + h\lambda_1 + h^{1/2}\lambda_2)$$

where C is independent of ε, h, λ_1 *and* λ_2.

Proof: Let $\sigma_{h,I}^{*(1)}$ and $a_{h,I}^{*(1)}$ be the piecewise linear interpolates of $a^* \nabla u^*$ and a^* on Δ_h. Now

$$J(\sigma_{p,h}, a_{p,h}) \leq J(\sigma_{h,I}^{*(1)}, a_{h,I}^{*(1)}),$$

and so

$$(J(\sigma_{p,h}, a_{p,h}))^{1/2} \leq \|\sigma_{h,I}^{*(1)} - a_{h,I}^{*(1)} \nabla u^m\|_{L_2} + h\|\operatorname{div} \sigma_{h,I}^{*(1)} + f^m\|_{L_2} + h^{1/2}\|\sigma_{h,I}^{*(1)} \cdot n - g^m\|_{L_2(\partial\Omega)}.$$

Now

$$\begin{aligned}\|\sigma_{h,I}^{*(1)} - a_{h,I}^{*(1)} \nabla u^m\|_{L_2} &\leq \|\sigma_{h,I}^{*(1)} - a^* \nabla u^*\|_{L_2} + \|(a^* - a_{h,I}^{*(1)}) \nabla u^*\|_{L_2} \\ &\quad + \|a_{h,I}^{*(1)} \nabla (u^* - u^m)\|_{L_2} \\ &\leq C(h^2 + \varepsilon) \quad \text{by (1}c\text{)};\\ \|\operatorname{div} \sigma_{h,I}^{*(1)} + f^m\|_{L_2} &\leq \|\operatorname{div} (\sigma_{h,I}^{*(1)} - a^* \nabla u^*)\|_{L_2} + \|f^* - f^m\|_{L_2} \\ &\leq C(h + \lambda_1) \quad \text{by (1}c\text{)};\end{aligned}$$

and

$$\begin{aligned}\|\sigma_{h,I}^{*(1)} \cdot n - g^m\|_{L_2(\partial\Omega)} &\leq \\ &\leq \|\sigma_{h,I}^{*(1)} \cdot n - a^* \nabla u^* \cdot n\|_{L_2(\partial\Omega)} + \|g^* - g^m\|_{L_2(\partial\Omega)} \\ &\leq C\left\{h^{-1/2}\|\sigma_{h,I}^{*(1)} - a^* \nabla u^*\|_{L_2} + h^{1/2}\|\sigma_{h,I}^{*(1)} - a^* \nabla u^*\|_{H_1} \right. \\ &\quad \left. + \|g^* - g^m\|_{L_2(\partial\Omega)}\right\} \qquad \text{by (1}a\text{)} \\ &\leq C\{h^{-1/2}(Ch^2) + h^{1/2}(Ch)\} + \lambda_2 \qquad \text{by (1}c\text{)} \\ &\leq C\{h^{3/2} + \lambda_2\}.\end{aligned}$$

Consequently,

$$\begin{aligned}(J(\sigma_{p,h}, a_{p,h}))^{1/2} &\leq C(h^2 + \varepsilon) + h(C(h + \lambda_1)) + h^{1/2}(C(h^{3/2} + \lambda_2)) \\ &\leq C(h^2 + \varepsilon + h\lambda_1 + h^{1/2}\lambda_2).\end{aligned}$$

LEMMA 3 :

$$\|\sigma_{p,h} - a_{p,h}\nabla u^m\|_{L_2} \leqslant (J(\sigma_{p,h}, a_{p,h}))^{1/2} \leqslant C(h^2 + \varepsilon + h\lambda_1 + h^{1/2}\lambda_2)$$

$$\|\operatorname{div}\sigma_{p,h} + f^m\|_{L_2} \leqslant h^{-1}(J(\sigma_{p,h}, a_{p,h}))^{1/2} \leqslant C(h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2}\lambda_2)$$

$$\|\sigma_{p,h}\cdot n - g^m\|_{L_2(\partial\Omega)} \leqslant h^{-1/2}(J(\sigma_{p,h}, a_{p,h}))^{1/2}$$
$$\leqslant C(h^{3/2} + \varepsilon h^{-1/2} + h^{1/2}\lambda_1 + \lambda_2).$$

This follows immediately from the definition of $J(\sigma, a)$ and Lemma 2.

LEMMA 4 : *Suppose* $\inf_\Omega \max\{|\nabla u^*|, \Delta u^*\} > 0$. *Then there exist constants* $\lambda > 0$ *and* $\mu > 0$ *such that* $|\nabla u^*|^2 + \lambda\,\Delta u^* \geqslant \mu$ *on* Ω.

Proof: The condition $\inf_\Omega \max\{|\nabla u^*|, \Delta u^*\} > 0$ is equivalent to the existence of Ω_1, Ω_2, k_1, k_2 with

$$\Omega = \Omega_1 \cup \Omega_2$$
$$|\nabla u^*| \geqslant k_1 > 0 \quad \text{on} \quad \Omega_1$$
$$\Delta u^* \geqslant k_2 > 0 \quad \text{on} \quad \Omega_2.$$

The lemma follows easily by choosing λ sufficiently small.

We now begin the proof of Theorem 1.

Let $\phi = a_{p,h} - a^*$, $v = \phi w$ where $w = e^{-u^*/\lambda}$, λ being defined by Lemma 4. Calculation gives

$$\int_\Omega \phi\,\nabla u^*\,.\,\nabla v = \int_\Omega \phi^2\,\nabla u^*\,.\,\nabla w + \int_\Omega \phi\,\nabla\phi\,.\,w\,\nabla u^*.$$

Using the identity $\phi\,\nabla\phi = \frac{1}{2}\nabla\phi^2$, and integrating by parts,

$$\int_\Omega \phi\,\nabla\phi\,.\,w\,\nabla u^* = \frac{1}{2}\int_\Omega \nabla\phi^2\,.\,w\,\nabla u^*$$
$$= -\frac{1}{2}\int_\Omega \phi^2\,\nabla\,.\,(w\,\nabla u^*) + \frac{1}{2}\int_{\partial\Omega}\phi^2\,w\,\frac{\partial u^*}{\partial n}.$$

This in turn gives

$$\int_\Omega \phi\,\nabla u^*\,.\,\nabla v = \int_\Omega \phi^2\,\nabla u^*\,.\,\nabla w - \frac{1}{2}\int_\Omega \phi^2\,\nabla\,.\,(w\,\nabla u^*) + \frac{1}{2}\int_{\partial\Omega}\phi^2\,w\,\frac{\partial u^*}{\partial n}$$
$$= \frac{1}{2}\int_\Omega \phi^2(\nabla u^*\,.\,\nabla w - w\,\Delta u^*) + \frac{1}{2}\int_{\partial\Omega}\phi^2\,w\,\frac{\partial u^*}{\partial n}.$$

Now $\nabla u^* \cdot \nabla w - w\,\Delta u^* = -\dfrac{1}{\lambda} e^{-u^*/\lambda}(|\nabla u^*|^2 + \lambda\,\Delta u^*)$, and so we have

$$\int_\Omega \phi\,\nabla u^* \cdot \nabla v = -\frac{1}{2\lambda}\int_\Omega \phi^2 e^{-u^*/\lambda}(|\nabla u^*|^2 + \lambda\,\Delta u^*) + \frac{1}{2}\int_{\partial\Omega} \phi^2 w \frac{\partial u^*}{\partial n}.$$

Lemma 4 gives

$$\left(\min_{\bar\Omega} e^{-u^*/\lambda}\right) \mu \int_\Omega \phi^2 \leqslant \int_\Omega \phi^2 e^{-u^*/\lambda}(|\nabla u^*|^2 + \lambda\,\Delta u^*)$$
$$= 2\lambda \left\{\frac{1}{2}\int_{\partial\Omega} \phi^2 w \frac{\partial u^*}{\partial n} - \int_\Omega \phi\,\nabla u^* \cdot \nabla v\right\},$$

from which it follows that

$$\|\phi\|_{L_2}^2 \leqslant \frac{2\lambda}{\mu}\left(\min_{\bar\Omega} e^{-u^*/\lambda}\right)^{-1}\left\{\frac{1}{2}\left|\int_{\partial\Omega} \phi^2 w \frac{\partial u^*}{\partial n}\right| + \left|\int_\Omega \phi\,\nabla u^* \cdot \nabla v\right|\right\}. \tag{3.2}$$

We shall now estimate the right hand side of (3.2).

