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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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AN ANALYSIS OF THE B.P.M. APPROXIMATION OF THE HELMHOLTZ EQUATION IN AN OPTICAL FIBER (*)

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Abstract. — *Beam propagation in an optical fiber is usually described by the Helmholtz equation. Feit and Fleck have proposed a splitting method algorithm (the Beam Propagation Method or B.P.M.) in order to approximate this equation. In this work, we discuss their procedure and derive the equation which is consistent with this algorithm.*

We give a functionnal framework that allows one to solve this equation. After describing some qualitative features of the solutions, we show the convergence of the splitting method. Finally we make some remarks in the nonlinear case where the refractive index of the fiber depends on the intensity of the beam (Kerr-effect).

Résumé. — *La propagation d'un faisceau monochromatique dans une fibre optique est usuellement décrite par l'équation d'Helmholtz. Feit et Fleck ont proposé un algorithme d'approximation de cette équation par une méthode de décomposition. Dans ce travail, nous discutons ce procédé et dérivons l'équation qui est consistante avec cet algorithme. Nous donnons un cadre fonctionnel qui permet de résoudre cette équation et, après avoir décrit quelques propriétés qualitatives de ses solutions, nous montrons la convergence de l'algorithme de décomposition. Enfin nous faisons quelques remarques dans le cas non linéaire où l'indice de réfraction de la fibre dépend de l'intensité du faisceau lumineux (effet Kerr).*

CONTENTS

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1. INTRODUCTION

Let us consider the propagation of a single frequency light in an optical fiber. We assume that it can be described by the Helmholtz equation

$$\frac{\partial^2 E}{\partial x_1^2} + \frac{\partial^2 E}{\partial x_2^2} + \frac{\partial^2 E}{\partial x_3^2} + \frac{\omega^2}{c^2} n^2 E = 0 \tag{1.1}$$

where E denotes one of the components of the electric field, c the speed of light in a vacuum and $n(\omega, x)$ the refractive index ($x = (x_1, x_2, x_3)$ is the generic point in $\mathbb{R}^3$).

In what follows we discuss the method proposed by M. D. Feit and J. A. Fleck [3] for solving (1.1) in the case where n has small variations from a reference value n_0:

$$n(\omega, x) \simeq n_0 .$$

The field E is given at the entry of the fiber ($x_3 = 0$)

$$E(x_1, x_2, 0) = E_0(x_1, x_2) \tag{1.2}$$

and satisfies the radiation condition at $+\infty$ (see (1.7)).

Since the propagation is studied for a range of x_3 at least a million times the wave length, this hypothesis is reasonable.

The electric field is sought in the form

$$E(x, \omega) = A(x, \omega)\, e^{-ik_0 x_3} \tag{1.3}$$

where

$$k_0 = \frac{\omega}{c} n_0 . \tag{1.4}$$

The envelope function $A(x, \omega)$ is supposed to vary slowly over lengths of the order of $\lambda_0 = \frac{2\pi}{k_0}$. Substituting (1.3) into (1.1) leads to the following equation for A

$$-2ik_0 \frac{\partial A}{\partial x_3} + \frac{\partial^2 A}{\partial x_3^2} + \Delta_\perp A + \frac{\omega^2}{c^2}(n^2 - n_0^2) A = 0 , \tag{1.5}$$

where $\Delta_\perp$ denotes the transverse laplacian

$$\Delta_\perp A = \frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2} . \tag{1.6}$$

The boundary conditions are

$$(1.7) \qquad \begin{cases} A(x_1, x_2, 0) = A_0(x_1, x_2) = E_0(x_1, x_2)\ ; \\ \text{and } A(x_1, x_2, x_3) \text{ satisfies the radiation condition } (^1) \end{cases}$$

Before giving the method proposed by Feit and Fleck for solving (1.5), let us recall the classical *parabolic approximation* (or Fresnel approximation). This method consists in droping the term $\frac{\partial^2 A}{\partial x_3^2}$ in (1.5) because of the slow variation of A with respect to x_3:

$$(1.8) \qquad -2\,ik_0 \frac{\partial A}{\partial x_3} + \Delta_\perp A + \frac{\omega^2}{c^2}(n^2 - n_0^2)\,A = 0\,.$$

Then, the Cauchy problem (1.7)-(1.8) is solved numerically with a splitting method (Split Step Fourier Transform) proposed by R. A. Fisher and W. K. Bischel [4] which is very popular in optics. It is based on a discretization in the x_3 direction :

1st step : On an interval of length $\Delta x_3/2$, one solves

$$(1.9) \qquad -2\,ik_0 \frac{\partial A}{\partial x_3} + \Delta_\perp A = 0\,.$$

2nd step : On an interval of lenght Δx_3, one solves

$$(1.10) \qquad -2\,ik_0 \frac{\partial A}{\partial x_3} + \frac{\omega^2}{c^2}(n^2 - n_0^2)\,A = 0\,.$$

3rd step : On an interval of length $\Delta x_3/2$, one solves

$$(1.11) \qquad -2\,ik_0 \frac{\partial A}{\partial x_3} + \Delta_\perp A = 0\,.$$

At each step the initial condition is the terminal condition of the previous one. The great applicability of this method is due to the fact that the first and third steps are solved by Fourier transform with respect to (x_1, x_2) while the second step is a family of O.D.E.'S.

Feit and Fleck's idea is to use this algorithm to solve equation (1.5) with the boundary conditions (1.7) as follows :

1st step : On an interval of length $\Delta x_3/2$, one solves

$$(1.12) \qquad -2\,ik_0 \frac{\partial A}{\partial x_3} + \frac{\partial^2 A}{\partial x_3^2} + \Delta_\perp A = 0\,.$$

(1) This condition consists in dropping the k_3' term in (2.4).

