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A NOTE ON THE APPROXIMATION OF FREE BOUNDARIES BY FINITE ELEMENT METHODS (*)

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Abstract — *Under the sole assumption of non-degeneracy of the continuous solution and the knowledge of an L^p-error estimate, we prove rates of convergence in measure (and in distance) for the approximate free boundaries, they are defined as suitable level sets, hence easy to compute The results are applied to the primal and mixed formulation of the obstacle problem and to singular parabolic problems (the two-phase Stefan problem and the porous medium equation) in several variables*

Résumé — *Nous démontrons ici des résultats de convergence en mesure et en distance pour des frontières libres approchées, sous la seule hypothèse de non dégénérescence de la solution continue et en supposant connue l'estimation de l'erreur L^p, les frontières approchées sont définies comme des ensembles de niveau appropriés et donc faciles à calculer Les résultats sont appliqués aux formulations primale et mixte du problème de l'obstacle et à des problèmes paraboliques singuliers à plusieurs variables d'espace (le probleme de Stefan à deux phases et les équations des milieux poreux)*

I. INTRODUCTION

In the numerical approximation of free boundary problems (stationary or not) the determination of an approximate free boundary « close to the continuous one is frequently as important as the obtainment of the discrete solution itself. This happens, for instance, when one deals with models of physical problems (filtration problems, melting problems, diffusion problems, etc.), for which the free boundary has a precise (and relevant !) physical meaning. Besides, it is well-known that the finite element method provides a sequence of discrete solutions convergent to the continuous one, with theoretical rates measured in

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some Sobolev spaces. However, these error estimates do not yield, by themselves, any estimate on the rate of convergence of the free boundaries. In [6], F. Brezzi and L. Caffarelli have proved for the stationary obstacle problem that the difficulty can be overcome if the solutions (continuous and discrete) satisfy a *non-degeneracy property*. Roughly speeking, this means that the solution leaves the obstacle (or equivalently, the free boundary) with a certain « minimum speed ». Recently, this idea was applied to the one-phase Stefan problem, [28].

The purpose of the present paper is to prove *error estimates in measure and in distance* for the approximation of the exact free boundary without using the discrete non-degeneracy property. To this end we only assume to know :

1. an L^p-error estimate for the solutions ($p \leqslant \infty$),
2. a non-degeneracy property for the continuous solution.

Then, we define the « discrete free boundary » as a δ-level set of the discrete solution and, therefore, easy to compute (δ related to 1) and 2); this idea was introduced in [2, 3]). Consequently, we give a simple methodology for computing approximate free boundaries that converge to the continuous one with a rate that can be estimated.

The fact of avoiding the discrete non-degeneracy is very important in practice, because there are several problems whose discrete solutions do not satisfy it even if the continuous one does, and also our procedure does not imply an additional computational cost. Moreover, due to low regularity properties of certain problems, such as singular parabolic problems (see § 4, 5), L^p-error estimates for solutions have been proved, but $p < \infty$. In the sequel we discuss these features for some *model problems*, for which [6] does not apply.

The continuous non-degeneracy property for the *obstacle problem* (stationary or evolutionary) was obtained by L. Caffarelli and used for proving regularity of the free boundary, [10, 20]. In [6] (and also in [28]) this property was reproduced for discrete solutions, and next combined with quasi-optimal L^∞-error estimates for finite element approximations [2, 24], in order to obtain rates of convergence (in measure and in distance) for discrete free boundaries. We point out that the discrete non-degeneracy property as well as the L^∞-bounds rely strongly on the discrete maximum principle (D.M.P.). This principle is valid for C^0-piecewise linear splines if the simplices verify a geometrical restriction [16]. For C^0-piecewise quadratics splines [15], the D.M.P. is not satisfied. In this case optimal error estimates in H^1 are known [7, 9], but not in L^∞. The same happens for mixed methods, for which optimal L^2-error estimates were shown in [8]. For these finite element approximations, the D.M.P. is not available.

For *singular parabolic problems* the non-degeneracy of the continuous solution is also known under certain qualitative assumptions upon the data, [12, 13, 27]. Their numerical approach is usually carried out combining a regularization procedure with standard piecewise linear finite elements in space and backward differences in time. Actually, due to the regularization, we cannot expect to get a discrete non-degeneracy property : for instance, the solution of the regularized porous-medium equation is positive everywhere. Besides, L^p-error estimates are known, but $p < \infty$.