$$\int_{\partial\Omega} \phi^2 w \frac{\partial u^*}{\partial n} = \int_{\partial\Omega} w\phi(a_{p,h} - a^*)\frac{\partial u^*}{\partial n} = \int_{\partial\Omega} w\phi\left(a_{p,h}\frac{\partial u^*}{\partial n} - g^*\right).$$

Hence,

$$\left|\int_{\partial\Omega} \phi^2 w \frac{\partial u^*}{\partial n}\right| \leqslant |w|_{\infty,\partial\Omega}\,\|\phi\|_{L_2(\partial\Omega)}\left\|a_{p,h}\frac{\partial u^*}{\partial n} - g^*\right\|_{L_2(\partial\Omega)}. \tag{3.3}$$

Now,

$$\left\|a_{p,h}\frac{\partial u^*}{\partial n} - g^*\right\|_{L_2(\partial\Omega)} \leqslant \|(\sigma_{p,h} - a_{p,h}\nabla u^*)\cdot n\|_{L_2(\partial\Omega)}$$
$$+ \|\sigma_{p,h}\cdot n - g^m\|_{L_2(\partial\Omega)} + \|g^m - g^*\|_{L_2(\partial\Omega)}$$
$$\leqslant \|(\sigma_{p,h} - a_{p,h}\nabla u^*)\cdot n\|_{L_2(\partial\Omega)} \tag{3.4}$$
$$+ C(h^{3/2} + \varepsilon h^{-1/2} + h^{1/2}\lambda_1 + \lambda_2)$$

by Lemma 3. Moreover,

$$\|(\sigma_{p,h} - a_{p,h}\nabla u^*)\cdot n\|_{L_2(\partial\Omega)}^2 \leqslant \sum_{T\in\partial\Delta_h} \|\sigma_{p,h} - a_{p,h}\nabla u^*\|_{L_2(\partial T)}^2$$
$$\leqslant 2\sum_{T\in\partial\Delta_h}\left\{\|\sigma_{p,h} - a_{p,h}\nabla u^m\|_{L_2(\partial T)}^2\right.$$
$$\left. + \|a_{p,h}\nabla(u^m - u^*)\|_{L_2(\partial T)}^2\right\},$$

whence,

$$\|(\sigma_{p,h} - a_{p,h}\nabla u^*) \cdot n\|^2_{L_2(\partial\Omega)} \leq 2 \sum_{T \in \partial\Delta_h} \left\{ \|\sigma_{p,h} - a_{p,h}\nabla u^m\|^2_{L_2(\partial T)} + \beta^2 \|\nabla(u^m - u^*)\|^2_{L_2(\partial T)} \right\} . \tag{3.5}$$

Using (1*a*) and (1*b*), one easily verifies that

$$\|\sigma_{p,h} - a_{p,h}\nabla u^m\|_{L_2(\partial T)} \leq C h^{-1/2} \|\sigma_{p,h} - a_{p,h}\nabla u^m\|_{L_2(T)} . \tag{3.6}$$

For $u^{*(2)}_{h,I}$, the piecewise quadratic interpolate of u^* on Δ_h, using (1*b*) and (1*c*), one easily verifies that

$$\begin{aligned} \|u^* - u^m\|_{H_2(T)} &\leq \|u^* - u^{*(2)}_{h,I}\|_{H_2(T)} + \|u^{*(2)}_{h,I} - u^m\|_{H_2(T)} \\ &\leq C \left\{ h\|u^*\|_{H_3(T)} + h^{-1}\|u^* - u^m\|_{H_1(T)} \right\} . \end{aligned}$$

This, along with (1*a*) gives

$$\|\nabla(u^m - u^*)\|_{L_2(\partial T)} \leq C \left\{ h^{-1/2} \|u^* - u^m\|_{H_1(T)} + h^{3/2} \|u^*\|_{H_3(T)} \right\} . \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) gives

$$\|(\sigma_{p,h} - a_{p,h}\nabla u^*) \cdot n\|^2_{L_2(\partial\Omega)} \leq C \left\{ h^{-1} \|\sigma_{p,h} - a_{p,h}\nabla u^m\|^2_{L_2} + h^{-1} \|u^* - u^m\|^2_{H_1} + h^3 \|u^*\|^2_{H_3} \right\} .$$

Using Lemma 3 gives

$$\|(\sigma_{p,h} - a_{p,h}\nabla u^*) \cdot n\|_{L_2(\partial\Omega)} \leq C \left\{ h^{3/2} + \varepsilon h^{-1/2} + h^{1/2}\lambda_1 + \lambda_2 \right\} .$$

Combining this estimate with (3.3), (3.4) and (1*d*) gives

$$\left| \int_{\partial\Omega} \phi^2 w \frac{\partial u^*}{\partial n} \right| \leq C (h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2}\lambda_2)(\|\phi\|_{L_2} + h^2) . \tag{3.8}$$

We now estimate the term $\left| \int_\Omega \phi \nabla u^* \cdot \nabla v \right|$ in (3.2) :

$$\begin{aligned} -\int_\Omega \phi \nabla u^* \cdot \nabla v &= \int_\Omega (a^* - a_{p,h}) \nabla u^* \cdot \nabla v \\ &= \int_\Omega a^* \nabla u^* \cdot \nabla v - \int_\Omega a_{p,h} \nabla u^* \cdot \nabla v \\ &= \int_\Omega f^* v + \int_{\partial\Omega} g^* v - \int_\Omega a_{p,h} \nabla u^* \cdot \nabla v \end{aligned}$$

using the variational form of equation (3.1). Adding and subtracting various terms,

$$- \int_\Omega \phi \, \nabla u^* \, . \, \nabla v =$$

$$= \int_\Omega (f^* - f^m) \, v + \int_\Omega (f^m + \operatorname{div} \sigma_{p,h}) \, v - \int_\Omega (\operatorname{div} \sigma_{p,h}) \, v$$
$$+ \int_{\partial\Omega} (\sigma_{p,h} \, . \, n) \, v + \int_{\partial\Omega} (g^m - \sigma_{p,h} \, . \, n) \, v$$
$$+ \int_{\partial\Omega} (g^* - g^m) \, v - \int_\Omega a_{p,h} \nabla u^* \, . \, \nabla v \, .$$

By Green's Theorem,

$$\int_{\partial\Omega} (\sigma_{p,h} \, . \, n) \, v - \int_\Omega (\operatorname{div} \sigma_{p,h}) \, v = \int_\Omega \sigma_{p,h} \, . \, \nabla v \, ,$$

and so

$$- \int_\Omega \phi \, \nabla u^* \, . \, \nabla v = \int_\Omega (f^* - f^m) \, v + \int_\Omega (f^m + \operatorname{div} \sigma_{p,h}) \, v$$
$$+ \int_{\partial\Omega} (g^m - \sigma_{p,h} \, . \, n) \, v + \int_{\partial\Omega} (g^* - g^m) \, v$$
$$+ \int_\Omega (\sigma_{p,h} - a_{p,h} \nabla u^m) \, . \, \nabla v + \int_\Omega a_{p,h} \nabla (u^m - u^*) \, . \, \nabla v \, .$$

Therefore

$$\left| \int_\Omega \phi \, \nabla u^* \, . \, \nabla v \right| \leqslant (\| f^* - f^m \|_{L_2} + \| \operatorname{div} \sigma_{p,h} + f^m \|_{L_2}) \| v \|_{L_2}$$
$$+ (\| \sigma_{p,h} \, . \, n - g^m \|_{L_2(\partial\Omega)} + \| g^* - g^m \|_{L_2(\partial\Omega)}) \| v \|_{L_2(\partial\Omega)}$$
$$+ (\| \sigma_{p,h} - a_{p,h} \nabla u^m \|_{L_2} + | a_{p,h} |_{\infty, \Omega} \, \| u^* - u^m \|_{H_1}) \| \nabla v \|_{L_2} \, ,$$

whence using (1*a*)

$$\left| \int_\Omega \phi \, \nabla u^* \, . \, \nabla v \right| \leqslant c_0 \, \| v \|_{L_2} + c_1 \, \| \nabla v \|_{L_2} \tag{3.9}$$

where

$$\begin{aligned} c_0 = \; & \| f^* - f^m \|_{L_2} + \| \operatorname{div} \sigma_{p,h} + f^m \|_{L_2} \\ & + C h^{-1/2} (\| \sigma_{p,h} \, . \, n - g^m \|_{L_2(\partial\Omega)} + \| g^* - g^m \|_{L_2(\partial\Omega)}) \end{aligned} \tag{3.9a}$$

and

$$c_1 = \|\sigma_{p,h} - a_{p,h} \nabla u^m\|_{L_2} + \beta \|u^* - u^m\|_{H_1} + Ch^{1/2}(\|\sigma_{p,h} \cdot n - g^m\|_{L_2(\partial\Omega)} + \|g^* - g^m\|_{L_2(\partial\Omega)}) . \tag{3.9b}$$

