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CHEBYSHEV SPECTRAL APPROXIMATION OF NAVIER-STOKES EQUATIONS IN A TWO DIMENSIONAL DOMAIN (*)

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Résumé — On analyse dans cet article l'estimation de l'erreur commise dans l'approximation pseudo-spectrale de la solution des équations de Navier-Stokes homogènes posées sur un carré. La formulation en fonction de courant de ces équations a été choisie pour plusieurs raisons détaillées dans l'introduction. Les résultats de convergence sont optimaux c'est-à-dire du même ordre que la meilleure approximation polynomiale.

Abstract — We analyse here the convergence of a pseudo-spectral method for the approximation of the homogeneous Navier-Stokes equations over a square. We use the stream function formulation for various reasons detailed in the introduction. We prove optimal convergence rate i.e. of the same order as the best polynomial approximation.

I. INTRODUCTION

We present and analyse here a Chebyshev spectral method for non periodic, steady-state, 2-D incompressible Navier-Stokes equations.

A very large literature exists now concerning the numerical resolution of Navier-Stokes equations by spectral methods (see Voigt-Gottlieb-Hussaini [1] for a survey).

Generally the velocity pressure formulation of the equations is used

$$\begin{cases} \frac{\partial \underline{u}}{\partial t} - \nu \Delta \underline{u} + \underline{u} \cdot \nabla \underline{u} + \nabla p = \underline{f} & \text{in the domain,} \\ \operatorname{div} \underline{u} = 0 & \text{in the domain} \end{cases} \quad (1.0)$$

with often periodic boundary conditions in 1, 2 or 3 directions.

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The great problem with non periodic boundary conditions is the treatment of the pressure. In order to solve numerically (1.0), the various authors propose different strategies for handling separately the velocity and the pressure :

* The most obvious way to obtain an equation for pressure is to take the divergence of the equation of momentum to obtain the following Poisson equation :

$$\Delta p = \operatorname{div} \underline{f} - \operatorname{div} (\underline{u} \cdot \underline{\nabla} \underline{u}) .$$

Various boundary conditions can be imposed then : Orszag-Israeli-Deville [1] have analysed several boundary conditions on pressure. Their conclusion is that numerical instabilities lead to prefer conditions with no physical meaning. These methods are compared in Deville-Kleiser-Montigny [1].

The best suited type of boundary conditions is :

$$\operatorname{div} \underline{u} = 0 .$$

It has been used by Kleiser-Schumann [1] and Lequeré-Alziary de Roquefort [1] but requires the inversion of the so called influence-matrix which is large, full and ill-conditioned. This method is then difficult to use in 3-D non periodic situations.

* Another strategy, that can be used in 3-D non periodic curved domains is proposed by Métivet-Morchoisne [1] (see also Métivet [1]). It consists in an iterative method based on a minimization of divergence at each time step that does not involve boundary conditions over the pressure, this one being considered as a Lagrangien multiplicator. A method with no boundary conditions over the pressure is also proposed in Malik-Zang and Hussaini [1] for a 2-D periodic/non periodic problem.

* A last treatment consists in the elimination of the pressure. A clever choice of divergence free velocity expansion functions can be used then, if directions of periodicity exist (see Mozer-Moin-Leonard [1]). But it seems difficult to be generalized to pure non periodic boundary conditions.

In fact, the difficulty encountered here, due the compatibility conditions between velocity and pressure, is well known in finite element method and lies partially in a good choice for velocity and pressure discrete spaces. This condition is known as “ inf-sup condition ” (see Girault-Raviart [1]). This problem, in spectral framework, has been analysed in Bernardi-Maday-Métivet [2].

However, there exist two other presentations of 2-D incompressible Navier-Stokes equations that do not involve such a problem : the stream function-vorticity formulation that requires boundary conditions on vorticity, and

the stream function formulation. The last one has been used for numerical resolution in Morchoisne [1] and its extension to 3-D case is studied.

In this paper we analyse the 2-D approximation of this formulation :

Find Ψ , defined over $\Omega =]-1, 1[^2$, such that :

$$\begin{cases} \nu \Delta^2 \Psi + \frac{\partial}{\partial x_1} \left((-\Delta \Psi) \frac{\partial \Psi}{\partial x_2} \right) - \frac{\partial}{\partial x_2} \left((-\Delta \Psi) \frac{\partial \Psi}{\partial x_1} \right) = f, \text{ in } \Omega, \\ \Psi = \frac{\partial \Psi}{\partial n} = 0 \text{ over } \Gamma \equiv \partial \Omega. \end{cases} \quad (1.1)$$

Note that problem (1.1) is well fitted to spectral approximations since their accuracy increases with the regularity of the function (Ψ is more regular than u). We prove here that the collocation method leads to a discrete solution Ψ_N asymptotically as close to Ψ as the best polynomial approximation of Ψ .

For other theoretical analysis concerning spectral approximations of Navier-Stokes equations we refer to Maday-Quarteroni [1], Bernardi-Maday-Métivet [1], [2] and Bernardi-Canuto-Maday [1] for pure spectral methods and Canuto-Maday-Quarteroni [1] for a combined finite element and spectral method.

In section 2 we recall some theoretical results concerning the approximation by Chebyshev spectral methods.

In section 3 we consider the approximation of the Stokes problem and section 4 deals with the Navier-Stokes problem. Optimal error bounds are proved. The analysis is based on the use of results of Descloux-Rappaz [1] on the approximation of branches of non singular solution of P.D.E. . We indicate in the appendix a suitable version of their theorems 3.1, 3.2.

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II. NOTATIONS AND DEFINITIONS

Let $I =]-1, +1[$ and $\Omega = I \times I$. A point of I (resp. Ω) is denoted by x (resp. $\underline{x} = (x_1, x_2)$) and $\Gamma = \partial \Omega$.

Let $\Phi \in \mathcal{D}'(\Omega)$; then $D^\alpha \Phi$ means $\frac{\partial^{|\alpha|} \Phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \Phi$ for any $\alpha = (\alpha_1, \alpha_2)$ in \mathbb{N}^2 , and $|\alpha| = \alpha_1 + \alpha_2$.

We consider the weight function $\omega(x) = (1 - x^2)^{-1/2}$, $x \in I$, associated with the Chebyshev polynomials, and $\underline{\omega}(\underline{x}) = \omega(x_1) \omega(x_2)$, $\underline{x} \in \Omega$. We define the

weighted Sobolev spaces $H_{\underline{\omega}}^s(\Omega)$ as follows :

(i) $H_{\underline{\omega}}^0(\Omega) = L_{\underline{\omega}}^2(\Omega) \equiv \{ \Phi : \Omega \rightarrow \mathbb{R} \mid \Phi \text{ is measurable and } (\Phi, \Phi)_{0, \underline{\omega}} < \infty \}$,

where $(\Phi, \chi)_{0, \underline{\omega}} = \int_{\Omega} \Phi(\underline{x}) \chi(\underline{x}) \underline{\omega}(\underline{x}) \, d\underline{x}$,

(ii) for any $s \in \mathbb{N}^*$:

$$H_{\underline{\omega}}^s(\Omega) = \{ \Phi \in L_{\underline{\omega}}^2(\Omega) \mid D^\alpha \Phi \in L_{\underline{\omega}}^2(\Omega); \quad \forall \alpha \in \mathbb{N}^2, \quad |\alpha| \leq s \}.$$

These spaces are Hilbert spaces for the scalar product :

$$((\Phi, \chi))_{s, \underline{\omega}} = \sum_{|\alpha| \leq s} (D^\alpha \Phi, D^\alpha \chi)_{0, \underline{\omega}},$$

(iii) for any $s \in \mathbb{R}^+ \setminus \mathbb{N}$, $H_{\underline{\omega}}^s(\Omega)$ is defined by interpolation of index $s - \bar{s}$ between $H_{\underline{\omega}}^{\bar{s}}(\Omega)$ and $H_{\underline{\omega}}^{\bar{s}+1}(\Omega)$ where \bar{s} is the greatest integer $\leq s$ (see Bergh-Löfström [1] for the definition of interpolation).

The norm of $H_{\underline{\omega}}^s(\Omega)$ will be denoted by $\| \cdot \|_{s, \underline{\omega}}$ in the sequel. For any $s \in \mathbb{R}^+$ we denote by $H_{0, \underline{\omega}}^s(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H_{\underline{\omega}}^s(\Omega)$. Let us recall that :

* For any $s \in \mathbb{R}^+ \setminus \mathbb{N} \cup \left\{ \mathbb{N} + \frac{1}{4} \right\}$, $H_{0, \underline{\omega}}^s(\Omega)$ is the interpolate space of index $s - \bar{s}$ between $H_{0, \underline{\omega}}^{\bar{s}}(\Omega)$ and $H_{0, \underline{\omega}}^{\bar{s}+1}(\Omega)$.

* For any $s \in \mathbb{N}$:

$$\Phi \rightarrow \|\Phi\|_{s, \underline{\omega}} = \left(\sum_{|\alpha|=s} (D^\alpha \Phi, D^\alpha \Phi)_{0, \underline{\omega}} \right)^{1/2}, \tag{2.1}$$

is a norm over $H_{0, \underline{\omega}}^s(\Omega)$ equivalent to the norm $\| \cdot \|_{s, \underline{\omega}}$ (see Grisvard [1] for more details).

If X and Y are two Banach spaces such as $X \subset Y$ then $X \hookrightarrow Y$ (resp. $X \xhookrightarrow{c} Y$) will mean that the identity mapping is continuous (resp. compact) from X into Y . We recall now some rather simple properties (see Maday [3]) :

$$H_{\underline{\omega}}^s(\Omega) \xhookrightarrow{c} C^0(\overline{\Omega}) \quad \text{for any } s > 1, \tag{2.2}$$

$$H^{s+1/2}(\Omega) \xhookrightarrow{c} H_{\underline{\omega}}^s(\Omega) \quad \text{for any } s \geq 0, \tag{2.3}$$

$$H_{\underline{\omega}}^s(\Omega) \text{ is an algebra for any } s > 1. \tag{2.4}$$

Next we introduce some notations and results commonly used in spectral methods. Let $\mathbb{P}_N(I)$ (resp. $\mathbb{P}_N(\Omega)$) denote the space of all polynomials over \mathbb{R} of degree $\leq N$ (resp. over \mathbb{R}^2 of degree $\leq N$ in each variable).

The results we will mention now are valid for $\theta = I$ or $\theta = \Omega$, the various norm $\| \cdot \|_{s,\omega}$ and scalar product $(\cdot, \cdot)_{0,\omega}$ standing for the one previously defined if $\theta = \Omega$, and for those defined in the same way for $\theta = I$.

For any $(\mu, \nu) \in \mathbb{R}^2, 0 \leq \mu \leq \nu$, there exists a positive constant C such that : (inverse inequality)

$$\| \Phi \|_{\nu,\omega} \leq CN^{2(\nu-\mu)} \| \Phi \|_{\mu,\omega}, \quad \forall \Phi \in \mathbb{P}_N(\theta), \quad (2.5)$$

Besides, let $\Pi_{0,N}$ denote the orthogonal projection operator from $L^2_\omega(\theta)$ onto $\mathbb{P}_N(\theta)$, we have :

$$\| \Phi - \Pi_{0,N} \Phi \|_{0,\omega} \leq CN^{-\sigma} \| \Phi \|_{\sigma,\omega}, \quad \forall \Phi \in H^\sigma_\omega(\theta), \quad (2.6)$$

(see Canuto-Quarteroni [1] for the proofs of (2.5) and (2.6)). Let $V = H^2_{0,\omega}(\theta)$ and $V_N = V \cap \mathbb{P}_N(\theta)$. We recall that, for any real $r \geq 2$, there exists an operator $\Pi_{r,N}$ from $H^r_\omega(\theta) \cap V$ onto V_N such that :

$$\| \Phi - \Pi_{r,N} \Phi \|_{\mu,\omega} \leq CN^{\mu-\nu} \| \Phi \|_{\nu,\omega}, \quad \forall \Phi \in H^\nu_\omega(\theta) \cap V, \\ 0 \leq \mu \leq r \leq \nu. \quad (2.7)$$

(see Maday [1] and [3]).