The outline of the paper is the following. In § 2 we present the main results (theorems 2.1 and 2.2), whereas § 3, 4, 5 are devoted to apply them to the elliptic obstacle problem, the two-phase Stefan problem and the porous medium problem respectively. We finally point out that if an L^p-error estimate is known with $p < \infty$, then the rates of convergence for free boundaries are in measure (theorem 2.1); error estimates in distance are obtained for $p = \infty$ (theorem 2.2).

2. APPROXIMATION OF THE FREE BOUNDARIES

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$, and set $Q = \Omega \times (0, T)$ for $0 \leqslant T < \infty$. Let Ω be decomposed into finite elements, and let $k \in (\mathbb{R}^+)^I$ be a discretization parameter whose components tend to zero (k might be h (= mesh size) for a stationary problem; or $k = (h, \tau)$ for an evolutionary problem; or $k = (h, \varepsilon, \tau)$ for a regularized evolutionary problem; and so on). Let u and u_k be the continuous and discrete solutions respectively of some free boundary problem.

We define the *continuous free boundary* by

$$(2.1) \qquad F := \partial \{ x \in Q : u(x) > 0 \} \cap Q \qquad (= \partial Q^+ \cap Q),$$

and the *approximate free boundary* by

$$(2.2) \qquad F_k := \partial \{ x \in Q : u_k(x) > \delta \} \cap Q \qquad (= \partial Q_k^+ \cap Q),$$

where $\delta > 0$ is a constant to be determined later on (see theorems 2.1 and 2.2).

Let us consider the following hypotheses :

(H1) *there exists a function* $\sigma(k) : (\mathbb{R}^+)^I \to \mathbb{R}^+$, $\sigma(k) \downarrow 0$ *as* $k \downarrow 0^+$, *such that for some* p, $1 \leqslant p \leqslant \infty$,

$$\| u - u_k \|_{L^p(Q)} < \sigma(k),$$

(H2) *there exist two constants* α, $C^* > 0$, *such that*

$$\operatorname{meas}(\{ 0 < u < \varepsilon^\alpha \}) \leqslant C^*.\varepsilon, \quad \textit{for all} \quad 0 < \varepsilon < \varepsilon_0 .$$

Remark 2.1 : (H2) is the non-degeneracy property of the continuous solution u we talked about in the introduction.

Then we have the following *error estimate in measure* for the free boundaries.

THEOREM 2.1 : *Assume* (H1) *and* (H2), *and let* $\delta = \sigma(k)^{\alpha p/(1+\alpha p)}$. *Then there exists a constant* $C > 0$ *depending only on* α, *such that*

$$\operatorname{meas}(Q^+ \Delta Q_k^+) \leqslant C C^* \sigma(k)^{p/(1+\alpha p)},$$

where C^* *is from* (H2).

Proof : Since $Q^+ \Delta Q_k^+ = (Q^+ \setminus Q_k^+) \cup (Q_k^+ \setminus Q^+)$, we can analyse each set separately. For $Q^+ \setminus Q_k^+$ we have

$$Q^+ \setminus Q_k^+ = \{ x \in Q : 0 < u(x) < 2\delta \wedge u_k(x) \leqslant \delta \} \cup \{ x \in Q : u(x) \geqslant 2\delta \wedge u_k(x) \leqslant \delta \} = A \cup B .$$

Now, we observe that $A \subset \{ x \in Q : 0 < u(x) < 2\delta \}$ and, consequently, by (H2) we get $\operatorname{meas}(A) \leqslant C^*(2\delta)^{1/\alpha}$. For the set B, notice that if $x \in B$ then

$$u(x) - u_k(x) \geqslant \delta .$$

Thus, using the estimate of (H1) we can bound meas (B) as

$$\operatorname{meas}(B) \leqslant \frac{1}{\delta^p} \int_B | u - u_k |^p \leqslant \left(\frac{\sigma(k)}{\delta} \right)^p .$$

