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ASYMPTOTIC BEHAVIOUR FOR THE SOLUTION OF THE COMPRESSIBLE NAVIER-STOKES EQUATION, WHEN THE COMPRESSIBILITY GOES TO ZERO (*)

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Résumé. — *Nous étudions le comportement asymptotique des solutions* (u^λ, p^λ) *des équations de Navier-Stokes compressibles lorsque la compressibilité tend vers* 0 $(\lambda \to \infty)$:

$$\begin{cases} \rho^\lambda(u_t^\lambda + (u^\lambda \cdot \nabla) u^\lambda) - \nu \Delta u^\lambda = -\lambda^2 \nabla p^\lambda , \\ p_t^\lambda + (\nabla p^\lambda) \cdot u^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0 , \\ u^\lambda(x, 0) = u_0(x) + \nabla \Phi_0(x) + \dfrac{u_1(x)}{\lambda}, \quad \operatorname{div} u_0 = 0 , \\ p^\lambda(x, 0) = p_0 + \dfrac{p_1(x)}{\lambda^2}, p_0 = \text{Cte}, \textit{où } p = A\rho^\gamma \quad \textit{avec} \quad \gamma > 1 \textit{ et } A > 0 . \end{cases}$$

Nous établissons d'abord l'existence globale en temps des solutions (u^λ, p^λ), *les estimations obtenues étant uniformes en* λ.

Lorsque $\Phi_0 = 0$, *nous prouvons que* u^λ *converge fortement vers* u^∞, *solution des équations de Navier-Stokes incompressibles suivantes :*

$$\begin{cases} \rho_0(u_t^\infty + (u^\infty \cdot \nabla) u^\infty) - \nu \Delta u^\infty = -\nabla p^\infty , \\ \operatorname{div} u^\infty = 0 \quad \textit{et} \quad u^\infty(x, 0) = u_0(x) . \end{cases}$$

Lorsque $\Phi_0 \neq 0$, *nous mettons en évidence un phénomène de couche initiale. Plus précisément, nous prouvons que* $u^\lambda - u^\infty - v^\lambda$ *converge fortement vers* 0, *où* v^λ *est la solution de l'équation couplée suivante :*

$$\begin{cases} \rho_0 v_t^\lambda - \nu \Delta v^\lambda + \lambda \nabla q^\lambda = 0 , \\ q_t^\lambda + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0 , \\ v^\lambda(x, 0) = \nabla \Phi_0(x) , \quad q^\lambda(x, 0) = 0 . \end{cases}$$

Abstract. — *We study the asymptotic behaviour of the solutions* (u^λ, p^λ) *of compressible Navier-Stokes' equations when compressibility goes to zero* $(\lambda \to +\infty)$:

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$$\begin{cases} \rho^\lambda(u_t^\lambda + (u^\lambda \,.\, \nabla) u^\lambda) - \nu \,\Delta u^\lambda = -\lambda^2 \,\nabla p^\lambda , \\ p_t^\lambda + (\nabla p^\lambda) \,.\, u^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0 , \\ u^\lambda(x, 0) = u_0(x) + \nabla\Phi_0(x) + \dfrac{u_1(x)}{\lambda} , \quad \operatorname{div} u_0 = 0 , \\ p^\lambda(x, 0) = p_0 + \dfrac{p_1(x)}{\lambda^2} , p_0 = \text{Cte}, p = A\rho^\gamma \quad \textit{with}\ \gamma > 1 \textit{ and } A > 0 . \end{cases}$$

We first establish global existence in time of the solutions (u^λ, p^λ), the obtained estimates being uniform in λ.

When $\Phi_0 = 0$, we prove that u^λ strongly converges to u^∞, solution of the following Navier-Stokes' incompressible equations :

$$\begin{cases} \rho_0(u_t^\infty + (u^\infty \,.\, \nabla) u^\infty) - \nu \,\Delta u^\infty = -\nabla p^\infty , \\ \operatorname{div} u^\infty = 0 \quad et \quad u^\infty(x, 0) = u_0(x) . \end{cases}$$

When $\Phi_0 \neq 0$, an initial layer phenomenon arises.

More precisely, we prove that $u^\lambda - u^\infty - v^\lambda$ strongly converges to zero, where v^λ is the solution of the following coupled equation :

$$\begin{cases} \rho_0\, v_t^\lambda - \nu\, \Delta v^\lambda + \lambda\, \nabla q^\lambda = 0 , \\ q_t^\lambda + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0 , \\ v^\lambda(x, 0) = \nabla\Phi_0(x) , \quad q^\lambda(x, 0) = 0 . \end{cases}$$

I. INTRODUCTION

Our aim, in this paper, is to study the solutions of the equations of gases' dynamic :

$$(S) \quad \begin{cases} \rho \left(\dfrac{\partial u}{\partial t} + (u \,.\, \nabla) u \right) - \nu \,\Delta u = -\nabla p , \quad \nu > 0 , \\ \dfrac{\partial \rho}{\partial t} + \nabla \,.\, (\rho u) = 0 , \quad x \in \Omega \in \mathbb{R}^n , \quad t \in \mathbb{R}^+ , \\ u(x, 0) = u_0(x) , \quad \rho(x, 0) = \rho_0(x) , \end{cases}$$

where the velocity u and the density ρ are unknown, the pression p being a given function of ρ.

Klainerman and Majda in [1] have proved the local existence of a smooth solution (u, ρ) of the system (S) in the case where Ω is the torus T^n of $\mathbb{R}^n$. In [2], they show the local existence of a smooth solution of compressible Euler's equations (when $\nu = 0$) for the whole space $\mathbb{R}^n$.

On the other part, Nishida and Matsumura, in [3], have obtained a global in time result for the system (S) coupled with an evolution equation for the temperature. In their work, they consider the case where $\Omega = \mathbb{R}^3$, where the gas is perfect and polytropic, and they are led to impose to the initial data to be small enough in $H^3(\mathbb{R}^3)$ norm.

As far as we are concerned, we are going to study the compressible system (S) when compressibility goes to 0, for the whole space $\mathbb{R}^n$, in any dimension $n \geqslant 2$.

Let us consider ρ as a function of p.

A. Lagha, in [4], defines compressibility as the quantity :

$$\varepsilon = \left[\frac{\partial p}{\partial \rho} (\rho_0) \right]^{-1},$$

where ρ_0 represents a first approximation of the gases' density.

She obtains a relation of the shape :

$$\rho = \rho_0 + \varepsilon p ,$$

which leads her to study the following perturbed system :

$$(S^\varepsilon) \quad \begin{cases} \rho^\varepsilon \left(\dfrac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \, . \, \nabla) u^\varepsilon \right) - \nu \, \Delta u^\varepsilon = - \nabla p^\varepsilon , \quad x \in \mathbb{R}^n , \\ \varepsilon \dfrac{\partial p^\varepsilon}{\partial t} + \varepsilon u^\varepsilon \, . \, \nabla p^\varepsilon + \rho^\varepsilon \, \nabla u^\varepsilon = 0 , \quad t \in \mathbb{R}^+ , \\ u^\varepsilon(x, 0) = u_0(x) , \quad p^\varepsilon(x, 0) = p_0(x) . \end{cases}$$

Temam uses the same definition of compressibility in [5],but he works in a bounded open set Ω of $\mathbb{R}^n$.

On the other hand, Majda, in [6], takes a more physical definition of compressibility by considering the state equation of a perfect gas :

$$p = A\rho^\gamma , \qquad \gamma > 1 .$$

From the initial system :

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div} (\rho u) = 0 , \\ \rho \left(\dfrac{\partial u}{\partial t} + u \, . \, \nabla u \right) + \nabla p = 0 , \\ \rho(x, 0) = \rho_0(x) , \quad u(x, 0) = u_0(x) , \end{cases}$$

he is led to consider the following perturbed system :

$$\begin{cases} \dfrac{\partial \tilde{\rho}}{\partial t'} + \operatorname{div} (\tilde{\rho} \tilde{u}) = 0 , \\ \tilde{\rho} \left(\dfrac{\partial \tilde{u}}{\partial t'} + (\tilde{u} \, . \, \nabla) \tilde{u} \right) + \lambda^2 \, \nabla p(\tilde{\rho}) = 0 , \\ \tilde{\rho}(x, 0) = \dfrac{\rho_0(x)}{\rho_m} , \quad \tilde{u}(x, 0) = \dfrac{u_0(x)}{|u_m|} , \end{cases}$$

where

$$\tilde{\rho} = \frac{\rho}{\rho_m}, \quad \tilde{u} = \frac{u}{|u_m|}, \quad t' = |u_m|\, t,$$

$$\rho_m = \max \rho_0(x) \quad \text{and} \quad |u_m| = \max |u_0(x)| .$$

The compressibility is there given by $1/\lambda^2$, with

$$\lambda^2 = \left[\frac{\partial p}{\partial \rho}(\rho_m)/|u_m|^2 \right] (\gamma A)^{-1} .$$

Majda proves, for « small enough » initial data, the existence of a smooth solution for the system (S^λ), when λ is sufficiently large.

We have choosed to use this last definition of compressibility, while keeping the viscosity term : $-\nu\,\Delta u$.

This led us to consider a perturbed system, between those studied by A. Lagha and Majda, of the shape :

$$(S^\lambda)\quad \begin{cases} \rho^\lambda\left(\dfrac{\partial u^\lambda}{\partial t} + (u^\lambda\,.\,\nabla)\,u^\lambda\right) - \nu\,\Delta u^\lambda = -\lambda^2\,\nabla p^\lambda\,, \\ \dfrac{\partial p^\lambda}{\partial t} + (\nabla p^\lambda)\,.\,u^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0\,, \\ u^\lambda(x,0) = u_0(x) + \dfrac{u_1(x)}{\lambda}\,, \quad p^\lambda(x,0) = p_0 + \dfrac{p_1(x)}{\lambda^2}\,, \quad p_0 = \text{Cte}\,. \end{cases}$$

The shape of $u^\lambda(x,0) = u_0^\lambda(x)$ and $p^\lambda(x,0) = p_0^\lambda(x)$ issues from a formal asymptotic development (see [6]).

In the paragraph II, we have followed Lagha's way of proceeding which was taking its inspiration from Nishida and Matsumura's technics.

We introduce

$$E^\lambda(t) = |u^\lambda(t)|^2_{H^s} + |\lambda(p^\lambda - p_0)|^2_{H^s} \quad \text{where} \quad s > \left[\frac{n}{2}\right] + 1\,,$$

and we prove that, for sufficiently large λ and for « small enough » initial data, there exists some constant K_0, independent of λ, so that :

$$\forall t \in \mathbb{R}^+\,, \quad E^\lambda(t) + \int_0^t |\nabla u^\lambda(\tau)|^2_{H^s}\,d\tau + \int_0^t |\lambda\nabla(p^\lambda - p_0)|^2_{H^{s-1}}\,d\tau \leqslant K_0\,.$$

This result permits to conclude, in any dimension $n \geqslant 2$, that there exists a unic smooth global solution of the system (S^λ), for small enough initial data :

$$u^\lambda \in C_B(0,\infty,H^s) \cap C_B^1(0,\infty,H^{s-2})\,,$$

$$(p^\lambda - p_0) \in C_B(0,\infty,H^s) \cap C_B^1(0,\infty,H^{s-1})\,, \quad \text{where } s > \left[\frac{n}{2}\right] + 1\,.$$

In the following part of our work, we study the asymptotic behaviour of the solutions (u^λ, p^λ) of the system (S^λ) when the compressibility goes to zero, so when λ goes to infinity.

In paragraph III, we add the classical following hypothesis :

$$\operatorname{div} u_0 = 0 ,$$

and we study the convergence of the solutions (u^λ, p^λ) to the solution (u^∞, p^∞) of the incompressible Navier-Stokes equations :

$$(S^\infty) \quad \begin{cases} \rho_0\left(\dfrac{\partial u^\infty}{\partial t} + (u^\infty . \nabla) u^\infty\right) - \nu \Delta u^\infty = - \nabla p^\infty , \\ \operatorname{div} u^\infty = 0 , \quad u^\infty(x, 0) = u_0(x) . \end{cases}$$

We first obtain supplementary estimates concerning the time derivatives, independent of λ sufficiently large :

$$\forall t \in \mathbb{R}^+ , \quad |u_t^\lambda|_{H^{s-2}}^2 + |\lambda(p^\lambda - p_0)_t|_{H^{s-2}}^2 + \int_0^t |\nabla u_t^\lambda|_{H^{s-2}}^2 d\tau \leq M(t) ,$$

where $\quad s > \left[\dfrac{n}{2}\right] + 1 \quad$ and $\quad M(t) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+ , \mathbb{R}^+)$.

This leads us to state the following weak convergence result, obtained by Klainerman and Majda in the case of the torus of $\mathbb{R}^n$ and by A. Lagha in $\mathbb{R}^2$:

If $\Omega = \mathbb{R}^n$, with $n \geq 2$, then

$$\begin{aligned} &u^\lambda \to u^\infty \quad \text{in} \quad C_{\mathrm{loc}}(0, \infty, H^{s-1}_{\mathrm{loc}}) \quad \text{strongly} , \\ &\lambda^2 \nabla p^\lambda \to \nabla p^\infty \quad \text{in} \quad L^\infty_{\mathrm{loc}}(0, \infty, H^{s-2}) \quad \text{weak star (w.s.)} , \\ &\rho^\lambda \to \rho_0 \quad \text{in} \quad C_B(0, \infty, W^{\infty, s-2}) \quad \text{strongly} . \end{aligned}$$

However, Klainerman and Majda, in [2], prove the strong convergence of the solutions (u^λ, p^λ) of compressible Euler's equations :

$$\begin{cases} \rho^\lambda\left(\dfrac{\partial u^\lambda}{\partial t} + (u^\lambda . \nabla) u^\lambda\right) = - \lambda^2 \nabla p^\lambda , \\ \dfrac{\partial p^\lambda}{\partial t} + (\nabla p^\lambda) . u^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0 , \\ u^\lambda(x, 0) = u_0(x) + \dfrac{u_1(x)}{\lambda} , \quad p^\lambda(x, 0) = p_0 + \dfrac{p_1(x)}{\lambda^2} \\ p_0 = \mathrm{Cte} , \quad \operatorname{div} u_0 = 0 , \end{cases}$$

to the solution (u^∞, p^∞) of incompressible Euler's equations :

$$\begin{cases} \rho_0 \left(\dfrac{\partial u^\infty}{\partial t} + (u^\infty . \nabla) u^\infty \right) = - \nabla p^\infty , \\ \operatorname{div} u^\infty = 0 , \quad u^\infty(x, 0) = u_0(x) , \end{cases}$$

by imposing supplementary conditions to $|p^\infty|_{L^2}$ and $|p_t^\infty|_{L^2}$.

(It is, of course, a convergence on a finite time intervall.)

In paragraph IV, we take our inspiration from that technic. We impose to the solution (u^∞, p^∞) of the system (S^∞) to verifie the following hypothesis :

$$(H) \quad |p^\infty|_{L^2} + |p_t^\infty|_{L^2} \leqslant N(t), \quad \text{where} \quad N \in L^\infty_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+) .$$

Then, when the initial data $(u_0^\lambda, p_0^\lambda - p_0)$ are in $H^{s+2}(\mathbb{R}^n)$, we prove that there exists a locally bounded function $M(t)$ so that :

$$\forall t \in \mathbb{R}^+, \quad \forall \lambda \geqslant \lambda_0 ,$$

$$\lambda^2 |u^\lambda - u^\infty|^2_{H^s} + |\lambda^2 (p^\lambda - p_0) - p^\infty|^2_{H^s} + \lambda^2 \int_0^t |\nabla(u^\lambda - u^\infty)|^2_{H^s} d\tau \leqslant M(t) .$$

In paragraph V, we have studied what happens with the convergence of (u^λ, p^λ) to (u^∞, p^∞) when we cut out the fundamental hypothesis : $\operatorname{div} u_0 = 0$. So we consider the initial data with the following more general shape :

$$u_0^\lambda(x) = u_0(x) + \nabla \Phi_0(x) + \frac{u_1(x)}{\lambda}, \quad \text{with} \quad \operatorname{div} u_0 = 0 ,$$

$$p_0^\lambda(x) = p_0 + \frac{p_1(x)}{\lambda^2}, \quad p_0 = \text{Cte} .$$

In fact, an initial layer phenomenon appears.