Let $F_{\omega,N}^{GL} = \{ (\xi_i, \omega_i) \mid 0 \leq i \leq N \}$, (resp. $F_{\omega,N}^{GL} = \{ (\xi_{ij}, \omega_{ij}) \mid 0 \leq i, j \leq N \}$), be the set of abscissae and weights of the Gauss Lobatto quadrature formula of order $2N - 1$ associated with the weight ω (resp. ω).

From the definition we have the following properties :

$$\xi_{ij} = (\xi_i, \xi_j) \quad \text{and} \quad \omega_{ij} = \omega_i \omega_j, \quad 0 \leq i, j \leq N,$$

for any pair $(\Phi, \chi) \in C^0(\bar{\theta})^2$ such that $\Phi \chi \in \mathbb{P}_{2N-1}(\theta)$ we have :

$$\sum_{i \in I(N)} \Phi(\xi_i) \chi(\xi_i) \omega_i = \int_\theta \Phi(x) \chi(x) \omega(x) dx = (\Phi, \chi)_{0,\omega}, \quad (2.8)$$

(here, the notation $I(N)$ stands for $\{ 0, 1, \dots, N \}$ if $\theta = I$ and for $\{ 0, 1, \dots, N \} \times \{ 0, 1, \dots, N \}$ if $\theta = \Omega$, in this latter case i is an element of \mathbb{N}^2).

Let us set :

$$(\Phi, \chi)_{\omega,N} = \sum_{i \in I(N)} \Phi(\xi_i) \chi(\xi_i) \omega_i, \quad \forall (\Phi, \chi) \in C^0(\bar{\theta})^2. \quad (2.9)$$

This bilinear form is a scalar product over $\mathbb{P}_N(\theta)$. The associated norm is denoted $\| \cdot \|_{\omega,N}$, and verifies the following property (see Canuto-Quarteroni [1]) :

$$\| \Phi \|_{0,\omega} \leq \| \Phi \|_{\omega,N} \leq 4 \| \Phi \|_{0,\omega}, \quad \forall \Phi \in \mathbb{P}_N(\theta). \quad (2.10)$$

Let us now introduce the interpolation operator P_N (resp. \underline{P}_N) at the point ξ_i (resp. $\underline{\xi}_{ij}$) defined by :

$$\begin{aligned} P_N : C^0(\bar{I}) &\rightarrow \mathbb{P}_N(I) \quad (\text{resp. } \underline{P}_N : C^0(\bar{\Omega}) \rightarrow \mathbb{P}_N(\Omega)) \\ P_N(\varphi)(\xi_i) &= \varphi(\xi_i), \quad 0 \leq i \leq N \quad (\text{resp. } \underline{P}_N(\Phi)(\underline{\xi}_{ij}) = \Phi(\underline{\xi}_{ij}), \\ & \hspace{15em} 0 \leq i, j \leq N). \end{aligned} \quad (2.11)$$

These operators verify : (we use again an unique notation for the one and two dimension cases)

$$\| \Phi - P_N \Phi \|_{0,\omega} \leq CN^{-\sigma} \| \Phi \|_{\sigma,\omega}, \quad \forall \Phi \in H_\omega^\sigma(\theta), \quad (2.12)$$

with $\sigma > 1/2$ if $\theta = I$ and $\sigma > 1$ if $\theta = \Omega$.

Moreover, for any $\Phi \in C^0(\bar{\theta})$, $\chi \in \mathbb{P}_N(\theta)$:

$$|(\Phi, \chi)_{\omega,N} - (\Phi, \chi)_{0,\omega}| \leq C \| \chi \|_{0,\omega} (\| \Phi - P_N \Phi \|_{0,\omega} + \| \Phi - \Pi_{0,N-1} \Phi \|_{0,\omega}) \quad (2.13)$$

(see Maday-Quarteroni [3]).

III. SOME RESULTS CONCERNING THE CONTINUOUS PROBLEM

III.1. The biharmonic problem

Let us first consider the following biharmonic problem :

Given g , find Ψ such that :

$$\begin{cases} \Delta^2 \Psi = g & \text{in } \Omega \\ \Psi = 0, \quad \frac{\partial \Psi}{\partial n} = 0 & \text{on } \partial \Omega. \end{cases} \quad (3.1)$$

In order to analyse a spectral approximation of that problem we want to find a solution Ψ in $V = H_{0,\omega}^2(\Omega)$. A necessary condition for solving problem (3.1) is that g (being the laplacian of an element of L_ω^2) belongs to the space $\mathcal{H}_\omega^{-2}(\Omega)$ defined by :

$$\mathcal{H}_\omega^{-2}(\Omega) = \left\{ f = \sum_{|\alpha| \leq 2} D^\alpha g_\alpha, g_\alpha \in L_\omega^2(\Omega); \alpha \in \mathbb{N}^2 \right\}. \quad (3.2)$$

The main result of this section states that this condition is also sufficient.

Let us first recall a preliminary one-dimensional result which can be found in Maday [1].

LEMMA 3.1 : *There exist positive constants C_1 and C_2 such that, for any $\varphi \in H^2_{0,\omega}(I)$,*

$$\int_I \varphi^2 \omega^9 dx \leq C_1 \int_I \varphi'^2 \omega^5 dx \leq C_2 \int_I \varphi''^2 \omega dx. \tag{3.3}$$

Moreover there exist positive constants C'_1 and C'_2 such that, for any $(\varphi, \chi) \in \mathcal{D}^2(I)$:

$$\left| \int_I \varphi(\chi\omega)'' dx \right| \leq C'_1 \|\varphi\|_{0,\omega} \|\chi''\|_{0,\omega}, \tag{3.4}$$

$$\left| \int_I \varphi''(\varphi\omega)'' dx \right| \geq C'_2 \|\varphi''\|_{0,\omega}^2. \tag{3.5}$$

These results will now be extended to the 2-D case.

LEMMA 3.2 : *There exist two positive constants α and β such that, for any $(\Phi, \chi) \in V^2$:*

$$\int_{\Omega} \Delta\Phi \Delta(\chi\omega) d\underline{x} \leq \alpha \|\Phi\|_{2,\omega} \|\chi\|_{2,\omega}; \tag{3.6}$$

$$\int_{\Omega} \Delta\Phi \Delta(\Phi\omega) d\underline{x} \geq \beta \|\Phi\|_{2,\omega}^2. \tag{3.7}$$

Proof: Let Φ and χ be in $\mathcal{D}(\Omega)$. Then, $\Phi\omega$ and $\chi\omega$ belong also to $\mathcal{D}(\Omega)$ and we can write :

$$\int_{\Omega} \Delta\Phi \Delta(\chi\omega) d\underline{x} = A_{20} + 2 A_{11} + A_{02}, \tag{3.8}$$

where
$$A_{ij} = \int_{\Omega} D^{(i,j)} \Phi D^{(i,j)}(\chi\omega) d\underline{x}.$$

We have : (we precise here by ω_i the factor in ω which depends on $x_i, i=1, 2$)

$$A_{20} = \int_I \omega_2 \left[\int_I \frac{\partial^2 \Phi}{\partial x_1^2} \frac{\partial^2 (\chi\omega_1)}{\partial x_1^2} dx_1 \right] dx_2.$$

Using (3.4) we obtain :

$$|A_{20}| \leq C'_1 \int_I \omega_2 \left\| \frac{\partial^2 \Phi}{\partial x_1^2} \right\|_{0,\omega_1} \left\| \frac{\partial^2 \chi}{\partial x_1^2} \right\|_{0,\omega_1} dx_2,$$

so that, from the Cauchy-Schwarz's inequality we get :

$$|A_{20}| \leq C'_1 \|\Phi\|_{2,\underline{\omega}} \|\chi\|_{2,\underline{\omega}}. \quad (3.9)$$

Similarly, we find :

$$|A_{02}| \leq C'_1 \|\Phi\|_{2,\underline{\omega}} \|\chi\|_{2,\underline{\omega}}. \quad (3.10)$$

and :

$$|A_{11}| \leq \left\| \frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \right\|_{0,\underline{\omega}} \left\| \frac{1}{\underline{\omega}} \frac{\partial^2(\chi \underline{\omega})}{\partial x_1 \partial x_2} \right\|_{0,\underline{\omega}}. \quad (3.11)$$

Since $\frac{\partial \omega_i}{\partial x_i} = x_i \omega_i^3$ we obtain :

$$\begin{aligned} \left\| \frac{1}{\underline{\omega}} \frac{\partial^2(\chi \underline{\omega})}{\partial x_1 \partial x_2} \right\|_{0,\underline{\omega}} &\leq C \left[\left\| \frac{\partial^2 \chi}{\partial x_1 \partial x_2} \right\|_{0,\underline{\omega}} + \left(\int_{\Omega} \left(\frac{\partial \chi}{\partial x_2} \right)^2 \omega_1^5 \omega_2 \, d\underline{x} \right)^{1/2} + \right. \\ &\quad \left. + \left(\int_{\Omega} \left(\frac{\partial \chi}{\partial x_1} \right)^2 \omega_2^5 \omega_1 \, d\underline{x} \right)^{1/2} + \left(\int_{\Omega} \chi^2 \underline{\omega}^5 \, d\underline{x} \right)^{1/2} \right]. \end{aligned}$$

Now using (3.3) we find :

$$\left\| \frac{1}{\underline{\omega}} \frac{\partial^2(\chi \underline{\omega})}{\partial x_1 \partial x_2} \right\|_{0,\underline{\omega}} \leq C \left\| \frac{\partial^2 \chi}{\partial x_1 \partial x_2} \right\|_{0,\underline{\omega}}. \quad (3.12)$$

Thus, the estimate (3.6) follows from (3.8)-(3.12).

Next, in order to derive the ellipticity condition (3.7) we take $\chi = \Phi$ in (3.8). Using (3.5) we have :

$$\left\{ \begin{array}{l} A_{20} \geq C'_2 \int_I \omega_2 \left(\int_I \left(\frac{\partial^2 \Phi}{\partial x_1^2} \right)^2 \omega_1 \, dx_1 \right) dx_2 \geq C_2 \left\| \frac{\partial^2 \Phi}{\partial x_1^2} \right\|_{0,\underline{\omega}}^2, \\ \text{and} \\ A_{02} \geq C'_2 \int_I \omega_1 \left(\int_I \left(\frac{\partial^2 \Phi}{\partial x_2^2} \right)^2 \omega_2 \, dx_2 \right) dx_1 \geq C_2 \left\| \frac{\partial^2 \Phi}{\partial x_2^2} \right\|_{0,\underline{\omega}}^2. \end{array} \right. \quad (3.13)$$

Let us now check that the term A_{11} is nonnegative. We set :

$$J = \int_{\Omega} \left(\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \underline{\omega} + x_2 \frac{\partial \Phi}{\partial x_1} \omega_1 \omega_2^3 + x_1 \frac{\partial \Phi}{\partial x_2} \omega_1^3 \omega_2 + \frac{3}{2} x_1 x_2 \Phi \underline{\omega}^3 \right)^2 \underline{\omega}^{-1} d\underline{x}.$$

Since :

$$\frac{\partial \omega_i}{\partial x_i} = x_i \omega_i^3 \quad \text{and} \quad \frac{\partial^2 \omega_i}{\partial x_i^2} = (1 + 2 x_i^2) \omega_i^5, \tag{3.14}$$

an elementary calculation shows that :

$$\begin{aligned} J = A_{11} + \int_{\Omega} & \left[\frac{1}{4} x_1^2 x_2^2 + \frac{1}{8} x_1^2 + \frac{1}{8} x_2^2 - \frac{1}{2} \right] \Phi^2 \underline{\omega}^5 d\underline{x} + \\ & - \frac{1}{2} \int_{\Omega} \left[\frac{\partial \Phi}{\partial x_1} \omega_1^{1/2} \omega_2^{5/2} + \frac{3}{2} x_1 \Phi \underline{\omega}^{5/2} \right]^2 d\underline{x} + \\ & - \frac{1}{2} \int_{\Omega} \left[\frac{\partial \Phi}{\partial x_2} \omega_1^{5/2} \omega_2^{1/2} + \frac{3}{2} x_2 \Phi \underline{\omega}^{5/2} \right]^2 d\underline{x}. \end{aligned}$$