Therefore we have obtained the bound

$$\operatorname{meas}(Q^+ \setminus Q_k^+) \leqslant C^*(2\delta)^{1/\alpha} + \left(\frac{\sigma(k)}{\delta} \right)^p . \tag{2.3}$$

On the other hand, every $x \in Q_k^+ \setminus Q^+$ satisfies $u_k(x) - u(x) > \delta$, and then proceeding as in the analysis of B it follows immediately

$$\operatorname{meas}(Q_k^+ \setminus Q^+) \leqslant \left(\frac{\sigma(k)}{\delta} \right)^p . \tag{2.4}$$

Adding (2.3) and (2.4), and choosing $\delta = \sigma(k)^{\alpha p/(1+\alpha p)}$, we easily get the desired estimate. □

A more accurate description of the convergence of the discrete free boundaries requires stronger assumptions. For $K \subset\subset Q$ let us define

$$\mathcal{S}_\varepsilon(F \cap K) := \{ x \in Q : d(x, F \cap K) < C_K \varepsilon \}, \tag{2.5}$$

where $d(x, F)$ is the distance between x and F, and C_K is a suitable constant depending on K. Then we consider the hypothesis :

(H3) *there exists a constant* $C_K > 0$ *such that*

$$K \cap \{ 0 < u < \varepsilon^\alpha \} \subset \mathscr{S}_\varepsilon(F \cap K).$$

Now, we are in a position to prove the following *error estimate in distance* for the free boundaries.

THEOREM 2.2 : *Assume* (H1) *with* $p = \infty$ *and* (H3), *and in addition let* $\delta = \sigma(k)$. *Then* F_k *satisfies*

$$F_k \cap K \subset \mathscr{S}_{(2\sigma(k))^{1/\alpha}}(F \cap K). \tag{2.6}$$

Proof : Let $x \in F_k \cap K$. Then $u_k(x) = \delta = \sigma(k)$, and applying (H1) for $p = \infty$ we get,

$$0 = u_k(x) - \sigma(k) < u(x) < u_k(x) + \sigma(k) = 2\,\sigma(k).$$

Finally, by (H3) we obtain that

$$x \in \{ x \in K : 0 < u(x) < 2\,\sigma(k) \} \subset \mathscr{S}_{(2\sigma(k))^{1/\alpha}}(F \cap K),$$

which proves the theorem. □

Remark 2.2 : (2.6) says that the discrete free boundary lies in a $C_K(2\,\sigma(k))^{1/\alpha}$-neighborhood of the continuous one. Moreover, the proof of (2.6) also yields :

$$F_k \cap K \subset Q^+.$$

Remark 2.3 : clearly (2.6) is valid for $K = Q$ whenever this choice is admitted in (H3). For instance, this is true if the free boundary F is far away from $\partial\Omega \times [0, T]$.

3. THE OBSTACLE PROBLEM

In this section we consider, for the sake of simplicity, the following elliptic variational inequality :

$$\begin{cases} u \in K \\ \displaystyle\int_\Omega \nabla u . \nabla(v - u) \geqslant \int_\Omega f(v - u), \quad \text{for all } v \in K, \end{cases} \tag{3.1}$$

where $K = \{ v \in H^1(\Omega) : v \geqslant 0, v = g \text{ on } \Gamma \}$, and $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ. Let us assume that

$$f \in C^1(\overline{\Omega}), \quad g \in C^3(\overline{\Omega}) \quad \text{with} \quad g \geqslant 0 \text{ on } \Gamma\,; \tag{3.2}$$

then it is well-known that [4, 5, 20],

$$(3.3)\qquad u \in C^{1,1}(\Omega) \cap W^{s,p}(\Omega)\,, \quad 1 < p < \infty\,, \quad s < 2 + \frac{1}{p}.$$

We also assume

$$(3.4)\qquad f \leqslant C(f) < 0 \quad \text{in } \Omega.$$

Now, we recall some non-degeneracy properties on u proved by L. Caffarelli in [10] (see also [20]).