A fitting corrector term is given by the solution (v^λ, q^λ) of the following system (C^λ) :

$$(C^\lambda) \quad \begin{cases} \rho_0 \dfrac{\partial v^\lambda}{\partial t} - \nu \Delta v^\lambda = - \lambda \nabla q^\lambda , \\ \dfrac{\partial q^\lambda}{\partial t} + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0 , \\ v^\lambda(x, 0) = \nabla \Phi_0(x) , \quad q^\lambda(x, 0) = 0 . \end{cases}$$

We prove, in appendix, that if Φ_0 is choosen regular enough, then v^λ verifies the following inequalities :

$$|v^\lambda(.,t)|_{L^\infty} \leqslant \frac{C}{1+\lambda t} \qquad \text{if } n \geqslant 3\,,$$

$$|v^\lambda(.,t)|_{L^\infty} \leqslant \frac{C}{\sqrt{1+\lambda t}} \qquad \text{if } n = 2\,.$$

We obtain the following result :

If the solution (u^∞, p^∞) of the system (S^∞) satisfies to the hypothesis (H) and if the initial data are regular enough (we'll precise these assumptions later), there exists some locally bounded function $M(t)$ so that, for sufficiently large λ, we have :

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leqslant \frac{M(t)}{\lambda}\,(\text{Log}\,(1+\lambda t) + 1) \qquad \text{if } n \geqslant 3\,,$$

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leqslant \frac{M(t)}{\sqrt{\lambda}} \qquad \text{if } n = 2\,.$$

We then end by a remark concerning an initial layer's phenomenon in the compressible Euler's equations.

Notations :

— $|.|_{L^p}$ (or $|.|_p$), $|.|_{H^s}$ and $|.|_{W^{k,p}}$ will design respectively the norms $L^p(\mathbb{R}^n)$, $H^s(\mathbb{R}^n)$ and $W^{k,p}(\mathbb{R}^n)$.

— We'll call « C » different numerical constants and « K » different quantities only depending on initial data.

— Finally, $M(t)$ or $N(t)$ will design different increasing functions of $L^\infty_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+)$.

II. INDEPENDENT OF λ ESTIMATES. GLOBAL EXISTENCE

A. Independent of λ estimates

Let us consider the system (S^λ) :

$$\text{(2.1)} \qquad \rho^\lambda(u_t^\lambda + (u^\lambda\,.\,\nabla)\,u^\lambda) - \nu\,\Delta u^\lambda = -\,\lambda^2\,\nabla p^\lambda\,, \quad x \in \mathbb{R}^n\,,$$

$$\text{(2.2)} \qquad p_t^\lambda + \nabla p^\lambda\,.\,u^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0\,, \quad t \in \mathbb{R}^+\,,$$

$$\text{(2.3)} \qquad u^\lambda(x,0) = u_0^\lambda(x)\,, \quad p^\lambda(x,0) = p_0 + \frac{p_1(x)}{\lambda^2}\,, p_0 = \text{Cte}\,,$$

where $u_0^\lambda \in H^s$, $p_0 > 0$, $p_1 \in H^s$, s being an integer verifying $s > s_0 = \left[\frac{n}{2}\right] + 1$, and where $p = A\rho^\psi$, $\gamma > 1$.

Let us note that equation (2.2) may be written :

$$\rho_t^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 . \tag{2.4}$$

We are going to assume « a priori » that (u^λ, p^λ) satisfies the following $H(K, T)$ hypothesis :

There exists $T > 0$ and $K > 0$ so that (u^λ, p^λ) is a solution of (S^λ) on the intervall $[0, T]$, verifying :

$$\begin{aligned} &u^\lambda \in C([0,T], H^s) \cap C^1([0,T], H^{s-2}) , \\ &p^\lambda \in C([0,T], H^s) \cap C^1([0,T], H^{s-1}) \ \ \textit{and} \\ &\forall t \in [0,T], \quad E^\lambda(t) \leqslant K , \end{aligned}$$

where $E^\lambda(t)$ is defined by the relation :

$$E^\lambda(t) = |u^\lambda(t)|_{H^s}^2 + |\lambda(p^\lambda - p_0)|_{H^s}^2 .$$

We are going to prove that, in these conditions, there exists some constant $C_0(K)$, independent of T and λ, and there exists $\lambda_0 > 0$, so that :

$\forall t \in [0, T], \quad \forall \lambda \geqslant \lambda_0 ,$

$$E^\lambda(t) + \int_0^t |\nabla u^\lambda|_{H^s}^2 \, d\tau + \int_0^t |\lambda \nabla (p^\lambda - p_0)|_{H^{s-1}}^2 \, d\tau \leqslant C_0(K) \, . \, E_0^\lambda$$

(where $E_0^\lambda = E^\lambda(0)$).

First, let us make some preliminary remarks which will appreciably simplify the proof.

Let us note

$$\begin{aligned} &\tilde{p}^\lambda(x,t) = \lambda(p^\lambda(x,t) - p_0) \quad \text{and} \\ &\tilde{\rho}^\lambda(x,t) = \lambda(\rho^\lambda(x,t) - \rho_0) \quad \text{where} \quad p_0 = A\rho_0^\gamma . \end{aligned}$$

LEMMA 1 : *Under hypothesis $H(K, T)$, and if $\lambda \geqslant \lambda_1$, then there exists four strictly positive constants p_1, p_2, ρ_1, ρ_2, so that :*

$$\begin{aligned} \forall x \in \mathbb{R}^n , \quad \forall t \in [0,T] , \quad & 0 < p_1 \leqslant p^\lambda \leqslant p_2 \\ \textit{and} \quad & 0 < \rho_1 \leqslant \rho^\lambda \leqslant \rho_2 . \end{aligned}$$

In fact,

$$\begin{aligned} |p^\lambda - p_0|_\infty &\leqslant |p^\lambda - p_0|_{H^s} \quad \left(\text{since} \quad s > s_0 > \frac{n}{2}\right) \\ &\leqslant \frac{|\tilde{p}^\lambda|}{\lambda} H^s \leqslant \frac{K}{\lambda} . \end{aligned}$$

We have just to choose $\lambda_1 = \dfrac{2K}{p_0}$, which gives $p_1 = \dfrac{p_0}{2}$, $p_2 = \dfrac{3p_0}{2}$. Moreover, if $h(\rho) = A\rho^\gamma = p^\lambda$, then $0 < h^{-1}\left(\dfrac{p_0}{2}\right) \leq \rho^\lambda \leq h^{-1}\left(\dfrac{3p_0}{2}\right)$.

LEMMA 2 : *There exists two constants C_1 and C_2 and $\lambda_2 = \lambda_2(K)$, so that, if $\lambda \geq \lambda_2 \geq \lambda_1$, we get :*

$$\forall p \in [2, +\infty], \quad C_1|\tilde\rho^\lambda|_p \leq |\tilde p^\lambda|_p \leq C_2|\tilde\rho^\lambda|_p$$

and

$$C_1|D\tilde\rho^\lambda|_p \leq |D\tilde p^\lambda|_p \leq C_2|D\tilde\rho^\lambda|_p .$$

Let us note $k = h^{-1}$. Then there exists $p_\theta \in [p_0, p^\lambda]$, so that :

$$\tilde\rho^\lambda = \lambda[k(p^\lambda) - k(p_0)] = \lambda(p^\lambda - p_0) . k'(p_0) + \frac{\lambda}{2}(p^\lambda - p_0)^2 . k''(p_\theta) .$$

Then,

$$|\tilde\rho^\lambda - k'(p_0)\tilde p^\lambda|_p \leq \frac{1}{2\lambda}|\tilde p^\lambda|_p\,|\tilde p^\lambda|_\infty\,|k''(p_\theta)|_\infty \leq \frac{C}{\lambda}|\tilde p^\lambda|_p .$$

So, for large enough λ, $|\tilde\rho^\lambda|_p$ and $|\tilde p^\lambda|_p$ are comparable.

Moreover, $D\tilde\rho^\lambda = k'(p^\lambda) . D\tilde p^\lambda$; k and all its derivatives being locally bounded on $\mathbb{R}_+^*$, we may conclude with lemma 1.

LEMMA 3 :

(i) *$D^s\tilde\rho^\lambda$ may be written :*

$$D^s\tilde\rho^\lambda = k'(p^\lambda) . D^s\tilde p^\lambda + \frac{\chi}{\lambda} \quad \textit{where} \quad |\chi|_{L^2} \leq C|\nabla\tilde p^\lambda|_{H^{s-1}} .$$

In particular, $|D\tilde\rho^\lambda|_{H^{s-1}}$ and $|D\tilde p^\lambda|_{H^{s-1}}$ are comparable as soon as λ is sufficiently large, $\lambda \geq \lambda_3 \geq \lambda_2$.

(ii) $\left|D^{s-1}\left(\dfrac{1}{\rho^\lambda}\right)\right|_{L^2} \leq \dfrac{C}{\lambda}$, *as soon as λ is large enough.*

Proof:

(i)

$$D^s\tilde\rho^\lambda = k'(p^\lambda)\,D^s(\tilde p^\lambda) + \underbrace{+\sum_{p=2}^{s}\ \sum_{\substack{i_1+\cdots+i_s=p\\ i_1+2i_2+\cdots+(s-1)i_{s-1}=s}} C_{i_p,p}(D\tilde p^\lambda)^{i_1}\dots(D^{s-1}\tilde p^\lambda)^{i_{s-1}}\frac{k^{(p)}(p^\lambda)}{\lambda^{p-1}}}_{\chi/\lambda} .$$

If $\lambda \geqslant \max(K^2, 1)$, we deduce from hypothesis $H(K, T)$ that $|\chi|_{L^2} \leqslant C\,|\nabla \tilde{p}^\lambda|_{H^{s-1}}$. Then, since $0 < k'(p_2) \leqslant k'(p^\lambda) \leqslant k'(p_1)$, we get that $|D\tilde{\rho}^\lambda|_{H^{s-1}}$ and $|D\tilde{p}^\lambda|_{H^{s-1}}$ are comparable.

(ii) We just have to note that if $\phi(x) = x^{-1}$, then :

$$D^{s-1}\left(\frac{1}{\rho^\lambda}\right) = \sum_{p=1}^{s-1} \sum_{\substack{i_1 + \cdots + i_{s-1} = p \\ i_1 + \cdots + (s-1) i_{s-1} = s-1}} C_{i_\rho, p} (D\tilde{\rho}^\lambda)^{i_1} \ldots (D^{s-1}\tilde{\rho}^\lambda)^{i_{s-1}} \frac{\phi^{(p)}(\rho^\lambda)}{\lambda^p}.$$

We end with the assumption $H(K, T)$.

LEMMA 4 : *If u, v and w are smooth functions,*

$$\int (v \cdot \nabla) u \cdot w \, dx = - \int (v \cdot \nabla) w \cdot u \, dx - \int (u \cdot w) \operatorname{div} v \, dx .$$

In particular,

$$\int (v \cdot \nabla) u \cdot u \, dx = - \frac{1}{2} \int |u|^2 \operatorname{div} v \, dx .$$

LEMMA 5 [7] : *Let f and g be two smooth functions*

(2.5) $|D^k(fg) - f D^k g|_p \leqslant C\,|Df|_r\,|D^{k-1} g|_{r'} + C\,|D^k f|_s\,|g|_{s'}$;

(2.6) $|D^k(fg)|_p \leqslant C\,|f|_r\,|D^k g|_{r'} + C\,|D^k f|_s\,|g|_{s'}$,

where $k > 0, p \in [1, +\infty]$ *and* $\dfrac{1}{p} = \dfrac{1}{r} + \dfrac{1}{r'} = \dfrac{1}{s} + \dfrac{1}{s'}$.

We are now able to establish the desired « a priori » estimates.

First step : L^2-Norms of u^λ and p^λ.

Multiplying (2.1) by u^λ, and (2.4) by $\dfrac{|u^\lambda|^2}{2}$, we get :

$$\frac{|u^\lambda|^2}{2} \rho_t^\lambda + \frac{|u^\lambda|^2}{2} \operatorname{div}(\rho^\lambda u^\lambda) + \frac{\rho^\lambda}{2} |u^\lambda|_t^2 + \\ + (\rho^\lambda u^\lambda \nabla) u^\lambda \cdot u^\lambda - \nu\,\Delta u^\lambda \cdot u^\lambda = - \lambda\,\nabla \tilde{p}^\lambda \cdot u^\lambda .$$

Then, integrating on $\mathbb{R}^n$:

$$\frac{\partial}{\partial t} \int \frac{\rho^\lambda |u^\lambda|^2}{2} + \nu \int |\nabla u^\lambda|^2 + \int (\rho^\lambda u^\lambda \nabla) u^\lambda \cdot u^\lambda + \\ + \int \frac{|u^\lambda|^2}{2} \operatorname{div}(\rho^\lambda u^\lambda) = \lambda \int \tilde{p}^\lambda \operatorname{div} u^\lambda .$$

We deduce from lemma 4 that :

$$\frac{\partial}{\partial t}\int \frac{\rho^\lambda |u^\lambda|^2}{2}\,dx + \nu \int |\nabla u^\lambda|^2\,dx = \lambda \int (\tilde{p}^\lambda \operatorname{div} u^\lambda)\,dx\,. \tag{2.7}$$

Let us introduce

$$W(\rho^\lambda) = \int_{\rho_0}^{\rho^\lambda} \frac{\lambda \tilde{p}^\lambda(s)}{s^2}\,ds\,.$$

Multiplying (2.4) by $\frac{\partial}{\partial \rho}(\rho^\lambda W)$, we get :

$$\frac{\partial}{\partial t}\int \rho^\lambda W\,dx + \int \operatorname{div}(\rho^\lambda u^\lambda)\,W\,dx + \\ + \int (\rho^\lambda)^2 \operatorname{div} u^\lambda \frac{\partial W}{\partial \rho}\,dx + \int \rho^\lambda u^\lambda \cdot \nabla \rho^\lambda \frac{\partial W}{\partial \rho}\,dx = 0\,.$$

Now,

$$\int \operatorname{div}(\rho^\lambda u^\lambda)\,W\,dx = -\int \rho^\lambda u^\lambda \cdot \nabla W\,dx = -\int \rho^\lambda u^\lambda \cdot \nabla \rho^\lambda \frac{\partial W}{\partial \rho}\,dx\,,$$

and

$$\int (\rho^\lambda)^2 \operatorname{div} u^\lambda \frac{\partial W}{\partial \rho}\,dx = \lambda \int \tilde{p}^\lambda \operatorname{div} u^\lambda\,dx\,,$$

what gives us :

$$\frac{\partial}{\partial t}\int \rho^\lambda W\,dx + \lambda \int \tilde{p}^\lambda \operatorname{div} u^\lambda\,dx = 0\,. \tag{2.8}$$

We then can deduce from (2.7) and (2.8) that :

$$\frac{\partial}{\partial t}\left[\int \rho^\lambda W\,dx + \int \rho^\lambda \frac{|u^\lambda|^2}{2}\,dx\right] + \nu \int |\nabla u^\lambda|^2\,dx = 0\,,$$

and thanks to lemma 1 :

$$\int \rho^\lambda W\,dx + \frac{\rho_1}{2}\,|u^\lambda(t)|_2^2 + \nu \int_0^t |\nabla u^\lambda|_2^2\,d\tau \leqslant \int |\rho^\lambda W(0)|\,dx + \frac{\rho_2}{2}\,|u_0^\lambda|_2^2\,.$$

So we have to estimate $\int \rho^\lambda W\,dx$.

(i) *Minoration :* Let us consider

$$\Phi(\rho^\lambda) = \rho^\lambda \int_{\rho_0}^{\rho^\lambda} \frac{\lambda \tilde{p}^\lambda(s)}{s^2}\, ds .$$

The shape of Φ gives immediatly :

$$\Phi(\rho_0) = \Phi'(\rho_0) = 0 \quad \text{and} \quad \Phi''(\rho_0) = A\lambda^2 \gamma \rho_0^{\gamma-2}$$

and
$$\Phi'''(\rho) = A\gamma(\gamma-2)\,\lambda^2 \rho^{\gamma-3} .$$

So
$$\Phi(\rho^\lambda) = \frac{A}{2}(\rho^\lambda - \rho_0)^2 \lambda^2 \gamma \rho_0^{\gamma-2} + \frac{A}{6}\gamma(\gamma-2)\,\lambda^2(\rho^\lambda-\rho_0)^3 \rho_\theta^{\gamma-3}$$
$$= (\tilde{\rho}^\lambda)^2 \left[\frac{A}{2}\gamma\rho_0^{\gamma-2} + \frac{A}{6\lambda}\gamma(\gamma-2)\,\tilde{\rho}^\lambda \cdot \rho_\theta^{\gamma-3}\right],$$

with $\rho_\theta = \rho_0 + \theta(\rho^\lambda - \rho_0)$, $\theta \in [0,1]$.