Lemma 3 gives

$$\left| \int_\Omega \phi \nabla u^* \cdot \nabla v \right| \leqslant C(h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2} \lambda_2)(\|v\|_{L_2} + h \|\nabla v\|_{L_2}) . \tag{3.10}$$

Since $v = \phi w$, we have

$$\|v\|_{L_2} \leqslant C \|\phi\|_{L_2}$$

$$\|\nabla v\|_{L_2} \leqslant C \|\phi\|_{H_1} \leqslant C(h^{-1} \|\phi\|_{L_2} + h) ,$$

using $(1d)$ in the latter step. It follows that

$$\left| \int_\Omega \phi \nabla u^* \cdot \nabla v \right| \leqslant C(h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2} \lambda_2)(\|\phi\|_{L_2} + h^2) . \tag{3.11}$$

Combining (3.2), (3.8) and (3.11) gives

$$\|\phi\|^2_{L_2} \leqslant C(h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2} \lambda_2)(\|\phi\|_{L_2} + h^2) ,$$

whence

$$\|a_{p,h} - a^*\|_{L_2} \leqslant C\left\{h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2} \lambda_2\right\} ,$$

the principal assertion of the theorem. If $u^m = u^{*(2)}_{h,I}$, then $\|u^* - u^m\|_{H_1} \leqslant Ch^2$ by $(1c)$, and so

$$\|a_{p,h} - a^*\|_{L_2} \leqslant C\left\{h + \lambda_1 + h^{-1/2} \lambda_2\right\} .$$

The proof of Theorem 1 is now complete.

When ∇u^* vanishes in Ω, the uniqueness of the coefficient comes into doubt (although the recent results of Chicone and Gerlach apply in some cases [6]). Nevertheless, *even if* ∇u^* *vanishes on a set of positive measure,* it is still possible to get stability where $|\nabla u^*| \neq 0$. The best result is obtained by including a regularizing term $\nu \|\nabla a\|^2_{L_2}$ as part of the functional. The proof of the following result is strongly motivated by the work of Alessandrini [1].

THEOREM 2 : *Let u^m, f^m, and g^m be as in Theorem 1, with $\|u^* - u^m\|_{H_1} < \varepsilon$, $\|f^* - f^m\|_{L_2} < \lambda_1$, and $\|g^* - g^m\|_{L_2(\partial\Omega)} < \lambda_2$. Let $\sigma_{p,h} \in K_h$, $a_{p,h} \in A_h$ be such that*

$$J(\sigma_{p,h}, a_{p,h}) = \min_{\substack{\sigma \in K_h \\ a \in A_h}} J(\sigma, a)$$

$$= \min_{\substack{\sigma \in K_h \\ a \in A_h}} \Big\{ \|\sigma - a \nabla u^m\|^2_{L_2} + h^2 \|\operatorname{div} \sigma + f^m\|^2_{L_2}$$

$$+ h \|\sigma \cdot n - g^m\|^2_{L_2(\partial\Omega)} + \nu \|\nabla a\|^2_{L_2}\Big\}$$

where $\nu \sim (h^2 + \varepsilon + h\lambda_1 + h^{1/2}\lambda_2)^2$. Then

$$\int_\Omega |a_{p,h} - a^*| \, |\nabla u^*|^2 \leqslant C \left\{h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2}\lambda_2\right\}$$

where C is independent of ε, h, λ_1 and λ_2. If $u^m = u^{*(2)}_{h,I}$, the piecewise quadratic interpolate of u^* on Δ_h, then

$$\int_\Omega |a_{p,h} - a^*| \, |\nabla u^*|^2 \leqslant C \left\{h + \lambda_1 + h^{-1/2}\lambda_2\right\} .$$

Proof: Because $\nu \sim (h^2 + \varepsilon + h\lambda_1 + h^{1/2}\lambda_2)^2$, the bound

$$J(\sigma_{p,h}, a_{p,h})^{1/2} \leqslant C (h^2 + \varepsilon + h\lambda_1 + h^{1/2}\lambda_2) \tag{3.12}$$

still holds. In addition, we have

$$\|\nabla a_{p,h}\|_{L_2} \leqslant C . \tag{3.13}$$

By the same argument as used in Theorem 1, (3.9, 9*a*, 9*b*) leads to

$$\left| \int_\Omega \phi \nabla u^* \cdot \nabla v \right| \leqslant C (h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2}\lambda_2)(\|v\|_{L_2} + h \|\nabla v\|_{L_2}) \tag{3.14}$$

for all $v \in H_1$, where $\phi = a_{p,h} - a^*$.

We shall rewrite the left side of (3.14) as a sum of several terms. First, we approximate ϕ by a sequence of smooth functions $\psi_r \in C^\infty(\bar{\Omega})$ with

$$\|\psi_r - \phi\|_{H_1(\Omega)} \to 0 \quad \text{as} \quad r \to \infty . \tag{3.15}$$

Next, for $\delta > 0$ and r fixed, we consider

$$v_1 = \delta^{-1}(\psi_r^+ \wedge \delta)\, u^*, \quad v_2 = \delta^{-1}((-\psi_r)^+ \wedge \delta)\, u^*, \tag{3.16}$$

where $\psi_r^+ = \max \{\psi_r, 0\}$, $\psi_r^+ \wedge \delta = \min \{\psi_r^+, \delta\}$. Since $\|\psi_r\|_{H_1} \leq C$ as a consequence of (3.13) and (3.15), we have

$$\begin{aligned} \|v_i\|_{L_2} &\leq \|u^*\|_{L_2} \\ \|\nabla v_i\|_{L_2} &\leq C(\delta^{-1} + 1)\,. \end{aligned} \tag{3.17}$$

We claim that

$$\int_{\psi_r \geq \delta} \psi_r\, |\nabla u^*|^2 \leq C\delta + \left| \int_\Omega \psi_r\, \nabla u^* \,.\, \nabla v_1 \right| \tag{3.18a}$$

$$-\int_{\psi_r \leq -\delta} \psi_r\, |\nabla u^*|^2 \leq C\delta + \left| \int_\Omega \psi_r\, \nabla u^* \,.\, \nabla v_2 \right|. \tag{3.18b}$$

It will be enough to prove (3.18*a*), since the proof of (3.18*b*) is parallel in every respect. The definitions (3.16) give

$$\int_\Omega \psi_r\, \nabla u^* \,.\, \nabla v_1 = \int_{\psi_r \geq \delta} \psi_r\, |\nabla u^*|^2 + \\ + \delta^{-1} \int_{0 < \psi_r < \delta} \psi_r^2\, |\nabla u^*|^2 + \delta^{-1} \int_{0 < \psi_r < \delta} \psi_r\, \nabla\psi_r \,.\, u^*\, \nabla u^*\,.$$

The second term on the right satisfies

$$\delta^{-1} \int_{0 < \psi_r < \delta} \psi_r^2\, |\nabla u^*|^2 \leq C\delta\,. \tag{3.19}$$

In estimating the third term, we shall suppose (without loss of generality) that 0 and δ are regular values of ψ_r. Then

$$\begin{aligned} \delta^{-1} \int_{0 < \psi_r < \delta} \psi_r\, \nabla\psi_r \,.\, u^*\, \nabla u^* &= (4\,\delta)^{-1} \int_{0 < \psi_r < \delta} \nabla\psi_r^2 \,.\, \nabla(u^*)^2 \\ &= (4\,\delta)^{-1} \left\{ \int_{\partial\{0 < \psi_r < \delta\}} \psi_r^2 \frac{\partial (u^*)^2}{\partial n} - \int_{0 < \psi_r < \delta} \psi_r^2\, \Delta (u^*)^2 \right\}. \end{aligned} \tag{3.20}$$

Let us estimate the boundary term : we have $\partial\{0 < \psi_r < \delta\} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \{\psi_r = 0\} \cap \Omega, \quad \Gamma_2 = \{\psi_r = \delta\} \cap \Omega, \quad \Gamma_3 = \{0 \leq \psi_r \leq \delta\} \cap \partial\Omega\,.$$