THEOREM 3.1 [10] : *For all $K \subset\subset \Omega$ there exist constants $C_1, C_2 > 0$ such that for all $\varepsilon > 0$,*

$$(3.5)\qquad \operatorname{meas}\left(\{ x \in \Omega : | \nabla u(x) | < \varepsilon \} \cap K\right) \leqslant C_1\, \varepsilon\,,$$

$$(3.6)\qquad \{ x \in \Omega : 0 < u(x) < \varepsilon^2 \} \cap K \subset \{ x \in \Omega : | \nabla u(x) | \leqslant C_2\, \varepsilon \} \cap K\,.$$

Furthermore, if the number of singular points of $F \cap K$ is finite, then

$$(3.7)\qquad \{ x \in \Omega : 0 < u(x) < \varepsilon^2 \} \cap K \subset \mathscr{S}_\varepsilon(F \cap K)\,.$$

Remark 3.1 : (3.7) needs some explanation. L. Caffarelli and N. Rivière have studied in [14] the asymptotic behaviour of u and F near a singular point. Then, combining these results with the good behaviour of u and F near regular points [20], (3.7) follows.

For the *primal* finite element approximation of (3.1) by using C^0-piecewises quadratics elements of Lagrange type (the size not greater than h), the following error estimate was proved in [7, 9],

$$(3.8)\qquad \| u - u_h \|_{H^1(\Omega)} = O(h^{3/2-\beta}), \quad \text{for all } \beta > 0\,.$$

The obstacle constraint was imposed only at the element midpoints (not at the vertices). Using the inverse inequality in $\mathbb{R}^2$: $\| v_h \|_{L^\infty(\Omega)} \leqslant C\,|\log h\,|\, \| v_h \|_{H^1(\Omega)}$, (3.8) yields

$$(3.9)\qquad \| u - u_h \|_{L^\infty(\Omega)} = O(h^{3/2-\beta})\,, \quad \text{for all } \beta > 0\,.$$

Then we choose $\delta = h^{3/2-\beta}$ in (2.2), and easily get the following consequence of theorems 2.1, 2.2 and 3.1.

COROLLARY 3.1 : *For $\beta > 0$ and $K \subset\subset \Omega$, we have*

$$(3.10)\qquad \operatorname{meas}\left((\Omega^+ \,\Delta\Omega_h^+) \cap K\right) \leqslant C h^{3/4-\beta}\,.$$

Furthermore, if (3.7) is verified, then

$$(3.11)\qquad F_h \cap K \subset \mathscr{S}_{h^{3/4-\beta}}(F \cap K)\,.$$

Remark 3.2 : the results of § 2 also apply to the primal formulation with C^0-piecewise linear elements. If one knows that $\| u - u_h \|_{L^\infty(\Omega)} < \sigma(h)$, setting $\delta = \sigma(h)^{1/2}$ one easily gets that the approximate free boundaries converge in measure and in distance with a rate of order δ. In particular, if the discrete maximum principle (D.M.P.) is valid, it is known (under the regularity assumption : $u \in C^{1,1}(\overline{\Omega})$) that $\sigma(h) = ch^2 \mid \log h \mid$, [24]. In this case we obtain the *same* rates of convergence for the free boundaries as in [6]. However, the geometrical restrictions necessary for a D.M.P. are not easy to verify in practice, for instance when triangulations are generated *automatically*. Still, optimal H^1-error estimates of order h are known, [7]. Consequently, by using inverse inequalities in $\mathbb{R}^2$, theorems 2.1, 2.2 and 3.1 give

$$(3.12) \quad \text{meas}\,((\Omega^+ \,\Delta \Omega_h^+) \cap K) \leqslant C_K(h \mid \log h \mid)^{1/2} ;$$

and if (3.7) holds, then

$$(3.13) \quad F \cap K \subset \mathcal{S}_{c(h|\log h|)^{1/2}}(F \cap K) .$$

The *mixed method* of Raviart-Thomas was used in [8] to approximate the obstacle problem (3.1); the piecewise linear approximation p_h of $p = \nabla u$ satisfies

$$(3.14) \quad \| p_h - p \|_{L^2(\Omega)} = O(h^{3/2-\beta}), \quad \text{for all } \beta > 0 .$$

Now, we set $\delta = h^{1-\beta}$ and define the discrete free boundaries by means of p_h (that is, $F_h := \partial \{ x \in \Omega : \max_{i=1,2} (| p_h^i(x) |) > \delta \} \cap \Omega$). Then, applying theorem 2.1 and (3.5), we get :

COROLLARY 3.2 : $\text{meas}\,((\Omega^+ \,\Delta \Omega_h^+) \cap K) \leqslant Ch^{1-\beta}$, *for all* $\beta > 0$.