Now,

$$\left|\frac{A}{6\lambda}\gamma(\gamma-2)\,\tilde{\rho}^\lambda \cdot \rho_\theta^{\gamma-3}\right|_\infty \leqslant \frac{A}{6\lambda}\gamma(\gamma-2)\,K\rho_2^{\gamma-3} \leqslant \frac{CK}{\lambda} .$$

Since $C = \frac{A}{2}\gamma\rho_0^{\gamma-2} > 0$, we get that : for λ large enough, $\lambda \geqslant \lambda_4(K) \geqslant \lambda_3$, we have :

$$\int \Phi(\rho^\lambda)\, dx \geqslant \frac{C}{2}|\tilde{\rho}^\lambda|_2^2 \geqslant \frac{C}{2} C_1 |\tilde{p}^\lambda|_2^2 .$$

(ii) *Majoration :*

Since $\tilde{p}^\lambda(s) = \lambda A(s^\gamma - \rho_0^\gamma)$, then $\sup_{[\rho_0,\rho^\lambda]} |\tilde{p}^\lambda(s)| = |\tilde{p}^\lambda(\rho^\lambda)|$.

Then

$$\int \left|\rho^\lambda \int_{\rho_0}^{\rho^\lambda} \frac{\lambda\tilde{p}^\lambda(s)}{s^2}\right| dx \leqslant \int \rho^\lambda \lambda |\tilde{p}^\lambda(\rho^\lambda)| \left|\int_{\rho_0}^{\rho^\lambda} \frac{ds}{s^2}\right| dx$$
$$\leqslant \int \rho^\lambda \lambda |\tilde{p}^\lambda| \frac{|\rho^\lambda - \rho_0|}{\rho^\lambda \rho_0}\, dx \leqslant \frac{1}{\rho_0}\int |\tilde{\rho}^\lambda|\,|\tilde{p}^\lambda|\, dx .$$

So, thanks to lemma 2, we get that :

$$\int |\rho^\lambda W|\, dx \leqslant \frac{C_2}{\rho_0}|\tilde{p}^\lambda|_2^2 = C\,|\tilde{p}^\lambda|_2^2 .$$

Finally, we conclude that :

(2.9) *Under hypothesis $H(K, T)$, there exists $\lambda_4 = \lambda_4(K)$, and some constant C, independent of T, λ and K, so that, $\forall t \in [0, T]$, $\forall \lambda \geqslant \lambda_4$, we have :*

$$|u^\lambda|_2^2 + |\tilde{p}^\lambda|_2^2 + |\tilde{\rho}^\lambda|_2^2 + \nu \int_0^t |\nabla u^\lambda|_2^2 \, d\tau \leqslant C \, . \, E_0^\lambda \, ,$$

where $E_0^\lambda = |u_0^\lambda|_{H^s}^2 + |\tilde{p}_0^\lambda|_{H^s}^2$ *and* $\tilde{p}_0^\lambda(x) = \lambda(p^\lambda(x, 0) - p_0)$.

2nd Step : Estimate of $\int_0^t |D\tilde{p}^\lambda|_2^2 \, d\tau$.

Multiplying equation (2.1) by $-\dfrac{\nabla\tilde{\rho}^\lambda}{\lambda\rho^\lambda}$, and integrating in time and on $\mathbb{R}^n$, we get :

$$\int_0^t \int \frac{\nabla\tilde{\rho}^\lambda \, . \, \nabla\tilde{p}^\lambda}{\rho^\lambda} \, dx \, d\tau = - \int_0^t \int u_t^\lambda \frac{\nabla\tilde{\rho}^\lambda}{\lambda} \, dx \, d\tau - \int_0^t \int \frac{(u^\lambda \, . \, \nabla) \, u^\lambda \, . \, \nabla\tilde{\rho}^\lambda}{\lambda} \, dx \, d\tau + \nu \int_0^t \int \frac{\Delta u^\lambda \, . \, \nabla\tilde{\rho}^\lambda}{\lambda\rho^\lambda} \, dx \, d\tau \, .$$

Now

$$\int_0^t \int u_t^\lambda \frac{\nabla\tilde{\rho}^\lambda}{\lambda} \, dx \, d\tau = \left[\int u^\lambda \frac{\nabla\tilde{\rho}^\lambda}{\lambda} \, dx \right]_0^t - \int_0^t \int u^\lambda \, . \, \nabla\rho_t^\lambda \, dx \, d\tau$$
$$= \left[\int u^\lambda \frac{\nabla\tilde{\rho}^\lambda}{\lambda} \, dx \right]_0^t + \int_0^t \int \operatorname{div} u^\lambda \, . \, \operatorname{div} \, (\rho^\lambda u^\lambda) \, dx \, d\tau \, .$$

Finally,

$$(a) = \int_0^t \int \frac{\nabla\tilde{\rho}^\lambda \, . \, \nabla\tilde{p}^\lambda}{\rho^\lambda} \, dx \, d\tau$$
$$= \left[\int u^\lambda \frac{\nabla\tilde{\rho}^\lambda}{\lambda} \, dx \right]_0^t + \int_0^t \int \operatorname{div} u^\lambda \, . \, \operatorname{div} \, (\rho^\lambda u^\lambda) \, dx \, d\tau$$
$$+ \nu \int_0^t \int \frac{\Delta u^\lambda \, . \, \nabla\tilde{\rho}^\lambda}{\lambda\rho^\lambda} \, dx \, d\tau - \int_0^t \int \frac{(u^\lambda \, . \, \nabla) \, u^\lambda \, . \, \nabla\tilde{\rho}^\lambda}{\lambda} \, dx \, d\tau$$
$$= (b) + (c) + (d) + (e) \, .$$

(i) We get from lemma 1 :

$$(a) = \int_0^t \int \frac{(\nabla\tilde{p}^\lambda)^2}{\rho^\lambda} k'(p^\lambda) \, dx \, d\tau \geqslant \frac{k'(p_2)}{\rho_1} \int_0^t |\nabla\tilde{p}^\lambda|_2^2 d\tau \, ,$$

(ii) $|(b)| \leqslant |u^\lambda(t)|_2^2 + \frac{1}{\lambda^2}|D\tilde{\rho}^\lambda|_2^2 + |u^\lambda(0)|_2^2 + \frac{1}{\lambda^2}|D\tilde{\rho}^\lambda(0)|_2^2$

$$\leqslant C \, . \, E_0^\lambda + \frac{1}{\lambda^2}|D^s\tilde{p}^\lambda(t)|_2^2 \quad \text{as soon as } \lambda \geqslant \sup\ (\lambda_4, 1)\,.$$

(iii) $|(c)+(e)| \leqslant 2\int_0^t |u^\lambda|_\infty\, |Du^\lambda|_2 \frac{|D\tilde{\rho}^\lambda|_2}{\lambda}\, d\tau$

$$\leqslant \frac{4\,K}{\lambda}\int_0^t |Du^\lambda|_2^2\, d\tau + \frac{1}{\lambda}\int_0^t |D\tilde{\rho}^\lambda|_2^2\, d\tau \quad (|u|_\infty \leqslant \sqrt{K})$$

$$\leqslant \frac{4\,KC}{\lambda\nu}E_0^\lambda + \frac{1}{\lambda}\int_0^t |D\tilde{\rho}^\lambda|_2^2\, d\tau \quad (\text{by } (2.9))$$

(iv) $|(d)| \leqslant \frac{\nu^2}{\rho_1^2\,\lambda}\int_0^t |\nabla\tilde{\rho}^\lambda|_2^2\, d\tau + \frac{1}{\lambda}\int_0^t |D^2u^\lambda|_2^2\, d\tau$

$$\leqslant \frac{\nu^2}{\rho_1^2\,\lambda}\int_0^t |\nabla\tilde{\rho}^\lambda|_2^2\, d\tau + \frac{C}{\lambda\nu}E_0^\lambda + \frac{1}{\lambda}\int_0^t |D^{s+1}u^\lambda|_2^2\, d\tau \quad (\text{by } (2.9))\,.$$

We deduce from all above that :

$$\int_0^t |\nabla\tilde{p}^\lambda|_2^2\, d\tau \leqslant C\left(1+\frac{K}{\lambda}\right)E_0^\lambda + \frac{1}{\lambda^2}|D^s\tilde{p}^\lambda|_2^2 + \\ + \frac{1}{\lambda}\int_0^t |D^{s+1}u^\lambda|_2^2\, d\tau + \frac{C}{\lambda}\int_0^t |\nabla\tilde{p}^\lambda|_2^2\, d\tau\,.$$

We conclude from that :

(2.10) *Under hypothesis $H(K, T)$, there exists*

$$\lambda_5 = \lambda_5(K) \geqslant \max\ (\lambda_4, 1, K)$$

and some constant C, independent of λ, T, and K so that, $\forall t \in [0, T]\,,\quad \forall\lambda \geqslant \lambda_5\,,$

$$\int_0^t |\nabla\tilde{p}^\lambda(\tau)|_2^2\, d\tau \leqslant CE_0^\lambda + \frac{1}{\lambda^2}|D^s\tilde{p}^\lambda|_2^2 + \frac{1}{\lambda}\int_0^t |D^{s+1}u^\lambda|_2^2\, d\tau\,.$$

The norm $|u|_{H^s}$ being equivalent to the norm $(|u|_2^2 + |D^s u|_2^2)$, we go straitly to the :

3rd Step : L^2-Norm of the derivatives of order s.

Deriving equations (2.1) and (2.2) s times yields to :

(2.11) $\quad \partial^s(\rho^\lambda u_t^\lambda) + \partial^s(\rho^\lambda(u^\lambda . \nabla)u^\lambda) - \nu\,\Delta\,\partial^s u^\lambda = -\lambda\,\nabla\,\partial^s\tilde{p}^\lambda\,,$

(2.12) $\quad \partial^s\tilde{p}_t^\lambda + \partial^s(\nabla\tilde{p}^\lambda . u^\lambda) + \gamma\partial^s(\tilde{p}^\lambda \operatorname{div} u^\lambda) + \lambda\gamma p_0\,\partial^s \operatorname{div} u^\lambda = 0\,.$

The operation

$$\int \left[(2.11) \cdot \gamma p_0 \, \partial^s u^\lambda + (2.12) \cdot \partial^s \tilde{p}^\lambda + (2.4) \, \gamma p_0 \frac{(\partial^s u^\lambda)^2}{2}\right] dx$$

leads to the following equality :

$$\frac{\partial}{\partial t}\left[\frac{\gamma p_0}{2} |\sqrt{\rho^\lambda} \partial^s u^\lambda|_2^2 + \frac{1}{2} |\partial^s \tilde{p}^\lambda|_2^2\right] + \nu\gamma p_0 |\nabla \, \partial^s u^\lambda|_2^2 =$$

$$= - \gamma p_0 \int [\partial^s(\rho^\lambda u_t^\lambda) - \rho^\lambda \, \partial^s u_t^\lambda] \cdot \partial^s u^\lambda \, dx -$$

$$- \gamma p_0 \int \partial^s((\rho^\lambda u^\lambda \cdot \nabla) u^\lambda) \cdot \partial^s u^\lambda \, dx$$

$$- \gamma p_0 \int \operatorname{div} (\rho^\lambda u^\lambda) \frac{(\partial^s u^\lambda)^2}{2} dx - \int (\nabla(\partial^s \tilde{\rho}^\lambda) \cdot u^\lambda) \, \partial^s \tilde{\rho}^\lambda \, dx$$

$$- \int [\partial^s(\nabla \tilde{p}^\lambda \cdot u^\lambda) - (\partial^s \nabla \tilde{p}^\lambda) u^\lambda] \, \partial^s \tilde{p}^\lambda \, dx - \gamma \int \partial^s(\tilde{p}^\lambda \operatorname{div} u^\lambda) \, \partial^s \tilde{p}^\lambda \, dx$$

$$= (a) + (b) + (c) + (d) + (e) + (f) .$$

(i) Let us estimate (a). Thanks to (2.5), we may write :

$$|(a)| \leqslant C |D^s u^\lambda|_2 [|D\rho^\lambda|_\infty |D^{s-1} u_t^\lambda|_2 + |D^s \rho^\lambda|_2 |u_t^\lambda|_\infty]$$

$$\leqslant C |D^s u^\lambda|_2 \frac{|D\tilde{\rho}^\lambda|_\infty}{\lambda} |D^{s-1} u_t^\lambda|_2 + C |D^s u^\lambda|_2 |D^s \tilde{\rho}^\lambda|_2 \frac{1}{\lambda} |u_t^\lambda|_\infty .$$

Now (2.1) gives us :

$$u_t^\lambda = - \lambda \frac{\nabla \tilde{p}^\lambda}{\rho^\lambda} + \nu \frac{\Delta u^\lambda}{\rho^\lambda} - (u^\lambda \cdot \nabla) u^\lambda .$$

Thus

$$\frac{1}{\lambda} |u_t^\lambda|_\infty \leqslant \frac{1}{\rho_1} |D\tilde{p}^\lambda|_\infty + \frac{\nu}{\rho_1 \lambda} |\Delta u^\lambda|_\infty + \frac{\sqrt{K}}{\lambda} |Du^\lambda|_\infty .$$

We now use an inequality due to Gagliardo and Nirenberg [9].

So, with hypothesis $H(K, T)$, we can get that :

$$|D^s u^\lambda|_2 |D^s \tilde{\rho}^\lambda|_2 \frac{1}{\lambda} |u_t^\lambda|_\infty \leqslant$$

$$\leqslant C \sqrt{K} \left[|D\tilde{p}^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2 + |Du^\lambda|_2^2 + \frac{|D^{s+1} u^\lambda|_2^2}{\lambda} \right],$$

as soon as $\lambda \geqslant \lambda_5$.

On the other hand,

$$|D^{s-1} u_t^\lambda|_2 = \left| D^{s-1} \frac{\rho^\lambda u_t^\lambda}{\rho^\lambda} \right|_2 \leqslant$$
$$\leqslant C\, |\rho^\lambda u_t^\lambda|_\infty \left| D^{s-1} \frac{1}{\rho^\lambda} \right|_2 + C\, |D^{s-1}(\rho^\lambda u_t^\lambda)|_2 \cdot \left| \frac{1}{\rho^\lambda} \right|_\infty .$$

We know that (lemma 3 (ii)), as soon as λ is large enough,

$$\left| D^{s-1} \frac{1}{\rho^\lambda} \right|_2 \leqslant \frac{C}{\lambda} .$$

Moreover, using assertion (2.5) of lemma 5 and hypothesis $H(K, T)$, we get :

$$|D^{s-1}(\rho^\lambda u_t^\lambda)|_2 \leqslant \lambda\, |D^s \tilde{p}^\lambda|_2 + \nu\, |D^{s+1} u^\lambda|_2 + |D^{s-1}(\rho^\lambda u^\lambda . \nabla)\, u^\lambda|_2$$
$$\leqslant C\lambda\, |D^s \tilde{p}^\lambda|_2 + C\sqrt{K}\, |Du^\lambda|_2 + C\sqrt{K}\, |D^{s+1} u^\lambda|_2 .$$

So, when λ is large enough, $\lambda \geqslant \lambda_6(K) \geqslant \lambda_5$, we have :

$$|D^s u^\lambda|_2\, |D\tilde{p}^\lambda|_\infty \frac{1}{\lambda}\, |D^{s-1} u_t^\lambda|_2 \leqslant$$
$$\leqslant C\,(1 + K^{3/2}) \left[|D\tilde{p}^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2 + |Du^\lambda|_2^2 + \frac{|D^{s+1} u^\lambda|_2^2}{\lambda} \right] ,$$

and (a) verifies the same inequality.