Clearly

$$\int_{\Gamma_1} \psi_r^2 \frac{\partial (u^*)^2}{\partial n} = 0 .$$

As for Γ_2,

$$\begin{aligned}\int_{\Gamma_2} \psi_r^2 \frac{\partial (u^*)^2}{\partial n} &= \delta^2 \int_{\partial \{\psi_r < \delta\} \cap \Omega} \frac{\partial (u^*)^2}{\partial n} \\ &= \delta^2 \int_{\psi_r < \delta} \Delta (u^*)^2 - \delta^2 \int_{\partial \{\psi_r < \delta\} \cap \partial\Omega} \frac{\partial (u^*)^2}{\partial n}\end{aligned}$$

whence

$$\left| \int_{\Gamma_2} \psi_r^2 \frac{\partial (u^*)^2}{\partial n} \right| \leqslant C\delta^2 ,$$

using that $\partial (u^*)^2/\partial n$ is bounded on $\partial\Omega$. Similarly,

$$\left| \int_{\Gamma_3} \psi_r^2 \frac{\partial (u^*)^2}{\partial n} \right| \leqslant C\delta^2 .$$

The interior term in (3.20) is easy to estimate :

$$\left| \int_{0 < \psi_r < \delta} \psi_r^2 \, \Delta (u^*)^2 \right| \leqslant C\delta^2 .$$

Combining these results with (3.20) gives

$$- \delta^{-1} \int_{0 < \psi_r < \delta} \psi_r \, \nabla\psi_r \, . \, u^* \, \nabla u^* \leqslant C\delta , \tag{3.21}$$

which together with (3.19) establishes (3.18*a*).

Assembling (3.18*a*) and (3.18*b*) with the obvious result

$$\int_{|\psi_r| < \delta} |\psi_r| \, |\nabla u^*|^2 \leqslant C\delta ,$$

we conclude that

$$\int_{\Omega} |\psi_r| \, |\nabla u^*|^2 \leqslant C \left\{ \delta + \left| \int_{\Omega} \psi_r \, \nabla u^* \, . \, \nabla v_1 \right| + \left| \int_{\Omega} \psi_r \, \nabla u^* \, . \, \nabla v_2 \right| \right\} .$$

The same follows with ψ_r replaced by $\phi = a_{p,h} - a^*$, by passage to the limit

$r \to \infty$. Applying (3.14) with $v = v_1$ and $v = v_2$, and using (3.17), we conclude that

$$\int_\Omega |\phi| |\nabla u^*|^2 \leq C \{\delta + (h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2} \lambda_2)(1 + h\delta^{-1})\} .$$

The best choice for δ is $\delta \sim h$; it gives

$$\int_\Omega |a_{p,h} - a^*| |\nabla u^*|^2 \leq C \{h + \varepsilon h^{-1} + \lambda_1 + h^{-1/2} \lambda_2\} ,$$

the principal assertion of the theorem. If $u^m = u_{h,I}^{*(2)}$, then $\varepsilon \sim h^2$ and so

$$\int_\Omega |a_{p,h} - a^*| |\nabla u^*|^2 \leq C \{h + \lambda_1 + h^{-1/2} \lambda_2\} .$$

This concludes the proof of Theorem 2.

We note with regard to Theorem 2 that the constant C doesn't contain exponentials. Also, if u^* and a^* are sufficiently regular, the proof we have given works in any space dimension. Finally, it was not essential that $u^m \in Q_h^{(k)}$ for some k: in fact, the result holds for arbitrary $u^m \in H_1(\Omega)$.

As already noted, these two theorems give convergence only if $\|u^* - u^m\|_{H_1} = O(h^\alpha)$ with $\alpha > 1$. In applications, u^* is measured at well sites and u^m is most easily taken as the piecewise linear interpolate of the measured u^*. A convergence result for piecewise linear u^m is obtained by making a different choice of weights in the functional:

THEOREM 3: *Let $u^m \in Q_h^{(1)}$, f^m, g^m be as in Theorem 1 with $\|u^* - u^m\|_{H_1} < \varepsilon$, $\|f^* - f^m\|_{L_2} < \lambda_1$, and $\|g^* - g^m\|_{L_2(\partial\Omega)} < \lambda_2$. Let $\sigma_{p,h} \in K_h$, $a_{p,h} \in A_h$ be such that*

$$\begin{aligned} J(\sigma_{p,h}, a_{p,h}) &= \min_{\substack{\sigma \in K_h \\ a \in A_h}} J(\sigma, a) \\ &= \min_{\substack{\sigma \in K_h \\ a \in A_h}} \Big\{ \|\sigma - a \nabla u^m\|_{L_2}^2 + \|\operatorname{div} \sigma + f^m\|_{L_2}^2 \\ &\quad + h^{-1} \|\sigma . n - g^m\|_{L_2(\partial\Omega)}^2 + \nu \|\nabla a\|_{L_2}^2 \Big\} \end{aligned}$$

where $\nu \sim (\varepsilon + h + \lambda_1 + h^{-1/2} \lambda_2)^2$. Then

$$\int_\Omega |a_{p,h} - a^*| |\nabla u^*|^2 \leq C \{h^{1/2} + \varepsilon h^{-1/2} + h^{-1/2} \lambda_1 + h^{-1} \lambda_2\}$$

where C is independent of h, ε, λ_1 and λ_2. If $u^m = u_{h,I}^{*(1)}$, then

$$\int_\Omega |a_{p,h} - a^*|\,|\nabla u^*|^2 \leq C\left\{h^{1/2} + h^{-1/2}\lambda_1 + h^{-1}\lambda_2\right\}.$$

Proof: Proceeding as in Lemma 2, it is easily verified that $(J(\sigma_{p,h}, a_{p,h}))^{1/2} \leq C\left\{\varepsilon + h + \lambda_1 + h^{-1/2}\lambda_2\right\}$ and $\|\nabla a_{p,h}\|_{L_2} \leq C$. With v_i $(i = 1, 2)$ chosen as in Theorem 2, the method of Theorem 2 gives

$$\int_\Omega |a_{p,h} - a^*|\,|\nabla u^*|^2 \leq C\left\{\delta + (\varepsilon + h + \lambda_1 + h^{-1/2}\lambda_2)(1 + \delta^{-1})\right\}.$$

Choosing the optimal $\delta \sim h^{1/2}$ gives

$$\int_\Omega |a_{p,h} - a^*|\,|\nabla u^*|^2 \leq C\left\{h^{1/2} + \varepsilon h^{-1/2} + h^{-1/2}\lambda_1 + h^{-1}\lambda_2\right\}.$$

If $u^m = u_{h,I}^{*(1)}$, then $\varepsilon \sim h$ and

$$\int_\Omega |a_{p,h} - a^*|\,|\nabla u^*|^2 \leq C\left\{h^{1/2} + h^{-1/2}\lambda_1 + h^{-1}\lambda_2\right\}.$$

This concludes the proof of Theorem 3.

As in Theorem 2, it was not essential that $u^m \in Q_h^{(1)}$. All that is necessary is that $u^m \in H_1(\Omega)$.

In contrast to Falk's argument in [9], the proof of Theorem 1 does not involve the solution of a hyperbolic equation. However, we can do a little better than Theorem 3 if u^* satisfies the following condition :

$$\begin{array}{l}\text{For all } \psi \in H_1(\Omega), \text{ the equation } \nabla u^* \cdot \nabla v_\psi = \psi \\ \text{has a solution with } \|v_\psi\|_{H_1} \leq C\,\|\psi\|_{H_1}.\end{array} \tag{3.22}$$

If this condition holds (see Lemma 5), then estimate (3.9, 9*a*, 9*b*), with $v = v_\psi$ gives an estimate on $\|a_{p,h} - a^*\|_{H^{-1}}$ ($H^{-1} =$ dual of H_1). Indeed,

$$\left|\int_\Omega (a_{p,h} - a^*)\,\psi\right| \leq c_0\,\|v_\psi\|_{L_2} + c_1\,\|\nabla v_\psi\|_{L_2}$$

where the constants c_0 and c_1 are given by (3.9*a*) and (3.9*b*) respectively.