We can also obtain some information about the convergence in distance. Indeed, by using inverse inequalities, we easily achieve the bound : $\| p_h - p \|_{L^\infty(\Omega)} = O(h^{1/2-\beta})$. Next, setting $\delta = h^{1/2-\beta}$ in (2.2) and applying theorem 2.2 with (3.7), we have

COROLLARY 3.3 : *If* (3.7) *holds, then*

$$F_h \cap K \subset \mathcal{S}_{h^{1/2-\beta}}(F \cap K), \quad \textit{for all } \beta > 0 .$$

Remark 3.3 : the mixed approximation obtained by piecewise constant elements of Raviart-Thomas provides a sequence p_h so that $\| p - p_h \|_{L^2(\Omega)} = O(h)$, [8]. Then, by choosing $\delta = h^{2/3}$ (and defining the approximate free boundaries by means of p_h), theorems 2.1 and 3.1 yield

$$(3.15) \quad \text{meas}\,((\Omega^+ \,\Delta \Omega_h^+) \cap K) \leqslant C_K\, h^{2/3}.$$

Remark 3.4 : if $g > 0$ on Γ, we can take $K = \Omega$ in the previous results (see remark 2.3).

Remark 3.5 : Theorem 3.1 is also valid for parabolic variational inequalities (one-phase Stefan problem), [20]. Then the results of this section provide a way of approximating the continuous free boundary (see also [28]).

4. THE TWO-PHASE STEFAN PROBLEM

In this section we consider the *enthalpy formulation* of the multidimensional two-phase Stefan problem ($\Omega \subset \mathbb{R}^N, N \geqslant 1$) :

$$(4.1) \qquad \begin{cases} \gamma(u)_t - \Delta_x u + f(u) = 0 \ , & \text{in } Q \\ u = g \, , & \text{on } \Gamma \times (0, T) \\ \gamma(u) = \gamma_0 \, , & \text{on } \Omega \times \{ 0 \} \, , \end{cases}$$

where $\gamma(u)$ (the enthalpy) satisfies the constitutive relation $\gamma(u) = c(u) + H(u)$. Here H is the Heaviside graph and c is a strictly increasing Lipschitz continuous function. It is well-known that there is a unique solution u (physically, the temperature); moreover it has the global regularity properties [11, 17, 18, 19, 23, 31] :

$$(4.2) \qquad \begin{array}{l} \text{assume } u_0 := \gamma^{-1}(\gamma_0) \in C^{0,1}(\overline{\Omega}) \text{ and } f \in C^{0,1}(\mathbb{R}), \text{ then} \\ u \in C^0(\overline{Q}) \cap L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \, . \end{array}$$

In addition, an important physical feature is that the free boundary F could be irregular (for instance, F could vary in a discontinuous manner) even if the data are very smooth, [19]. Besides, for $f \neq 0$, F could degenerate in a set with positive $(N + 1)$-dimensional Lebesgue measure (mushy region). However, under some qualitative assumptions upon the data, the author proved in [27] a non-degeneracy property of u. Let us first state the basic assumptions :

(4.3) $\quad g_t \geqslant \alpha > 0$,

(4.4) $\quad F_0 = \{ x \in \Omega : u_0(x) = 0 \}$ is a Lipschitz manifold,

(4.5) $\quad \Delta u_0 - f(u_0) \geqslant \alpha(1 + \mu_{F_0})$, where μ_{F_0} is a Radon measure supported on F_0, so that $\dfrac{d\mu_{F_0}}{dH_{N-1}} \geqslant$ (H_{N-1} is the $(N - 1)$-Hausdorff measure on F_0)

(4.6) $\quad u_0^+(x) \geqslant \alpha d(x, F_0)$ for $x \in \Omega^+$.

Then, using (4.3)-(4.6) plus some regularity hypotheses, we were able to prove the following result.