(ii) Thanks to lemma 5 (2.6), lemma 3, and hypothesis $H(K, T)$, we deduce the following estimate for $(b) + (c)$: $(\beta \geqslant 1)$

$$|(b) + (c)| \leqslant C\,(1 + K^\beta)\, C(\alpha)\, |Du^\lambda|_2^2 +$$
$$+ \alpha\, |D^{s+1} u^\lambda|_2^2 + \frac{K}{\lambda} \left(|Du^\lambda|_2^2 + |D^{s+1} u^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2 \right) .$$

(We also need the inequality :

$$|D^{s+1} u|_2\, |D^s u|_2 \leqslant C\, |Du|_2^{1-a}\, |D^{s+1} u|_2^{a+1} \leqslant$$
$$\leqslant C(\alpha)\, |Du|_2^2 + \alpha\, |D^{s+1} u|_2^2) .$$

(iii) For (d), we just have to write :

$$\left| \int (\nabla\, \partial^s \tilde{p}^\lambda . u^\lambda)\, \partial^s \tilde{p}^\lambda\, dx \right| = \left| - \int \operatorname{div} u^\lambda \frac{(\partial^s \tilde{p}^\lambda)^2}{2}\, dx \right| \leqslant C\sqrt{K}\, |D^s \tilde{p}^\lambda|_2^2 .$$

(iv) Thanks again to lemma 3, to assertions (2.5) and (2.6) of lemma 5 and to $H(K, T)$, we finally estimate (e) and (f) in the following way :

$$|(e) + (f)| \leq (1 + K)\, C(\alpha)|D^s \tilde{p}^\lambda|_2^2 + \\ + \alpha |D^{s+1} u^\lambda|_2^2 + (1 + K)\, C(\alpha)|Du^\lambda|_2^2 .$$

So, taking into account these estimates and lemma 1, we find, integrating on $[0, T]$, that there exists $\beta > 1$ and $\lambda_6 = \lambda_6(K)$ so that :

$\forall t \in [0, T]$, $\forall \lambda \geq \lambda_6$,

$$|D^s u^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2 + \int_0^t |D^{s+1} u^\lambda|_2^2\, d\tau \leq$$

$$\leq CE_0^\lambda + C(\alpha)(1 + K^\beta) \int_0^t |Du^\lambda|_2^2\, d\tau + C(1 + K^{3/2}) \int_0^t |D\tilde{p}^\lambda|_2^2\, d\tau$$

$$+ C(\alpha)(1 + K^{3/2}) \int_0^t |D^s \tilde{p}^\lambda|_2^2\, d\tau + \left(\alpha + \frac{KC}{\lambda}\right) \int_0^t |D^{s+1} u^\lambda|_2^2\, d\tau .$$

Then, using results (2.9) and (2.10), choosing $\alpha = 1/4$, and $\lambda_7 = \max(\lambda_6, 4\,KC)$, we obtain the following result :

(2.11) *Under hypothesis* $H(K, T)$, *there exists* $\lambda_7 = \lambda_7(K) \geq \lambda_6 \geq \cdots \geq \lambda_1$, $\beta > 1$, *and some constant* C, *independent of* λ, K *and* T, *so that*: $\forall t \in [0, T]$, $\forall \lambda \geq \lambda_7$,

$$|D^s u^\lambda|_2^2 + |D^s \tilde{p}^\lambda|_2^2 + \int_0^t |D^{s+1} u^\lambda|_2^2\, d\tau \leq$$

$$\leq C(1 + K^\beta)\left(E_0^\lambda + \int_0^t |D^s \tilde{p}^\lambda|_2^2\, d\tau\right) .$$

We now have to estimate $\int_0^t |D^s \tilde{p}^\lambda|_2^2\, d\tau$, which is the aim of the :

4th Step : Estimate of $\int_0^t |D^s \tilde{p}^\lambda|_2^2\, d\tau$.

First, let us note that if we call $v^\lambda = \rho^\lambda u^\lambda$, equation (2.1) becomes :

$$v_t^\lambda + (v^\lambda \cdot \nabla)\, u^\lambda + u^\lambda \operatorname{div} v^\lambda - \nu\, \Delta u^\lambda = -\lambda\, \nabla \tilde{p}^\lambda . \tag{2.12}$$

Deriving $(s - 1)$ times in x this equation, multiplying by $-\dfrac{\nabla\, \partial^{s-1} \tilde{\rho}^\lambda}{\lambda}$, and integrating on $\mathbb{R}^n \times [0, T]$, we obtain :

$$\int_0^t \int \nabla\, \partial^{s-1}\tilde{p}^\lambda \,.\, \nabla\, \partial^{s-1}\tilde{p}^\lambda \,dx\,d\tau =$$

$$- \int_0^t \int \partial^{s-1} v_t^\lambda \cdot \frac{\nabla\, \partial^{s-1}\tilde{\rho}^\lambda}{\lambda}\,dx\,d\tau + \nu \int_0^t \int \Delta\, \partial^{s-1} u^\lambda \cdot \frac{\nabla\, \partial^{s-1}\tilde{\rho}^\lambda}{\lambda}\,dx\,d\tau$$

$$- \int_0^t \int \partial^{s-1}((v^\lambda \,.\, \nabla)\, u^\lambda)\, \frac{\nabla\, \partial^{s-1}\tilde{\rho}^\lambda}{\lambda}\,dx\,d\tau$$

$$- \int_0^t \int \partial^{s-1}(u^\lambda \operatorname{div} v^\lambda)\, \frac{\nabla\, \partial^{s-1}\tilde{\rho}^\lambda}{\lambda}\,dx\,d\tau = (a) + (b) + (c) + (d)\,.$$

(i) From lemma 3, we easily deduce that :

$$\int \nabla\, \partial^{s-1}\tilde{p}^\lambda \,.\, \nabla\, \partial^{s-1}\tilde{\rho}^\lambda\,dx \geqslant k'(p_2)|\nabla\, \partial^{s-1}\tilde{p}^\lambda|_2^2 - \frac{C}{\lambda}\,|\nabla \tilde{p}^\lambda|_{H^{s-1}}\,|\nabla\, \partial^{s-1}\tilde{p}^\lambda|_2\,.$$

It follows that there exists $\lambda_8 = \lambda_8(K)$ so that, for any $\lambda \geqslant \lambda_8$, we have :

$$\int_0^t \int \nabla\, \partial^{s-1}\tilde{p}^\lambda \,.\, \nabla\, \partial^{s-1}\tilde{\rho}^\lambda\,dx\,d\tau \geqslant$$

$$\geqslant \frac{k'(p_2)}{2} \int_0^t |\nabla\, \partial^{s-1}\tilde{p}^\lambda|_2^2\,d\tau - \frac{C}{\lambda}\int_0^t |D\tilde{p}^\lambda|_2^2\,d\tau\,.$$

(ii) *Estimate of (a).*

$$(a) = -\left[\int \partial^{s-1} v^\lambda\, \frac{\nabla\, \partial^{s-1}\tilde{\rho}^\lambda}{\lambda}\,dx\right]_0^t + \int_0^t \int \partial^{s-1} v^\lambda \,.\, \nabla\, \partial^{s-1}\rho_t^\lambda\,dx\,d\tau\,.$$

Now, $\rho_t^\lambda = -\operatorname{div}\,(\rho^\lambda u^\lambda) = -\operatorname{div} v^\lambda$. Then,

$$(a) = -\left[\int \partial^{s-1} v^\lambda\, \frac{\nabla\, \partial^{s-1}\tilde{\rho}^\lambda}{\lambda}\,dx\right]_0^t + \int_0^t |\operatorname{div}\, \partial^{s-1} v^\lambda|_2^2\,d\tau\,.$$

So,

$$|(a)| \leqslant \frac{C}{\lambda}\,E_0^\lambda + |D^{s-1} v^\lambda|_2\,\frac{|D^s \tilde{\rho}^\lambda|_2}{\lambda} + \int_0^t |D^s v^\lambda|_2^2\,d\tau\,.$$

On the other hand, thanks to lemma 5 and to hypothesis $H(K, T)$, we obtain :

$$(2.13)\quad \left|\begin{aligned} &|D^k v^\lambda|_2 \leqslant C\,|D^k u^\lambda|_2 + \frac{\sqrt{K}}{\lambda}\,|D^k \tilde{\rho}^\lambda|_2 \\ &\qquad \leqslant C\,|D^k u^\lambda|_2 + C\,\frac{\sqrt{K}}{\lambda}\,(|D^k \tilde{p}^\lambda|_2 + |D\tilde{p}^\lambda|_2)\,; \\ &|v^\lambda|_\infty \leqslant CK\,; \quad |Dv^\lambda|_\infty \leqslant CK\,. \end{aligned}\right.$$

What gives finally :

$$|(a)| \leqslant CE_0^\lambda + \frac{C}{\lambda}\,|D^s u^\lambda|_2^2 + \frac{C}{\lambda}\,|D^s \tilde{p}^\lambda|_2^2 + C(\alpha)\int_0^t |Du^\lambda|_2^2\,d\tau$$
$$+\,\alpha\int_0^t |D^{s+1}u^\lambda|_2^2\,d\tau + \frac{K}{\lambda^2}\int_0^t |D^s\tilde{p}^\lambda|_2^2\,d\tau + \frac{K}{\lambda^2}\int_0^t |D\tilde{p}^\lambda|_2^2\,d\tau\,.$$

(iii) It follows from lemma 5, (2.9) and (2.13) that :

$$|c+d| \leqslant C\int_0^t \frac{|D^s\tilde{\rho}^\lambda|_2}{\lambda}\,[K\,|Du^\lambda|_2 +$$
$$+\,K\,|D^{s+1}u^\lambda|_2 + \frac{K}{\lambda}\,|D\tilde{p}^\lambda|_2 + \frac{K}{\lambda}\,|D^s\tilde{p}^\lambda|_2\Big]\,d\tau\,,$$

and consequently,

$$|c+d| \leqslant \frac{K^2 C(\alpha)}{\lambda^2}\int_0^t |D^s\tilde{p}^\lambda|_2^2\,d\tau +$$
$$+\,\frac{K^2C(\alpha)}{\lambda^2}\int_0^t |D\tilde{p}^\lambda|_2^2\,d\tau + CE_0^\lambda + \alpha\int_0^t |D^{s+1}u^\lambda|_2^2\,d\tau\,.$$

(iv) At last, we get easily :

$$|b| \leqslant \alpha\int_0^t |D^{s+1}u^\lambda|_2^2\,d\tau + \frac{C(\alpha)}{\lambda^2}\int_0^t (|D^s\tilde{p}^\lambda|_2^2 + |D\tilde{p}^\lambda|_2^2)\,d\tau\,.$$

We deduce from the estimates above the following result :

$$(2.14)\quad \left|\begin{array}{l} \forall t\in[0,T]\,,\quad \forall\lambda \geqslant \lambda_8(K)\geqslant\lambda_7\,,\\ \displaystyle\int_0^t |D^s\tilde{p}^\lambda|_2^2\,d\tau \leqslant CE_0^\lambda + \frac{C}{\lambda}\,|D^s u^\lambda|_2^2\\ \displaystyle +\,\frac{C}{\lambda}\,|D^s\tilde{p}^\lambda|_2^2 + 3\,\alpha\int_0^t |D^{s+1}u^\lambda|_2^2\,d\tau\\ \displaystyle +\,\frac{(1+K^2)}{\lambda^2}\,C(\alpha)\int_0^t |D\tilde{p}^\lambda|_2^2\,d\tau + \frac{(1+K^2)}{\lambda^2}\,C(\alpha)\int_0^t |D^s\tilde{p}^\lambda|_2^2\,d\tau\,. \end{array}\right.$$

Choosing α small enough and putting together the results (2.9), (2.10), (2.13) and (2.14), we can conclude.

Namely :

PROPOSITION (2.15) : *Under hypothesis $H(K,T)$, there exists some constants $N\in\mathbb{N}^*$ and $C\geqslant 1$, independent of λ, K and T, and $\lambda_9=\lambda_9(K)$, independent of T, so that :*

$$\forall t\in[0,T]\,,\quad \forall\lambda\geqslant\lambda_9\,,$$

$$|u^\lambda(t)|^2_{H^s} + |\tilde{p}^\lambda(t)|^2_{H^s} + \int_0^t |\nabla u^\lambda|^2_{H^s} d\tau + \int_0^t |\nabla \tilde{p}^\lambda|^2_{H^{s-1}} d\tau \leq C(1+K)^N . E_0^\lambda$$

and $|\tilde{\rho}^\lambda(t)|^2_{H^s} + \int_0^t |\nabla\tilde{\rho}^\lambda|^2_{H^{s-1}} d\tau \leq C(1+K)^N . E_0^\lambda$.

COROLLARY : *Under the same assumptions, the following estimate is verified :*

$$|u_t^\lambda|^2_{H^{s-2}} + |\tilde{p}_t^\lambda|^2_{H^{s-1}} \leq C\lambda(1+K)^M E_0^\lambda \quad (\textit{for some } M \in \mathbb{N}^*) .$$

(It is a consequence of (2.15)).

B. Global existence

We first have to see that there really exists K and T verifying hypothesis $H(K, T)$.

Taking our inspiration from Nishida and Matsumura's technic in [3], we get the following local existence's result :

PROPOSITION (2.16) : *Let* $(u_0^\lambda, p_1) \in (H^s(\mathbb{R}^n))^2$, *and* $p_0^\lambda(x) = p_0 + \dfrac{p_1(x)}{\lambda^2}$.
Let $E_0^\lambda = |u_0^\lambda|^2_{H^s} + |\lambda(p_0^\lambda(x) - p_0)|^2_{H^s}$, *where* $s > \left[\dfrac{n}{2}\right] + 1$.

Then, for large enough λ, $\lambda \geq \lambda_{10}$, *there exists a unic solution of the system* (S^λ) *on some interval* $[0, T^\lambda(E_0^\lambda)]$, *verifying :*

(i) $T^\lambda(E_0)$ *is an decreasing function of* E_0 ;

(ii) *The solution* (u^λ, p^λ) *satisfies :*

$$\forall t \in [0, T^\lambda(E_0^\lambda)], E^\lambda(t) = |u^\lambda(t)|^2_{H^s} + |\lambda(p^\lambda(t) - p_0)|^2_{H^s} \leq \phi(E_0^\lambda) . E_0^\lambda ,$$

where ϕ *is an increasing function, independent of* $\lambda \geq \lambda_{10}$, *so that* $\phi \geq 1$.

Now, we are going to put together proposition (2.15) and the above result to prove the global existence as soon as λ is large enough.

Let us introduce K_0 realizing the maximum of the function $\Psi(K)$:

$$\Psi(K) = \frac{K}{C(1+K)^N . \phi[C(1+K)^N]} .$$

Let us note $\lambda_0 = \max(\lambda_9(K_0), \lambda_{10})$.

Choosing E_0^λ so that $E_0^\lambda \leqslant \Psi(K_0) < 1$, we get :

$$\phi(E_0^\lambda)\, E_0^\lambda \leqslant \phi(1)\, E_0^\lambda \leqslant \phi(1) \leqslant \phi[C(1+K_0)^N] \leqslant \frac{K_0}{C(1+K_0)^N} \leqslant K_0 .$$

Let us note $T_0^\lambda = T^\lambda(C(1+K_0)^N\, E_0^\lambda) \leqslant T^\lambda(E_0^\lambda)$.

Thus, we deduce that hypothesis $H(K_0, T_0^\lambda)$ is verified as soon as $\lambda \geqslant \lambda_0$.

It yields, from (2.15), that :

$$\forall t \in [0, T_0^\lambda], \quad \forall \lambda \geqslant \lambda_0, \quad E^\lambda(t) \leqslant C(1+K_0)^N \,.\, E_0^\lambda .$$

In particular, $E^\lambda(T_0^\lambda) \leqslant C(1+K_0)^N \,.\, E_0^\lambda$.

Now, let us apply the result (2.16), taking T_0^λ as initial instant. Since $E^\lambda(T_0^\lambda) \leqslant C(1+K_0)^N \,.\, E_0^\lambda$, then $T_0^\lambda \leqslant T^\lambda(E^\lambda(T_0^\lambda))$.

So, it follows that :

$$\forall t \in [T_0^\lambda, 2\, T_0^\lambda], \quad \forall \lambda \geqslant \lambda_0, \quad E^\lambda(t) \leqslant \phi(E^\lambda(T_0^\lambda)) \,.\, E^\lambda(T_0^\lambda) .$$

Now, by construction :

$$\begin{aligned} \phi(E^\lambda(T_0^\lambda)) \,.\, E^\lambda(T_0^\lambda) &\leqslant \phi(C(1+K_0)^N \,.\, E_0^\lambda) \,.\, C(1+K_0)^N \,.\, E_0^\lambda \\ &\leqslant \phi(C(1+K_0)^N) \,.\, C(1+K_0)^N \,.\, \Psi(K_0) \leqslant K_0 . \end{aligned}$$

So, $\forall t \in [0, 2\, T_0^\lambda]$, $\forall \lambda \geqslant \lambda_0$, $E^\lambda(t) \leqslant K_0$.

Iterating the process, we get the global existence.

Namely :

THEOREM 1 : *There exists $\lambda_0 > 0$ and $K_0 > 0$ so that : If $E_0^\lambda \leqslant K_0$ and $\lambda \geqslant \lambda_0$, then the system (S^λ) admits a unic global solution (u^λ, p^λ) verifying :*

$$u^\lambda \in C_B(0, \infty, H^s) \cap C_B^1(0, \infty, H^{s-2}) ,$$
$$(p^\lambda - p_0) \in C_B(0, \infty, H^s) \cap C_B^1(0, \infty, H^{s-1}) ,$$

and $$\forall t \geqslant 0 , \quad \forall \lambda \geqslant \lambda_0 ,$$

$$|u^\lambda|_{H^s}^2 + |\lambda(p^\lambda - p_0)|_{H^s}^2 + \int_0^\infty |\nabla u^\lambda|_{H^s}^2\, d\tau + + \int_0^\infty |\lambda \nabla(p^\lambda - p_0)|_{H^{s-1}}^2\, d\tau \leqslant K_0 .$$

Moreover, $|\partial_t p^\lambda|_{H^{s-1}}$ and $|\partial_t \rho^\lambda|_{H^{s-1}}$ are bounded, independently of $\lambda \geqslant \lambda_0$.