2nd step : On an interval of length Δx_3, one solves

$$- 2\, i k_0 \frac{\partial A}{\partial x_3} + \frac{\omega^2}{c^2} (n^2 - n_0^2) A = 0 . \tag{1.13}$$

3rd step : On an interval of lenght $\Delta x_3/2$, one solves

$$- 2\, i k_0 \frac{\partial A}{\partial x_3} + \frac{\partial^2 A}{\partial x_3^2} + \Delta_{\perp} A = 0 . \tag{1.14}$$

As before the initial condition of each step is the terminal one of the previous step. But steps 1 and 3 are still underdetermined since we have to solve a second order equation in the x_3 variable. These steps can be viewed as a second order O.D.E. in x_3 (recall that these steps are solved by Fourier transform w.r. to x_1, x_2). Therefore the solution A can be written as the sum of two exponentials and M. D. Feit and J. A. Fleck erase the one which propagates in the $- x_3$ direction (see (2.7)). In a certain sense they apply the vanishing condition at $+ \infty$. The first and third steps are actually solved by Fourier transform and it is this method which is called B.P.M. Let us point out that besides the fact that from a computational point of view the Fourier transform is very performent (F.F.T.), the B.P.M. method gives at the same time the signal and its Fourier transform which is of great importance for physical reasons (energy spectrum,...).

In fact the Split Step Fourier Transform applied to the parabolic approximation and the B.P.M. method can be implemented in a very similar way. In particular they can be performed as two close versions of a computer program. In a subsequent work we shall report on the comparison of these two procedures with respect to the original problem which was to solve the Helmholtz equation in an optical fiber.

In this paper we study the B.P.M. method. In the following section we derive formally the continuous equation, the B.P.M. equation (i.e. the equation obtained by letting $\Delta x_3 \to 0$) which is consistent with the B.P.M. algorithm (1.12), (1.14). In the third section we give some mathematical results on the B.P.M. equation. Then in the fourth section, we show that the B.P.M. algorithm is consistent with B.P.M. equation. Finally, in the last section we give some remarks on the nonlinear B.P.M. equation which corresponds to the case where the refractive index depends on the electrical field (Kerr-effect).

In a preliminary version of this paper, [1], we have given in full detail the proofs of the results presented here. We refer to this report for some missing points, however this article is intended to be self contained. We also refer to [3] for complementary references on the physical and computational point of view.

2. THE B.P.M. EQUATION

In this section we derive the B.P.M. equation. Then we give some comparisons with the Helmholtz equation and with the parabolic approximation.

2.1. Derivation of the equation

As already noticed, equation (1.12) is solved by means of a Fourier transform with respect to the x_1 and x_2 variables. Let us denote by

$$\hat{A}(\xi_1, \xi_2, x_3) = \int_{\mathbb{R}^2} A(x_1, x_2, x_3) \exp(-i(x_1 \xi_1 + x_2 \xi_2))\, dx_1\, dx_2 \tag{2.1}$$

the Fourier transform of A ; then equation (1.12) reads

$$-2ik_0 \frac{\partial \hat{A}}{\partial x_3} + \frac{\partial^2 \hat{A}}{\partial x_3^2} - (\xi_1^2 + \xi_2^2)\hat{A} = 0 . \tag{2.2}$$

The associated caracteristic equation is

$$k^2 - 2k_0 k + \xi_1^2 + \xi_2^2 = 0 \tag{2.3}$$

and if k_3 and k_3' denote its solutions, the solutions of (2.2) are given by

$$\hat{A} = \hat{a}\, e^{ik_3 x_3} + \hat{a}'\, e^{ik_3' x_3} . \tag{2.4}$$

1st case : $\xi_1^2 + \xi_2^2 < k_0^2$.

We have :

$$\begin{cases} k_3 = k_0 - (k_0^2 - \xi_1^2 - \xi_2^2)^{1/2} , \\ k_3' = k_0 + (k_0^2 - \xi_1^2 - \xi_2^2)^{1/2} , \end{cases} \tag{2.5}$$

and k_3 corresponds to a forward propagation while k_3' corresponds to a backward one. As explained in the Introduction, the B.P.M. consists in choosing the k_3-term in (2.4). It can also be noticed that when $\xi_1^2 + \xi_2^2 \ll k_0^2$, k_3 is close to 0 and k_3' is close to $2k_0$. We have chosen the part of the wave which has the slowest variation in x_3, this is consistent with the envelope hypothesis (see (1.3)).

2nd case : $\xi_1^2 + \xi_2^2 > k_0^2$.

Here

$$\begin{cases} k_3 = k_0 + i(\xi_1^2 + \xi_2^2 - k_0^2)^{1/2} , \\ k_3' = k_0 - i(\xi_1^2 + \xi_2^2 - k_0^2)^{1/2} , \end{cases} \tag{2.6}$$

and k_3 produces a vanishing wave in the x_3 direction while k_3' gives rise to an amplified wave. The B.P.M. consists in choosing the vanishing wave, droping the amplified one which is physically irrelevent.

Hence in both cases, the k_3'-term in (2.4) is dropped and instead of (1.12) and (1.14) we take

$$\frac{\partial \hat{A}}{\partial x_3} = i k_3 \hat{A} , \tag{2.7}$$

where k_3 is given by (2.5)-(2.6).

Equation (2.7) is written in Fourier space ; in order to return to the physical space we introduce the function $\hat{G}$, plotted in figure 2.1,

$$\hat{G}(\xi_1, \xi_2) = \begin{cases} k_0 - (k_0^2 - \xi_1^2 - \xi_2^2)^{1/2} , & \text{for } \xi_1^2 + \xi_2^2 \leqslant k_0^2 , \\ k_0 + i(\xi_1^2 + \xi_2^2 - k_0^2)^{1/2} , & \text{for } \xi_1^2 + \xi_2^2 \geqslant k_0^2 , \end{cases} \tag{2.8}$$