THEOREM 4.1 [27] : *Assume $f(0) \geqslant 0$. Then there exists a Lipschitz function $S : \bar{\Omega} \to [0, T]$ such that*

$$F = \{ (x, t) \in Q : t = S(x) \}. \tag{4.7}$$

Moreover there exists a constant $C > 0$ depending only on the data such that

$$u(x, S(x) + r) \geqslant Cr\,, \quad \text{for} \quad 0 < r \leqslant T - S(x)\,. \tag{4.8}$$

Due to the low regularity of the problem, the numerical approximation of (4.1) is usually carried out by using a regularization procedure as a first step [21, 25, 26] ($\varepsilon > 0$ will denote the regularization parameter). The second step consists in a standard discretization procedure obtained by using C^0-piecewise linear finite elements in space and backward differences in time (h and τ will denote the sizes of the spatial and time decompositions respectively). Let $k = (h, \varepsilon, \tau)$ and let u_k be the approximate solution of (4.1). Let us assume the following non-degeneracy property,

$$\text{meas}\,(\{ (x, t) \in Q : 0 \leqslant u(x, t) \leqslant r \}) \leqslant Cr\,, \quad \text{for all } r > 0\,. \tag{4.9}$$

Clearly, (4.8) implies (4.9). Then, the error estimate for temperatures

$$\| u - u_k \|_{L^2(Q)} \leqslant Ch\,, \tag{4.10}$$

was proved in [25] under the relations $h \sim \varepsilon \sim \tau^{2/3}$.

Consequently, by choosing $\delta = h^{2/3}$ in (2.2) and using theorem 2.1 we obtain the following *error estimate in measure* for F.

COROLLARY 4.1 : $\text{meas}\,(Q^+ \Delta Q_k^+) \leqslant Ch^{2/3}$.

Remark 4.1 : estimates like (4.9) and (4.10) are also true for other boundary conditions, including either linear or non-linear flux through Γ [25, 26]. Therefore the previous result is extended to this case.

If (4.9) is not satisfied, the error estimate (4.10) becomes

$$\| u - u_k \|_{L^2(Q)} \leqslant Ch^{2/3}\,, \tag{4.11}$$

provided that $h^{4/3} \sim \varepsilon \sim \tau$ [21, 25]. Then, in general, we have « two » free boundaries : $F^+ = \partial(\{ u > 0 \}) \cap Q$, $F^- = \partial(\{ u < 0 \}) \cap Q$. Finally, if we know that u leaves either F^+ or F^- with a « minimum speed » property, we can still obtain a result in measure like Corollary 4.1.

5. THE POROUS MEDIUM PROBLEM

In this section we consider the *porous medium equation* in several space variables ($\Omega \subset \mathbb{R}^N$, $N \geqslant 1$) :

$$(5.1) \quad \begin{cases} u_t - \Delta_x u^m = 0\,, & \text{in } Q\,, \\ u = 0\,, & \text{on } \partial\Omega \times (0, T)\,, \\ u = u_0\,, & \text{on } \Omega \times \{0\}\,, \end{cases}$$

where $u\,(\geqslant 0)$ represents the density of a gas that diffuses in Ω, $v = \dfrac{m}{m-1}\,u^{m-1}$ is the (normalized) pressure, $m > 1$ and $u_0 \geqslant 0$ is a smooth function (say, $u_0^m \in C^{0,1}(\bar{\Omega})$). The most striking manifestation of the singular character of (5.1) is that the interface $F(t) = \partial\Omega^+(t)$ has a finite speed of propagation ($\Omega^+(t) := \operatorname{supp} u(.,t)$). Moreover, $\Omega^+(t)$ expands as t increases and this expansion is strict after a possible waiting time that is related to the shape of u_0 near $F_0 = \partial\Omega^+(0)$. Indeed, let us assume [13, 20, 22]

$$(5.2) \qquad F_0 \in C^2 \quad \text{and} \quad u_0(x) \geqslant C^*\, d(x, F_0)^{\mu/(m-1)} \text{ for some } \mu < 2\,.$$