We are now going to establish some independent of λ estimates on derivatives in time, in order to obtain some convergence's results. This leads us to consider an initial data u_0^λ of the shape :

$$u_0^\lambda(x) = u_0(x) + \frac{1}{\lambda} u_1(x) , \quad \text{where} \quad \operatorname{div} u_0(x) = 0 .$$

III. A WEAK CONVERGENCE'S RESULT

Hence, we consider the system (S^λ):

(2.1) $$\rho^\lambda u_t^\lambda + \rho^\lambda (u^\lambda . \nabla) u^\lambda - \nu \Delta u^\lambda = - \lambda \nabla \tilde{p}^\lambda ,$$

(2.2) $$\tilde{p}_t^\lambda + u^\lambda . \nabla \tilde{p}^\lambda + \gamma \tilde{p}^\lambda . \operatorname{div} u^\lambda + \lambda \gamma p_0 \operatorname{div} u^\lambda = 0 ,$$

(2.3) $$u^\lambda(x, 0) = u_0(x) + \frac{1}{\lambda} u_1(x) , \quad p^\lambda(x, 0) = p_0 + \frac{1}{\lambda^2} p_1(x) ,$$

with the supplementary condition :

(3.1) $$\operatorname{div} u_0(x) = 0 .$$

The operation $\partial_t(2.1) \times \gamma p_0 u_t^\lambda + \partial_t(2.2) \times \tilde{p}_t^\lambda$ gives, after integration on $\mathbb{R}^n$ and thanks to lemma 4 :

$$\frac{d}{dt} \left[\frac{\gamma}{2} p_0 \left| \sqrt{\rho^\lambda} u_t^\lambda \right|_2^2 + \frac{1}{2} \left| \tilde{p}_t^\lambda \right|_2^2 \right] + \nu \gamma p_0 \left| \nabla u_t^\lambda \right|_2^2 + \gamma p_0 \int \rho_t^\lambda (u^\lambda . \nabla) u^\lambda . u_t^\lambda \, dx$$
$$+ \gamma p_0 \int \rho^\lambda (u_t^\lambda . \nabla) u^\lambda . u_t^\lambda \, dx + \gamma p_0 \int \rho_t^\lambda \left| u_t^\lambda \right|^2 dx + \int u_t^\lambda . \nabla \tilde{p}^\lambda \tilde{p}_t^\lambda \, dx$$
$$+ \left(\gamma - \frac{1}{2} \right) \int \left| \tilde{p}_t^\lambda \right|^2 \operatorname{div} u^\lambda \, dx + \gamma \int \tilde{p}^\lambda \tilde{p}_t^\lambda \operatorname{div} u_t^\lambda \, dx = 0 .$$

We deduce from that, thanks to lemmas 1 and 3, and to the results of theorem 1, the following inequality :

$$\left| u_t^\lambda \right|_2^2 + \left| \tilde{p}_t^\lambda \right|_2^2 + \int_0^t \left| \nabla u_t^\lambda \right|_2^2 d\tau \leqslant$$
$$\leqslant C \left[\left| u_t^\lambda(0) \right|_2^2 + \left| \tilde{p}_t^\lambda(0) \right|_2^2 + \int_0^t \left(\left| u_t^\lambda \right|_2^2 + \left| \tilde{p}_t^\lambda \right|_2^2 \right) d\tau \right] .$$

This part of the reasoning clearly shows the necessity to introduce the assumption (3.1). As a matter of fact, it permits to obtain that, under the hypothesis of theorem 1 :

$$\left| u_t^\lambda(., 0) \right|_2 \leqslant \left| u_0^\lambda . \nabla u_0^\lambda \right|_2 + \frac{\nu}{\rho_1} \left| \Delta u_0^\lambda \right|_2 + \frac{\nu}{\rho_1} \left| \nabla p_1 \right|_2 \leqslant C$$

and

$$|\tilde{p}_t^\lambda(.,0)|_2 \leq \left|\gamma\left(p_0+\frac{p_1(.)}{\lambda}\right)\operatorname{div} u_1\right|_2 + \left|\frac{\nabla p_1}{\lambda}u_0^\lambda\right|_2 \leq C .$$

So, for λ large enough, we have the following result :

$$\forall t \geq 0 , \quad |u_t^\lambda|_2^2 + |\tilde{p}_t^\lambda|_2^2 + \int_0^t |\nabla u_t^\lambda|_2^2 \, d\tau \leq C \, e^{Ct} . \tag{3.2}$$

Using of the same methods for the derivatives of order $(s-2)$ in x, we get the equality :

$$\frac{d}{dt}\left[\frac{\gamma p_0}{2}\left|\sqrt{\rho^\lambda}D^{s-2}u_t^\lambda\right|_2^2 + \frac{1}{2}\left|D^{s-2}\tilde{p}_t^\lambda\right|_2^2\right] + \nu\gamma p_0\left|D^{s-1}u_t^\lambda\right|_2^2 =$$

$$= -\gamma p_0 \int [D^{s-2}(\rho^\lambda u_{tt}^\lambda) - \rho^\lambda D^{s-2}u_{tt}^\lambda] \, . \, D^{s-2}u_t^\lambda \, dx$$

$$+ \gamma p_0 \int \frac{1}{2}\rho_t^\lambda (D^{s-2}u_t^\lambda)^2 \, dx$$

$$- \gamma p_0 \int D^{s-2}(\rho_t^\lambda u_t^\lambda) \, D^{s-2}u_t^\lambda \, dx - \gamma p_0 \int D^{s-2}(\rho_t^\lambda u^\lambda \, . \, \nabla u^\lambda) \, D^{s-2}u_t^\lambda \, dx$$

$$- \gamma p_0 \int D^{s-2}(\rho^\lambda u_t^\lambda \, . \, \nabla u^\lambda) \, D^{s-2}u_t^\lambda \, dx$$

$$- \gamma p_0 \int D^{s-2}(\rho^\lambda u^\lambda \, . \, \nabla u_t^\lambda) \, D^{s-2}u_t^\lambda \, dx$$

$$+ \int D^{s-2}(\nabla\tilde{p}^\lambda u_t^\lambda) \, D^{s-2}\tilde{p}_t^\lambda \, dx$$

$$+ \frac{1}{2}\int \operatorname{div} u^\lambda (D^{s-2}\tilde{p}_t^\lambda)^2 \, dx + \gamma \int D^{s-2}(\tilde{p}^\lambda \operatorname{div} u_t^\lambda) \, D^{s-2}\tilde{p}_t^\lambda \, dx$$

$$+ \int (D^{s-2}(\nabla\tilde{p}_t^\lambda u^\lambda) - u^\lambda D^{s-2}\nabla\tilde{p}_t^\lambda) \, D^{s-2}\tilde{p}_t^\lambda \, dx$$

$$- \gamma \int D^{s-2}(\tilde{p}_t^\lambda \operatorname{div} u^\lambda) \, D^{s-2}\tilde{p}_t^\lambda \, dx .$$

Except the first term of the right member, all the (numerous !) terms of this equality can be estimated by the technics developped all along the preceeding paragraph (lemma 5 and estimates of theorem 1).

Let us study this particular term a little more attentively.

Let us write :

$$\int_0^t \int [D^{s-2}(\rho^\lambda u_{tt}^\lambda) - \rho^\lambda D^{s-2} u_{tt}^\lambda] D^{s-2} u_t^\lambda \, dx \, d\tau \leq$$

$$\leq \int_0^t [|D\rho^\lambda|_\infty |D^{s-3} u_{tt}^\lambda|_2 + |D^{s-2}\rho^\lambda|_r |u_{tt}^\lambda|_{r'}] |D^{s-2} u_t^\lambda|_2 \, d\tau .$$

Taking $(r, r') = (\infty, 2)$ when $n = 2$ or 3, and $(r, r') = \left(\frac{2n}{n-2}, \frac{n}{2}\right)$ when $n \geq 4$, we get :

$$|D^{s-2}\rho^\lambda|_r \leq \frac{1}{\lambda} |\tilde{\rho}^\lambda|_{H^s} \leq \frac{K_0}{\lambda} \quad \text{and} \quad |u_{tt}^\lambda|_{r'} \leq |u_{tt}^\lambda|_{H^{s-3}} .$$

So, we just have to estimate $\int_0^t \frac{1}{\lambda^2} |u_{tt}^\lambda|_{H^{s-3}}^2 \, d\tau$.

Let us note $\chi = |u_t^\lambda|_{H^{s-2}}^2 + |\tilde{p}_t^\lambda|_{H^{s-2}}^2$, and let us derive in time the equation (2.1).

Proceeding by the now classical method, and using lemma 4, lemma 5 and the results of theorem 1, we get :

$$\int_0^t \frac{1}{\lambda^2} (|u_{tt}^\lambda|_{H^{s-3}}^2) \, d\tau \leq C \int_0^t \chi(\tau) \, d\tau + \frac{C}{\lambda^2} \int_0^t |\nabla u_t^\lambda|_{H^{s-2}}^2 \, d\tau .$$

Which yields, for λ large enough, to the following Gronwald's inequality :

$$\chi(t) + \int_0^t |\nabla u_t^\lambda|_{H^{s-2}}^2 \, d\tau \leq C\chi(0) + C \int_0^t \chi(\tau) \, d\tau .$$

We then can state the obtained result in the :

PROPOSITION : *If* $u_0^\lambda(x) = u_0(x) + \frac{u_1(x)}{\lambda} \in H^s$, *with* $\operatorname{div} u_0 = 0$, *If* $p_0^\lambda(x) = p_0 + \frac{p_1(x)}{\lambda^2}$, *with* $p_1 \in H^s$ *and* $s > \left[\frac{n}{2}\right] + 1$, *then, under the assumptions of theorem 1, the solutions* (u^λ, p^λ) *of* (S^λ) *verify, as soon as* λ *is large enough, in addition to the already obtained estimates :*

$$|u_t^\lambda|_{H^{s-2}}^2 + |\tilde{p}_t^\lambda|_{H^{s-2}}^2 + \int_0^t |\nabla u_t^\lambda|_{H^{s-2}}^2 \, d\tau \leq M(t) . \tag{3.3}$$

In particular,

$$(3.4)\quad \left| \begin{array}{l} |p_t^\lambda|^2_{H^{s-2}} + |\rho_t^\lambda|^2_{H^{s-2}} \leq \frac{1}{\lambda^2} M(t) , \\ \\ |\nabla \tilde{p}^\lambda|^2_{H^{s-2}} + |\operatorname{div} (\rho^\lambda u^\lambda)|^2_{H^{s-2}} + |\operatorname{div} u^\lambda|^2_{H^{s-2}} \leq \frac{1}{\lambda^2} M(t) , \end{array} \right.$$

where $M(t) \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}_+)$.

Now, we have got all that is necessary to prove that the sequence (u^λ, p^λ) weakly converges (in a sense that will be precised), to the solution (u^∞, p^∞) of the viscous incompressible fluid's equation :

$$(S^\infty) \quad \begin{cases} \rho_0(u_t^\infty + (u^\infty . \nabla) u^\infty) - \nu \Delta u^\infty = - \nabla p^\infty , \\ \operatorname{div} u^\infty = 0 , \quad u^\infty(x, 0) = u_0(x) . \end{cases}$$

Remark : We'll now write « u^λ » for any subsequence of u^λ. In fact, this notation is justified : the unicity of the solutions (u^λ, p^λ) and (u^∞, p^∞) shows, a posteriori, that this is really the sequence (u^λ, p^λ) that converges and not any subsequence.

From the estimates of theorem 1 and from (3.3), we deduce that there exists u^∞ verifying :

$$u^\infty \in C_B(0, \infty, H^s) \cap C_B^1(0, \infty, H^{s-2}) ,$$

so that :

$$(3.5)\quad \left| \begin{array}{lll} u^\lambda \to u^\infty & \text{in} & L^\infty(0, \infty, H^s) \text{ w.s.} , \\ \\ u_t^\lambda \to u_t^\infty & \text{in} & L^\infty_{\text{loc}}(0, \infty, H^{s-2}) \text{ w.s.} , \end{array} \right.$$

and,

$$(3.6)\quad \left| \begin{array}{lll} \nabla u^\lambda \to \nabla u^\infty & \text{in} & L^2(0, \infty, H^s) \text{ w.s.} , \\ \\ \nabla u_t^\lambda \to \nabla u_t^\infty & \text{in} & L^2_{\text{loc}}(0, \infty, H^{s-2}) \text{ w.s.} . \end{array} \right.$$

Moreover, from the inequality :

$$|\lambda(\rho^\lambda - \rho_0)|_{H^s} \leq C K_0 ,$$

we deduce :

$$(3.7)\quad \rho^\lambda \to \rho_0 \quad \text{in} \quad C_B(0, \infty, W^{\infty, s-2}) \quad \text{strongly} .$$

Then, $\quad \rho^\lambda u_t^\lambda \to \rho_0 u_t^\infty \quad \text{in} \quad L^\infty_{\text{loc}}(0, \infty, H^{s-2}) \text{ w.s.} .$

From (3.5), we get that :

$$u^{\lambda} \to u^{\infty} \quad \text{in} \quad L^{\infty}_{\text{loc}}(0, \infty, H^{s-1}_{\text{loc}}) \quad \text{strongly and almost everywhere} .$$

These last points lead to the following result :

$$\rho^{\lambda}(u^{\lambda} . \nabla) u^{\lambda} \to \rho_0 (u^{\infty} . \nabla) u^{\infty} \quad \text{in} \quad D'(0, \infty, H^{s-1}) .$$

Let us now consider ϕ in $D(0, T, H^{s-2})$, so that div $\phi = 0$. Then,

$$(\rho^{\lambda}(u_t^{\lambda} + (u^{\lambda} . \nabla) u^{\lambda}) - \nu \, \Delta u^{\lambda}, \phi) = 0 .$$

Making λ go to $+\infty$, we deduce from the above results that :

$$(\forall \phi \in D(0, T, H^{s-2})) ,$$
$$(\operatorname{div} \phi = 0 \Rightarrow (\rho_0 u_t^{\infty} + \rho_0 (u^{\infty} . \nabla) u^{\infty} - \nu \, \Delta u^{\infty}, \phi) = 0) .$$

So, we have shown that there exists some function p^{∞} verifying :

$$\rho_0 u_t^{\infty} + \rho_0 (u^{\infty} . \nabla) u^{\infty} - \nu \, \Delta u^{\infty} = - \nabla p^{\infty} .$$

By construction, it is clear that :

$$\nabla p^{\infty} \in C(0, \infty, H^{s-2}) ,$$

and

$$\lambda \, \nabla \tilde{p}^{\lambda} \to \nabla p^{\infty} \quad \text{in} \quad L^{\infty}_{\text{loc}}(0, \infty, H^{s-2}) \text{ w.s.} .$$

We can gather all these results in the following theorem :

THEOREM 2 : *Let us consider initial data of the shape :*

$$u_0^{\lambda}(x) = u_0(x) + \frac{1}{\lambda} u_1(x) , \quad p_0^{\lambda}(x) = p_0 + \frac{1}{\lambda^2} p_1(x) ,$$
$$\operatorname{div} u_0 = 0 , \quad p_0 = \text{Cte} ;$$

$$(u_0, u_1, p_1) \in [H^s(\mathbb{R}^n)]^3 , \quad \textit{with} \quad s > \left[\frac{n}{2} \right] + 1 , \quad \textit{and} \quad |u_0|^2_{H^s} < K_0 .$$

Then, the sequence $(u^{\lambda}, p^{\lambda})$ converges to (u^{∞}, p^{∞}), solution of the system (S^{∞}), in the following sense :

$$u^{\lambda} \to u^{\infty} \quad \textit{in} \quad C_{\text{loc}}(0, \infty, H^{s-1}_{\text{loc}}(\mathbb{R}^n)) \quad \textit{strongly} ,$$
$$\lambda \, \nabla \tilde{p}^{\lambda} \to \nabla p^{\infty} \quad \textit{in} \quad L^{\infty}_{\text{loc}}(0, \infty, H^{s-2}(\mathbb{R}^n)) \; \textit{w.s.} .$$

In addition, $u^{\infty} \in C_B(0, \infty, H^s) \cap C^1(0, \infty, H^{s-2})$ and

$$\nabla u^{\infty} \in L^2(0, \infty, H^s) .$$

Remark : We have shown a double stability for the system (S^λ) :

— On one hand, stability of the estimates towards λ large enough.

— On the other hand, stability of the limit (u^∞, p^∞) towards the initial data (u_1, p_1) smooth enough.

In particular, to obtain the results we need concerning the derivatives in time of u^∞ and p^∞, we can choose $u_1 = p_1 = 0$.

In this case, taking u_0 smooth enough and deriving once more in time the equations (2.1) and (2.2), we just have to proceed as usual to get uniform in λ estimates on u_{tt}^λ and $\tilde{p}_{tt}^\lambda$.