Hence,

$$\left|\int_\Omega (a_{p,h} - a^*)\,\psi\right| \leq (c_0 + c_1)\|v_\psi\|_{H_1} \leq C\,(c_0 + c_1)\|\psi\|_{H_1}$$

and so

$$\|a_{p,h} - a^*\|_{H^{-1}} \leq C(c_0 + c_1).$$

Using the inequality $\|a_{p,h} - a^*\|_{L_2} \leq C \|a_{p,h} - a^*\|_{H^{-1}}^{1/2} \|a_{p,h} - a^*\|_{H_1}^{1/2}$, we have :

THEOREM 4 : *Let* u^m, f^m, *and* g^m *be measurements of* u^*, f^*, *and* g^* *with* $\|u^* - u^m\|_{H_1} < \varepsilon$, $\|f^* - f^m\|_{L_2} < \lambda_1$, *and* $\|g^* - g^m\|_{L_2(\partial\Omega)} < \lambda_2$. *Let* $\sigma_{p,h} \in K_h$, $a_{p,h} \in A_h$ *be such that*

$$J(\sigma_{p,h}, a_{p,h}) = \min_{\substack{\sigma \in K_h \\ a \in A_h}} \left\{ \|\sigma - a\nabla u^m\|_{L_2}^2 + \|\operatorname{div}\sigma + f^m\|_{L_2}^2 \right.$$

$$\left. + h^{-1}\|\sigma . n - g^m\|_{L_2(\partial\Omega)}^2 + \nu \|\nabla a\|_{L_2}^2 \right\}$$

where $\nu \sim (\varepsilon + h + \lambda_1 + h^{-1/2}\lambda_2)^2$. *Then if condition* (3.22) *is satisfied,*

$$\|a_{p,h} - a^*\|_{L_2} \leq C(\varepsilon + h + \lambda_1 + h^{-1/2}\lambda_2)^{1/2}. \tag{3.23a}$$

If $f^m = f_{h,l}^{*(1)}$, $g^m = g_{h,l}^{*(1)}$, *and*

$$J(\sigma_{p,h}, a_{p,h}) = \min_{\substack{\sigma \in K_h,\, a \in A_h \\ \sigma . n = g_{h,l}^{*(1)} \text{ on } \partial\Omega}} \left\{ \|\sigma - a\nabla u^m\|_{L_2}^2 + \|\operatorname{div}\sigma + f^m\|_{L_2}^2 + \nu\|\nabla a\|_{L_2}^2 \right\}$$

then

$$\|a_{p,h} - a^*\|_{L_2} \leq C(\varepsilon + h)^{1/2}. \tag{3.23b}$$

If $u^m = u_{h,l}^{*(1)}$, *then* $\|a_{p,h} - a^*\|_{L_2} \leq Ch^{1/2}$.

Proof: Proceeding as in Lemma 2, it is easily verified that

$$c_0 \leq C(\varepsilon + h + \lambda_1 + h^{-1/2}\lambda_2), \quad c_1 \leq C(\varepsilon + h + \lambda_1 + h^{-1/2}\lambda_2)$$

for the first functional and $c_0 \leq C(\varepsilon + h)$, $c_1 \leq C(\varepsilon + h)$ for the second functional. In each case, $\|\nabla a_{p,h}\|_{L_2} \leq C$. Hence,

$$\begin{aligned}\|a_{p,h} - a^*\|_{L_2} &\leq C\|a_{p,h} - a^*\|_{H^{-1}}^{1/2}\|a_{p,h} - a^*\|_{H_1}^{1/2} \\ &\leq C\|a_{p,h} - a^*\|_{H^{-1}}^{1/2}(\|a_{p,h}\|_{H_1} + \|a^*\|_{H_1})^{1/2} \\ &\leq C\|a_{p,h} - a^*\|_{H^{-1}}^{1/2},\end{aligned}$$

which gives (3.23*a*) and (3.23*b*). If $u^m = u_{h,I}^{*(1)}$ in the second functional $J(\sigma, a)$, then $\varepsilon \sim h$ and $\|a_{p,h} - a^*\|_{L_2} \leqslant Ch^{1/2}$, concluding the proof.

We note that if $\|u^* - u^m\|_{H_1} = O(h^\alpha)$, $\lambda_1 = 0$ and $\lambda_2 = 0$, then Theorem 3 guarantees $a_{p,h} \to a^*$ if $\alpha > \frac{1}{2}$, while Theorem 4 guarantees $a_{p,h} \to a^*$ if $\alpha > 0$. Hence, Theorem 4 gives better stability than Theorem 3.

We close this section with a proof of (3.22) for a reasonably large class of u^*.

LEMMA 5 : *Suppose $u^* \in C^2(\bar{\Omega})$ and $|\nabla u^*| \neq 0$ on $\bar{\Omega}$. Then for each $\psi \in H_1(\Omega)$, there exists a $v_\psi \in H_1(\Omega)$ satisfying*

$$\nabla u^* \cdot \nabla v_\psi = \psi \quad in\ \Omega \tag{3.24}$$

and the estimate

$$\|v_\psi\|_{H_1} \leqslant C \|\psi\|_{H_1}, \tag{3.24a}$$

where the constant C is independent of ψ.

Proof: Theorem 4.1 of [8] shows that for $\xi \in C^1(\bar{\Omega})$ and λ sufficiently large, the equation

$$\lambda w + \xi \cdot \nabla w = \phi \tag{3.25}$$

has a solution $w \in H_1(\Omega)$ for each $\phi \in H_1(\Omega)$, with

$$\|w\|_{H_1} \leqslant C \|\phi\|_{H_1}. \tag{3.25a}$$

The constant C depends on ξ and λ, but not on ϕ. (The solution w is of course not unique, since no boundary condition has been specified. Although the statement of the result in [8] assumes that $\partial\Omega$ is C^1, the proof also works for Lipschitz domains.)

Our equation (3.24) is transformed into one of the form (3.25) by the substitution

$$v_\psi = e^{\lambda u^*} w .$$

This gives

$$e^{\lambda u^*}(\nabla u^* \cdot \nabla w + \lambda |\nabla u^*|^2 w) = \nabla u^* \cdot \nabla v_\psi ,$$

so (3.24) holds if and only if

$$\lambda w + \frac{\nabla u^*}{|\nabla u^*|^2} \cdot \nabla w = (e^{-\lambda u^*}/|\nabla u^*|^2)\, \psi .$$

If w solves this equation and satisfies (3.25a), then v_ψ is easily seen to satisfy (3.24a).

4. NUMERICAL PERFORMANCE OF THE VARIATIONAL METHOD

This section discusses the performance of the variational method on some specific examples. We consider the unit square $\Omega = (0, 1) \times (0, 1)$, and the triangulation Δ_N composed of triangles $\left(\frac{i-1}{N}, \frac{j-1}{N}\right)$, $\left(\frac{i}{N}, \frac{j-1}{N}\right)$, $\left(\frac{i}{N}, \frac{j}{N}\right)$ and $\left(\frac{i-1}{N}, \frac{j-1}{N}\right)$, $\left(\frac{i-1}{N}, \frac{j}{N}\right)$, $\left(\frac{i}{N}, \frac{j}{N}\right)$, $1 \leq i$, $j \leq N$. We choose to minimize the functional

$$J_N(\sigma, a) = \|\sigma - a\, \nabla u^m\|_{L_2}^2 + \|\operatorname{div} \sigma + f^m\|_{L_2}^2 + \frac{1}{N^2}\|\nabla a\|_{L_2}^2,$$

where σ has continuous, piecewise linear components on Δ_N, and a is continuous and piecewise linear on Δ_N with a priori bounds $\alpha \leq a \leq \beta$. Furthermore, σ is constrained by $\sigma \cdot n|_{\partial\Omega} = g^m$, where the measurements u^m, f^m, and g^m are continuous and piecewise linear. The numerical minimum, σ_N, a_N of J_N, was computed by a relaxation method, iteratively minimizing over σ, a at successive nodes while keeping the value at all other nodes fixed. This algorithm was chosen for its simplicity ; probably the conjugate gradient method would be faster for very fine triangulations. Our goal was to evaluate how well the numerically reconstructed coefficient a_N approximates the true coefficient a^* in practice.

We present two examples with " perfect measurements ", i.e. in which $u^m = u^*$, $f^m = f^*$, and $g^m = g^*$ at nodal points. Each used $a^* = 1 + y^2$, $f^* = -\nabla \cdot (a^* \nabla u^*)$, $g^* = a^* \frac{\partial u^*}{\partial n}$. The " solution " u^* was

$$u^* = x + y + \frac{1}{3}(x^3 + y^3) \qquad \text{in Example 1},$$

and

$$u^* = \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \quad \text{in Example 2}.$$

In particular, $|\nabla u^*| \neq 0$ in Example 1, whereas $|\nabla u^*| = 0$ at a single point in Example 2 (but the condition $\inf_\Omega \max\{|\nabla u^*|, \Delta u^*\} > 0$ is satisfied).