Then, the free boundary $F(t)$ is strictly increasing at $t = 0$ (and also for all $t > 0$), i.e. $\Omega^+(t) \supset \bar{\Omega}^+(0)$ for $t > 0$. For $N = 1$ this was proved by B. Knerr [22]; the proof for $N > 1$ is similar [13, 20]. Moreover, if $u_0(x) \leqslant C^*.\, d(x, F_0)^{\mu/(m-1)}$, $\mu \geqslant 2$ and $\Omega^+(0)$ is convex there exists $t^* > 0$ (waiting time) such that $\Omega^+(t) = \Omega^+(0)$, for $t \leqslant t^*$, [1, 20, 22]. The local behavior of F and u was studied by L. Caffarelli and A. Friedman in [13] (see also [20]). They have proved :

THEOREM 5.1 [13] : *If* (5.2) *holds, then there exist constants* $0 < \gamma < 1$ *and* $C_i > 0$ $(1 \leqslant i \leqslant 4)$ *depending only on* $t_0 > 0$ *such that* :

(5.3) $F = \{(x, t) \in Q : t = S(x)\}$, *where S is uniformly Hölder continuous (exponent γ) in every set* $\{x \in \Omega : S(x) \geqslant t_0\}$,

(5.4) $\Omega^+(t + r)$ *contains a* $(C_1\, r^{1/\gamma})$*-neighborhood of* $\Omega^+(t)$ *provided* $0 < r < 1$, $t \geqslant t_0$,

(5.5) $\Omega^+(t + r)$ *is contained in a* $(C_2\, r^{1/2})$*-neighborhood of* $\Omega^+(t)$ *provided* $0 < r < 1$, $t \geqslant t_0$,

$$(5.6) \qquad v(x, S(x) + r) \geqslant C_3\, r^{(2-\gamma)/\gamma} \quad \text{if} \quad S(x) \geqslant t_0, r \leqslant C_4\,.$$

Remark 5.1 : (5.6) deserves a comment because it was not explicitly stated in [13], neither in [20]. The proof is sketch in the appendix, but relies on lemmas of intrinsic interest regarding the manner in which the gas is expanding [13, 20].

In one space dimension much more is known about the solution and the interface. Indeed, if $\Omega^+(0)$ is an interval, the following result is well-known [12] (see also [1, 20, 22]).

THEOREM 5.2 [12] : *Suppose that* (5.2) *holds and* $N = 1$. *Then the free boundary is described by two strictly monotone functions* ζ_i $(i = 1, 2)$ *such that*

(5.7) $\quad \zeta_i \in C^1(0, T), \zeta_1 < \zeta_2,$

(5.8) $\quad v_x(\zeta_i(t), t) = -\zeta_i'(t) \quad$ *for all* $0 < t \leqslant T$.

Furthermore, for all $t_0 > 0$ *there exists* $C(t_0) > 0$ *such that*

(5.9) $\quad v(\zeta_i(t) - (-1)^i r, t) \geqslant C(t_0).r, \quad$ *for all* $t \geqslant t_0$ $(r > 0)$.

Consequently, under the assumption (5.2) we can assume the non-degeneracy property :

(5.10) $\quad \text{meas}\,(\{(x, t) \in Q : t \geqslant t_0 > 0, 0 < u(x, t) < \varepsilon^\alpha\}) \leqslant C(t_0)\,\varepsilon$.

Clearly, (5.6) and (5.9) imply (5.10) with $\alpha = (2-\gamma)/\gamma(m-1)$ and $\alpha = 1/(m-1)$ respectively.

The numerical approximation of (5.1) was given in [30], the approximation scheme being similar to that of the previous section. We only point out that the regularization procedure consists in replacing $\gamma(s) = s^{1/m}$ by a function with maximal slope equal to $1/\varepsilon$ ($\varepsilon = \beta^{m-1}$, β regularization parameter in [30]). Let us call u_k the approximate solution, where $k = (h, \varepsilon, \tau)$. Then it is known that [30] :

(5.11) $\quad \| u - u_k \|_{L^{m+1}(Q)} \leqslant Ch^\sigma$,

where

$$(5.12) \qquad \sigma = \begin{cases} 2/m, & \text{if } \varepsilon \sim h^{(m-1)\sigma}, \quad \Delta t \sim h^2 \qquad (N = 1) \\ \dfrac{2(m+1)}{m^2 + 2m - 1}, & \text{if } \varepsilon \sim h^{(m-1)\sigma}, \quad \Delta t \sim h^{m\sigma} \quad (N > 1). \end{cases}$$

When $N > 1$, $m > 2$ the convergence rates are probably not sharp [30]. The original error estimates in [30] contain a logarithmic factor that may be avoided using a recent result in [29].

