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STUDY OF A BOUNDARY LAYER PROBLEM IN ELASTIC COMPOSITE MATERIALS (*)

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Abstract — *We consider an elastic stratified material with a periodic structure, we propose to define and calculate the local stresses in the neighbourhood of a free boundary Assuming that the microscopic displacement and stresses are periodic in the direction of the stratification, an homogenization method gives an approximation of the micro-stresses within the material Since this approximation is not valid near the boundary, we define the micro-stresses as the sum of the microscopic stresses of the classical homogenization and boundary layers stresses which are periodic parallel to the boundary By using a result due to Tartar, we prove that the additional stresses satisfy a well-posed problem and decrease exponentially with the orthogonal boundary variable We present some numerical results which show the improvement in the stress calculation, due to the boundary layers terms, near the boundaries This method applies to other composite materials and other types of boundary conditions*

Résume — *On considère un matériau élastique stratifié à structure périodique, on se propose de définir et de calculer les contraintes locales au voisinage d'un bord libre d'effort Une methode d'homogeneisation permet d'obtenir une approximation des microcontraintes au sein du matériau en supposant le déplacement et les contraintes microscopiques périodiques suivant la direction de la stratification Mais cette approximation tombe en défaut dans les régions voisines du bord, on est donc conduit a définir les microcontraintes au voisinage du bord comme la somme des contraintes microscopiques de l'homogénéisation classique et des contraintes de couche limite périodiques parallelement à la frontiere étudiée Nous utilisons un résultat de Tartar pour prouver que ces contraintes additionnelles satisfont un problème bien posé et qu'elles décroissent exponentiellement suivant la variable orthogonale au bord libre Nous présentons des résultats numériques qui mettent en évidence la contribution des termes de couche limite au voisinage d'une frontière libre Cette méthode s'applique également à d'autres composites ainsi qu'à d'autres types de bord*

1. INTRODUCTION

The homogenization theory, which allows to analyse the behaviour of the composite materials with periodic structures, has been the subject of intensive studies, (see for example : [1], [6], [9], [11], [13], ...). We remind that it consists in

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substituting a non homogeneous material for an homogeneous material with equivalent mechanic properties. Among the various methods to approach this theory, we choose in this paper the asymptotic expansion technique which is shortly presented in the section 2. This method consists in expanding the displacement and stresses in periodic terms according to the periodicity of the structure and it gives the global displacement and stresses within the material at the macroscopic scale. But in practice, we need a more precise analysis of the local stress field, at the microscopic scale of the heterogeneities, specially near the boundaries. It is, for example, necessary to study the initiation of local damage in the forms of delamination fracture whose the micro-stresses are certainly responsible in the neighbourhood of the boundaries in laminated composites. After solving the homogenized problem, the asymptotic expansion technique allows to obtain an approximation of the micro-stresses within the material by a localization method. But in this way, the micro-stresses do not satisfy the boundary conditions of Neumann, in addition they are supposed periodic as the structure and this hypothesis must be discussed near a boundary. Consequently, the approximation obtained by the classical homogenization theory, is not very satisfactory in the neighbourhood of a Neumann boundary.

The main of this paper is to define an approximation of the local stresses near a Neumann boundary for an elastic material with a periodic structure. Here, the method is applied to a material composed of homogeneous isotropic layers near an any free boundary; but it remains valid for other composites and other boundaries. The local study, presented in the section 3, is inspired by the results of [1], [2], [9] concerning the homogenization of the diffusion equation near a Dirichlet boundary. To the classical terms of the expansion of the displacement and stresses of the homogenization theory, we add boundary layers terms which are supposed periodic parallel to the boundary. The introduction of Lekhniskii's stress potentials [8] into the problem verified by the boundary layers terms, allows us to study the static formulation of this problem. In this way, it divides into two independant scalar problems posed on a strip, which is semi-infinite in the perpendicular direction to the free edge. Such as the diffusion equation [9], we applied a lemma due to Tartar [9] to these two elliptic Dirichlet problems, one of them being of the 4th order. This lemma allows us to prove that they are well-posed and that the boundary layers stresses decrease exponentially according to the orthogonal variable to the free boundary. We study also the kinematic formulation of the boundary layer problem where the unknown is the boundary layer displacement. This formulation leads to consider a second order elliptic system with boundary Neumann conditions, which we study by using the above- mentioned lemma. This method is illustrated by some numerical results. The microscopic stresses of the classical homogenization, the boundary layer stresses and the stresses prevailing in the heterogeneous material are computed near a free boundary supposed to be an inclined plane at angle $\pi/4$ to the direction of the stratification. A detailed analysis of these computations is given in [5]. In the section 7, we confine ourselves to present some curves where it is obvious that the boundary layers stresses decrease exponentially and that their sum with the classical micro-stresses are a better approximation of the real stresses near a free boundary than are micro-stresses only.

I am indebted to E. Sanchez-Palencia for the fruitful conversations that we had and for his careful reading of this study.