But we note that, for $\underline{x} \in \Omega$:

$$\left[\frac{1}{4} x_1^2 x_2^2 + \frac{1}{8} x_1^2 + \frac{1}{8} x_2^2 - \frac{1}{2} \right] \leq 0.$$

Since J is non negative we obtain :

$$A_{11} \geq 0. \tag{3.15}$$

Thus it follows from (3.8) and (3.13) that :

$$\int_{\Omega} \Delta \Phi \Delta(\Phi \underline{\omega}) d\underline{x} \geq C_2 \left(\left\| \frac{\partial^2 \Phi}{\partial x_1^2} \right\|_{0, \underline{\omega}}^2 + \left\| \frac{\partial^2 \Phi}{\partial x_2^2} \right\|_{0, \underline{\omega}}^2 \right). \tag{3.16}$$

For proving (3.7), it remains to check that there exists $C' \geq 0$ such that :

$$\int_{\Omega} \left(\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \right)^2 \underline{\omega} d\underline{x} \leq C' \left(\int_{\Omega} \left[\left(\frac{\partial^2 \Phi}{\partial x_1^2} \right)^2 \underline{\omega} + \left(\frac{\partial^2 \Phi}{\partial x_2^2} \right)^2 \underline{\omega} \right] d\underline{x} \right). \tag{3.17}$$

In fact, integrating by parts yields :

$$\int_{\Omega} \left(\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \right)^2 \underline{\omega} \underline{dx} = \int_{\Omega} \frac{\partial}{\partial x_1} \left(\frac{\partial \Phi}{\partial x_1} \omega_1 \right) \frac{\partial}{\partial x_2} \left(\frac{\partial \Phi}{\partial x_2} \omega_2 \right) \underline{dx},$$

hence, we have :

$$\left\{ \int_{\Omega} \left(\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \right)^2 \underline{\omega} \underline{dx} \leq \frac{1}{2} \left(\int_{\Omega} \left[\frac{\partial}{\partial x_1} \left(\frac{\partial \Phi}{\partial x_1} \omega_1 \right) \right]^2 \frac{\omega_2}{\omega_1} \underline{dx} + \int_{\Omega} \left[\frac{\partial}{\partial x_2} \left(\frac{\partial \Phi}{\partial x_2} \omega_2 \right) \right]^2 \frac{\omega_1}{\omega_2} \underline{dx} \right). \quad (3.18) \right.$$

Using (3.14) we find :

$$\frac{\partial}{\partial x_1} \left(\frac{\partial \Phi}{\partial x_1} \omega_1 \right) = \frac{\partial^2 \Phi}{\partial x_1^2} \omega_1 + \frac{\partial \Phi}{\partial x_1} x_1 \omega_1^3,$$

hence we have :

$$\begin{aligned} \int_{\Omega} \left[\frac{\partial}{\partial x_1} \left(\frac{\partial \Phi}{\partial x_1} \omega_1 \right) \right]^2 \frac{\omega_2}{\omega_1} \underline{dx} &= \int_{\Omega} \left[\left(\frac{\partial^2 \Phi}{\partial x_1^2} \right)^2 \underline{\omega} + 2 \frac{\partial^2 \Phi}{\partial x_1^2} \frac{\partial \Phi}{\partial x_1} x_1 \omega_1^3 \omega_2 + \right. \\ &\quad \left. + x_1^2 \left(\frac{\partial \Phi}{\partial x_1} \right)^2 \omega_1^5 \omega_2 \right] \underline{dx} \\ &= \int_{\Omega} \left[\left(\frac{\partial^2 \Phi}{\partial x_1^2} \right)^2 \underline{\omega} + \frac{\partial}{\partial x_1} \left(\frac{\partial \Phi}{\partial x_1} \right)^2 x_1 \omega_1^3 \omega_2 + x_1^2 \left(\frac{\partial \Phi}{\partial x_1} \right)^2 \omega_1^5 \omega_2 \right] \underline{dx}. \end{aligned}$$

Integrating by parts and using (3.14) once more we obtain :

$$\begin{aligned} \int_{\Omega} \left[\frac{\partial}{\partial x_1} \left(\frac{\partial \Phi}{\partial x_1} \omega_1 \right) \right]^2 \frac{\omega_2}{\omega_1} \underline{dx} &= \int_{\Omega} \left[\left(\frac{\partial^2 \Phi}{\partial x_1^2} \right)^2 \underline{\omega} \right. \\ &\quad \left. - (1 + x_1^2) \left(\frac{\partial \Phi}{\partial x_1} \right)^2 \omega_1^5 \omega_2 \right] \underline{dx}. \quad (3.19) \end{aligned}$$

Similarly we find :

$$\int_{\Omega} \left[\frac{\partial}{\partial x_2} \left(\frac{\partial \Phi}{\partial x_2} \omega_2 \right) \right]^2 \frac{\omega_1}{\omega_2} \underline{dx} = \int_{\Omega} \left[\left(\frac{\partial^2 \Phi}{\partial x_2^2} \right)^2 \underline{\omega} - (1 + x_2^2) \left(\frac{\partial \Phi}{\partial x_2} \right)^2 \omega_1 \omega_2^5 \right] \underline{dx}. \quad (3.19')$$

We derive the estimate (3.17) with $C' = \frac{1}{2}$ from (3.18)-(3.19). The ellipticity condition (3.7) follows from (3.16), (3.17) and the property (2.1). ■

Let us now consider an element \hat{g} of $H_{\underline{\omega}}^{-2}(\Omega) = (H_{0,\underline{\omega}}^2(\Omega))'$. It follows from lemma 3.2 and Lax-Milgram theorem that there exists a unique solution Ψ of problem :

find $\Psi \in V$ such that :

$$a(\Psi, \chi) = \langle \hat{g}, \chi \rangle, \quad \forall \chi \in V, \tag{3.20}$$

where $a(\Psi, \chi) = \int_{\Omega} \Delta \Psi \Delta(\chi \underline{\omega}) \, d\underline{x}$ and $\langle \cdot, \cdot \rangle$ stands for the $H_{\underline{\omega}}^{-2}(\Omega) \times H_{0,\underline{\omega}}^2(\Omega)$ duality pairing.

Taking $\chi \in \mathcal{D}(\Omega)$ in (3.20) we deduce that the solution Ψ of (3.20) verifies :

$$(\Delta^2 \Psi) \underline{\omega} = \hat{g},$$

(this equality holds in $\mathcal{D}'(\Omega)$). Hence, for any \hat{g} in $H_{\underline{\omega}}^{-2}(\Omega)$, there exists a unique Ψ of V such that

$$\Delta^2 \Psi = \frac{\hat{g}}{\underline{\omega}}, \tag{3.21}$$

this equality holds in $\mathcal{D}'(\Omega)$ and more precisely in $\mathcal{H}_{\underline{\omega}}^{-2}(\Omega)$.

For any $g \in \mathcal{H}_{\underline{\omega}}^{-2}(\Omega)$ we know that there exists $(g_{\alpha})_{|\alpha| \leq 2}$, $g_{\alpha} \in L_{\underline{\omega}}^2(\Omega)$ such that $g = \sum_{|\alpha| \leq 2} D^{\alpha} g_{\alpha}$. This element can be associated with \tilde{g} of $H_{\underline{\omega}}^{-2}(\Omega)$ as follows (remind (3.4)) :

$$\langle \tilde{g}, \Phi \rangle = \sum_{|\alpha| \leq 2} \int_{\Omega} (-1)^{|\alpha|} g_{\alpha} D^{\alpha}(\Phi \underline{\omega}) \, d\underline{x}, \quad \forall \Phi \in V, \tag{3.22}$$

\tilde{g} is independent of the decomposition $g = \sum_{|\alpha| \leq 2} D^{\alpha} g_{\alpha}$. Indeed if there exist g_{α} , g'_{α} in $L_{\underline{\omega}}^2(\Omega)$, $|\alpha| \leq 2$ such that :

$$\sum_{|\alpha| \leq 2} D^{\alpha} g_{\alpha} = \sum_{|\alpha| \leq 2} D^{\alpha} g'_{\alpha},$$

this equality holding in $\mathcal{D}'(\Omega)$; then, for any χ in $\mathcal{D}(\Omega)$

$$\sum_{|\alpha| \leq 2} \int_{\Omega} (-1)^{|\alpha|} g_{\alpha} D^{\alpha} \chi \, d\underline{x} = \sum_{|\alpha| \leq 2} \int_{\Omega} (-1)^{|\alpha|} g'_{\alpha} D^{\alpha} \chi \, d\underline{x}$$

this result remains true if we take $\chi = \Phi_{\underline{\omega}}$ with $\Phi \in \mathcal{D}(\Omega)$; by density we then derive that \tilde{g} is independent of the decomposition of g .

The previous remark and (3.21) prove that the mapping of $\mathcal{D}'(\Omega)$:

$$h \mapsto \frac{h}{\underline{\omega}},$$

defines an isomorphism from $H_{\underline{\omega}}^{-2}(\Omega)$ onto $\mathcal{H}_{\underline{\omega}}^{-2}(\Omega)$; hence, we can solve problem (3.1) in V iff $g \in \mathcal{H}_{\underline{\omega}}^{-2}(\Omega)$. This enables us to define an operator $T : g \in \mathcal{H}_{\underline{\omega}}^{-2}(\Omega) \mapsto Tg \in V$ by :

$$\alpha(Tg, \chi) = \langle \tilde{g}, \chi \rangle, \quad \forall \chi \in V. \tag{3.23}$$

THEOREM 3.1 : *The linear operator T is bounded from $\mathcal{H}_{\underline{\omega}}^{-2}(\Omega)$ into V and is a linear compact operator from $H^{-s}(\Omega)$ into V for any s such that $0 < s < \frac{3}{2}$.*

Remark 3.1 : The space $\mathcal{H}_{\underline{\omega}}^{-2}(\Omega)$ is now equipped with the norm :

$$\|g\|_{\mathcal{H}_{\underline{\omega}}^{-2}(\Omega)} = \|\tilde{g}\|_{H_{\underline{\omega}}^{-2}(\Omega)}.$$

Proof : The first part of this theorem is an easy consequence of Lax-Milgram theorem.

Let $g \in H^{-s}(\Omega)$. It follows from Grisvard [2] or more easily from Bernardi-Raugel [1] that there exists an f in $H^{4-s}(\Omega) \cap H_0^2(\Omega)$ such that :

$$\Delta^2 f = g, \tag{3.24}$$

and

$$\|f\|_{H^{4-s}(\Omega)} \leq C \|g\|_{H^{-s}(\Omega)}. \tag{3.25}$$

Hence T is continuous from $H^{-s}(\Omega)$ into $H^{4-s}(\Omega)$. It is well known (see Adams [1]) that $H^{4-s}(\Omega) \xrightarrow{c} H^{5/2}(\Omega)$. From (2.3) we derive that

$$H^{5/2}(\Omega) \xrightarrow{c} H_{\underline{\omega}}^2(\Omega).$$

T is then compact from $H^{-s}(\Omega)$ into $H_{\underline{\omega}}^2(\Omega)$ and the theorem is proved.