Which, passing to the limit, allowds to enonce the following properties :

PROPOSITION : *Let us suppose that* $|u_0|^2_{H^{s+k}} < K_0$ $(k \geq 1)$. *Then :*

$$|u_{tt}^\infty|^2_{H^{s+k-4}} + \int_0^t |\nabla u_{tt}^\infty|^2_{H^{s+k-4}} d\tau + \int_0^t |\nabla \tilde{p}_t^\infty|^2_{H^{s+k-3}} d\tau \leq M(t) .$$

Such a result naturally raises the following question :

« Could we get a best convergence by adding new fitting assumptions ? ».

IV. STRONG CONVERGENCE

Like it often happens, to establish strong convergence's results, we have to give more regularity to the initial data.

Moreover, we have an estimate of $|\nabla p^\infty|_{H^k}$ and $|\nabla p_t^\infty|_{H^{k-2}}$, but we don't know anything about $|p^\infty|_2$ and $|p_t^\infty|_2$.

So, like Klainerman and Majda [2], we are going to impose to $|p^\infty|_2$ and $|p_t^\infty|_2$ to be locally bounded.

We then get the following result :

THEOREM 3 : *Let us consider the system* (S^λ) *with initial data :*

$$u^\lambda(x, 0) = u_0(x) + \frac{1}{\lambda} u_1(x) , \quad p^\lambda(x, 0) = p_0 + \frac{1}{\lambda^2} p_1(x) ,$$

$$\operatorname{div} u_0 = 0 , \quad p_0 > 0 ,$$

$$(u_0, u_1, p_1) \in [H^{s+2}(\mathbb{R}^n)]^3 , \quad \textit{with} \quad s > \left[\frac{n}{2}\right] + 1 , \quad \textit{and} \quad |u_0|^2_{H^{s+2}} < K_0 .$$

Let us suppose, in addition, that the following assumption (H) *is true :*

$$(H) \quad |p^\infty(t)|_2 + |p_t^\infty(t)|_2 \leq M(t) , \quad \textit{where} \quad M(t) \in L^\infty_{\text{loc}}(\mathbb{R}_+ , \mathbb{R}_+) .$$

Then, there exists $\lambda_0 \geq 0$, *so that :*

$$\forall t \geq 0\,,\quad \forall \lambda \geq \lambda_0\,,\quad \lambda^2 |u^\lambda - u^\infty|^2_{H^s} + |\lambda^2(p^\lambda - p_0) - p^\infty|^2_{H^s} + \\ + \lambda^2 \int_0^t |\nabla(u^\lambda - u^\infty)|^2_{H^s}\, d\tau \leq M(t)\,.$$

Remark : The assumption $|u_0|^2_{H^{s+2}} \leq K_0$ is necessary to assure global existence of (u^λ, p^λ) and (u^∞, p^∞), as soon as λ is large enough (see theorem 1). Before going on, let us sum up the results that we have already got, in the case where the initial data are in H^{s+k}, with $k \in \mathbb{N}^*$:

(4.1) $$|u^\lambda|^2_{H^{s+k}} + |\tilde{p}^\lambda|^2_{H^{s+k}} + \int_0^\infty |\nabla u^\lambda|^2_{H^{s+k}}\, d\tau + \int_0^\infty |\nabla \tilde{p}^\lambda|^2_{H^{s+k-1}}\, d\tau \leq K_0\,;$$

(4.2) $$|u_t^\lambda|^2_{H^{s+k-2}} + |\tilde{p}_t^\lambda|^2_{H^{s+k-2}} + \int_0^t |\nabla u_t^\lambda|^2_{H^{s+k-2}}\, d\tau \leq M(t) \quad (t \geq 0)\,;$$

(4.3) $$|\nabla \tilde{p}^\lambda|^2_{H^{s+k-2}} + |\operatorname{div} u^\lambda|^2_{H^{s+k-2}} \leq \frac{M(t)}{\lambda^2}\,;$$

(4.4) $$|\tilde{\rho}|^2_{H^{s+k}} \leq C K_0\,,\quad |\tilde{\rho}_t^\lambda|^2_{H^{s+k-2}} \leq M(t)\,;$$

(4.5) $$|p^\lambda - p_0|^2_{W^{\infty, s+k-2}} + |\rho^\lambda - \rho_0|^2_{W^{\infty, s+k-2}} \leq \frac{K_0}{\lambda^2}\,;$$

(4.6) $$|u^\infty|^2_{H^{s+k}} + \int_0^\infty |\nabla u^\infty|^2_{H^{s+k}}\, d\tau \leq K_0\,;$$

(4.7) $$|u_t^\infty|^2_{H^{s+k-2}} + |\nabla p^\infty|^2_{H^{s+k-2}} + \int_0^t |\nabla u_t^\infty|^2_{H^{s+k-2}}\, d\tau \leq M(t) \quad (t \geq 0)\,;$$

(4.8) $$\int_0^t |\nabla p_t^\infty|^2_{H^{s+k-3}}\, d\tau \leq M(t)\,.$$

Having got all these important results, we are now going to use the usual technics to prove the result of the theorem.

Let us note

$$\hat{u} = \lambda(u^\lambda - u^\infty) \quad \text{and} \quad \hat{p} = \lambda^2(p^\lambda - p_0) - p^\infty\,.$$

(N.B. : It follows from hypothesis (H) that $\hat{p} \in L^2$ and $\hat{p}_t \in L^2$.)

Then the couple $(\hat{u}, \hat{p})$ is a solution of the following system :

(4.9) $$\rho_0\, \hat{u}_t + \tilde{\rho}^\lambda\, u_t^\lambda + \tilde{\rho}^\lambda (u^\lambda \,.\, \nabla)\, u^\lambda + \rho_0 (u^\lambda \,.\, \nabla)\, \hat{u} + \\ + \rho_0 (\hat{u} \,.\, \nabla)\, u^\infty - \nu\, \Delta \hat{u} = -\lambda\, \nabla \hat{p}\,,$$

(4.10) $\hat{\rho}_t + \lambda\, \nabla \tilde{p}^\lambda \,.\, u^\lambda + \gamma \tilde{p}^\lambda \operatorname{div} \hat{u} + \lambda \gamma p_0 \operatorname{div} \hat{u} = -p_t^\infty\,, \quad (\operatorname{div} u^\infty = 0)$

(4.11) $\hat{u}(x,0) = u_1(x)\,, \quad \hat{p}(x,0) = p_1(x) - p^\infty(x,0)\,.$

1st Step : L^2-Norms of $\hat{u}$ and $\hat{p}$.

Multiplying equation (4.9) by $\gamma p_0 \hat{u}$ and equation (4.10) by $\hat{p}$, and integrating on $\mathbb{R}^n$, we get :

$$\frac{d}{dt}\left[\frac{\gamma p_0 \rho_0}{2}\,|\hat{u}|_2^2 + \frac{1}{2}\,|\hat{p}|_2^2\right] + \nu\gamma p_0 |\nabla \hat{u}|_2^2 =$$

$$- \gamma p_0 \int \tilde{\rho}^\lambda (u_t^\lambda + u^\lambda \nabla u^\lambda)\,\hat{u}\,dx - \gamma p_0 \rho_0 \int (u^\lambda \nabla)\,\hat{u}\,.\,\hat{u}\,dx$$

$$- \gamma p_0 \rho_0 \int (\hat{u}\,\nabla u^\infty)\,\hat{u}\,dx$$

$$- \int u^\lambda(\lambda\, \nabla \tilde{p}^\lambda)\,\hat{p}\,dx - \int \gamma \tilde{p}^\lambda \operatorname{div} \hat{u}\,\hat{p}\,dx - \int p_t^\infty\,\hat{p}\,dx\,.$$

Thanks to estimates (4.1) to (4.7), the right member can be majored by :

$$M(t) + |\hat{u}|_2^2 + \frac{\nu\gamma p_0}{2}\,|\nabla \hat{u}|_2^2 + |\hat{p}|_2^2 + |p_t^\infty|_2^2\,.$$

Using the supplementary condition on p_t^∞, it yields :

$$\forall t \geqslant 0\,, \quad |\hat{u}(t)|_2^2 + |\hat{p}(t)|_2^2 + \int_0^t |\nabla \hat{u}(\tau)|_2^2\,d\tau \leqslant M(t)\,.$$

2nd Step : L^2-Norms of $D^s u$ and $D^s p$.

Let us derive s times the equations (4.9) and (4.10), multiply the first obtained equation by $\gamma p_0\, \partial^s \hat{u}$, the second by $\partial^s \hat{p}$, and integrate on $\mathbb{R}^n \times [0,t]$. Using the results (4.1) to (4.8) (for $k = 2$), and the usual technics to estimate the obtained terms, we get :

$$|D^s \hat{u}|_2^2 + |D^s \hat{p}(t)|_2^2 + \int_0^t |\nabla D^s \hat{u}|_2^2\,d\tau \leqslant M(t) + C\,|\hat{u}(0)|_{H^s}^2 + |\hat{p}(0)|_{H^s}^2 +$$

$$+ C \int_0^t (|D^s \hat{u}(\tau)|_2^2 + |D^s \hat{p}(\tau)|_2^2)\,d\tau + C \int_0^t |\nabla D^{s-1} p_t^\infty(\tau)|_2^2\,d\tau\,.$$

So, $$\forall t \geqslant 0\,, \quad |\hat{u}(t)|_{H^s}^2 + |\hat{p}(t)|_{H^s}^2 + \int_0^t |\nabla \hat{u}(\tau)|_{H^s}^2\,d\tau \leqslant M(t)\,.$$

Remark : We can get « good » principle parts by scaling non linear terms.

V. AN INITIAL LAYER PHENOMENON WHEN div $u_0 \neq 0$

Hence we consider the solution (u^λ, p^λ) of the system (S^λ) :

$$\begin{cases} \rho^\lambda(u_t^\lambda + (u^\lambda . \nabla) u^\lambda) - \nu \Delta u^\lambda = - \lambda \nabla \tilde{p}^\lambda , \\ \tilde{p}_t^\lambda + u^\lambda . \nabla \tilde{p}^\lambda + \gamma \tilde{p}^\lambda \operatorname{div} u^\lambda + \lambda \gamma p_0 \operatorname{div} u^\lambda = 0 , \\ u^\lambda(x, 0) = u_0(x) + \frac{1}{\lambda} u_1(x) , \quad p^\lambda(x, 0) = p_0 + \frac{1}{\lambda^2} p_1(x) , \end{cases}$$

with now div $u_0 \neq 0$.

Let us write :

$$u_0 = v_0 + \nabla \phi_0 , \quad \text{with} \quad \operatorname{div} v_0 = 0 . \tag{5.1}$$

Since the solution (u^∞, p^∞) of the system (S^∞) verifies the condition : Div $u^\infty = 0$, it clearly appears an initial layer's phenomenon.

A fitting corrector term is provided by the solution (v^λ, q^λ) of the linear following system :

$$(C^\lambda) \quad \begin{cases} (5.2) \quad \rho_0 v_t^\lambda - \nu \Delta v^\lambda = - \lambda \nabla q^\lambda , \\ (5.3) \quad q_t^\lambda + \lambda \gamma p_0 \operatorname{div} v^\lambda = 0 , \\ (5.4) \quad v^\lambda(x, 0) = \nabla \phi_0(x) , \quad q^\lambda(x, 0) = 0 . \end{cases}$$

We'll establish, in an appendix, the following result :

PROPOSITION (5.5) : *If $\phi_0 \in W^{s+n+4}(\mathbb{R}^n)$, then v^λ verifies the following $L^\infty - L^1$ estimate :*

$$|v^\lambda|_{W^{s,\infty}} \leq \frac{C}{(1 + \lambda t)} |\phi_0|_{W^{1,s+n+4}} \quad if \quad n \geq 3 ,$$

$$|v^\lambda|_{W^{s,\infty}} \leq \frac{C}{\sqrt{1 + \lambda t}} |\phi_0|_{W^{1,s+6}} \quad if \quad n = 2 .$$

Let us consider the solution (u^∞, p^∞) of the system (S^∞) :

$$(S^\infty) \quad \begin{cases} \rho_0(u_t^\infty + (u^\infty . \nabla) u^\infty) - \nu \Delta u^\infty = - \nabla p^\infty , \\ \operatorname{div} u^\infty = 0 , \quad u^\infty(x, 0) = v_0(x) . \end{cases}$$

Like in paragraph 4, we'll impose, in the whole part left, to p^∞ to verify :

$$(H) \quad |p^\infty|_2^2 + |p_t^\infty|_2^2 \leq M(t) , \quad \text{where} \quad M(t) \in L^\infty_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+) .$$

We then prove the :

THEOREM 4 : *Let us consider the system* (S^λ) *with the initial data :*

$$u^\lambda(x,0) = v_0(x) + \nabla\phi_0(x) + \frac{1}{\lambda} u_1(x) , \quad \textit{with} \quad \operatorname{div} v_0(x) = 0 ,$$

$$p^\lambda(x,0) = p_0 + \frac{1}{\lambda^2} p_1(x) , \quad p_0 > 0 ,$$

$$(v_0, u_1, p_1) \in [H^{s+2}(\mathbb{R}^n)]^3 \quad \textit{and} \quad \phi_0 \in W^{1,s+n+5} \subset H^{s+3}\left(s > \left[\frac{n}{2}\right] + 1\right),$$

and $|v_0 + \nabla\phi_0|^2_{H^{s+2}} < K_0$.

Let us suppose, in addition, that hypothesis (H) is verified.

Then, there exists $\lambda_0 \geqslant 0$, *so that :*

$$\forall t > 0 , \quad \forall \lambda \geqslant \lambda_0 ,$$

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leqslant M(t)\,\frac{(1 + \operatorname{Log}\,(1 + \lambda t))}{\lambda} \qquad if \quad n \geqslant 3 ,$$

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leqslant \frac{1}{\sqrt{\lambda}} M(t) \qquad if \quad n = 2 .$$

Proof: Let us note $w = u^\lambda - u^\infty - v^\lambda$ and $b = \tilde{p}^\lambda - \frac{1}{\lambda} p^\infty - q^\lambda$.

Considering the equations satisfied by (u^λ, p^λ), (u^∞, p^∞) and (v^λ, q^λ), we find that (w, b) is a solution of the following system :

$$\text{(5.6)} \quad \rho^\lambda w_t + \rho_0 w \nabla u^\infty + \rho_0 u^\lambda \nabla w - \nu \Delta w + \frac{\tilde{\rho}^\lambda}{\lambda} v_t^\lambda + \\ + \frac{\tilde{\rho}^\lambda}{\lambda} (u_t^\infty + u^\lambda \nabla u^\lambda) + \rho_0(v^\lambda \nabla u^\infty + u^\lambda \nabla v^\lambda) = -\lambda \nabla b ,$$

$$\text{(5.7)} \quad b_t + u^\lambda \nabla b + \gamma \tilde{p}^\lambda \operatorname{div} w + \lambda \gamma p_0 \operatorname{div} w + \\ + \left(\frac{p_t^\infty}{\lambda} + \frac{u^\lambda \nabla p^\infty}{\lambda} + \frac{\nu u^\lambda \Delta v^\lambda}{\lambda} \right) + \gamma \tilde{p}^\lambda \operatorname{div} v^\lambda - \rho_0 v_t^\lambda \frac{u^\lambda}{\lambda} = 0 ,$$

$$\text{(5.8)} \quad w(x,0) = \frac{1}{\lambda} u_1(x) , \quad b(x,0) = \frac{1}{\lambda} (p_1(x) - p^\infty(x,0)) .$$

Let us note :

$$\text{(5.9)} \quad h(x,t) = w_t + \frac{\lambda \nabla b}{\rho^\lambda} \quad \text{and} \quad k(x,t) = b_t + \lambda \gamma p_0 \operatorname{div} w + \frac{1}{\lambda} p_t^\infty .$$

Thanks to estimates (4.1), (4.5), (4.6) and (4.7), we deduce from the smoothness of the initial data ($k = 2$), that :

$$\forall t \geqslant 0 , \quad |h(t)|_{H^s} + |k(t)|_{H^s} \leqslant M(t) . \tag{5.10}$$

Let us also note that equations (5.6) and (5.7) can be written as follows :

$$\rho^\lambda w_t + \rho_0 (w \nabla) u^\infty + \rho_0 (u^\lambda \nabla) w - \nu \Delta w + \frac{\tilde{\rho}^\lambda}{\lambda} v_t^\lambda + f^\lambda = - \lambda \nabla b , \tag{5.11}$$

$$\left(1 - \frac{\tilde{p}^\lambda}{\lambda p_0}\right) b_t + u^\lambda . \nabla b + \lambda \gamma p_0 \operatorname{div} w - \rho_0 v_t^\lambda \frac{u^\lambda}{\lambda} + g^\lambda = 0 , \tag{5.12}$$

where $f^\lambda = \frac{\tilde{\rho}^\lambda}{\lambda} (u_t^\infty + (u^\lambda \nabla) u^\lambda) + \rho_0 ((v^\lambda \nabla) u^\infty + (u^\lambda \nabla) v^\lambda)$,

and $g^\lambda = \frac{1}{\lambda} \left(p_t^\infty + u^\lambda \nabla p^\infty + \nu u^\lambda \Delta v^\lambda + \frac{\tilde{p}^\lambda}{p_0} k - \frac{\tilde{p}^\lambda}{\lambda p_0} p_t^\infty \right) + \gamma \tilde{p}^\lambda \operatorname{div} v^\lambda$.