The numerical minimum σ_N, a_N of J_N was determined for values of N between 10 and 50. The upper and lower bounds on a_N, $\alpha = 0.1$ and $\beta = 4.0$, were never active at the solution. The convergence of

$\|a_N - a^*\|_{L_2}$, $\|\operatorname{div} \sigma_N + f^m\|_{L_2}$, and $\|\sigma_N - a_N \nabla u^m\|_{L_2}$ is graphed in figures 1 and 2.

Not surprisingly, the variational method performed better when $|\nabla u^*| \neq 0$. Indeed, the observed convergence was $\|a_N - a^*\|_{L_2} \sim O(N^{-1.6})$ in Example 1, but only $\|a_N - a^*\|_{L_2} \sim O(N^{-1})$ in Example 2. Moreover, in Example 2, different behavior was observed for even and odd N, the results being better when N is odd. This is presumably due to the fact that ∇u^* vanishes at a node when N is even. In both of these examples, the observed convergence is better than we are able to prove theoretically : our best convergence estimate, Theorem 4, establishes that $\|a_N - a^*\|_{L_2} \sim O(N^{-1/2})$.

We observe with respect to Example 1 that $\|\sigma_N - a_N \nabla u^m\|_{L_2} \sim O(N^{-1})$ and $\|\operatorname{div} \sigma_N + f^m\|_{L_2} \sim O(N^{-1})$. In the proof of Theorem 4, the term $\|a_N - a^*\|^2_{L_2}$ is bounded in terms of $\|\sigma_N - a_N \nabla u^m\|_{L_2}$ and $\|\operatorname{div} \sigma_N + f^m\|_{L_2}$ and other $O(N^{-1})$ expressions. Thus Theorem 4 was doomed to give less than optimal results, as we bounded the rapidly converging $\|a_N - a^*\|^2_{L_2}$ in terms of $O(N^{-1})$ expressions.

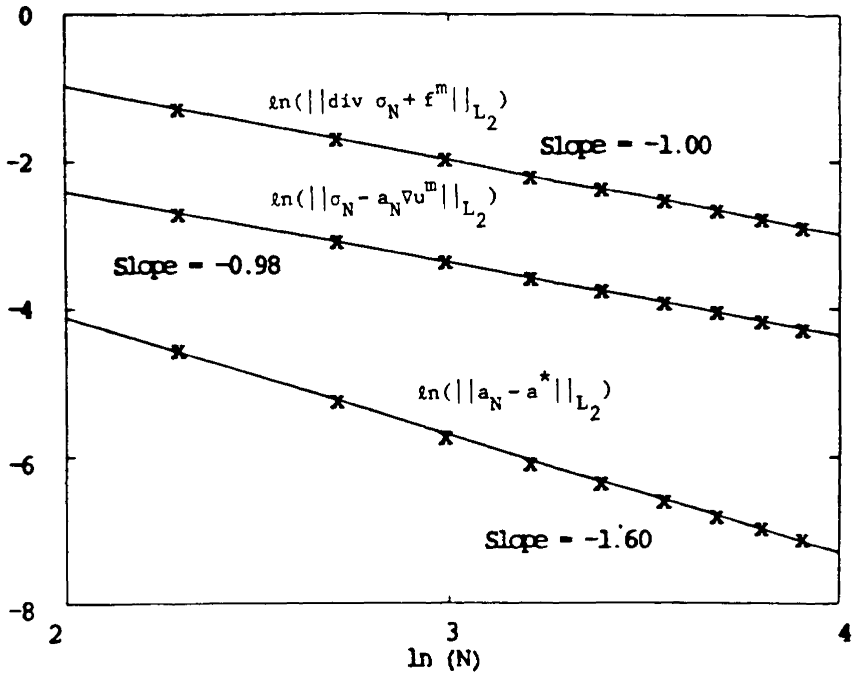

Figure 1. — Convergence results for Example 1.

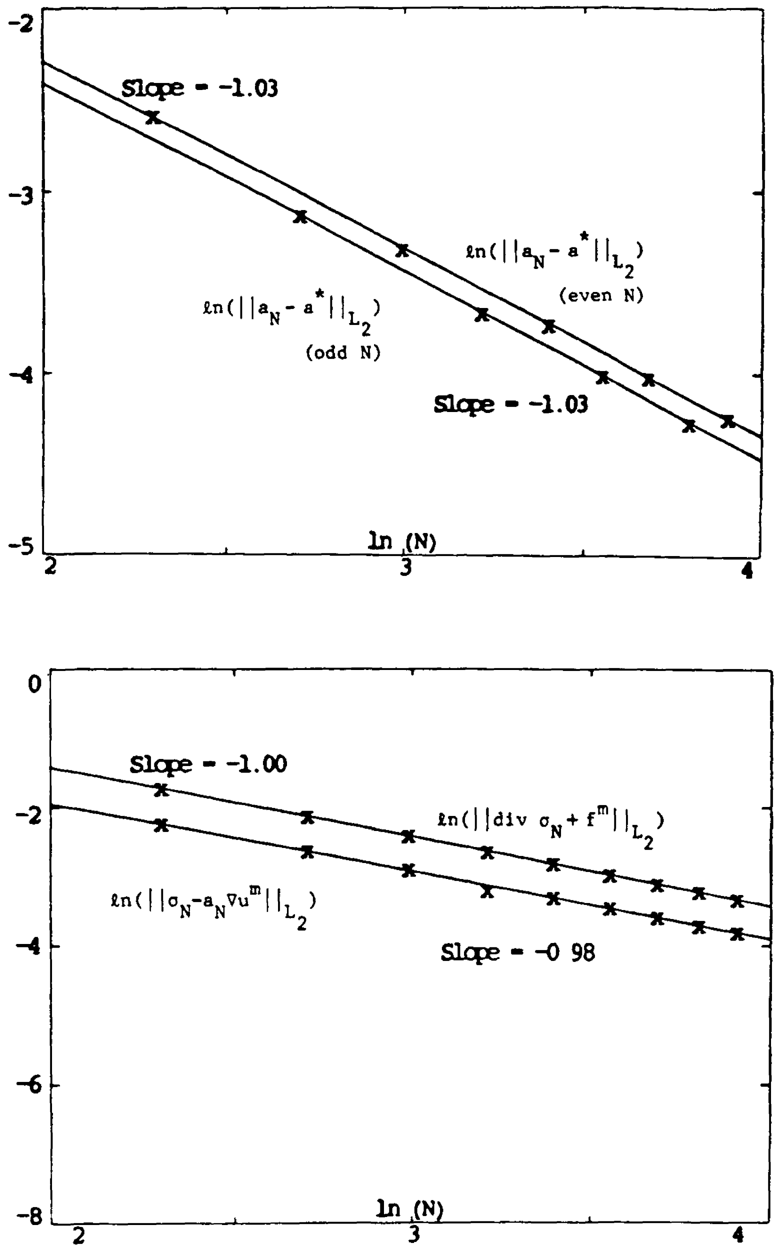

Figure 2. — Convergence results for Example 2.

Of course, one wants a_N to be close to a^* even in the pointwise sense In the example considered, this is true except for a layer near the outflow boundaries (i e $\partial u^*/\partial n > 0$) and (in Example 2) a neighborhood of where $|\nabla u^*| = 0$, see tables 1 and 2

TABLE 1

Nodal values of $a_N - a^*$ in Example 1 with $N = 10$

071	050	050	052	052	051	050	048	044	039	051
-032	-012	-012	-011	-011	-011	-011	-011	-010	-027	036
-002	003	006	006	004	003	003	001	003	-008	035
-015	-001	001	000	-001	-002	-002	-009	001	-008	034
-013	001	002	001	000	-001	-005	-001	002	-008	032
-014	001	001	000	-001	-004	-001	-001	002	-008	030
-012	001	001	000	-003	-001	000	-001	002	-008	028
-010	002	001	-001	-001	-001	000	-001	002	-008	026
-007	002	001	-002	-001	-001	000	-001	002	-008	025
-004	004	-002	000	000	000	000	-001	001	-007	024
001	-002	-001	-001	-001	-001	-001	-003	002	-015	035

TABLE 2

Nodal values of $a_N - a^*$ in Example 2, with $N = 10$

194	038	106	110	131	138	135	120	100	082	110
018	-046	-034	-049	-055	-058	-054	-045	-032	-060	077
077	-029	006	-005	-002	-003	-002	-004	000	-031	087
069	-043	-012	-034	-044	-049	-036	-039	-009	-038	094
080	-044	-012	-045	-097	-173	-089	-038	-006	-042	093
081	-044	-016	-054	-187	-690	-170	-045	-008	-042	084
076	050	-015	-043	-092	-170	-087	-038	-006	-037	069
066	-032	-015	-036	-035	-046	-037	-026	-007	-031	050
053	-024	-009	-010	-011	-011	-008	-008	002	-020	045
041	-036	-020	-028	-032	-034	-032	-028	-018	-026	009
052	040	048	057	064	066	063	053	053	019	097

We also did a series of tests to evaluate how measurement error influences the performance of the method. For this purpose, we used " measurements " $u_\varepsilon^m = u_{N,I}^{*(1)} + 2\varepsilon v$, where $u_{N,I}^{*(1)}$ is the piecewise linear interpolate of u^* on Δ_N, and v is a piecewise linear function on Δ_N, whose nodal values are independent random variables, uniformly distributed between 0 and 1. The parameter ε controls the size of the " measurement error ".