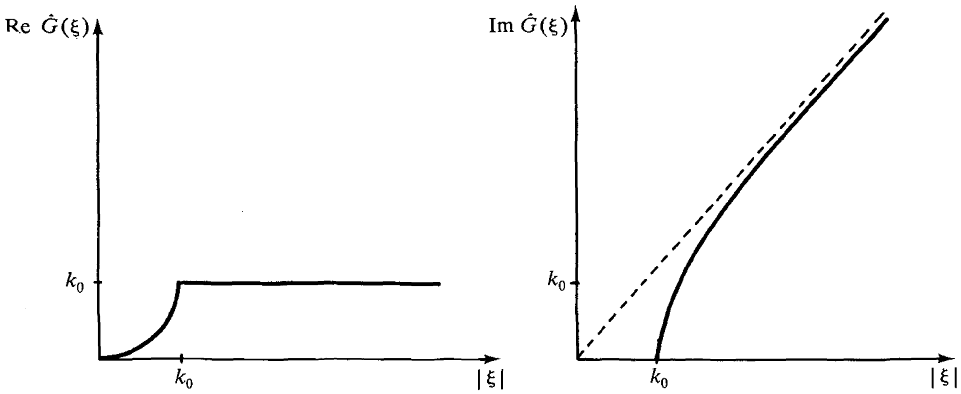

Figure 2.1. — Real and imaginary part of the function $\hat{G}(\xi)$, $\xi = (\xi_1, \xi_2)$ *versus* $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2}$.

Equation (2.7) reads

$$\frac{\partial \hat{A}}{\partial x_3} = i \hat{G}(\xi_1, \xi_2) \hat{A} , \tag{2.9}$$

or by inverse Fourier transform w.r.t. ξ_1, ξ_2 ;

$$\frac{\partial A}{\partial x_3} = i G * A \tag{2.10}$$

where G denotes the inverse Fourier transform of $\hat{G}$, and the convolution $*$ acts on the x_1 and x_2 variables.

Remark 2.1 : We set $K(x) = \frac{1}{4\pi} \frac{e^{-ik_0|x|}}{|x|}$, where $x = (x_1, x_2)$, $|x| = (x_1^2 + x_2^2)^{1/2}$. By classical calculus (see e.g. [9]) we have

$$\hat{G}(\xi) = k_0 - 2i(k_0^2 - \xi_1^2 - \xi_2^2)\hat{K}(\xi) \tag{2.11}$$

and

$$G(x) = \frac{1}{2\pi}\left\{k_0 - \frac{1}{2\pi}\left(\frac{k_0}{|x|^2} - \frac{i}{|x|^3}\right)e^{-ik_0|x|}\right\}. \tag{2.12}$$

The function G is not locally integrable in $\mathbb{R}^2$, the equation (2.10) is formal, and the convolution is understood in the sense of principal values. We shall give in the third section a rigorous meaning to equation (2.10), see also Remark 3.1. □

Let us now return to the B.P.M. algorithm (1.12) to (1.14), where equation (1.12) and (1.14) are replaced by (2.10). In the fourth section we show that, when $\Delta x_3 \to 0$, the discrete approximations converge to the solution of the continuous equation

$$i\frac{\partial A}{\partial x_3} + G * A + k_0 \frac{n_0^2 - n^2}{2n_0^2} A = 0, \tag{2.13}$$

which is referred to in the sequel as the *Beam Propagation Method equation* (or B.P.M. equation).

2.2. Comparison with the Helmholtz equation

In the particular case where the refractive index $n(\omega, x)$ is independent of x, i.e. $n(\omega, x) = n_0(\omega)$, the B.P.M. equation (2.13) and the Helmholtz equation (1.5)-(1.7) have the same solutions. Indeed the radiation condition at $x_3 = +\infty$ is exactly the argument that leads to the choice of k_3 in (2.4).

In the case where $n(\omega, x)$ depends on x, the B.P.M. equation is in general an approximation of Helmholtz equation. One of the advantages of the B.P.M. equation is that it is an evolution equation, while Helmholtz's equation includes a boundary condition at $x_3 = +\infty$ which is more difficult to implement. On the other hand, the former equation is an envelope equation for a function which varies slowly, in application, in a range of order $\lambda_0 = \frac{2\pi}{k_0}$, the wave lenght. This allows one to take the step of discretization Δx_3 of the order of λ_0. It is also worthwhile to note that, althought a non local operator occurs in the B.P.M. equation, this difficulty is avoided by using the Fourier transform. However we must note that this method does not propagate without x_3-deformation the guided modes, which are the non-zero solutions of the equation

$$\Delta_{\perp} A + \frac{\omega^2}{c^2}(n^2 - n_0^2) A = 0 \tag{2.14}$$

(here $n(\omega, x) = n(\omega, x_1, x_2)$ i.e. $\frac{\partial n}{\partial x_3} \equiv 0$). Indeed these solutions are stationary solutions of Helmholtz's equation (1.15), but if $n \not\equiv n_0$ they are not stationary solutions of the B.P.M. equation. We refer to [1], where we propose a modified version of the B.P.M. equation, which conserves the solutions of (2.14).

Remark 2.2 : The parabolic approximation (1.8) can also be viewed as an approximation of the B.P.M. equation for small wave numbers. Indeed the expansion of $\hat{G}(\xi_1, \xi_2)$, for $\frac{|\xi|^2}{k_0^2} \ll 1$, gives

$$\hat{G}(\xi_1, \xi_2) \simeq \hat{G}_p(\xi_1, \xi_2) \equiv \frac{\xi_1^2 + \xi_2^2}{2\, k_0}, \tag{2.15}$$

and by replacing G by G_p in (2.13) we obtain (1.8). We stress the fact that the parabolic approximation is a good approximation of Helmholtz's equation only for small wave numbers, while the B.P.M. approximate this last equation for arbitrarily large wave numbers (and is exact for constant refractive index). See also Remark 3.4. □

3. SOME MATHEMATICAL PROPERTIES OF THE B.P.M. EQUATION

In this section we give a mathematical framework that allows one to prove existence and uniqueness of a solution to the B.P.M. equation. Then we study the long time behavior of these solutions and finally the dependence of the solutions on the refractive index is addressed.