Let $a^\lambda(t)$ be the quantity :

$$a^\lambda(t) = \int_0^t (|f^\lambda(\tau)|_{H^s} + |g^\lambda(\tau)|_{H^s} + |v^\lambda(\tau)|_{W^{\infty, s+1}}) \, d\tau .$$

We are going to need the following lemma :

LEMMA (5.13) :

$$a^\lambda(t) \leqslant \frac{M(t)}{\sqrt{\lambda}} \quad \textit{if} \quad n = 2 \quad \textit{and}$$

$$a^\lambda(t) \leqslant \frac{M(t)}{\lambda} (1 + \operatorname{Log} (1 + \lambda t)) \quad \textit{if} \quad n \geqslant 3 .$$

It is immediatly deduced from proposition (5.5) and from the assumptions of theorem 4.

1st Step : Estimate of w and b in L^2-norm.

Let us multiply equation (5.11) by $\gamma p_0 w$ and equation (5.12) by b. The only true difficulty lays in the terms :

$$\frac{\tilde{\rho}^\lambda}{\lambda} v_t^\lambda w \quad \text{and} \quad \frac{u^\lambda}{\lambda} v_t^\lambda b ,$$

because we just know that $\frac{v_t^\lambda}{\lambda}$ is bounded.

To avoid this difficulty, we just have to integrate by part, using (5.9). So, we obtain :

$$\frac{d}{dt}\left[\int\left(\gamma p_0\,\rho^\lambda\frac{w^2}{2}+\left(1-\frac{\tilde{p}^\lambda}{\lambda p_0}\right)\frac{b^2}{2}+\gamma p_0\frac{\tilde{\rho}^\lambda}{\lambda}v^\lambda w-\rho_0\frac{v^\lambda u^\lambda b}{\lambda}\right)dx\right]$$

$$+\nu\gamma p_0 = |\nabla w|_2^2 = \int\left(\gamma p_0\,\rho_t^\lambda\frac{w^2}{2}-p_t^\lambda\frac{b^2}{2p_0}-\gamma\rho_0 p_0(w\,\nabla)\,u^\infty\,.\,w\right.$$

$$\left.+\frac{\gamma}{2}\rho_0 p_0\operatorname{div}u^\lambda|w|^2+\operatorname{div}u^\lambda\frac{|b|^2}{2}\right)dx$$

$$+\int\left(\gamma p_0\frac{\tilde{\rho}_t^\lambda}{\lambda}v^\lambda w-\gamma p_0 f^\lambda w-\gamma\rho_0 p_0\nabla(u^\lambda v^\lambda)\,w\right.$$

$$\left.+\gamma p_0\operatorname{div}\left(\frac{\tilde{\rho}^\lambda v^\lambda}{\rho^\lambda}\right)b-\rho_0\frac{v^\lambda}{\lambda}u_t^\lambda b-g^\lambda b\right)dx$$

$$+\int\left(\gamma p_0\,\tilde{\rho}^\lambda h\frac{v^\lambda}{\lambda}-\rho_0 u^\lambda\frac{v^\lambda}{\lambda}k+\rho_0 u^\lambda v^\lambda\frac{p_t^\infty}{\lambda^2}\right)dx$$

$$= I_1(t)+I_2(t)+I_3(t)\,.$$

Let $\chi_0^2(t)=\operatorname*{Sup}_{[0,t]}\left(|w(\tau)|^2+|b(\tau)|^2\right)$.

From the results of theorem 1 (§ II), we easily deduce the following estimates

$$\int_0^t|I_1(\tau)|\,d\tau\leqslant K\int_0^t\chi_0^2\,d\tau\,,$$

$$\int_0^t|I_2(\tau)|\,d\tau\leqslant K\chi_0\,a^\lambda(t)\,,$$

$$\int_0^t|I_3(\tau)|\,d\tau\leqslant K\frac{a^\lambda(t)}{\lambda}\,.$$

Let us also note

$$I_4(t)=\int\left(\gamma p_0\frac{\tilde{\rho}^\lambda}{\lambda}v^\lambda w-\rho_0\frac{v^\lambda u^\lambda b}{\lambda}\right)dx\,.$$

Then I_4 verifies :

$$|I_4(t)|\leqslant\frac{K}{\lambda}\chi_0\,.$$

Now, thanks to hypothesis (H) and (5.8), we deduce that : $\chi_0(0)\leqslant\dfrac{K}{\lambda}$. Thus, we get the following inequality :

$$\chi_0^2(t)\leqslant\frac{K}{\lambda^2}+\frac{K}{\lambda}\chi_0+\frac{K}{\lambda}a^\lambda(t)+K\chi_0\,a^\lambda(t)+K\int_0^t\chi_0^2(\tau)\,d\tau\,,$$

so

$$\chi_0^2(t) \leq K\left(|a^\lambda(t)|^2 + \int_0^t \chi_0^2(\tau)\, d\tau\right). \tag{5.14}$$

2nd Step : Estimate of $D^s w$ and $D^s b$ in L^2-norm

We'll use the technics developped in paragraph II (pp. 16-18), the difficulty raised in the first step being solved by integrating by parts again. (We shall use in particular the inequalities (2.5) and (2.6)).

The operation

$$\int D^s(5.6)\, \gamma p_0\, D^s w\, dx + \int D^s(5.7)\, .\, D^s b\, dx$$

hence gives :

$$\frac{d}{dt}\left[\int \gamma p_0\, \rho^\lambda \frac{(D^s w)^2}{2} + \left(1 - \frac{\tilde{p}^\lambda}{\lambda p_0}\right)\frac{(D^s b)^2}{2} + \gamma p_0 \frac{D^s(\tilde{\rho}^\lambda v^\lambda)}{\lambda} D^s w\right.$$

$$\left. - \rho_0 \frac{D^s(u^\lambda v^\lambda)}{\lambda} D^s b\right] + \nu \rho_0 p_0 |\nabla D^s w|_2^2 = \gamma \rho_0 p_0 \int \left(\frac{\rho_t}{\rho_0}\frac{(D^s w)^2}{2}\right.$$

$$- D^s(w\, .\, \nabla u^\infty)\, D^s w$$

$$\left. + \operatorname{div} u^\lambda \frac{(D^s w)^2}{2} - [D^s(u^\lambda\, .\, \nabla w) - u^\lambda D^s \nabla w]\, .\, D^s w\, dx\right.$$

$$+ \int \left(\operatorname{div} u^\lambda \frac{(D^s b)^2}{2} - \frac{p_t^\lambda}{p_0}\frac{(D^s b)^2}{2}\right.$$

$$\left. - [D^s(u^\lambda\, .\, \nabla b) - u^\lambda(D^s \nabla b)]\, .\, D^s b\right) dx$$

$$+ \gamma p_0 \int \left(D^s\left(\tilde{\rho}_t^\lambda \frac{v^\lambda}{\lambda}\right) D^s w + D^{s+1}(\tilde{\rho}^\lambda v^\lambda)\, D^{s-1}\left(\frac{\nabla b}{\rho^\lambda}\right)\right.$$

$$\left. - D^s f^\lambda\, .\, D^s w - \rho_0 D^s \nabla (u^\lambda v^\lambda)\, D^s w\right) dx$$

$$- \int \left(\rho_0 D^s\left(u_t^\lambda \frac{v^\lambda}{\lambda}\right) D^s b + D^s g^\lambda\, .\, D^s b\right) dx$$

$$+ \int \left(\gamma p_0 D^s(\tilde{\rho}^\lambda v^\lambda) \frac{D^s h}{\lambda} - \rho_0 D^s(u^\lambda v^\lambda) \frac{D^s k}{\lambda} + \rho_0 D^s(u^\lambda v^\lambda) \frac{D^s(p_t^\infty)}{\lambda^2}\right) dx$$

$$+ \int \left(\gamma p_0 [D^s(\rho^\lambda w_t) - \rho^\lambda (D^s w_t)]\, .\, D^s w\right.$$

$$\left. + \left[D^s\left(1 - \frac{\tilde{\rho}^\lambda}{\lambda p_0}\right) b_t - \left(1 - \frac{\tilde{p}^\lambda}{\lambda p_0}\right) D^s b_t\right] .\, D^s b\right) dx$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6\, .$$

Let $$\chi_s^2(t) = \operatorname*{Sup}_{[0,t]} \left(|D^s w(\tau)|_2^2 + |D^s b(\tau)|_2^2\right).$$

Thanks to the lemma 5 and the results of theorem 1 in particular, the integral

$$\left|\int_0^t (I_1 + I_2 + I_3 + I_4 + I_5)(\tau)\, d\tau\right|$$

is majored, as in the first step, by :

$$K\left[\int_0^t (\chi_0^2(\tau) + \chi_s^2(\tau))\, d\tau + a^\lambda(t)(\chi_0 + \chi_s) + a^\lambda(t)^2\right].$$

Also, if we note

$$I_7(t) = \int \left(\gamma p_0 \frac{D^s(\tilde{\rho}^\lambda v^\lambda)}{\lambda} D^s w - \rho_0 \frac{D^s(u^\lambda v^\lambda)}{\lambda} D^s b\right) dx\,,$$

then, $$|I_7(t)| \leqslant \frac{K}{\lambda}(\chi_0 + \chi_s) \leqslant a^\lambda(t) \cdot (\chi_0 + \chi_s)\,.$$

Now, we have to estimate I_6. Using (5.9) and (2.5), we get :

$$I_6(t) \leqslant \frac{1}{\lambda}\chi_s + (\chi_0 + \chi_s)\cdot\chi_s + \frac{1}{\lambda^2}|p_t^\infty|_{H^s}\chi_s\,.$$

Thus, we get the following inequality for χ_s :

$$\chi_s^2(t) \leqslant K\left[(\chi_0 + \chi_s)\, a^\lambda(t) + a^\lambda(t)^2 + \int_0^t (\chi_0^2 + \chi_s^2)(\tau)\, d\tau\right],$$

what, added (!) to (5.14), leads to a Gronwald's inequality verified by $\chi_0^2 + \chi_s^2$. Hence,

$$|w|_{H^s}^2 + |b|_{H^s}^2 = \chi_0^2 + \chi_s^2 \leqslant KM(t)\,.\,a^\lambda(t)^2\,.$$

Finally, let us remark that :

$$|\lambda(p^\lambda - p_0) - q^\lambda|_{H^s}^2 \leqslant |b|_{H^s}^2 + \frac{1}{\lambda^2}|p^\infty|_{H^s}^2\,.$$

So, the theorem is proven.

Remark : As in paragraph 4, we can find a principal part of $u^\lambda - u^\infty - v^\lambda$, which, in fact, is the same than in the case : div $u_0 = 0$.

A REMARK CONCERNING EULER'S EQUATIONS

In [2], Klainerman and Majda study the compressible Euler's equations

$$(E^\lambda)\quad \begin{cases} \rho^\lambda \left(\dfrac{\partial u^\lambda}{\partial t} + (u^\lambda \cdot \nabla) u^\lambda \right) = -\lambda^2 \nabla p^\lambda , \\ \dfrac{\partial p^\lambda}{\partial t} + u^\lambda \cdot \nabla p^\lambda + \gamma p^\lambda \operatorname{div} u^\lambda = 0 , \\ u^\lambda(x, 0) = u_0^\lambda(x) , \quad p^\lambda(x, 0) = p_0^\lambda(x) , \end{cases}$$

with again : $p = A\rho^\gamma$, $\gamma > 1$.

First, they consider initial data :

$$u_0^\lambda \in H^s(\mathbb{R}^n) , \quad (p_0^\lambda - p_0) \in H^s(\mathbb{R}^n) \quad \text{with} \quad s > \left[\frac{n}{2} \right] + 1 .$$

Then, they obtain, on a finite time intervall, estimations of the same type than the ones obtained in paragraph 2 (by completly different methods).

More precisely, they prove that there exists a finite time intervall $[0, T]$, depending only on initial data, and a constant $\Delta_s > 0$, so that, for $\lambda \geq 1$, there exists a classical solution (u^λ, p^λ) in $C^1([0, T] \times \mathbb{R}^n)$ for the system (E^λ), satisfying :

$$\forall t \in [0, T] , \quad |u^\lambda|_{H^s} + |\lambda(p^\lambda - p_0)|_{H^s} \leq \Delta_s .$$

If the initial data verify in addition :

$$(5.15)\quad \begin{aligned} & u_0^\lambda(x) = u_0(x) + \frac{1}{\lambda} u_1(x) , \quad \text{with} \quad \operatorname{div} u_0 = 0 \\ & p_0^\lambda(x) = p_0 + \frac{1}{\lambda^2} p_1(x) , \quad p_0 = \text{Cte} , \quad (u_1, p_1) \in H^s , \end{aligned}$$

they obtain, as we did, estimates on derivatives in time of (u^λ, p^λ).

So, they prove a weak convergence of the solutions (u^λ, p^λ) to the solution (u^∞, p^∞) of incompressible Euler's equations :

$$(E^\infty)\quad \begin{cases} \rho_0(u_t^\infty + (u^\infty \cdot \nabla) u^\infty) = -\nabla p^\infty , \\ \operatorname{div} u^\infty = 0 , \quad u^\infty(x, 0) = u_0(x) . \end{cases}$$

(this solution living on an intervall $[0, T^*[$, see [10]).

Finally, introducing the supplementary condition :

$$\forall T_0 < T^* , \quad \forall t \in [0, T_0] , \quad |p^\infty|_2 + |p_t^\infty|_2 \leq M(t) ,$$

they show the following strong convergence's result : there exists $\lambda(T_0)$ so that, for $\lambda \geq \lambda(T_0)$, the system (E^λ) with initial data (5.15) has a unic classical solution (u^λ, p^λ) verifying :

$$\forall t \leq T_0 , \quad |u^\lambda - u^\infty|_{H^s} + \frac{1}{\lambda} |u_t^\lambda - u_t^\infty|_{H^{s-1}} \leq \frac{C}{\lambda} ,$$

$$\lambda |p^\lambda - p^\infty|_{H^s} + |p_t^\lambda|_{H^{s-1}} \leq \frac{C}{\lambda} \quad (C > 0) .$$

They also show a principal part.

Their results and ours were so similar that we decided to study the initial layer's problem appearing in this case, if we no more suppose :

$$\text{Div}\, u_0 = 0 , \quad \text{but} : u_0(x) = v_0(x) + \nabla\phi_0(x) , \quad \text{with div}\, v_0 = 0 .$$

Precisely, we get the :

PROPOSITION : *Let us consider the system (E^λ) with initial data :*

$$u^\lambda(x, 0) = v_0(x) + \nabla\phi_0(x) + \frac{1}{\lambda} u_1(x) ,$$

$$\text{div}\, v_0 = 0 , \quad p^\lambda(x, 0) = p_0 + \frac{1}{\lambda^2} p_1(x) ,$$

$$(v_0, u_1, p_1) \in [H^{s+1}(\mathbb{R}^n)]^3 , \quad \phi_0 \in W^{1, s+n+2}(\mathbb{R}^n)$$

and $s > \left[\frac{n}{2}\right] + 1 \quad (n \geq 2)$.