III.2. The Navier-Stokes problem

Let B be defined by : for any $(\Phi, \chi) \in \mathcal{D}(\Omega)^2$:

$$B(\Phi, \chi) = -\frac{1}{2} \left[\frac{\partial}{\partial x_1} \left(\Delta \Phi \frac{\partial \chi}{\partial x_2} + \Delta \chi \frac{\partial \Phi}{\partial x_2} \right) - \frac{\partial}{\partial x_2} \left(\Delta \Phi \frac{\partial \chi}{\partial x_1} + \Delta \chi \frac{\partial \Phi}{\partial x_1} \right) \right], \tag{3.26}$$

and, for any $(\lambda, \Phi) \in \mathbb{R} \times \mathcal{D}(\Omega), f \in \mathcal{D}'(\Omega)$:

$$H_f(\lambda, \Phi) = \lambda[B(\Phi, \Phi) - f]. \tag{3.27}$$

From the next lemma we see that, the Navier-Stokes equations (1.1) consist in finding $\Psi \in V$ such that :

$$\Delta^2 \Psi = - H_f(\lambda_0, \Psi), \tag{3.28}$$

with

$$\lambda_0 = \frac{1}{\nu}.$$

LEMMA 3.3 : Let us assume that $1 < s < 3/2$; then :

i) There exists a constant $\gamma > 0$ such that, for any (Φ, χ) in $\mathcal{D}(\Omega)^2$:

$$\| B(\Phi, \chi) \|_{H^{-s}(\Omega)} \leq \gamma \| \Phi \|_{2, \underline{\omega}} \| \chi \|_{2, \underline{\omega}}. \tag{3.29}$$

ii) B can be extended in a continuous mapping from V^2 into $H^{-s}(\Omega)$.

Proof : Let (Φ, χ, Ξ) in $\mathcal{D}(\Omega)^3$, from (3.26) we obtain :

$$\left\{ \begin{aligned} \int_{\Omega} 2 B(\Phi, \chi) \Xi \, d\underline{x} &= \int_{\Omega} \frac{\partial^2 \Phi}{\partial x_1^2} \frac{\partial \chi}{\partial x_2} \frac{\partial \Xi}{\partial x_1} \, d\underline{x} + \int_{\Omega} \frac{\partial^2 \Phi}{\partial x_2^2} \frac{\partial \chi}{\partial x_2} \frac{\partial \Xi}{\partial x_1} \, d\underline{x} + \\ &+ \int_{\Omega} \frac{\partial^2 \chi}{\partial x_1^2} \frac{\partial \Phi}{\partial x_2} \frac{\partial \Xi}{\partial x_1} \, d\underline{x} + \int_{\Omega} \frac{\partial^2 \chi}{\partial x_2^2} \frac{\partial \Phi}{\partial x_2} \frac{\partial \Xi}{\partial x_1} \, d\underline{x} \\ &- \int_{\Omega} \frac{\partial^2 \Phi}{\partial x_1^2} \frac{\partial \chi}{\partial x_1} \frac{\partial \Xi}{\partial x_2} \, d\underline{x} - \int_{\Omega} \frac{\partial^2 \Phi}{\partial x_2^2} \frac{\partial \chi}{\partial x_1} \frac{\partial \Xi}{\partial x_2} \, d\underline{x} \\ &- \int_{\Omega} \frac{\partial^2 \chi}{\partial x_1^2} \frac{\partial \Phi}{\partial x_1} \frac{\partial \Xi}{\partial x_2} \, d\underline{x} - \int_{\Omega} \frac{\partial^2 \chi}{\partial x_2^2} \frac{\partial \Phi}{\partial x_1} \frac{\partial \Xi}{\partial x_2} \, d\underline{x}. \end{aligned} \right. \tag{3.30}$$

We shall only consider the first term; the others can be treated in the same way.

Let $q = \frac{2}{s-1}$; then, due to the hypothesis on s , we have $4 < q < \infty$. Therefore, from Hölder's inequality we have :

$$\left\{ \begin{aligned} \left| \int_{\Omega} \frac{\partial^2 \Phi}{\partial x_1^2} \frac{\partial \chi}{\partial x_2} \frac{\partial \Xi}{\partial x_1} \, d\underline{x} \right| &\leq \left| \int_{\Omega} \left| \frac{\partial^2 \Phi}{\partial x_1^2} \right|^2 \, d\underline{x} \right|^{1/2} \left| \int_{\Omega} \left| \frac{\partial \chi}{\partial x_2} \right|^q \, d\underline{x} \right|^{1/q} \\ &\quad \left| \int_{\Omega} \left| \frac{\partial \Xi}{\partial x_1} \right|^{2q/(q-2)} \, d\underline{x} \right|^{(q-2)/2q} \end{aligned} \right. \tag{3.31}$$

Moreover it is well known that (see Adams [1]) :

$$\left| \int_{\Omega} \left| \frac{\partial \chi}{\partial x_2} \right|^q d\underline{x} \right|^{1/q} \leq C \| \chi \|_{H^2(\Omega)},$$

$$\left| \int_{\Omega} \left| \frac{\partial \Xi}{\partial x_1} \right|^{2q/(q-2)} d\underline{x} \right|^{(q-2)/2q} \leq C \| \Xi \|_{H^s(\Omega)}.$$

By (3.31) and the obvious imbedding $H_{\underline{\omega}}^2(\Omega) \hookrightarrow H^2(\Omega)$, we get :

$$\left| \int_{\Omega} \frac{\partial^2 \Phi}{\partial x_1^2} \frac{\partial \chi}{\partial x_2} \frac{\partial \Xi}{\partial x_1} d\underline{x} \right| \leq C \| \Phi \|_{2,\underline{\omega}} \| \chi \|_{2,\underline{\omega}} \| \Xi \|_{H^s(\Omega)},$$

and (3.29) follows. The end of the lemma is derived by a classical density argument. ■

With the previous notations, we have the following result.

LEMMA 3.4 : *Let us consider the problem : given $f \in \mathcal{H}_{\underline{\omega}}^{-2}(\Omega)$,*

$$\begin{cases} \text{find } (\lambda, \Psi) \in \mathbb{R} \times V \text{ such that :} \\ F(\lambda, \Psi) \equiv \Psi + TH_f(\lambda, \Psi) = 0. \end{cases} \tag{3.32}$$

Then, $\Psi \in V$ is a solution of (3.28) if and only if (λ_0, Ψ) is a solution of (3.32).

Using the same kind of arguments as in Lions [1], we can prove that there exists a λ_0 in \mathbb{R} and a compact interval $\Lambda = [\lambda_0 - \delta, \lambda_0 + \delta]$, $\delta > 0$, such that, for any $\lambda \in \Lambda$, problem (3.32) has exactly one solution $(\lambda, \Psi(\lambda))$. We shall denote $\Psi_0 = \Psi(\lambda_0)$ in the sequel, and shall assume :

$$f \in H_{\underline{\omega}}^{\rho}(\Omega), \quad \rho > 1, \tag{3.33}$$

so that $f \in C^0(\overline{\Omega})$ (see (2.2)).

IV. APPROXIMATION OF NAVIER-STOKES EQUATIONS BY A PSEUDO-SPECTRAL METHOD

IV.1. Formulation of the approximate problem

A straightforward calculation gives, from (3.26)

$$B(\Phi, \chi) = -\frac{1}{2} \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \left(\frac{\partial \Phi}{\partial x_1} \frac{\partial \chi}{\partial x_2} + \frac{\partial \Phi}{\partial x_2} \frac{\partial \chi}{\partial x_1} \right) + \left(\frac{\partial^2}{\partial x_1 \partial x_2} \right) \left(\frac{\partial \Phi}{\partial x_1} \frac{\partial \chi}{\partial x_1} - \frac{\partial \Phi}{\partial x_2} \frac{\partial \chi}{\partial x_2} \right),$$

We set, for Φ and χ in $\mathbb{P}_N(\Omega)$ and λ in \mathbb{R} :

$$\left\{ \begin{aligned} \mathring{B}_N(\Phi, \chi) = & -\frac{1}{2} \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \left(\underline{P}_N \left(\frac{\partial \Phi}{\partial x_1} \frac{\partial \chi}{\partial x_2} \right) + \underline{P}_N \left(\frac{\partial \Phi}{\partial x_2} \frac{\partial \chi}{\partial x_1} \right) \right) + \\ & + \frac{\partial^2}{\partial x_1 \partial x_2} \left(\underline{P}_N \left(\frac{\partial \Phi}{\partial x_1} \frac{\partial \chi}{\partial x_1} \right) - \underline{P}_N \left(\frac{\partial \Phi}{\partial x_2} \frac{\partial \chi}{\partial x_2} \right) \right), \end{aligned} \right. \quad (4.1)$$

and :

$$\mathring{H}_{N,f}(\lambda, \Phi) = \lambda(\mathring{B}_N(\Phi, \Phi) - \underline{P}_N f). \quad (4.2)$$

Then, we define the approximate problem as follows : find $\Psi_N \in V_N$ such that, for any Φ in $V_N = P_N^2(\Omega) \cap V$:

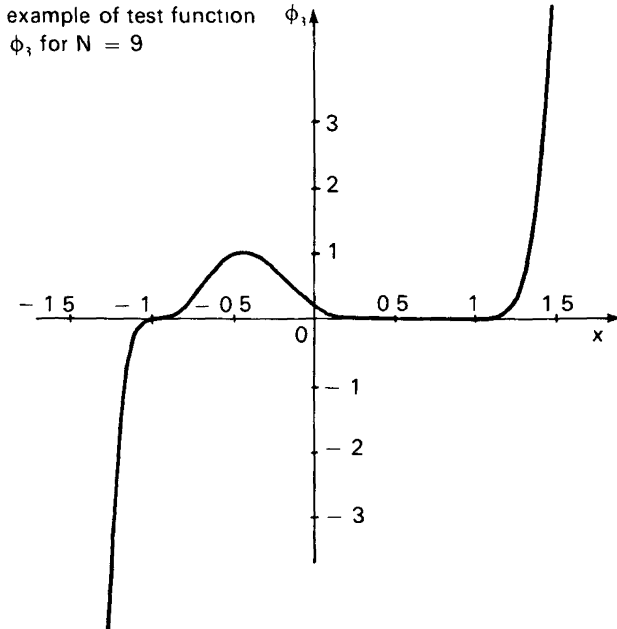
$$(\Delta^2 \Psi_N, \Phi)_{\underline{\omega}, N} + (\mathring{H}_{N,f}(\lambda_0, \Psi_N), \Phi)_{\underline{\omega}, N} = 0. \quad (4.3)$$

Remark 4.1 : Interpretation of the scheme as a collocation method.

Let us define $\Phi_i(x)$, for $i = 2, \dots, N - 2$ as being the element of $\mathbb{P}_N(I)$ such that :

$$\Phi_i(\xi_j) = \delta_{ij}, \quad j = 1, \dots, N - 1, \quad j \neq i - 1 \quad \text{and} \quad j \neq i + 1,$$

and $\Phi_i/(1 - x^2)^2 \in \mathbb{P}_{N-4}(I)$.



We define then Φ_{ij} over Ω by

$$\Phi_{ij}(\underline{x}) = \Phi_i(x_1) \Phi_j(x_2) \text{ for any } i, j, \quad 2 \leq i, j \leq N - 2.$$

This set of elements of $\mathbb{P}_N(\Omega)$ is a basis of V_N and using this basis as test function in (4.3) leads to the following equivalent problem :

$$\begin{cases} \text{find } \Psi_N \in V_N \text{ such that :} \\ \mathcal{S}(\Delta^2 \Psi_N + \mathring{H}_{N,f}(\lambda_0, \Psi_N))(\underline{\xi}_{ij}) = 0 \quad 2 \leq i, j \leq N - 2, \end{cases} \quad (4.3')$$

where for any function χ in $C^0(\bar{\Omega})$ we have posed

$$\mathcal{S}\chi(\underline{\xi}_{ij}) = \sum_{\substack{k=i-1, i, i+1 \\ l=j-1, j, j+1}} \Phi_{ij}(\underline{\xi}_{kl}) \chi(\underline{\xi}_{kl}) \omega_{kl}.$$

The scheme (4.3') would be a collocation scheme for the equation (3.28) if the values $\Phi_{ij}(\underline{\xi}_{kl})$ were equal to zero for $(k, l) \neq (i, j)$. Since it is not the case, it is a collocation-like method involving nine points at the same time.