3.1. Existence and uniqueness of solutions to the B.P.M. equation

To begin with, we make some changes in the notations. We denote by t the x_3-variable because of the evolutionary character of the problem, and we normalize the refractive index function and set

$$q(x_1, x_2) = \frac{k_0}{2} \frac{n^2(x_1, x_2) - n_0^2}{n_0^2}.$$

Moreover we assume, for the sake of simplicity, that n does not depend on x_3 (or equivalently t), see also Remark 3.2. Finally we denote by u the envelope function which was denoted A.

Functional setting

As usual we denote by $H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, the Sobolev fractional space of order s on $\mathbb{R}^2$ $(s \geq 0)$:

$$H^s(\mathbb{R}^2) = \left\{ u \in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} |\xi|^{2s} |u(\xi)|^2 \, d\xi < +\infty \right\} \tag{3.1}$$

where $L^r(\mathbb{R}^2)$, $1 \leq r \leq +\infty$, is the space of measurable functions u, defined on $\mathbb{R}^2$ with values in $\mathbb{C}$, for which

$$|u|_{L^r(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |u|^r \, dx \right)^{1/r} < +\infty \tag{3.2}$$

for $1 \leq r < +\infty$, and for $r = +\infty$ those which are essentially bounded on $\mathbb{R}^2$:

$$|u|_{L^\infty(\mathbb{R}^2)} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^2} |u(x)| < +\infty . \tag{3.3}$$

Let κ denote a positive number (which has the dimension of a wave number, it can be for instance the unit in which k_0 is measured), the space $H^s(\mathbb{R}^2)$ is a Hilbert space with scalar product

$$(u, v)_s = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} (\kappa^2 + |\xi|^2)^s \, \hat{u}(\xi) \, \bar{\hat{v}}(\xi) \, d\xi \tag{3.4}$$

and norm $|u|_s = (u, u)_s^{1/2}$. The space $L^2(\mathbb{R}^2)$ is identified with its antidual and then $H^{-s}(\mathbb{R}^2)$ is identified with the antidual of $H^s(\mathbb{R}^2)$.

We introduce the sesquilinear form

$$a_q(u, v) = -\frac{i}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{G}(\xi) \, \hat{u}(\xi) \, \bar{\hat{v}}(\xi) \, d\xi + i \int_{\mathbb{R}^2} q(x) \, u(x) \, \bar{v}(x) \, dx \tag{3.5}$$

where the function $\hat{G}$ is given in (2.8) and q is a given real valued function with

$$q \in L^r(\mathbb{R}^2) , \quad \text{for some } r , \quad 2 \leq r \leq +\infty . \tag{3.6}$$

Let us mention that this last hypothesis on q is motivated by the fact that in practice the refractive index, and therefore q, can be discontinuous, a piecewise constant for example.

It follows readily from the expression (2.8) of $\hat{G}$ and the Sobolev imbeddings, $H^{1/2}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for $2 \leq p \leq 4$ with continuous injection, that a_q is continuous on $H^{1/2}(\mathbb{R}^2) \times H^{1/2}(\mathbb{R}^2)$. Moreover we have

$$(\kappa^2 + k_0^2)^{1/2} \, |v|_0^2 + \operatorname{Re} a_q(v, v) \geq |v|_{1/2}^2 , \quad \forall v \in H^{1/2}(\mathbb{R}^2) , \tag{3.7}$$

which shows that a_q is *coercive* on $H^{1/2}(\mathbb{R}^2)$. We introduce the linear and continuous operator from $H^{1/2}(\mathbb{R}^2)$ into $H^{-1/2}(\mathbb{R}^2)$, A_q, defined by

$$\langle A_q v, w\rangle_{-1/2,1/2} \equiv a_q(v, w), \quad \forall v, w \in H^{1/2}(\mathbb{R}^2). \tag{3.8}$$

with this notation, the B.P.M. equation (2.13) reads

$$\frac{du}{dt} + A_q u = 0, \quad t \geqslant 0, \tag{3.9}$$

or equivalently

$$\frac{du}{dt} + A_0 u + qu = 0, \quad t \geqslant 0, \tag{3.9'}$$

with the initial condition

$$u(0) = u_0. \tag{3.10}$$

Remark 3.1 : (Continuation of Remark 2.1) : From (2.11), it follows that

$$\begin{aligned} a_0(u, v) = &- i k_0 \int_{\mathbb{R}^2} u\bar{v}\, dx - 2 k_0^2 \int_{\mathbb{R}^2} (K * u)\, \bar{v}\, dx + \\ &+ 2 \int_{\mathbb{R}^2} \left(K * \frac{\partial u}{\partial x_1}\right) \frac{\partial \bar{v}}{\partial x_1}\, dx + 2 \int_{\mathbb{R}^2} \left(K * \frac{\partial u}{\partial x_2}\right) \frac{\partial \bar{v}}{\partial x_2}\, dx \end{aligned} \tag{3.11}$$

where

$$(K * w)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^2} w(x-y)\, \frac{e^{-ik_0|y|}}{|y|}\, dy. \tag{3.12}$$

In opposition whith the equation (2.10) where the convolution was understood in the sense of principal values, since G is not locally integrable, the expression (3.11) is in term of classical functions since (3.12) is a convolution with a locally integrable kernel. The conterpart being that now the x-derivatives of u are involved in (3.11). □

Existence and uniqueness

The continuity and coercivity properties of A_q, allow one to solve the problem (3.9)-(3.10) by the classical methods for linear parabolic equations (see e.g. J. L. Lions and E. Magenes [7], or A. Friedman [5]) :

PROPOSITION 3.1 : *For every* u_0

$$u_0 \in L^2(\mathbb{R}^2), \tag{3.13}$$

the problem (3.9)-(3.10) *possesses a unique solution*

(3.14) $u \in C(\mathbb{R}_+ ; L^2(\mathbb{R}^2)) \cap L^2(0, T ; H^{1/2}(\mathbb{R}^2))$ (1) , $\forall T < +\infty$.