Figure 3 shows the results obtained for various choices of ε, with $N = 15$ and $0.02 \leqslant \varepsilon N \leqslant 0.3$, using the same u^* and a^* as in Example 1. Examination of the graph shows that $\|a_N^\varepsilon - a^*\|_{L_2}$ is roughly linear in ε within this range.

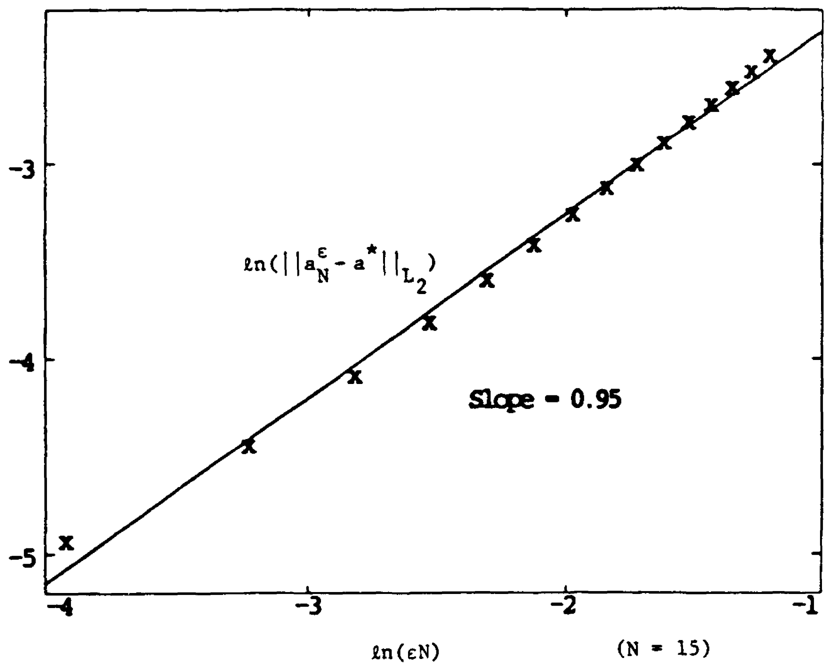

Figure 3. — The effect of measurement error in Example 1.

5. A VARIATIONAL RECONSTRUCTION ALGORITHM FOR MATRIX CONDUCTIVITIES

This section presents a technique for the reconstruction of a matrix conductivity $A^*(x)$, where $A^*(x)$ is a 2×2 positive definite matrix. It is the analogue of the variant of our scalar method, described in the introduction, in which the functional

$$J(\sigma, a) = \frac{1}{2} \int_{\Omega} |a^{-1/2} \sigma - a^{1/2} \nabla u^*|^2 \, dx \tag{5.1}$$

is minimized over coefficients $a(x) \geqslant 0$ (by hand), then over vector fields σ subject to div $\sigma = -f^*$, $\sigma \cdot n = g^*$ (numerically). The advantage of this version over others is that the minimization over coefficients is relatively easy. Moreover, the functional that emerges is *convex* even in the matrix case. A similar minimization problem arises in [13], as the relaxation of a method for finding a scalar conductivity by impedance computed tomography. Numerical or theoretical convergence results are not established in the matrix case ; these remain directions for further study.

A single boundary-value problem

$$\begin{aligned} -\nabla \cdot (A^*(x) \nabla u^*) &= f^* \quad \text{in} \quad \Omega \\ A^*(x) \nabla u^* \cdot n &= g^* \quad \text{on} \quad \partial\Omega \end{aligned} \tag{5.2}$$

obviously does not suffice to specify a matrix conductivity A^* uniquely. Therefore we shall suppose that measurements have been made on several boundary value problems, say N of them. The state equation is thus generalized from (5.2) to

$$\left.\begin{aligned} -\nabla \cdot (A^*(x) \nabla u_i^*) &= f_i^* \quad \text{in} \quad \Omega \\ A^*(x) \nabla u_i^* \cdot n &= g_i^* \quad \text{on} \quad \partial\Omega \end{aligned}\right. \quad (1 \leqslant i \leqslant N) \,. \tag{5.3}$$

The analogue of (5.1) for the matrix case with N measurements is

$$J(\sigma_1, \dots, \sigma_N, A) = \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega} |A^{-1/2} \sigma_i - A^{1/2} \nabla u_i^*|^2 \, dx$$

where div $\sigma_i = -f_i^*$ and $\sigma_i \cdot n = g_i^*$ $(1 \leqslant i \leqslant N)$. Expanding $|A^{-1/2} \sigma_i - A^{1/2} \nabla u_i^*|^2$, and noting that

$$\begin{aligned} \int_{\Omega} \langle \sigma_i, \nabla u_i^* \rangle \, dx &= -\int_{\Omega} (\operatorname{div} \sigma_i) \, u_i^* \, dx + \int_{\partial\Omega} (\sigma_i \cdot n) \, u_i^* \, ds \\ &= \int_{\Omega} f_i^* \, u_i^* \, dx + \int_{\partial\Omega} g_i^* \, u_i^* \, ds \,, \end{aligned}$$

we have

$$\begin{aligned} J(\sigma_1, \dots, \sigma_N, A) = \sum_{i=1}^{N} \Bigg\{ &\int_{\Omega} \frac{1}{2} \left(\langle A^{-1} \sigma_i, \sigma_i \rangle + \langle A \nabla u_i^*, \nabla u_i^* \rangle \right) dx \\ &- \int_{\Omega} f_i^* \, u_i^* \, dx - \int_{\partial\Omega} g_i^* \, u_i^* \, ds \Bigg\} . \end{aligned} \tag{5.4}$$

We seek to minimize (5.4) over all matrices $A > 0$ explicitly ; this will leave a new functional $E(\sigma_1, \dots, \sigma_N)$, the analogue of (1.12), to be minimized numerically. Because the term $\int_\Omega f_i^* u_i^* \, dx + \int_{\partial\Omega} g_i^* u_i^* \, ds$ is a known and fixed quantity, we focus our attention on the rest, which will be denoted by J_1 :

$$J_1(\sigma_1, \dots, \sigma_N, A) = \sum_{i=1}^{N} \left\{ \int_\Omega \frac{1}{2} \left(\langle A^{-1} \sigma_i, \sigma_i \rangle + \langle A \, \nabla u_i^*, \nabla u_i^* \rangle \right) dx \right\}. \quad (5.5)$$

As a first step we rewrite (5.5) in terms of the 2×2 nonnegative matrices

$$L = \sum_{i=1}^{N} (\nabla u_i^*) (\nabla u_i^*)^T \quad \text{and} \quad K = K_{\sigma_1, \dots, \sigma_N} = \sum_{i=1}^{N} \sigma_i \, \sigma_i^T .$$

It is easy to see that

$$\sum_{i=1}^{N} \langle A^{-1} \sigma_i, \sigma_i \rangle = \operatorname{tr} (A^{-1} K) \quad \text{and} \quad \sum_{i=1}^{N} \langle A \, \nabla u_i^*, \nabla u_i^* \rangle = \operatorname{tr} (AL) ,$$

so that

$$J_1(\sigma_1, \dots, \sigma_N, A) = \frac{1}{2} \int_\Omega [\operatorname{tr} (A^{-1} K) + \operatorname{tr} (AL)] \, dx . \quad (5.6)$$