Let us suppose in addition that :

$$\forall T_0 < T , \quad \forall t \in [0, T_0] , \quad |p^\infty(t)|_2 + |p_t^\infty(t)|_2 \leq M(t) .$$

Then, there exists $\lambda(T_0) > 0$, so that :

$\forall \lambda \geq \lambda(T_0) , \quad \forall t \in [0, T_0] ,$

$$|u^\lambda - u^\infty - v^\lambda|_{H^s} + |\lambda(p^\lambda - p_0) - q^\lambda|_{H^s} \leq \frac{C}{\sqrt{\lambda}} \quad \textit{if} \quad n = 2$$

$$\frac{C}{\lambda} (1 + \log (1 + \lambda t)) \quad \textit{if} \quad n = 3$$

$$\frac{C}{\lambda} \left(1 + (1 + \lambda t)^{-\frac{n-3}{2}}\right) \quad \textit{if} \quad n = 4$$

where $(v^{\lambda}, q^{\lambda})$ *is the solution of the waves equation :*

$$\begin{cases} \rho_0 v_t^{\lambda} + \lambda \nabla q^{\lambda} = 0 , \\ q_t^{\lambda} + \lambda \gamma p_0 \operatorname{div} v^{\lambda} = 0 , \\ v^{\lambda}(x, 0) = \nabla \phi_0(x) , \quad q^{\lambda}(x, 0) = 0 . \end{cases}$$

The demonstration of this result is exactly the same than the one of theorem 4 but, in this case, the initial layer's properties are well known. As a matter of fact, Klainerman proves in [8] the following property, which is here fundamental :

PROPOSITION : *If* $\phi_0 \in W^{1, s+n+1}$, *we have the following* $L^{\infty} - L^1$ *estimate :*

$$|v^{\lambda}(t)|_{W^{\infty, s}} \leq C (1 + \lambda t)^{-\frac{n-1}{2}} |\nabla \phi_0|_{W^{1, s+n}} \quad (\forall n \geq 2) .$$

APPENDIX

Our purpose here is to study the decreasing with λ of $|D^s v^{\lambda}|_{\infty}$, where $(v^{\lambda}, q^{\lambda})$ is the solution of the following linear system :

$$(C^{\lambda}) \quad \begin{cases} \rho_0 \dfrac{\partial v^{\lambda}}{\partial t} - \nu \, \Delta v^{\lambda} = - \lambda \nabla q^{\lambda} , \\ \dfrac{\partial q^{\lambda}}{\partial t} + \lambda \gamma p_0 \operatorname{div} v^{\lambda} = 0 , \\ v^{\lambda}(x, 0) = \nabla \phi_0(x) , \quad q^{\lambda}(x, 0) = 0 . \end{cases}$$

The choice of the initial data $(v^{\lambda}(x, 0) = \nabla \phi_0(x))$, and the regularity of ϕ_0, permit to write the solution $(v^{\lambda}, q^{\lambda})$ in the form $(\nabla \phi^{\lambda}, q^{\lambda})$, where the couple $(\phi^{\lambda}, q^{\lambda})$ verifies the following equations :

$$(D^{\lambda}) \quad \begin{cases} \rho_0 \dfrac{\partial \phi^{\lambda}}{\partial t} - \nu \, \Delta \phi^{\lambda} = - \lambda q^{\lambda} , \\ \dfrac{\partial q^{\lambda}}{\partial t} + \lambda \gamma p_0 \, \Delta \phi^{\lambda} = 0 , \\ \phi^{\lambda}(x, 0) = \phi_0(x) , \quad q^{\lambda}(x, 0) = 0 . \end{cases}$$

We then obtain the following result :

THEOREM : *Let us suppose that* $\phi_0 \in W^{1, k+n+3}$ $(k \in \mathbb{N})$. *Then, for* λ *large enough, the following estimates are verified :*

$$|\phi^\lambda(.,t)|_{W^{\infty,k}} \leq \frac{C}{(1+\lambda t)} |\phi_0|_{W^{1,k+n+3}} \quad if\ n \geq 3 ,$$
$$|\phi^\lambda(.,t)|_{W^{\infty,k}} \leq \frac{C}{\sqrt{1+\lambda t}} |\phi_0|_{W^{1,k+5}} \quad if\ n = 2 .$$

Remark : Since $W^{1,n}(\mathbb{R}^n) \subset H^{\left[\frac{n}{2}\right]}(\mathbb{R}^n)$, we also have :

$$\gamma\rho_0 p_0 |\nabla\phi^\lambda(.,t)|^2_{H^h} + |q^\lambda(.,t)|^2_{H^h} \leq$$
$$\leq \gamma\rho_0 p_0 |\nabla\phi_0|^2_{H^h}, \quad \text{for any } h \leq \left[\frac{n}{2}\right] + 2 + k .$$

COROLLARY : *If* $\phi_0 \in W^{1,k+n+4}(\mathbb{R}^n)$, *then* :

$$|v^\lambda(.,t)|_{W^{\infty,k}} \leq \frac{C}{(1+\lambda t)} |\phi_0|_{W^{1,k+n+4}} \quad if\ n \geq 3 ,$$
$$|v^\lambda(.,t)|_{W^{\infty,k}} \leq \frac{C}{\sqrt{1+\lambda t}} |\phi_0|_{W^{1,k+6}} \quad if\ n = 2 .$$

Remark : If we had chosen initial data under the shape :

$$v^\lambda(x,0) = v_0(x) + \nabla\phi_0(x) \quad \text{with} \quad \operatorname{div} v_0 = 0 \quad \text{and} \quad v_0 \neq 0 ,$$

we couldn't have obtained these basic decreasing of v^λ results.

As a matter of fact, we would have obtained : $v^\lambda = w + \nabla\phi^\lambda$, where ϕ^λ is the solution of the system (D^λ), and w the solution of the heath equation :

$$\begin{cases} w_t - \nu\,\Delta w = 0 \\ w(x,0) = v_0(x) . \end{cases}$$

w being independent of λ, there is no more decreasing with λ.

Proof of the theorem : The function ϕ^λ being a solution of the system (D^λ), it verifies the following equation :

$$\begin{cases} \rho_0\,\phi^\lambda_{tt} - \nu\,\Delta\phi^\lambda_t - \lambda^2\,\gamma p_0\,\Delta\phi^\lambda = 0 , \\ \phi^\lambda(x,0) = \phi_0(x) , \quad \phi^\lambda_t(x,0) = \dfrac{\nu}{\rho_0}\Delta\phi_0(x) . \end{cases}$$

To make the calculations simpler, we shall suppose that :

$$\rho_0 = 1 , \quad \nu = 2 , \quad \gamma p_0 = 1 .$$

Hence, let us consider ϕ^λ solution of

$$\begin{cases} \phi^\lambda_{tt} - 2\,\Delta\phi^\lambda_t - \lambda^2\,\Delta\phi^\lambda = 0 , \\ \phi^\lambda(x,0) = \phi_0(x) , \quad \phi^\lambda_t(x,0) = 2\,\Delta\phi_0(x) . \end{cases}$$

We then find that the Fourier Transform in x, $\hat{\phi}^\lambda$, of ϕ^λ verifies :

$$\hat{\phi}^\lambda_{tt} + 2|\xi|^2 \hat{\phi}^\lambda_t + \lambda^2 |\xi|^2 \hat{\phi}^\lambda = 0 , \quad \xi \in \mathbb{R}^n , \quad t \in \mathbb{R}^+ ,$$
$$\hat{\phi}^\lambda(\xi, 0) = \hat{\phi}_0(\xi) , \quad \hat{\phi}^\lambda_t(\xi, 0) = -2|\xi|^2 \hat{\phi}_0(\xi) .$$

So we obtain ϕ^λ in the form :

$$\phi^\lambda(x,t) = \int_{\mathbb{R}} e^{ix\cdot\xi} \hat{\phi}_0(\xi)\, d\xi$$
$$= \int_{|\xi|<\lambda} e^{ix\cdot\xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times$$
$$\times \left[\cos\left(t|\xi| \sqrt{\lambda^2 - |\xi|^2}\right) - \frac{|\xi|}{\sqrt{\lambda^2 - |\xi|^2}} \sin\left(t|\xi| \sqrt{\lambda^2 - |\xi|^2}\right) \right] d\xi$$
$$+ \int_{|\xi|>\lambda} e^{ix\cdot\xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times$$
$$\times \left[\operatorname{ch}\left(t|\xi| \sqrt{|\xi|^2 - \lambda^2}\right) - \frac{|\xi|}{\sqrt{|\xi|^2 - \lambda^2}} \operatorname{sh}\left(t|\xi| \sqrt{|\xi|^2 - \lambda^2}\right) \right] d\xi .$$

So, we shall write :

$$\phi^\lambda(x,t) = \int_{|\xi|<\sqrt{\lambda}} e^{ix\cdot\xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times \cos\left(t|\xi| \sqrt{\lambda^2 - |\xi|^2}\right) d\xi$$
$$+ \int_{\sqrt{\lambda}<|\xi|<\lambda} e^{ix\cdot\xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times$$
$$\times \left[\cos\left(t|\xi| \sqrt{\lambda^2 - |\xi|^2}\right) - \frac{|\xi|}{\sqrt{\lambda^2 - |\xi|^2}} \sin\left(t|\xi| \sqrt{\lambda^2 - |\xi|^2}\right) \right] d\xi$$
$$- \int_{|\xi|<\sqrt{\lambda}} e^{ix\cdot\xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \frac{|\xi|}{\sqrt{\lambda^2 - |\xi|^2}} \sin\left(t|\xi| \sqrt{\lambda^2 - |\xi|^2}\right) d\xi$$
$$+ \int_{|\xi|>\lambda} e^{ix\cdot\xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \times$$
$$\times \left[\operatorname{ch}\left(t|\xi| \sqrt{|\xi|^2 - \lambda^2}\right) - \frac{|\xi|}{\sqrt{|\xi|^2 - \lambda^2}} \operatorname{sh}\left(t|\xi| \sqrt{|\xi|^2 - \lambda^2}\right) \right] d\xi$$
$$= I_1 + I_2 + I_3 + I_4 .$$

(i) *Majoration of* I_1 :

This term represents, in a way, the « principal » part of $\phi^\lambda(x,t)$. Let us call S the waves equation's semi-group, and K the heat equation's Kernel.

Then, let us split up I_1 :

$$I_1 = \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \cos (t|\xi| \lambda) d\xi$$
$$- \int_{\sqrt{\lambda} < |\xi|} e^{ix \cdot \xi} e^{-t|\xi|^2} \hat{\phi}_0(\xi) \cos (t|\xi| \lambda) d\xi$$
$$+ \int_{|\xi| < \sqrt{\lambda}} e^{ix \cdot \xi} e^{-|\xi|^2 t} \hat{\phi}_0(\xi) \left[\cos t|\xi| \lambda \sqrt{1 - \frac{|\xi|^2}{\lambda^2}} - \cos t|\xi| \lambda\right] d\xi$$
$$= I_5 + I_6 + I_7 .$$

We recognize in I_5 the following expression : $I_5 = S(\lambda t)(K * \phi_0)$.

Thanks to the properties of the solutions of the waves and heat equations, we deduce from that :

$$\text{(A.1)} \quad |I_5| \leqslant C |K * \phi_0|_{W^{1,n}} (1 + \lambda t)^{-\frac{n-1}{2}} \leqslant C |\phi_0|_{W^{1,n}} (1 + \lambda t)^{-\frac{n-1}{2}} .$$

Remark : In the case where $\nu = 0$, that is to say for Euler's equations, $\phi^\lambda(x, t)$ is reduced to integral I_5, and we obtain :

$$|\phi^\lambda(x, t)|_\infty \leqslant C |\phi_0|_{W^{1,n}} (1 + \lambda t)^{-\frac{n-1}{2}} .$$

We are now going to estimate separatly $I_2 + I_6$, $I_3 + I_7$ and I_4.

For that, we shall need the following auxiliary results :

LEMMA :

$$\text{(A.2)} \qquad \forall u \in [0, 1] , \quad 1 - u \leqslant \sqrt{1 - u} \leqslant 1 - \frac{u}{2} ;$$

$$\text{(A.3)} \qquad \forall u \geqslant 0 , \quad \sin u \leqslant u , \quad \operatorname{sh} u \leqslant u \,.\, e^u, \operatorname{ch} u \leqslant e^u ;$$

$$\text{(A.4)} \qquad \forall u \geqslant 0 , \quad (1 + u) \,.\, e^{-u} \leqslant C \exp\left(-\frac{u}{2}\right) .$$

(ii) *Majoration of* $|I_2 + I_6|$.

Using the inequalities (A.3) and (A.5), we easily obtain :

$$|I_2 + I_6| \leqslant C \int_{\sqrt{\lambda} < |\xi|} e^{-|\xi|^2 t} |\hat{\phi}_0(\xi)| (1 + t|\xi|^2) d\xi$$
$$\leqslant \int_{\sqrt{\lambda} < |\xi|} e^{-\frac{\lambda t}{2}} |\xi|^{n+1} |\hat{\phi}_0(\xi)| \frac{d\xi}{|\xi|^{n+1}} ,$$

that is to say :

$$\text{(A.5)}\ |I_2 + I_6| \leqslant C \exp\left(-\frac{\lambda t}{2}\right) |\phi_0|_{W^{1,n+1}} \leqslant C |\phi_0|_{W^{1,n+1}} (1 + \lambda t)^{-\frac{n-1}{2}} .$$

(iii) *Majoration of* $|I_7| + |I_3|$.

We can write :

$$|I_7| \leqslant C \int_{|\xi| < \sqrt{\lambda}} e^{-|\xi|^2 t} |\hat{\phi}_0(\xi)| \left| \sin \frac{t|\xi|\lambda}{2} \left(1 - \sqrt{1 - \frac{|\xi|^2}{\lambda^2}}\right) \right| \times$$
$$\times \left| \sin \frac{t|\xi|\lambda}{2} \left(1 - \sqrt{1 + \frac{|\xi|^2}{\lambda^2}}\right) \right| d\xi .$$

Thanks to the lemma, we deduce from that :

$$\begin{aligned}
|I_7| &\leqslant C \int_{|\xi| < \sqrt{\lambda}} e^{-|\xi|^2 t} |\hat{\phi}_0(\xi)| \frac{t|\xi|^3}{2\lambda} d\xi \\
&\leqslant C \int_{|\xi| < \sqrt{\lambda}} \exp\left(-\frac{|\xi|^2 t}{2}\right) |\hat{\phi}_0(\xi)| \frac{|\xi|}{\lambda} d\xi \\
&\leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} \frac{|\xi|^m}{|\xi|^{m-1}} \exp\left(-\frac{|\xi|^2 t}{2}\right) |\hat{\phi}_0(\xi)| d\xi \\
&\leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} (1 + |\xi|^a) |\hat{\phi}_0(\xi)| \exp\left(-\frac{|\xi|^2 t}{2}\right) \frac{d\xi}{|\xi|^{m-1}},
\end{aligned}$$

where $a = m$, if m is even, $a = m + 1$ if m is odd.

Choosing $m = n - 1$, we find :

$$|I_7| \leqslant \frac{C}{\lambda} |\phi_0|_{W^{1,n}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\xi|^2 t}{2}\right) \frac{d\xi}{|\xi|^{n-2}} .$$

So,

$$\text{(A.6)} \qquad |I_7| \leqslant \frac{C}{\lambda t} |\phi_0|_{W^{1,n}} .$$

On the other hand, since $|\xi| < \sqrt{\lambda} \underset{\infty}{\leqslant} \lambda$, we get :

$$|I_3| \leqslant \int_{|\xi| < \sqrt{\lambda}} \exp(-|\xi|^2 t) |\hat{\phi}_0(\xi)| \frac{C|\xi|}{\lambda} d\xi .$$

So, as above :

$$\text{(A.7)} \qquad |I_3| \leqslant \frac{C}{\lambda t} |\phi_0|_{W^{1,n}} .$$

(iv) *Majoration of* I_4.

Thanks to the inequalities (A.2) and (A.3), we have :

$$|I_4| \leqslant \int_{|\xi|>\lambda} \exp(-|\xi|^2 t)(1+|\xi|^2 t) \times$$
$$\times \exp\left(|\xi|^2 t \sqrt{1-\frac{\lambda^2}{|\xi|^2}}\right) |\hat{\phi}_0(\xi)| \, d\xi$$
$$\leqslant \exp\left(-\frac{\lambda^2 t}{2}\right)(1+t) \int_{|\xi|>\lambda} (1+|\xi|^2)|\xi|^{n+1} \left|\hat{\phi}_0(\xi)\right| \frac{d\xi}{|\xi|^{n+1}} .$$

What finally gives the following inequality :

$$\text{(A.8)} \qquad |I_4| \leqslant C \exp\left(-\frac{\lambda^2 t}{2}\right)(1+t)|\phi_0|_{W^{1,n+3}} .$$

(v) At last, let us remark that :

$$|\phi^\lambda|_\infty \leqslant |\phi^\lambda|_{H^{\left[\frac{n}{2}\right]+1}} \leqslant |\phi_0|_{W^{1,n+2}} .$$

We then easily deduce from (A.1), (A.5), (A.6), (A.7) and (A.8) the following result :

$$|\phi^\lambda|_\infty \leqslant \frac{C}{\sqrt{1+\lambda t}} |\phi_0|_{W^{1,n+3}} \quad \text{if} \quad n \geqslant 3 ,$$
$$|\phi^\lambda|_\infty \leqslant \frac{C}{1+\lambda t} |\phi_0|_{W^{1,5}} \quad \text{if} \quad n = 2 .$$

In order to estimate the derivatives in x of ϕ^λ, we just have to do the same work after deriving the linear system (D^λ).

So the theorem is proven.

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