Moreover

$$\frac{1}{2}\frac{d}{dt}|u|_0^2 + \operatorname{Re} a_0(u, u) = 0 , \quad \text{for a.e. } t \geq 0 . \quad \square \tag{3.15}$$

It will be usefull to introduce the semi-group notation to represent the solution u to (3.9)-(3.10). Therefore we define a linear and continuous operator on $L^2(\mathbb{R}^2)$ by

$$e^{-A_q t} : u_0 \mapsto e^{-A_q t} u_0 \equiv u(t) . \tag{3.16}$$

The continuity of this operator relies on the fact that (3.15) can be written as

$$\int_{\mathbb{R}^2} |u(x, t)|^2 \, dx + \int_0^t \int_{|\xi| \geq k_0} (|\xi|^2 - k_0^2)^{1/2} \, |u(\xi, t)|^2 \, d\xi \, ds = \tag{3.17}$$
$$= \int_{\mathbb{R}^2} |u_0(x)|^2 \, dx$$

thus

$$\left|e^{-A_q t} u_0\right|_0 \leq |u_0|_0 , \quad \forall t \geq 0 . \tag{3.18}$$

Remark 3.2 : It has been assumed that q does not depend on t. In fact Theorem 3.1 is still valid for time (t) dependent q, for instance for $q \in L^\infty_{\text{loc}}(\mathbb{R}_+ ; L^r(\mathbb{R}^2))$, $2 \leq r \leq +\infty$. □

When the function q is more regular, further regularity results on u can be obtained (the following results are well-known, see [1] for details). We recall that $W^{m,r}(\mathbb{R}^2)(m \in \mathbb{N}, 1 \leq r \leq +\infty)$ is the set of elements of $L^r(\mathbb{R}^2)$ whose distribution derivatives up to the order m are in $L^r(\mathbb{R}^2)$.

PROPOSITION 3.2 : *For* $q \in W^{m-1,\infty}(\mathbb{R}^2)$, $m \geq 1$ *and* $u_0 \in H^m(\mathbb{R}^2)$, *there exists a constant* C_m *which depends only on* $|q|_{W^{m-1,\infty}(\mathbb{R}^2)}$, m, κ *and* k_0 *such that*

$$\sup_{t \geq 0} \left|e^{-A_q t} u_0\right|_m \leq C_m |u_0|_m . \quad \square \tag{3.19$_m$}$$

Remark 3.3 : By the smoothing effect of parabolic equation, if $q \in W^{m,\infty}(\mathbb{R}^2)$ for every m, and $u_0 \in L^2(\mathbb{R}^2)$, the solution u is $\mathscr{C}^\infty$ in $]0, +\infty[\times \mathbb{R}^2$. □

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(1) For vector valued distributions we refer to L. Schwartz [8].

3.2. Some qualitative properties of the solutions

Long time behavior of the solutions

We are going to show that the wave number k_0 plays the role of a cutoff. Indeed, Theorem 3.1 hereafter shows that the energy contained in the wave numbers larger than k_0 is damped and tends to zero. This is due to the fact the imaginary part of $\hat{G}$ is positive for $|\xi| > k_0$ (see *fig.* 2.1).

THEOREM 3.1 : *Let u_0 be given in $L^2(\mathbb{R}^2)$, and let u be the solution to the B.P.M. equation, with q satisfying* (3.6). *Then*

$$\lim_{t \to +\infty} \int_{|\xi| \geq k_0} (|\xi|^2 - k_0^2)^{1/2} \, |\hat{u}(\xi, t)|^2 \, d\xi = 0 . \tag{3.20}$$

Proof: Let us first assume that $u_0 \in H^1(\mathbb{R}^2)$. According to (3.17) the function ϕ

$$\phi(t) = \int_{|\xi| \geq k_0} (|\xi|^2 - k_0^2)^{1/2} \, |\hat{u}(\xi, t)|^2 \, d\xi$$

belongs to $L^1(0, +\infty)$. Since $u_0 \in H^1(\mathbb{R}^2)$, the function $\frac{\partial u}{\partial t}$ is also a solution to the B.P.M. equation with initial condition $-A_q u_0 \in L^2(\mathbb{R}^2)$. Therefore the function

$$\psi(t) = \int_{|\xi| \geq k_0} (|\xi|^2 - k_0^2)^{1/2} \left| \frac{\partial \hat{u}}{\partial t} (\xi, t) \right|^2 d\xi$$

belongs also to $L^1(0, +\infty)$. On the other hand

$$\frac{d\phi}{dt}(t) = 2 \operatorname{Re} \int_{|\xi| \geq k_0} (|\xi|^2 - k_0^2)^{1/2} \, \hat{u} \, \frac{\partial \hat{u}}{\partial t} \, d\xi$$

and thanks to the Cauchy-Schwarz inequality,

$$\left| \frac{d\phi}{dt}(t) \right| \leq \phi(t) + \psi(t)$$

hence $\frac{d\phi}{dt} \in L^1(0, \infty)$. Since $\phi \in L^1(0, \infty)$, (3.20) follows. In the general case, i.e. when $u_0 \in L^2(\mathbb{R}^2)$, by the smoothing effect $u(t_0) \in \mathrm{H}^1(\mathbb{R}^2)$ for every $t_0 > 0$ and the previous proof applies again. □

Remark 3.4 : This decay property is not satisfied by the parabolic approximation (1.8) since for their solutions $\int_{\mathbb{R}^2} |\hat{u}(\xi, t)|^2 \, d\xi$ and

$\int_{\mathbb{R}^2} |\xi|^2 \, |\hat{u}(\xi, t)|^2 \, d\xi$ remain constant in time. This also show that this approximation of Helmholtz equation is not realistic for large wave numbers. □

Dependence of the solutions on the index q

We are given two functions q_1, q_2 with (3.20) $q_1, q_2 \in L^r(\mathbb{R}^2)$ for some r, $2 \leqslant r \leqslant +\infty$. The following results show that the solution to the B.P.M. equation depends continuously on the function q. It will be usefull for proving the convergence of the algorithm in the case of a non smooth refractive index. For its proof we refer to [1].