Some indications for the numerical treatment of the scheme.

First we want to point out that the expression of the nonlinear term we have used is not only a theoretic tool but has been preferred by Basdevant [1] for the approximation of the evolutionary Navier-Stokes equations with periodic boundary conditions. In this latter case the formulation provides an economy of C.P.U. time and input-output.

For the numerical resolution of the problem we suggest an iterative method treating explicitly the non linear terms : Ψ_N is the limit of a sequence $(\Psi_N^n)_{n \in \mathbb{N}}$.

The following method :

$$\mathcal{S}(\Delta^2 \Psi_N^{n+1})(\underline{\xi}_{ij}) = \mathcal{S}[\mathring{H}_{N,f}(\lambda_0, \Psi_N^n)](\underline{\xi}_{ij}) \quad 2 \leq i, j \leq N - 2,$$

would involve the inversion of a full and ill conditioned system. Hence, as it is preconised in the litterature we prefer the use of a finite difference preconditionning Δ_{FD}^2 of the operator Δ^2 (see Orszag [1], for example)

$$\mathcal{S}(\Delta_{FD}^2(\Psi_N^{n+1} - \Psi_N^n))(\underline{\xi}_{ij}) = \varepsilon \mathcal{S}[\Delta^2 \Psi_N^n + \mathring{H}_{N,f}(\lambda_0, \Psi_N^n)](\underline{\xi}_{ij}) \quad 2 \leq i, j \leq N - 2, \quad (4.3'')$$

where ε is a relaxation parameter.

The evaluation of the explicit right hand side of (4.3'') involves two types of calculus : first the values of a product of two functions on the set of $(N + 1)^2$ points $\underline{\xi}_{ij}$; secondly the values of the derivative of a function on this set. The first calculus is very easy to perform in $O(N^2)$ operations. The second one is very expensive ($O(N^4)$ operations) directly, in the " physical space ". It can be done in $O(N^2)$ operations, using recurrence formulae in the space of components of

the functions in the bases of Chebyshev polynomials (the spectral space). The passage from one space the other one is performed via the F.F.T. algorithm in $O(N^2 \log N)$ operations (see Gottlieb-Orszag [1] for more details about this procedure). ■

Let us introduce a discrete bilinear form α_N , defined over V_N by :

$$\alpha_N(\Phi, \chi) = (\Delta^2 \Phi, \chi)_{\omega, N}, \quad \forall (\Phi, \chi) \in V_N^2; \tag{4.4}$$

and an operator $L : \mathbb{P}_N(\Omega) \rightarrow \mathbb{P}_N(\Omega)$ defined by :

$$\int_{\Omega} (L\Phi) \chi \omega \, d\underline{x} = (\Phi, \chi)_{\omega, N}, \quad \forall (\Phi, \chi) \in V_N^2.$$

Let us set :

$$B_N = LB_N^{\circ}, \quad H_{N,f} = LH_{N,f}^{\circ}. \tag{4.5}$$

Then, an equivalent formulation of problem (4.3) is :

$$\left\{ \begin{array}{l} \text{Find } \Psi_N \in V_N, \text{ such that, for any } \Phi \in V_N : \\ \alpha_N(\Psi_N, \Phi) + \langle \widetilde{H_{N,f}(\lambda_0, \Psi_N)} \rangle, \Phi \rangle = 0. \end{array} \right. \tag{4.6}$$

(see (3.22) for the notation $\widetilde{\cdot}$).

We first prove that α_N is continuous and elliptic.

LEMMA 4.1 : *There exist two positive constants $\tilde{\alpha}$ and $\tilde{\beta}$ independent of N such that, for any $(\Phi, \chi) \in V_N^2$:*

$$| \alpha_N(\Phi, \chi) | \leq \tilde{\alpha} \| \Phi \|_{2, \omega} \| \chi \|_{2, \omega}, \tag{4.7}$$

$$\alpha_N(\Phi, \Phi) \geq \tilde{\beta} \| \Phi \|_{2, \omega}^2. \tag{4.8}$$

Proof : We first prove (4.7). Let $(\Phi, \chi) \in V_N^2$, we have, as in (3.8) :

$$\alpha_N(\Phi, \chi) = A_{20,N} + 2 A_{11,N} + A_{02,N},$$

with :

$$A_{ij,N} = \sum_{k,l=0}^N \frac{\partial^4}{\partial x_i^{2i} \partial x_j^{2j}} \Phi(\xi_{kl}) \chi(\xi_{kl}) \omega_{kl}.$$

Using (2.8) we obtain :

$$A_{20,N} = \sum_{l=0}^N \left(\int_I \frac{\partial^4 \Phi}{\partial x_1^4}(x_1, \xi_l) \chi(x_1, \xi_l) \omega_1(x_1) \, dx_1 \right) \omega_l,$$

so that, by integration by parts :

$$A_{20,N} = \sum_{l=0}^N \left(\int_I \frac{\partial^2 \Phi}{\partial x_1^2} (x_1, \xi_l) \frac{\partial^2}{\partial x_1^2} (\chi(x_1, \xi_l) \omega_1(x_1)) dx_1 \right) \omega_l. \quad (4.9)$$

Thus, it follows from (3.4) that :

$$\begin{aligned} |A_{20,N}| &\leq C_1 \sum_{l=0}^N \omega_l \left[\left(\int_I \left(\frac{\partial^2 \Phi}{\partial x_1^2} \right)^2 (x_1, \xi_l) \omega_1(x_1) dx_1 \right)^{1/2} \times \right. \\ &\quad \left. \times \left(\int_I \left(\frac{\partial^2 \chi}{\partial x_1^2} \right)^2 (x_1, \xi_l) \omega_1(x_1) dx_1 \right)^{1/2} \right], \\ &\leq C_1 \sum_{l=0}^N \omega_l \|\Phi(\cdot, \xi_l)\|_{H_{\omega}^2(I)} \|\chi(\cdot, \xi_l)\|_{H_{\omega}^2(I)}. \end{aligned}$$

Hence, by (2.10) we find :

$$|A_{20,N}| \leq C \|\Phi\|_{2,\underline{\omega}} \|\chi\|_{2,\underline{\omega}}. \quad (4.10)$$

Similarly we get :

$$|A_{02,N}| \leq C \|\Phi\|_{2,\underline{\omega}} \|\chi\|_{2,\underline{\omega}}. \quad (4.11)$$

Finally, from (2.8) we have :

$$A_{11,N} = \int_{\Omega} \frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} \chi \underline{\omega} d\underline{x} = A_{11}. \quad (4.12)$$

Therefore, (4.7) is a consequence of (4.10) - (4.12), (3.11), (3.12).

Let us prove now (4.8). We keep the above notations but with $\chi = \Phi$. Due to (3.5) and (4.9) we have :

$$A_{20,N} \geq C'_2 \sum_{l=0}^N \left[\int_I \left(\frac{\partial^2 \Phi}{\partial x_1^2} \right)^2 (x_1, \xi_l) \omega_1(x_1) dx_1 \right] \omega_l$$

so that, by (2.10), we derive :

$$A_{20,N} \geq C \left\| \frac{\partial^2 \Phi}{\partial x_1^2} \right\|_{0,\underline{\omega}}^2. \quad (4.13)$$

Similarly, we have :

$$A_{02,N} \geq C \left\| \frac{\partial^2 \Phi}{\partial x_2^2} \right\|_{0,\underline{\omega}}^2. \tag{4.14}$$

Moreover (4.12) and (3.15) give :

$$A_{11,N} = A_{11} \geq 0. \tag{4.15}$$

Thus, (4.8) is a consequence of (3.17), (4.13)-(4.15) and the property (2.1). ■

As in the continuous case (see (3.23)) we can introduce an operator $T_N : \mathcal{H}_{\underline{\omega}}^{-2}(\Omega) \rightarrow V_N$ by :

$$a_N(T_N g, \Phi) = \langle \tilde{g}, \Phi \rangle \quad \forall \Phi \in V_N, \quad \forall g \in \mathcal{H}_{\underline{\omega}}^{-2}(\Omega). \tag{4.16}$$

Problem (4.6) is equivalent to the following one :

$$\begin{cases} \text{Find } \Psi_N \in V_N \text{ such that :} \\ \Psi_N + T_N H_{N,f}(\lambda_0, \Psi_N) = 0. \end{cases}$$

We consider now a slightly more general problem :

$$\begin{cases} \text{Find } (\lambda, \Psi_N) \in \mathbb{R} \times V_N \text{ such that :} \\ F_N(\lambda, \Psi_N) \equiv \Psi_N + T_N H_{N,f}(\lambda, \Psi_N) = 0. \end{cases} \tag{4.17}$$

IV.2. Existence of a solution of the approximate problem and error bound

We shall use now the general theorem concerning the approximation of problems stated as (3.32) by problems stated as (4.17) developed in Descloux-Rappaz [1]. We have recalled a suitable version of it in the appendix.

First we shall assume in the sequel that the solution Ψ_0 of (3.28) satisfies the following property :

$$(\lambda_0, \Psi_0) \text{ is a regular point of (3.32).}$$

Let us now prove that $\lim_{N \rightarrow \infty} T_N = T$. We introduce the projection operators Π_N and $\overset{\circ}{\Pi}_N$ from V onto V_N by :

$$a(\Pi_N \Phi, \chi) = a(\Phi, \chi), \quad \forall \chi \in V_N, \quad \forall \Phi \in V. \tag{4.18}$$

$$a_N(\overset{\circ}{\Pi}_N \Phi, \chi) = a(\Phi, \chi), \tag{4.19}$$

Using lemma 3.2 and well-known technics upon projection operators we get :

$$\| \Phi - \Pi_N \Phi \|_{2, \underline{\omega}} \leq C \inf_{\Phi_N \in \mathcal{V}_N} \| \Phi - \Phi_N \|_{2, \underline{\omega}}.$$

From (2.7), with $r = 2$ we derive that, for any Φ in $H_{\underline{\omega}}^{\sigma}(\Omega) \cap V$ ($\sigma \geq 2$) :

$$\| \Phi - \Pi_N \Phi \|_{2, \underline{\omega}} \leq CN^{2-\sigma} \| \Phi \|_{\sigma, \underline{\omega}}. \quad (4.20)$$

Let us remark now that $T_N = \overset{\circ}{\Pi}_N \circ T$. The property :

$$\lim_{N \rightarrow \infty} T_N = T,$$

will be a consequence of an estimate concerning $\overset{\circ}{\Pi}_N$ analogous to (4.20).