Since $A(x)$ is unconstrained (except for being a positive, symmetric matrix), the minimization over it can be done pointwise. The answer is provided by the following lemma :

LEMMA 5.1 : *If K and L are non negative, symmetric, 2×2 matrices, then*

$$\inf_{A > 0} [\operatorname{tr} (A^{-1} K) + \operatorname{tr} (AL)] = 2 [\operatorname{tr} (KL) + 2 (\det KL)^{1/2}]^{1/2} , \quad (5.7)$$

the minimization in (5.7) *being over all positive, 2×2 symmetric matrices A.*

Proof: We consider first the case in which K and L are both positive. Then

$$\operatorname{tr} (A^{-1} K) + \operatorname{tr} (AL) \to \infty$$

if any eigenvalue of A tends to 0 or ∞. Therefore the infimum in (5.7) is achieved, say at A_0. Taking the first variation leads to

$$\operatorname{tr} (\delta A (L - A_0^{-1} K A_0^{-1})) = 0$$

for all variations δA, and therefore

$$A_0 L A_0 = K . \quad (5.8)$$

It follows that

$$(A_0 L)^2 = KL \quad \text{and} \quad (A_0^{-1} K)^2 = LK . \tag{5.9}$$

The formula

$$\operatorname{tr} C = [\operatorname{tr} (C^2) + 2(\det C)]^{1/2}$$

holds for any 2×2 matrix C with positive trace. Using it, (5.9) yields

$$\operatorname{tr} (A_0 L) + \operatorname{tr} (A_0^{-1} K) = 2[\operatorname{tr} (KL) + 2(\det KL)^{1/2}]^{1/2} ,$$

verifying (5.7) in this case.

If K or L is singular, we argue by continuity. The desired formula (5.7) is known to hold when K and L are replaced by

$$K_\varepsilon = K + \varepsilon I , \quad L_\varepsilon = L + \varepsilon I$$

for any $\varepsilon > 0$. Moreover, the right hand side is continuous in ε as $\varepsilon \to 0$. Therefore we need only show that the left side,

$$g(\varepsilon) = \inf_{A > 0} [\operatorname{tr} (AL_\varepsilon) + \operatorname{tr} (A^{-1} K_\varepsilon)]$$

is continuous as $\varepsilon \to 0^+$. Since $K_\varepsilon \geqslant K$, $L_\varepsilon \geqslant L$, it is clear that

$$g(\varepsilon) \geqslant g(0) \quad \text{for all} \quad \varepsilon > 0 .$$

On the other hand, to any positive matrix A with eigenvalues a_1, a_2, we associate the matrix A_ε with the same eigenvectors, and eigenvalues

$$a_i^\varepsilon = \begin{cases} \varepsilon^{-1/2} & \text{if} \quad a_i > \varepsilon^{-1/2} \\ \varepsilon^{1/2} & \text{if} \quad a_i < \varepsilon^{1/2} \\ a_i & \text{otherwise} . \end{cases}$$

Evidently

$$A_\varepsilon - A \leqslant \varepsilon^{1/2} I , \quad A_\varepsilon^{-1} - A^{-1} \leqslant \varepsilon^{1/2} I ,$$

so that

$$\begin{aligned} \operatorname{tr} (A_\varepsilon L_\varepsilon) - \operatorname{tr} (AL) &= \operatorname{tr} ((A_\varepsilon - A) L) + \operatorname{tr} (A_\varepsilon (L_\varepsilon - L)) \\ &\leqslant 2 \varepsilon^{1/2} (\operatorname{tr} L) + 2 \varepsilon (\operatorname{tr} A_\varepsilon) \leqslant 2 \varepsilon^{1/2} (\operatorname{tr} L + 2) \end{aligned}$$

and

$$\begin{aligned} \operatorname{tr} (A_\varepsilon^{-1} K_\varepsilon) - \operatorname{tr} (A^{-1} K) &= \operatorname{tr} ((A_\varepsilon^{-1} - A^{-1}) K) + \operatorname{tr} (A_\varepsilon^{-1} (K_\varepsilon - K)) \\ &\leqslant 2 \varepsilon^{1/2} (\operatorname{tr} K) + 2 \varepsilon (\operatorname{tr} A_\varepsilon^{-1}) \leqslant 2 \varepsilon^{1/2} (\operatorname{tr} K + 2) . \end{aligned}$$

Therefore

$$g(\varepsilon) \leqslant \operatorname{tr}(A_\varepsilon L_\varepsilon) + \operatorname{tr}(A_\varepsilon^{-1} K_\varepsilon) \leqslant \operatorname{tr}(AL) + \operatorname{tr}(A^{-1}K) + C\varepsilon^{1/2}$$

with $C = 2(\operatorname{tr} K + \operatorname{tr} L + 4)$. Minimizing over all $A > 0$ gives

$$g(\varepsilon) \leqslant g(0) + C\varepsilon^{1/2}.$$

This establishes the continuity of g, completing the proof.

We return to the problem of minimizing J or (equivalently) J_1. By Lemma 5.1 and (5.6),

$$\inf_{A(x) > 0} J_1(\sigma_1, \dots, \sigma_N, A) = \int_\Omega [\operatorname{tr} KL + 2(\det KL)^{1/2}]^{1/2}\, dx,$$

in which $K = K_{\sigma_1, \dots, \sigma_N}$ is a function of the test fields $\{\sigma_i\}$. Our proposal for finding a matrix conductivity is to minimize this functional, or (putting back the boundary terms)

$$E(\sigma_1, \dots, \sigma_N) = \int_\Omega [\operatorname{tr} KL + 2(\det KL)^{1/2}]^{1/2} - \sum_{i=1}^{N} \langle \sigma_i, \nabla u_i^* \rangle\, dx.$$

It is evident from the preceding that $E \geqslant 0$, and $E = 0$ only if $\sigma_i = A_0 \nabla u_i^*$, where A_0 is the (unique, nonnegative, symmetric) solution of (5.3). If $\sigma_1^p, \dots, \sigma_N^p$ are the computed minimizers of E in some finite element approximation, then the predicted conductivity is the solution A_p of

$$A_p L A_p = K_p, \quad K_p = \sum_{i=1}^{N} \sigma_i^p (\sigma_i^p)^T.$$

The minimum of a family of convex functions is not in general convex. Therefore it was a pleasant surprise to discover that, as in the scalar case (1.12), the functional $E(\sigma_1, \dots, \sigma_N)$ is convex :

PROPOSITION 5.2 : *Let L be a nonnegative, symmetric,* 2×2 *matrix. For* $(\sigma_1, \dots, \sigma_N) \in (\mathbb{R}^2)^N$, *let* $K = \sum_{i=1}^{N} \sigma_i \sigma_i^T$. *Then the map*

$$(\sigma_1, \dots, \sigma_N) \mapsto [\operatorname{tr}(KL) + 2(\det KL)^{1/2}]^{1/2}$$

is a convex function from $(\mathbb{R}^2)^N$ *to* $\mathbb{R}$.

Proof: We generalize an argument of [12]. An elementary calculation gives

$$\det K = \sum_{i<j} [\det(\sigma_i, \sigma_j)]^2.$$

For $\alpha = \{\alpha_{ij}\}$ with

$$\sum_{1 \leq i < j \leq N} \alpha_{i,j}^2 = 1 ,$$

define

$$Q_\alpha = \operatorname{tr}(KL) + 2\left(\sum_{i<j} \alpha_{i,j} \det(\sigma_i, \sigma_j)\right)(\det L)^{1/2} .$$

Then

$$\begin{aligned} Q_\alpha &\geq \operatorname{tr}(KL) - 2\left(\sum_{i<j} \alpha_{i,j}^2\right)^{1/2}\left(\sum_{i<j} [\det(\sigma_i, \sigma_j)]^2\right)^{1/2}(\det L)^{1/2} \\ &= \operatorname{tr}(KL) - 2(\det KL)^{1/2} \geq 0 . \end{aligned}$$

Thus Q_α is a nonnegative quadratic form on $(\mathbb{R}^2)^N$. It follows that Q_α is a sum of squares of linear functions, and hence that $Q_\alpha^{1/2}$ is convex in $(\sigma_1, \ldots, \sigma_N)$. Now

$$[\operatorname{tr} KL + 2(\det KL)^{1/2}]^{1/2} = \sup_{\sum_{i<j} \alpha_{i,j}^2 = 1} Q_\alpha^{1/2}$$

is the supremum of convex functions, so it, too, is convex.

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