THEOREM 3.2: *For every T, $0 < T < \infty$, there exists a constant $C(T)$ which depends only on κ, k_0 and T such that, for every $u_0 \in L^2(\mathbb{R}^2)$,*

$$\sup_{0 \leqslant t \leqslant T} \left| e^{-iA_{q_1} t} u_0 - e^{-iA_{q_2} t} u_0 \right|_0 \leqslant C(T) |q_1 - q_2|_{L^r(\mathbb{R}^2)} |u_0|_0 , \tag{3.21}$$

$$\int_0^T \left| e^{-iA_{q_1} t} u_0 - e^{-iA_{q_2} t} u_0 \right|_{1/2}^2 dt \leqslant C(T) |q_1 - q_2|_{L^r(\mathbb{R}^2)} |u_0|_0 . \tag{3.22}$$ □

4. CONVERGENCE OF THE SPLITTING ALGORITHM

We study first a two step algorithm very similar to (1.12)-(1.14) in order to avoid technicalities. The convergence is obtained for smooth refractive index ($q \in W^{1,\infty}(\mathbb{R}^2)$) and then extended to non smooth ones ($q \in L^r(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for some r, $2 < r < +\infty$). Finally we state the convergence result for the original algorithm. The proofs are based on technics similar to that of J. T. Beale and A. Majda [2]. (Concerning general results on splitting algorithms, we refer to R. Temam [10]).

A two step algorithm

We are given T, $0 < T < \infty$ and $u_0 \in H^{1/2}(\mathbb{R}^2)$. We wish to approximate equation (2.13) by the following procedure. For $N \geqslant 1$, we set $\tau = \dfrac{T}{2N}$ and *initialize* the scheme by taking

$$u^0(0) = u_0 . \tag{4.1}$$

We suppose that the u^{2k} ($\simeq u(k\tau)$) are known for $0 \leqslant k \leqslant m$ and we compute u^{2m+1} by

$$\frac{dw}{dt} + A_0 w = 0 , \quad \text{on } [0, \tau] , \tag{4.2}$$

$$w(0) = u^{2m}, \tag{4.3}$$

$$u^{2m+1} \equiv w(\tau) = e^{-A_0 \tau} u^{2m}. \tag{4.4}$$

Then we compute u^{2m+2} by

$$\frac{dv}{dt} + iqv = 0, \quad \text{on } [0, \tau], \tag{4.5}$$

$$v(0) = u^{2m+1}, \tag{4.6}$$

$$u^{2m+2} \equiv v(\tau) = e^{-iq\tau} u^{2m+1}. \tag{4.7}$$

Convergence in the case of a smooth refractive index

We have the following convergence result.

THEOREM 4.1 : *Under the assumption that*

$$q \in W^{1,\infty}(\mathbb{R}^2), \tag{4.8}$$

there exists a constant $\tilde{C}$ *independent of* T *and* N *such that for every* $u_0 \in H^2(\mathbb{R}^2)$,

$$\operatorname*{Sup}_{0 \leq m \leq N} \left| u^{2m} - e^{-A_q m\tau} u_0 \right|_0 \leq \tilde{C} \frac{T}{2N} |u_0|_2 . \tag{4.9}$$

Proof: From (4.1), (4.4) and (4.6) we have

$$u^{2m} - e^{-A_q m\tau} u_0 = \left\{ (e^{-iq\tau} e^{-A_0 \tau})^m - (e^{-(A_0 + iq)\tau})^m \right\} u_0 . \tag{4.10}$$

The difference of the two operators appearing in the right hand side of (4.10) is equal to

$$\sum_{k=0}^{m-1} (e^{-iq\tau} e^{-A_0 \tau})^{m-1-k} (e^{-iq\tau} e^{-A_0 \tau} - e^{-A_q \tau}) e^{-A_q k\tau} \tag{4.11}$$

and since the semi-groups e^{-iqt} and $e^{-A_q t}$ contract in $L^2(\mathbb{R}^2)$ (see (3.18)), we deduce that

$$\left| u^{2m} - e^{-A_q m\tau} u_0 \right| \leq \sum_{k=0}^{m-1} \left| (e^{-iq\tau} e^{-A_0 \tau} - e^{-A_q \tau}) v_k \right|_0 , \tag{4.12}$$

where $v_k = e^{-A_q k\tau} u_0$.

We claim that for $v \in H^2(R^2)$,

$$\left| (e^{-iqt} e^{-A_0 t} - e^{-(A_0 + iq)t}) v \right|_0 \leq \tilde{K} t^2 |v|_2 , \quad \forall t \geq 0 , \tag{4.13}$$

where $\tilde{K}$ depends on q, κ, k_0 but not on t and v. Then (4.9) is a simple consequence of (4.12), (4.13) and $(3.19)_2$ which applies since $q \in W^{1,\infty}(\mathbb{R}^2)$. The theorem is proved.

It remains to show (4.13). We set (see (4.2))

$$\delta(t) \equiv e^{-iqt} w(t) - u(t) = e^{-iqt} e^{-A_0 t} v - e^{-(A_0 + iq)t} v \tag{4.14}$$

for $v \in H^2(\mathbb{R}^2)$ and $q \in W^{1,\infty}(\mathbb{R}^2)$. According to $(3.19)_m$, $m = 0, 1, 2$, there exists a constant $C_0(q)$ such that

$$\sum_{k=0}^{2} |e^{-A_q t} v|_k \leqslant C_0(q) |v|_2 , \quad \forall t \geqslant 0 . \tag{4.15}$$

Using this property with q and $q = 0$, it follows that there exists $C_1(q)$ such that

$$\sum_{k=0}^{2} \left(\left| \frac{d^k u}{dt^k}(t) \right|_0 + \left| \frac{d^k w}{dt^k}(t) \right| \right) \leqslant C_1(q) , \tag{4.16}$$

therefore there exists $C_2(q)$ such that

$$\operatorname*{Sup}_{t \geqslant 0} \left| \frac{d^2 \delta}{dt^2}(t) \right|_0 \leqslant C_2(q) . \tag{4.17}$$

Since $\delta(0)$ and $\dot{\delta}(0)$ vanish, (4.13) follows from (4.17) thanks to Taylor's formula at $t = 0$.