LEMMA 4.2 : *There exists a positive constant C , independent of N such that, for any $\sigma \geq 2$ and any $\Phi \in H_{\underline{\omega}}^{\sigma}(\Omega) \cap V$;*

$$\| \Phi - \overset{\circ}{\Pi}_N \Phi \|_{2, \underline{\omega}} \leq CN^{2-\sigma} \| \Phi \|_{\sigma, \underline{\omega}}. \quad (4.21)$$

Proof : Let Φ be in $V \cap H_{\underline{\omega}}^{\sigma}(\Omega)$, using (4.8) we have :

$$\| (\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi \|_{2, \underline{\omega}}^2 \leq \tilde{\beta}^{-1} \alpha_N ((\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi, (\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi).$$

Next we deduce from (4.18) and (4.19) that :

$$\left\{ \begin{aligned} \| (\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi \|_{2, \underline{\omega}}^2 &\leq \tilde{\beta}^{-1} [\alpha((\Pi_N - \Pi_{\sigma, N}) \Phi, (\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi) + \\ &+ |(\alpha - \alpha_N) (\Pi_{\sigma, N} \Phi, (\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi)|]. \end{aligned} \right. \quad (4.22)$$

Besides, as in the proof of (4.20), we get :

$$| \alpha((\Pi_N - \Pi_{\sigma, N}) \Phi, (\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi) | \leq CN^{2-\sigma} \| \Phi \|_{\sigma, \underline{\omega}} \| (\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi \|_{2, \underline{\omega}} \quad (4.23)$$

On the other hand, we define for $1 \leq i, j \leq 2$:

$$\left\{ \begin{aligned} J_{ij} &= \int_{\Omega} \frac{\partial^4}{\partial x_i^2 \partial x_j^2} (\Pi_{\sigma, N} \Phi) [(\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi] \underline{\omega} \\ &- \sum_{k, l} \frac{\partial^4}{\partial x_i^2 \partial x_j^2} (\Pi_{\sigma, N} \Phi) [(\overset{\circ}{\Pi}_N - \Pi_{\sigma, N}) \Phi] (\xi_{kl}) \omega_{kl}. \end{aligned} \right. \quad (4.24)$$

Then, we have :

$$(\alpha - \alpha_N) (\Pi_{\sigma,N} \Phi, (\hat{\Pi}_N - \Pi_{\sigma,N}) \Phi) = J_{11} + 2 J_{12} + J_{22} . \quad (4.25)$$

Setting $\hat{\Phi}_N = (\hat{\Pi}_N - \Pi_{\sigma,N}) \Phi$ and using (2.8) (see (4.9)) :

$$J_{11} = \int_I \left[\left(\frac{\partial^2}{\partial x_1^2} (\Pi_{\sigma,N} \Phi) (x_1, \cdot), \frac{\partial^2}{\partial x_1^2} (\hat{\Phi}_N \omega_1) (x_1, \cdot) \right)_{0, \omega_2} - \left(\frac{\partial^2}{\partial x_1^2} (\Pi_{\sigma,N} \Phi) (x_1, \cdot), \frac{\partial^2}{\partial x_1^2} (\hat{\Phi} \omega_1) (x_1, \cdot) \right)_{\omega_2, N} \right] dx_1 .$$

Using (2.6) and (2.13), noticing that $\Pi_{\sigma,N} \Phi(x_1, \cdot) \in \mathbb{P}_N(I)$ and that P_N reduces to the identity mapping over $\mathbb{P}_N(I)$ we get :

$$\begin{aligned} |J_{11}| &\leq CN^{2-\sigma} \int_I \omega_1 \left\| \frac{\partial^2}{\partial x_1^2} (\Pi_{\sigma,N} \Phi) (x_1, \cdot) \right\|_{\sigma-2, \omega_2} \\ &\quad \times \left\| \frac{1}{\omega_1} \frac{\partial^2}{\partial x_1^2} (\hat{\Phi}_N \omega_1) (x_1, \cdot) \right\|_{0, \omega_2} dx_1 \\ &\leq CN^{2-\sigma} \left\| \frac{\partial^2}{\partial x_1^2} (\Pi_{\sigma,N} \Phi) \right\|_{\sigma-2, \underline{\omega}} \left\| \frac{1}{\omega_1} \frac{\partial^2}{\partial x_1^2} (\hat{\Phi}_N \omega_1) \right\|_{0, \underline{\omega}} . \end{aligned}$$

Following the same lines as in the proof of (3.12) we obtain :

$$\left\| \frac{1}{\omega_1} \frac{\partial^2}{\partial x_1^2} (\hat{\Phi}_N \omega_1) \right\|_{0, \underline{\omega}} \leq C \left\| \frac{\partial^2 \hat{\Phi}_N}{\partial x_1^2} \right\|_{0, \underline{\omega}} , \quad (4.26)$$

so that :

$$|J_{11}| \leq CN^{2-\sigma} \|\Pi_{\sigma,N} \Phi\|_{\sigma, \underline{\omega}} \|\hat{\Phi}_N\|_{2, \underline{\omega}} . \quad (4.27)$$

Similarly we obtain :

$$|J_{22}| \leq CN^{2-\sigma} \|\Pi_{\sigma,N} \Phi\|_{\sigma, \underline{\omega}} \|\hat{\Phi}_N\|_{2, \underline{\omega}} . \quad (4.28)$$

Finally, noticing that :

$$\frac{\partial^4}{\partial x_1^2 \partial x_2^2} (\Pi_{\sigma,N} \Phi) \hat{\Phi}_N \in \mathbb{P}_{2N-2}(\Omega) ,$$

and using (2.8) we check that $J_{12} = 0$. Hence (4.21) follows from (2.7), (4.22), (4.23), (4.25) - (4.28). ■

According to (4.20) the hypothesis (A.2) of Theorem A.1 holds if we choose $\mathcal{D}_N \equiv \Pi_N$. In order to prove that hypothesis (A.1) and (A.3) holds we compute the derivatives of F and F_N . Using (3.26), (3.27) and (3.32) we have, for any (λ, Φ) , $(\mu_i, \chi_i) \in \mathbb{R} \times V$ ($i = 1, 2, 3$) :

$$\begin{aligned} F^{(1)}[\lambda, \Phi](\mu_1, \chi_1) &= \\ &= \chi_1 + 2 T(\lambda B(\Phi, \chi_1)) + T(\mu_1(B(\Phi, \Phi) - f)) \end{aligned} \quad (4.29)$$

$$\begin{aligned} F^{(2)}[\lambda, \Phi]((\mu_1, \chi_1), (\mu_2, \chi_2)) &= \\ &= 2 T(\lambda B(\chi_1, \chi_2)) + 2 T(\mu_1 B(\Phi, \chi_2) + \mu_2 B(\Phi, \chi_1)) \end{aligned} \quad (4.30)$$

$$\begin{aligned} F^{(3)}[\lambda, \Phi]((\mu_1, \chi_1), (\mu_2, \chi_2), (\mu_3, \chi_3)) &= \\ &= 2 T(\mu_1 B(\chi_2, \chi_3) + \mu_2 B(\chi_3, \chi_1) + \mu_3 B(\chi_1, \chi_2)) \end{aligned} \quad (4.31)$$

$$F^{(k)}[\lambda, \Phi] \equiv 0 \quad \forall k \geq 4. \quad (4.32)$$

Similar formulae are obtained for the derivatives of F_N , replacing T by T_N and B by B_N and (A.1) is clearly verified.

Moreover it can be checked that (A.3) is a consequence of the following property :

$$\lim_{N \rightarrow +\infty} \|TB(\Phi, \chi) - T_N B_N(\Pi_N \Phi, \Pi_N \chi)\|_{2, \underline{\omega}} = 0, \quad \forall (\Phi, \chi) \in V^2. \quad (4.33)$$

The following lemma and (4.20) will imply (4.33).

LEMMA 4.3 : *There exist two positive constants C and η independent of N such that, for any $(\Phi, \chi) \in V^2$:*

$$\left\{ \begin{aligned} \|TB(\Phi, \chi) - T_N B_N(\Pi_N \Phi, \Pi_N \chi)\|_{2, \underline{\omega}} &\leq C[N^{-\eta} \|\Phi\|_{2, \underline{\omega}} \|\chi\|_{2, \underline{\omega}} + \\ &+ \|\Phi - \Pi_N \Phi\|_{2, \underline{\omega}} \|\chi\|_{2, \underline{\omega}} + \|\chi - \Pi_N \chi\|_{2, \underline{\omega}} \|\Phi\|_{2, \underline{\omega}}]. \end{aligned} \right. \quad (4.34)$$

Proof : Let (Φ, χ) be in $\mathcal{D}(\Omega)^2$. We have :

$$\left\{ \begin{aligned} \|TB(\Phi, \chi) - T_N B_N(\Pi_N \Phi, \Pi_N \chi)\|_{2, \underline{\omega}} &\leq \|(T - T_N)B(\Phi, \chi)\|_{2, \underline{\omega}} + \\ &+ \|T_N(B(\Phi, \chi) - B(\Pi_N \Phi, \Pi_N \chi))\|_{2, \underline{\omega}} + \|T_N(B - B_N)(\Pi_N \Phi, \Pi_N \chi)\|_{2, \underline{\omega}}. \end{aligned} \right. \quad (4.35)$$

From (3.29) and the regularity of T we have, if $1 < s < 3/2$:

$$\|TB(\Phi, \chi)\|_{H^{4-s}(\Omega)} \leq C \|\Phi\|_{2, \underline{\omega}} \|\chi\|_{2, \underline{\omega}},$$

so that, by (2.3) :

$$\|TB(\Phi, \chi)\|_{7/2-s, \omega} \leq C \|\Phi\|_{2, \omega} \|\chi\|_{2, \omega}.$$

Using (4.21) and the equality $T_N = \overset{\circ}{\Pi}_N \circ T$, we obtain, for any $s, 1 < s < 3/2$:

$$\|(T - T_N)B(\Phi, \chi)\|_{2, \omega} \leq CN^{s-3/2} \|\Phi\|_{2, \omega} \|\chi\|_{2, \omega}. \tag{4.36}$$

Let us consider now the next term in the right hand side of (4.35). We get :

$$\left\{ \begin{aligned} \|T_N[B(\Phi, \chi) - B(\Pi_N \Phi, \Pi_N \chi)]\|_{2, \omega} &\leq \|T_N\|_{\mathcal{L}(H^{-s}(\Omega), \mathcal{V})} \times \\ &\times \|B(\Phi, \chi) - B(\Pi_N \Phi, \Pi_N \chi)\|_{H^{-s}(\Omega)}. \end{aligned} \right. \tag{4.37}$$

Using Theorem 3.1 and (4.21) we obtain :

$$\|T_N\|_{\mathcal{L}(H^{-s}(\Omega), \mathcal{V})} \leq C. \tag{4.38}$$

Moreover it follows from lemma 3.3 that :

$$\begin{aligned} \|B(\Phi, \chi) - B(\Pi_N \Phi, \Pi_N \chi)\|_{H^{-s}(\Omega)} &\leq \|B(\Phi, \chi - \Pi_N \chi)\|_{H^{-s}(\Omega)} + \\ &+ \|B(\Phi - \Pi_N \Phi, \Pi_N \chi)\|_{H^{-s}(\Omega)} \leq \\ &\leq \gamma(\|\Phi\|_{2, \omega} \|\chi - \Pi_N \chi\|_{2, \omega} + \|\Phi - \Pi_N \Phi\|_{2, \omega} \|\Pi_N \chi\|_{2, \omega}). \end{aligned}$$

Hence combining (4.37), (4.38) and the use of (4.20) with $\sigma = 2$, we get :

$$\left\{ \begin{aligned} \|T_N[B(\Phi, \chi) - B(\Pi_N \Phi, \Pi_N \chi)]\|_{2, \omega} &\leq (C \|\Phi\|_{2, \omega} \|\chi - \Pi_N \chi\|_{2, \omega} + \\ &+ \|\chi\|_{2, \omega} \|\Phi - \Pi_N \Phi\|_{2, \omega}). \end{aligned} \right. \tag{4.39}$$

For studying the last term in (4.35) we first notice that, by (4.8) we have, for any $g \in H^{-s}(\Omega)$:

$$\|T_N g\|_{2, \omega} \leq \frac{1}{\beta} \sup_{\Xi \in \mathcal{V}_N} \left| \frac{a_N(T_N g, \Xi)}{\|\Xi\|_{2, \omega}} \right|.$$

Hence, from (4.16) and (4.5) we have :

$$\left\{ \begin{aligned} \|T_N[B(\Pi_N \Phi, \Pi_N \chi) - B_N(\Pi_N \Phi, \Pi_N \chi)]\|_{2, \omega} &\leq \\ &\leq \frac{1}{\beta} \sup_{\Xi \in \mathcal{V}_N} \left(\frac{\langle \widetilde{B(\Pi_N \Phi, \Pi_N \chi)}, \Xi \rangle - (\overset{\circ}{B}_N(\Pi_N \Phi, \Pi_N \chi), \Xi)_{\omega, N}}{\|\Xi\|_{2, \omega}} \right). \end{aligned} \right. \tag{4.40}$$

Moreover, according to the definition of B and \hat{B}_N , the right hand side of (4.40) can be written as the sum of six expressions.