Finally, the results of § 2 may be applied to the porous medium equation. Let $p = m + 1$ and set $\delta = h^{\sigma\alpha p/(1+\alpha p)}$, with α given by (5.10). Then, taking (5.10)-(5.11) into account and using theorem 2.1, we get the following *error estimate in measure* for the free boundaries.

COROLLARY 5.1 : $\text{meas}\,((Q^+ \Delta Q_k^+) \cap \{ t > t_0 \}) \leqslant C(t_0)\, h^{\sigma \frac{p}{1+\alpha p}}$.

Remark 5.2 : In the previous result we may take $t_0 = 0$ whenever this choice is possible in (5.10) ; for instance, provided $N = \mu = 1$ [1].

APPENDIX

This part of the work is devoted to prove the non-degeneracy property (5.6), which is an immediate consequence of the results of [13]. In the sequel we reformulate two fundamental lemmas proved in [13] that describe the manner in which the gas expands in a porous medium. Although they are stated in a somewhat different way, the proof may be found in [13]. We shall denote :

$$\fint_{B_R(x)} v = (\operatorname{meas}(B_R(x)))^{-1} \int_{B_R(x)} v .$$

LEMMA A.1 : *There exist two constants $\gamma_1, \gamma_2 > 0$ depending only on m and N such that if*

$$v(x, t_0) = 0 \quad \textit{for} \quad x \in B_R(x_0), \quad \textit{and}$$

$$\fint_{B_R(x_0)} v(x, t_0 + \sigma) \leqslant \gamma_1 \frac{R^2}{\sigma}, \quad \textit{for} \quad \sigma \leqslant \gamma_2 t_0 ,$$

then $v(x, t_0 + \sigma) = 0$ *for* $x \in B_{R/6}(x_0)$.

LEMMA A.2 : *Let $\nu > 0$ be given, then there exist constants $\lambda, \gamma_3, \gamma_4 > 0$ depending only on ν, m and N such that if*

$$\fint_{B_R(x_0)} v(x, t_0) \geqslant \nu \frac{R^2}{\sigma} \quad \textit{for} \quad \sigma \leqslant \gamma_3 t_0 ,$$

then $$v(x_0, t_0 + \lambda\sigma) \geqslant \gamma_4 \frac{R^2}{\sigma} .$$

Proof of (5.6) : Let $x_0 \in \Omega$ so that $S(x_0) \geqslant t_0 > 0$. Then, recalling that S is uniformly Hölder continuous in $\{ t \geqslant t_0/2 \}$ (with exponent $\gamma = \gamma(t_0)$ and constant $C = C(t_0)$) and taking $r \leqslant t_0/2$, we easily get

$$v(x, S(x_0) - r) = 0 , \quad \text{for} \quad x \in B_{(r/C)^{1/\gamma(x_0)}}(x_0) .$$

Next, let $\tilde{C} = \frac{1}{2} \min(1, \gamma_2, \gamma_3)$ and $r \leqslant \tilde{C} t_0 \left(\leqslant \gamma_2 \frac{t_0}{2} \right)$.

Since $(x_0, S(x_0)) \in F$, lemma A 1 yields

$$\fint_{B_{(r/C)^{1/\gamma}}(x_0)} v(x, S(x_0)) > \gamma_1\, C^{-2/\gamma}\, r^{(2-\gamma)/\gamma}$$

Now, use lemma A 2 to obtain

$$v(x_0, S(x_0) + \lambda r) \geqslant \gamma_4\, C^{-2/\gamma}\, r^{(2-\gamma)/\gamma},$$

because $r \leqslant \gamma_3 \frac{t_0}{2}$ Finally, defining $C_3 = \gamma_4\, C^{-2/\gamma}\, \lambda^{(\gamma-2)/\gamma}$ and $C_4 = \lambda \tilde{C} t_0$, the last inequality implies (5 6) completing the proof □

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