Convergence for non smooth refractive index

As mentionned previously, the refractive index can be discontinuous, piecewise constant for example. In what follows we consider the case where the (strong) hypothesis $q \in W^{1,\infty}$ is relaxed. More precisely we assume that

$$q \in L^r(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \tag{4.18}$$

for some r, $2 < r < +\infty$.

THEOREM 4.2 : *Under the previous hypotheses on q, and with the notation of Theorem* 4.1,

$$\lim_{N \to +\infty} \operatorname*{Sup}_{0 \leqslant m \leqslant N} \left| u^{2m} - e^{-A_q m\tau} u_0 \right|_0 = 0 . \quad \square \tag{4.19}$$

We are going to sketch briefly the main steps of the proof of this result (the details are given in [1]), but first we make the following remark.

Remark 4.1 : In the case of a smooth refractive index the convergence was shown to be of order 1 ($0(1/N)$). The convergence obtained in Theorem 4.2 has no order of convergence. □

Sketch of the *proof of Theorem* 4.2. We introduce a regularized family $\{q_\varepsilon\}_{\varepsilon>0}$ which satisfies

$$(4.20)\qquad \begin{cases} q_\varepsilon \in W^{1,\infty}(\mathbb{R}^2) \cap L^r(\mathbb{R}^2)\,, \\ \operatorname*{Sup}_{\varepsilon>0} |q_\varepsilon|_{L^\infty(\mathbb{R}^2)} < +\infty\,, \\ q_\varepsilon \text{ converges to } q \text{ in } L^r(\mathbb{R}^2) \text{ as } \varepsilon \to 0\,. \end{cases}$$

Let u_0 be given in $H^2(\mathbb{R}^2)$, denote by

$$(4.21)\qquad \delta_m = u^{2m} - e^{-A_q m\tau} u_0\,,$$

and note that it is the sum of the three terms :

$$(4.22)\qquad \delta_m^1 = (e^{-(A_0+iq_\varepsilon)m\tau} - e^{-(A_0+iq)m\tau})\, u_0\,,$$

$$(4.23)\qquad \delta_m^2 = \sum_{k=0}^{m-1} (e^{-iq\tau} e^{-A_0\tau})^{m-k-1} \times$$
$$\times\, (e^{-iq\tau} e^{-A_0\tau} - e^{-iq_\varepsilon\tau} e^{-A_0\tau})\, e^{-(A_0+iq_\varepsilon)k\tau} u_0\,,$$

$$(4.24)\qquad \delta_m^3 = \sum_{k=0}^{m-1} (e^{-iq\tau} e^{-A_0\tau})^{m-k-1} \times$$
$$\times\, (e^{-iq_\varepsilon\tau} e^{-A_0\tau} - e^{-(A_0+iq_\varepsilon)\tau})\, e^{-(A_0+iq_\varepsilon)k\tau} u_0\,.$$

The term (4.22) is estimated thanks to Theorem 3.2, equation (3.21). The two terms (4.23) and (4.24) are bounded by the method we have used for (4.11). Then (4.20) allows us to conclude the proof of (4.19). □

Convergence of the B.P.M. algorithm

We finish this section by giving a result of convergence for the B.P.M. algorithm (1.12)-(1.14). Let us point out that this algorithm is totally similar to the one in two steps (4.1)-(4.6) analyzed previously. It differs only in that the O.D.E. step (4.5) is integrated on $[0, \tau/2]$ in the B.P.M. algorithm instead of on $[0, \tau]$ as in the two steps method. From a computational point of view they are almost identical.

We note that if we set $\tau = T/2N$ then the solution of one integration of the B.P.M. algorithm (3 steps : τ, 2τ, τ) with initialization u_0 reads in concise notation : $e^{-A_0\tau} e^{-2iq\tau} e^{-A_0\tau} u_0$.

THEOREM 4.3 : *Under the assumption that*

$$(4.25)\qquad q \in W^{2,\infty}(\mathbb{R}^2)\,,$$

there exists a constant $\tilde{C}(T)$, *independent of* N *such that for every* $u_0 \in H^3(\mathbb{R}^2)$,

$$(4.26) \quad \sup_{0 \leqslant m \leqslant N} \left| (e^{-A_0 \tau} e^{-2 iq\tau} e^{-A_0 \tau})^m u_0 - e^{-2(A_0 + iq) m\tau} u_0 \right| \leqslant \leqslant \frac{\tilde{C}(T)}{N^2} |u_0|_3 . \quad \square$$

The proof of this result is very similar to that of Theorem 4.2 ; therefore, we omit it. The order of convergence is improved (order 2). We can say that the B.P.M. algorithm is consistent with the B.P.M. equation.

5. REMARKS IN THE NONLINEAR CASE

It has been assumed previously that the refractive index, besides its dependence on ω, $x = (x_1, x_2)$ and t, was independent of the electric field. For certain media this is not a good approximation and one must take into account the influence of the electric field (autoinfluence). By physical arguments (objectivity of the refractive index) it can be show that the refractive index depends only on the intensity of the beam, which is proportional to the square of the norm of the electric field. A first order expansion reads

$$(5.1) \qquad n(\omega, E) = n_0(\omega) + n_2(\omega) |E|^2 ,$$

with $|n_2(\omega)| \ll n_0(\omega)$. The sign of n_2 is related to the nature of the nonlinearity : a positive sign corresponds to a self-focusing nonlinearity, a negative one to a self-defocusing nonlinearity. The first order expansion (5.1) leads to

$$(5.2) \qquad q = k_0 \frac{n^2 - n_0^2}{2 n_0^2} \simeq \frac{k_0}{n_0} n_2 |E|^2 \# \varepsilon |u|^2$$

where $\#$ means proportional and ε is the sign of n_2.