A typical one is the following :

$$I(\Phi, \chi, \Xi) = \int_{\Omega} \frac{-\partial}{\partial x_1} (\Pi_N \Phi) \frac{\partial}{\partial x_2} (\Pi_N \chi) \frac{\partial^2}{\partial x_1^2} (\Xi \omega) \, d\underline{x} - \\ - \sum_{i,j} \left(\frac{\partial^2}{\partial x_1^2} \left[\underline{P}_N \left(\frac{-\partial}{\partial x_1} (\Pi_N \Phi) \frac{\partial}{\partial x_2} (\Pi_N \chi) \right) \right] \Xi(\xi_{ij}) \omega_{ij} \right)$$

In order to estimate I , we note that, by (2.8) and an integration by parts in the x_1 direction :

$$\left\{ \begin{aligned} I(\Phi, \chi, \Xi) &= \int_{\Omega} \frac{-\partial}{\partial x_1} (\Pi_N \Phi) \frac{\partial}{\partial x_2} (\Pi_N \chi) \frac{\partial^2}{\partial x_1^2} (\Xi \omega) \, d\underline{x} \\ &- \int_I \sum_j \omega_j \underline{P}_N \left[\left(\frac{-\partial}{\partial x_1} \right) (\Pi_N \Phi) \frac{\partial}{\partial x_2} (\Pi_N \chi) \right] \frac{\partial^2}{\partial x_1^2} (\Xi \omega_1)(x_1, \xi_j) \, dx_1. \end{aligned} \right. \quad (4.41)$$

Since $\Xi \in V_N$, then $\Xi = (1 - x_1^2)^2 \Lambda$ with $\Lambda(\cdot, y) \in \mathbb{P}_{N-4}(I)$.

It follows from (3.14) that :

$$\frac{1}{\omega_1} \frac{\partial^2}{\partial x_1^2} (\Xi \omega_1)(\cdot, \xi_j) \in \mathbb{P}_{N-2}(I), \quad 0 \leq j \leq N, \quad (4.42)$$

hence we obtain :

$$I(\Phi, \chi, \Xi) = \int_{\Omega} \frac{-\partial}{\partial x_1} (\Pi_N \Phi) \frac{\partial}{\partial x_2} (\Pi_N \chi) \frac{\partial^2}{\partial x_1^2} (\Xi \omega_1) \frac{1}{\omega_1} \omega(\underline{x}) \, d\underline{x} \\ - \sum_{i,j} \left[\frac{-\partial}{\partial x_1} (\Pi_N \Phi) \frac{\partial}{\partial x_2} (\Pi_N \chi) \frac{\partial^2}{\partial x_1^2} (\Xi \omega_1) \frac{1}{\omega_1} \right] (\xi_{ij}) \omega_{ij}.$$

Applying the estimate (2.13) and (4.26) (with $\hat{\Phi}_N = \Xi$) we obtain :

$$|I(\Phi, \chi, \Xi)| \leq C \|\Xi\|_{2,\omega} \left[\left\| (I - \underline{P}_N) \left(\frac{\partial}{\partial x_1} (\Pi_N \Phi) \frac{\partial}{\partial x_2} (\Pi_N \chi) \right) \right\|_{0,\omega} + \right. \\ \left. + \left\| (I - \Pi_{0,N-1}) \left(\frac{\partial}{\partial x_1} (\Pi_N \Phi) \frac{\partial}{\partial x_2} (\Pi_N \chi) \right) \right\|_{0,\omega} \right]. \quad (4.43)$$

Using (2.4), (2.12) and (2.5) we derive that, for any $\varepsilon > 0$:

$$\begin{aligned} \left\| (I - \underline{P}_N) \left(\frac{\partial}{\partial x_1} \Pi_N \Phi \frac{\partial}{\partial x_2} \Pi_N \chi \right) \right\|_{0,\omega} &\leq CN^{-1-\varepsilon} \|\Pi_N \Phi\|_{2+\varepsilon,\omega} \times \\ &\times \|\Pi_N \chi\|_{2+\varepsilon,\omega} \leq CN^{-1+3\varepsilon} \|\Pi_N \Phi\|_{2,\omega} \|\Pi_N \chi\|_{2,\omega}. \end{aligned}$$

Similarly, using (2.6) we get :

$$\begin{aligned} \left\| (I - \Pi_{0,N-1}) \left(\frac{\partial}{\partial x_1} \Pi_N \Phi \frac{\partial}{\partial x_2} \Pi_N \chi \right) \right\|_{0,\omega} &\leq CN^{-1+3\varepsilon} \|\Pi_N \Phi\|_{2,\omega} \times \\ &\times \|\Pi_N \chi\|_{2,\omega}. \end{aligned} \tag{4.45}$$

Therefore, combining (4.43)-(4.45) and setting $\eta = \inf(3/2 - s, 1 - 3\varepsilon)$ ($0 < \varepsilon < \frac{1}{3}$) we obtain :

$$|I(\Phi, \chi, \Xi)| \leq CN^{-\eta} \|\Phi\|_{2,\omega} \|\chi\|_{2,\omega} \|\Xi\|_{2,\omega}.$$

Similar estimates for the other expressions in (4.40) leads to :

$$\|T_N(B(\Pi_N \Phi, \Pi_N \chi) - B_N(\Pi_N \Phi, \Pi_N \chi))\|_{2,\omega} \leq CN^{-\eta} \|\Phi\|_{2,\omega} \|\chi\|_{2,\omega}. \tag{4.46}$$

The desired estimate (4.34) then follows from (4.35), (4.36), (4.39) and (4.46). ■

The hypothesis (A.4) of Theorem A.1 is checked through :

LEMMA 4.4 : *There exist monotonically increasing functions*

$$C_k : \mathbb{R}^+ \rightarrow \mathbb{R}^+, k \in \mathbb{N}, \text{ such that, for any } (\lambda, \Phi) \in \mathbb{R} \times V_N :$$

$$\|F_N^{(k)}[\lambda, \Phi]\|_{\mathcal{L}((\mathbb{R} \times V_N)^k, V_N)} \leq C_k(|\lambda| + \|\Phi\|_{2,\omega}). \tag{4.47}$$

Proof : By using the expressions of the derivatives of F_N (see (4.29)-(4.32)) and the upper bound (see (3.33)) :

$$\|\underline{P}_N f\|_{0,\omega} \leq C \|f\|_{p,\omega},$$

we shall get (4.47) by the proof of the existence of a constant $C > 0$, such that, for any $(\Phi, \chi) \in V_N^2$:

$$\|T_N B_N(\Phi, \chi)\|_{2,\omega} \leq C \|\Phi\|_{2,\omega} \|\chi\|_{2,\omega}. \tag{4.48}$$

First we have

$$\| T_N B_N(\Phi, \chi) \|_{2, \underline{\omega}} \leq \| TB(\Phi, \chi) \|_{2, \underline{\omega}} + \| TB(\Phi, \chi) - T_N B_N(\Phi, \chi) \|_{2, \underline{\omega}}. \quad (4.49)$$

Since $\Pi_N \Phi = \Phi$ and $\Pi_N \chi = \chi$ we have, by (4.34) :

$$\| TB(\Phi, \chi) - T_N B_N(\Phi, \chi) \|_{2, \underline{\omega}} \leq CN^{-\eta} \| \Phi \|_{2, \underline{\omega}} \| \chi \|_{2, \underline{\omega}}. \quad (4.50)$$

Due to (3.29) and Theorem 3.1 we find :

$$\| TB(\Phi, \chi) \|_{2, \underline{\omega}} \leq C \| \Phi \|_{2, \underline{\omega}} \| \chi \|_{2, \underline{\omega}}, \quad (4.51)$$

so that (4.48) is a consequence of (4.49)-(4.51).

Finally let us check the hypothesis (A.5) of Theorem A.1. We compute for $(\lambda, \Phi) \in \mathbb{R} \times V_N$,

$$F^{(1)}[\lambda_0, \Psi_0](\lambda, \Phi) - F_N^{(1)}[\lambda_0, \Pi_N \Psi_0](\lambda, \Phi) \equiv A(\lambda, \Phi).$$

Using the expression of derivatives of F and F_N (see (4.29)), we obtain :

$$\left\{ \begin{aligned} A(\lambda, \Phi) &= 2 [T(\lambda_0 B(\Psi_0, \Phi)) - T_N(\lambda_0 B_N(\Pi_N \Psi_0, \Pi_N \Phi))] + \\ &+ [T(\lambda(B(\Psi_0, \Psi_0) - f)) - T_N(\lambda(B_N(\Pi_N \Psi_0, \Pi_N \Psi_0) - \underline{P}_N f))]. \end{aligned} \right. \quad (4.52)$$

Note that :

$$\| Tf - T_N \underline{P}_N f \|_{2, \underline{\omega}} \leq \| (T - T_N) f \|_{2, \underline{\omega}} + \| T_N(f - \underline{P}_N f) \|_{2, \underline{\omega}}. \quad (4.53)$$

Since $f \in H_{\underline{\omega}}^{\rho}(\Omega)$, $\rho > 1$ we deduce from (3.24) and (3.25) that $Tf \in H^3(\Omega)$ which is included in $H_{\underline{\omega}}^{5/2}(\Omega)$; from the equality $T_N = \overset{\circ}{\Pi}_N \circ T$ we get :

$$\lim_{N \rightarrow +\infty} \| (T - T_N) f \|_{2, \underline{\omega}} = 0. \quad (4.54)$$

Using the continuity of $\overset{\circ}{\Pi}_N$ from V into V_N , of T from $L_{\underline{\omega}}^2(\Omega)$ into V we derive, using (2.12) :

$$\| T_N(f - \underline{P}_N f) \|_{2, \underline{\omega}} \leq C \| f - \underline{P}_N f \|_{0, \underline{\omega}} \leq CN^{-\rho} \| f \|_{\rho, \underline{\omega}}, \quad (4.55)$$

hence :

$$\lim_{N \rightarrow +\infty} \| Tf - T_N \underline{P}_N f \|_{2, \underline{\omega}} = 0. \quad (4.56)$$

Finally (A.5) is derived from (4.52)-(4.56), (4.20) and (4.34). ■

Now we have proved that F_N is an approximation of F verifying the hypothesis of Descloux-Rappaz [1], we can state the main result of this paper, consequence of Theorem A.1.