The corresponding Helmholtz equation (known as the Helmholtz equation in a Kerr-type medium) reads

$$(5.3) \qquad \frac{\partial^2 E}{\partial^2 x_1^2} + \frac{\partial^2 E}{\partial x_2^2} + \frac{\partial^2 E}{\partial x_3^2} + \frac{\omega^2}{c^2} \left(n_0^2 + \frac{\omega}{c} n_2 |E|^2 \right) E = 0 ,$$

this equation is solved numerically by the B.P.M. algorithm (1.12)-(1.14) where (1.13) is replaced by

$$(5.4) \qquad -2 i k_0 \frac{\partial A}{\partial x_3} + \frac{\omega^3}{c^3} n_2 |A|^2 A = 0$$

which is again an explicit O.D.E. (multiply (5.4) by $i\bar{A}$ and take the real part).

With the notations of Sections 3 and 4, we introduce the *nonlinear B.P.M. equation*

$$\frac{du}{dt} + A_0 u + i\varepsilon |u|^2 u = 0 , \tag{5.5}$$

which is supplemented as before with the initial condition

$$u(0) = u_0 , \tag{5.6}$$

(note that u has been renormalized in order to keep $\varepsilon = \pm 1$).

THEOREM 5.1 : *For every* $u_0 \in L^2(\mathbb{R}^2)$, *there exists a function* u

$$u \in L^\infty(\mathbb{R}_+ ; L^2(\mathbb{R}^2)) \cap L^2(0, T ; H^{1/2}(\mathbb{R}^2)) , \quad \forall T < +\infty \tag{5.7}$$

solution of (5.5)-(5.6). □

This result shows that the nonlinear B.P.M. equation possesses a *global* weak solution (i.e. in the distribution sense). We do not know whether this solution is unique or even more regular for $t > 0$ (see also Proposition 5.1 and Theorem 5.2 below). The proof of Theorem 5.1 is based on standard energy and compactness methods ([6]), we refer to [1] for the details. Let us simply mention that if u satisfies (5.5) and (5.7), then

$$\frac{du}{dt} \in L^1(0, T ; L^1(\mathbb{R}^2)) + L^2(0, T ; H^{-1/2}(\mathbb{R}^2)) , \quad \forall T < \infty ,$$

and (5.6) makes sense. □

Concerning the uniqueness of solutions of the B.P.M. equation we have the

PROPOSITION 5.1 : *There exists at most one solution* u

$$u \in L^\infty(0, T ; H^{1/2}(\mathbb{R}^2)) \cap L^2(0, T ; H^1(\mathbb{R}^2)) , \quad \forall T < +\infty \tag{5.8}$$

of (5.5)-(5.6). □

Due to Sobolev imbeddings, if u satisfies (5.5) and (5.8), then $\frac{du}{dt} \in L^2(0, T ; L^2(\mathbb{R}^2))$, $\forall T < +\infty \cdot$ It follows ([7]) that

$$u \in \mathscr{C}(I ; H^{1/2}(\mathbb{R}^2)) \cap L^2_{\text{loc}}(I ; H^1(\mathbb{R}^2)) , \tag{5.9}$$

for every interval I of $\mathbb{R}_+$.

Functions u satisfying (5.5) and (5.9) are called *strong solutions* on I of the nonlinear B.P.M. equation. According to Proposition 5.1 there exists at most one strong solution to this equation. Concerning their existence, we are going to state a result (Theorem 5.2), which shows that, for small u_0 in $H^{1/2}(\mathbb{R}^2)$, there exists a (unique) global strong solution (i.e. for which $I = \mathbb{R}_+$) to the nonlinear B.P.M. equation. And for arbitrarily large u_0, there exists $T(u_0) > 0$ and a (unique) strong solution on $I = [0, T(u_0)]$ to the nonlinear B.P.M. equation. The proof of these results are slightly technical but classic, we refer to [1] for the details.

THEOREM 5.2 : (i) *There exists a constant* K_0 *depending on* κ *and* k_0 *such that for every* $u_0 \in H^{1/2}(\mathbb{R}^2)$,

$$|u_0|_{1/2} \leq K_0 , \tag{5.10}$$

the nonlinear B.P.M. equation (5.5)-(5.6) *possesses a unique strong solution on* $\mathbb{R}_+$.

(ii) *For given* $\delta > 0$ *and*

$$u_0 \in H^{1+\delta}(\mathbb{R}^2) , \tag{5.11}$$

there exists a positive real number $T_\delta = T_\delta(|u_0|_{1+\delta})$ *such that the nonlinear B.P.M. equation possesses a unique strong solution on* $[0, T_\delta]$. *Moreover*

$$u \in \mathscr{C}([0, T_\delta] ; H^{1+\delta}(\mathbb{R}^2)) \cap L^2\left([0, T_\delta] ; H^{\frac{3}{2}+\delta}(\mathbb{R}^2)\right) . \quad \square \tag{5.12}$$

Remark 5.1 : (i) We know neither part (i) of this result is still valid for arbitrarily large $u_0 \in H^{1/2}(\mathbb{R}^2)$ nor whether the solution obtained in part (ii) exists for arbitrarily large time, t.

(ii) Concerning hypothesis (5.11), we recall that $H^{1+\delta}(\mathbb{R}^2)$ is an algebra for $\delta > 0$, and this allows us to prove the following estimate for $|u|_{1+\delta}$ where u denotes the solution to (5.5) :

$$\frac{1}{2}\frac{d}{dt}|u|_{1+\delta}^2 + |u|_{3/2+\delta}^2 \leq m|u|_{1+\delta}^2 + C_\delta |u|_{1+\delta}^4 , \tag{5.13}$$

where m and C_δ are constants which do not depend on u or u_0. By comparison with the O.D.E.

$$\begin{cases} \dfrac{dy}{dt} = 2\,my + 2\,C_\delta\, y^2 \\ y(0) = |u_0|_{1+\delta}^2 \end{cases} \tag{5.14}$$

we obtain the following a priori estimate on $[0, T_\delta]$,

$$T_\delta(|u_0|_{1+\delta}) = \frac{1}{2\,m}\,\mathrm{Log}\left(\frac{1}{2} + \frac{m}{2\,C_\delta |u_0|_{1+\delta}^2}\right) ,$$

(5.15)
$$|u(t)|_{1+\delta}^2 \leq y(t) \leq \frac{m}{C_\delta},$$

hence the result. □

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