THEOREM 4.1 :

i) *There exist two positive constants γ and $\delta' \leq \delta$ and, for $N \geq N_0$ large enough, a unique C^∞ mapping $\Psi_N : [\lambda_0 - \delta', \lambda_0 + \delta'] \rightarrow V_N$ such that, $F_N(\lambda, \Psi_N(\lambda)) = 0$ and $\|\Psi_N(\lambda) - \Pi_N \Psi_0\|_{2,\omega} \leq \gamma$ for any $\lambda \in [\lambda_0 - \delta', \lambda_0 + \delta']$.*

ii) *Moreover, if there exist a positive constant M and a real $\sigma > 2$ such that :*

$$\forall \lambda \in [\lambda_0 - \delta', \lambda_0 + \delta'], \quad \|f\|_{\sigma-2,\omega} + \|\Psi(\lambda)\|_{\sigma,\omega} \leq M, \quad (4.57)$$

then, there exists a positive constant C , independent of N , such that :

$$\sup_{|\lambda - \lambda_0| \leq \delta'} \|\Psi(\lambda) - \Psi_N(\lambda)\|_{2,\omega} \leq CN^{2-\sigma}. \quad (4.58)$$

Proof : The point i) is a direct consequence of Theorem A.1. With regard to ii), we derive from Theorem A.1 that there exists a positive constant C , such that, for any $\lambda \in [\lambda_0 - \delta', \lambda_0 + \delta']$:

$$\|\Psi(\lambda) - \Psi_N(\lambda)\|_{2,\omega} \leq C(\|\Psi(\lambda) - \Pi_N \Psi(\lambda)\|_{2,\omega} + \|F_N(\lambda, \Pi_N \Psi(\lambda))\|_{2,\omega}).$$

From (4.20), we get immediately :

$$\|\Psi(\lambda) - \Pi_N \Psi(\lambda)\|_{2,\omega} \leq CN^{2-\sigma} \|\Psi(\lambda)\|_{\sigma,\omega}. \quad (4.59)$$

Besides, from the definitions of $\Psi(\lambda)$, F and F_N we obtain :

$$\left\{ \begin{aligned} &\|F_N(\lambda, \Pi_N \Psi(\lambda))\|_{2,\omega} \leq \|F_N(\lambda, \Pi_N \Psi(\lambda)) - F(\lambda, \Psi(\lambda))\|_{2,\omega} \leq \\ &\leq \|(I - \Pi_N) \Psi(\lambda)\|_{2,\omega} + |\lambda| \|(T - T_N)[B(\Psi(\lambda), \Psi(\lambda)) - f]\|_{2,\omega} + \\ &+ |\lambda| \|T_N[B(\Psi(\lambda), \Psi(\lambda)) - B(\Pi_N \Psi(\lambda), \Pi_N \Psi(\lambda))]\|_{2,\omega} + \\ &+ |\lambda| \|T_N(B - B_N)(\Pi_N \Psi(\lambda), \Pi_N \Psi(\lambda))\|_{2,\omega} + |\lambda| \|T_N(f - \underline{P}_N f)\|_{2,\omega}. \end{aligned} \right. \quad (4.60)$$

The first term on the right hand side of (4.60) is bounded thanks to (4.59). The second one is studied easily if we note that :

$$T[B(\Psi(\lambda), \Psi(\lambda)) - f] = -\frac{1}{\lambda} \Psi(\lambda).$$

The third one uses the continuity of B (see (4.39)). The last one has been studied

in (4.55). Finally denoting $\Pi_N \Psi(\lambda)$ by $\bar{\Psi}(\lambda)$ we can prove that :

$$\| T_N(B - B_N)(\bar{\Psi}(\lambda), \bar{\Psi}(\lambda)) \|_{2,\omega} \leq CN^{2-\sigma} \| \Psi(\lambda) \|_{\sigma,\omega}^2. \quad (4.61)$$

For this, we are led, as in (4.40), to study expressions as $I(\bar{\Psi}(\lambda), \bar{\Psi}(\lambda), \Xi)$ defined at (4.41) for $\Xi \in V_N$. Due to (4.43) we have :

$$\left\{ \begin{array}{l} | I(\bar{\Psi}(\lambda), \bar{\Psi}(\lambda), \Xi) | \leq C \| \Xi \|_{2,\omega} \left[\left\| (I - \underline{P}_N) \left(\frac{\partial}{\partial x_1} \bar{\Psi}(\lambda) \frac{\partial}{\partial x_2} \bar{\Psi}(\lambda) \right) \right\|_{0,\omega} + \right. \\ \left. + \left\| (I - \Pi_{0,N-1}) \left(\frac{\partial}{\partial x_1} \bar{\Psi}(\lambda) \frac{\partial}{\partial x_2} \bar{\Psi}(\lambda) \right) \right\|_{0,\omega} \right], \end{array} \right. \quad (4.62)$$

each term, on the right hand side of (4.62) are studied similarly by introducing $\frac{\partial}{\partial x_1} \Psi(\lambda) \frac{\partial}{\partial x_2} \Psi(\lambda)$.

$$\left\{ \begin{array}{l} \left\| (I - \underline{P}_N) \left(\frac{\partial}{\partial x_1} \bar{\Psi}(\lambda) \frac{\partial}{\partial x_2} \bar{\Psi}(\lambda) \right) \right\|_{0,\omega} \leq \left\| (I - \underline{P}_N) \left(\frac{\partial}{\partial x_1} \bar{\Psi}(\lambda) \frac{\partial}{\partial x_2} \bar{\Psi}(\lambda) \right) \right\|_{0,\omega} \\ + \left\| (I - P_N) \left(\frac{\partial}{\partial x_1} \bar{\Psi}(\lambda) \frac{\partial}{\partial x_2} \bar{\Psi}(\lambda) - \frac{\partial}{\partial x_1} \Psi(\lambda) \frac{\partial}{\partial x_2} \Psi(\lambda) \right) \right\|_{0,\omega} \end{array} \right. \quad (4.63)$$

If we choose $\mu < \frac{7}{3}$ such that $2 < \mu < \sigma$ we get from (2.12) and (2.4) :

$$\begin{aligned} \left\| (I - \underline{P}_N) \left(\frac{\partial}{\partial x_1} \bar{\Psi}(\lambda) \frac{\partial}{\partial x_2} \bar{\Psi}(\lambda) - \frac{\partial}{\partial x_1} \Psi(\lambda) \frac{\partial}{\partial x_2} \Psi(\lambda) \right) \right\|_{0,\omega} &\leq \\ &\leq CN^{1-\mu} [\| \bar{\Psi}(\lambda) \|_{\mu,\omega} \| (\Psi - \bar{\Psi})(\lambda) \|_{\mu,\omega} + \\ &\quad + \| \Psi(\lambda) \|_{\mu,\omega} \| (\Psi - \bar{\Psi})(\lambda) \|_{\mu,\omega}]. \end{aligned}$$

Using now the following inequality :

$$\| \bar{\Psi}(\lambda) - \Psi(\lambda) \|_{\mu,\omega} \leq \| \bar{\Psi}(\lambda) - \Pi_{\mu,N} \Psi(\lambda) \|_{\mu,\omega} + \| \Pi_{\mu,N} \Psi(\lambda) - \Psi(\lambda) \|_{\mu,\omega}$$

and the inverse inequality we derive after some calculation :

$$\begin{aligned} \left\| (I - \underline{P}_N) \left(\frac{\partial}{\partial x_1} \bar{\Psi}(\lambda) \frac{\partial}{\partial x_2} \bar{\Psi}(\lambda) - \frac{\partial}{\partial x_1} \Psi(\lambda) \frac{\partial}{\partial x_2} \Psi(\lambda) \right) \right\|_{0,\omega} &\leq \\ &\leq CN^{2-\sigma} \| \Psi(\lambda) \|_{\sigma,\omega}^2. \end{aligned}$$

The last term in the right hand side of (4.63) is bounded by the same quantity so that (4.58) is proved since $\|\Psi(\lambda)\|_{\sigma, \underline{\omega}}$ can be bounded independently of λ due to the compactness of Λ .

APPENDIX

THEOREM A.1 : *Let V be a Banach space over \mathbb{R} and $F : \mathbb{R} \times V \rightarrow V$. We assume that :*

- $F : \mathbb{R} \times V \rightarrow V$ is a C^p mapping with $p \geq 2$,
- $(\lambda_0, \Psi_0) \in \mathbb{R} \times V$ verifies $F(\lambda_0, \Psi_0) = 0$ and (λ_0, Ψ_0) is a regular point i.e. $D_{\Phi} F(\lambda_0, \Psi_0)$ is an homeomorphism from V onto V .

Then there exist positive numbers $\bar{\lambda}_0, \alpha$ and a map :

$$\Psi : \lambda \in]\lambda_0 - \bar{\lambda}_0, \lambda_0 + \bar{\lambda}_0[\rightarrow \Psi(\lambda) \in V$$

satisfying the condition :

$$F(\lambda, \Psi(\lambda)) = 0 \quad \text{and} \quad \|\Psi(\lambda) - \Psi_0\|_V \leq \alpha, \quad \forall \lambda \in]\lambda_0 - \bar{\lambda}_0, \lambda_0 + \bar{\lambda}_0[.$$

Furthermore Ψ is of class C^p .

In order to approximate the branch $\{\Psi(\lambda), \lambda \in]\lambda_0 - \bar{\lambda}_0, \lambda_0 + \bar{\lambda}_0[\}$ we introduce a family of finite dimensional subspaces of V , denoted by $V_N, N \in \mathbb{N}$ and a family of mappings $F_N : \mathbb{R} \times V_N \rightarrow V_N$ which shall approximate F .

We assume that :

$$F_N : \mathbb{R} \times V_N \rightarrow V_N \text{ are } C^p \text{ mappings.} \tag{A.1}$$

We are interested in solving the equation $F_N(\lambda, \Psi_N) = 0$ in a neighbourhood of the branch $\{\Psi(\lambda), \lambda \in]\lambda_0 - \bar{\lambda}_0, \lambda_0 + \bar{\lambda}_0[\}$.

We suppose that the following hypotheses are satisfied :

- (i) *For any N , there exists a projection operator $\mathcal{P}_N : V \rightarrow V_N$:*

$$\lim_{N \rightarrow +\infty} \|\Phi - \mathcal{P}_N \Phi\|_V = 0, \quad \forall \Phi \in V, \tag{A.2}$$

- (ii) *for any $0 \leq k \leq p - 1$ and any fixed $(\lambda, \Phi), (\lambda_1, \Phi_1), \dots, (\lambda_k, \Phi_k)$ in $\mathbb{R} \times V$, we have :*

$$\lim_{N \rightarrow \infty} \|F^{(k)}(\lambda, \Phi)(\lambda_1, \Phi_1, \lambda_2, \Phi_2, \dots, \lambda_k, \Phi_k) - F_N^{(k)}(\lambda, \mathcal{P}_N \Phi)(\lambda_1, \mathcal{P}_N \Phi_1, \dots, \lambda_k, \mathcal{P}_N \Phi_k)\|_V = 0 \tag{A.3}$$

(iii) there exist positive constants η, C_1, \dots, C_p such that $\forall N \in \mathbb{N}$,

$\forall k \in \{1, \dots, p\}, \forall (\lambda, \Phi) \in \mathbb{R} \times V_N$ with $(|\lambda - \lambda_0| + \|\Phi - \mathcal{P}_N \Phi\|_V) \leq \eta$,

$$\|F_N^{(k)}(\lambda, \Phi)\|_{\mathcal{L}((\mathbb{R} \times V_N)^k, V_N)} \leq C_k. \quad (\text{A.4})$$

(iv)

$$\lim_{N \rightarrow \infty} \sup_{\substack{(\lambda, \Phi) \in \mathbb{R} \times V_N \\ |\lambda| + \|\Phi\|_V = 1}} \|F^{(1)}(\lambda_0, \Phi_0)(\lambda, \Phi) - F^{(1)}(\lambda_0, \mathcal{P}_N \Phi_0)(\lambda, \Phi)\|_V = 0.$$

(A.5)

Then there exist $N_0 \in \mathbb{N}^*$, positive constants $\bar{\lambda}_0, \alpha', \beta$ and for $N \geq N_0$, a unique mapping $\Psi_N : \lambda \in]\lambda_0 - \bar{\lambda}'_0, \lambda_0 + \bar{\lambda}'_0[\rightarrow \Psi_N(\lambda) \in V_N$ satisfying the conditions :

$\forall \lambda \in]\lambda_0 - \bar{\lambda}'_0, \lambda_0 + \bar{\lambda}'_0[, F_N(\lambda, \Psi_N(\lambda)) = 0$ and $\|\Psi_N(\lambda) - \mathcal{P}_N \Psi_0\|_V \leq \alpha'$

Ψ_N is of class C^p with bounded derivatives uniformly with respect to λ and N .

Furthermore, $\bar{\lambda}'_0 \leq \bar{\lambda}_0$ and we have :

$$\|\psi(\lambda) - \Psi_N(\lambda)\|_V \leq \beta(\|F_N(\lambda, \mathcal{P}_N \psi(\lambda))\|_V + \|\psi(\lambda) - \mathcal{P}_N \psi(\lambda)\|_V),$$

$$\forall \lambda \in]\lambda_0 - \bar{\lambda}'_0, \lambda_0 + \bar{\lambda}'_0[\quad \blacksquare$$

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