Contents

2. AN APPROXIMATION OF THE MICRO-STRESSES WITHIN THE MATERIAL

We consider in $\mathbb{R}^3$ a bounded domain Ω, which smooth boundary $\partial\Omega$. Let us set $x = (x_1, x_2, x_3)$ the Cartesian coordinates of a point of Ω, with respect to the axes $R = (0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$. An elastic body occupies the domain Ω under the classical hypothesis of linear small deformation [7]. The body is subjected to forces of density $f = (f_1, f_2, f_3)$, is fixed for example on a portion Γ_1 of its boundary and we assume that the remainder Γ_2 of its boundary is free. The material is a laminated composite with periodic structure in the $\vec{e}_3$ direction. Each layer is elastic, homogeneous and isotropic. One denotes $\varepsilon - Z$ the period of the material, ε is a small parameter which represents the size of the basic period $Z =]0, Z^*[$ and which should be converge to zero. The elastic behaviour of the composite is characterized by the functions :

$$\mathscr{A}(z) = \{ a_{ijkh}(z) \},$$

defined on Z and extended to the whole space by Z periodicity. We assume that they satisfy the following properties :

$$\left.\begin{array}{l}\bullet\ a_{ijkh}(z) \in L^{\infty}(Z) \\ \bullet\ a_{ijkh} = a_{khij} = a_{jikh} \\ \bullet\ a_{ijkh}\, e_{ij}\, e_{kh} \geqslant c e_{ij}\, e_{ij}\, (c > 0)\, (\forall e_{ij} = e_{ji})\end{array}\right\} \tag{2.1}$$

(the convention of summation on the repeated indices is used in all this paper.)

The local elastic coefficients are given by :

$$\mathscr{A}^{\varepsilon}(x) = \mathscr{A}\left(\frac{x_3}{\varepsilon}\right) = \left\{ a_{ijkh}\left(\frac{x_3}{\varepsilon}\right)\right\}.$$

One sets, P^{ε}, the equilibrium problem defined by :

$$\left.\begin{array}{l}\bullet\ \mathrm{Div}_x\,(\boldsymbol{\sigma}^{\varepsilon}(x)) = f, \quad \text{in} \quad \Omega \\ \bullet\ \boldsymbol{\sigma}^{\varepsilon}(x) = \mathscr{A}^{\varepsilon}(x)\, \varepsilon(u^{\varepsilon}(x)), \quad \text{in} \quad \Omega \\ \bullet\ u^{\varepsilon}(x) = 0, \quad \text{on} \quad \Gamma_1 \\ \bullet\ \boldsymbol{\sigma}^{\varepsilon}(x).n = 0, \quad \text{on} \quad \Gamma_2\end{array}\right\} \tag{2.2}$$

where Div_x represents the operator of the divergence : $(\mathrm{Div}_x(A))_i = \dfrac{\partial A_{ij}}{\partial x_j}$, $\boldsymbol{\sigma}^{\varepsilon}$ is the Cauchy tensor, u^{ε} the displacement, $\boldsymbol{\varepsilon}$ the strain tensor :

$$\varepsilon_{ij}\,(V) = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)$$

and n is the outside unit normal of Ω.

We shall look for the solution $(u^{\varepsilon}, \boldsymbol{\sigma}^{\varepsilon})$ of P^{ε} in the following ansatzs :

$$u^{\varepsilon}(x) = u^0(x) + \varepsilon u^1(x, z) + \varepsilon^2 \ldots \tag{2.3}$$

$$\boldsymbol{\sigma}^{\varepsilon}(x) = \boldsymbol{\sigma}^0(x, z) + \varepsilon \boldsymbol{\sigma}^1(x, z) + \varepsilon^2 \ldots \tag{2.4}$$

where in (2.3) and (2.4) $\qquad z = \dfrac{x_3}{\varepsilon} \qquad$ (2.5)

and where the functions $u^j(x, z)$ and $\boldsymbol{\sigma}^k(x, z)\,(j \geqslant 1)\,(k \geqslant 0)$ are :

$$\left.\begin{array}{l}\bullet\ \text{defined for } x \in \Omega \text{ and } z \in Z \\ \bullet\ Z \text{ periodic in } z.\end{array}\right\} \tag{2.6}$$

One can show that :

$$\left.\begin{array}{ll} & \sigma_{ij}{}^0(x, z) = b_{ijkh}(z)\, \varepsilon_{pq}(u^0) \\ \text{with} & b_{ijkh}(z) = a_{ijkh}(z) - a_{ijpq}(z)\, e_{pq}(\chi^{kh}),\end{array}\right\} \tag{2.7}$$

where the functions $\chi^{kh}(z)$ satisfy :

$$\begin{cases} \bullet\ \chi^{kh}(z) \text{ are } Z\text{-periodic} \\ \bullet\ \dfrac{d}{dz}\left[a_{i3pq}(z)\, e_{pq}(\chi^{kh}(z))\right] = -\dfrac{d}{dz}\left[a_{i3kh}(z)\right], \text{ in } Z \end{cases}$$

the not-zero components of e being :

$$e_{\alpha 3}(v(z)) = \frac{1}{2}\frac{d}{dz}(v_\alpha)\ (\alpha = 1, 2),\ e_{33}(v(z)) = \frac{d}{dz}(v_3)\,.$$

We know ([1], [6], [9], [11], [13] ...), that the elastic homogenized behaviour — being the result of the convergence to zero of the coefficient ε — is obtained by solving the following problem :

$$\left.\begin{aligned} &\bullet\ \mathrm{Div}_x\left(\langle \boldsymbol{\sigma}^0(x, z) \rangle_Z\right) = f \\ &\bullet\ \langle \boldsymbol{\sigma}^0(x, z) \rangle_Z = \langle b_{ijkh}(z) \rangle_Z\, \varepsilon_{kh}(u^0) \end{aligned}\right\} \tag{2.8}$$

$$\bullet\ u^0(x) = 0, \text{ on } \Gamma_1$$

$$\bullet\ \langle \sigma^0(x, z) \rangle_Z . n = 0, \text{ on } \Gamma_2 \tag{2.9}$$

where $\langle \cdot \rangle_Z$ denotes the mean value on Z : $\langle g \rangle_Z = \dfrac{1}{Z^*}\displaystyle\int_0^{Z^*} g(z)\, dz.$

In practice, we get an approximation of the microscopic stresses from (2.7), once homogenized problem (2.8), (2.9) is solved.

This approximation is correct far enough from the boundary of Ω, but it is not very satisfactory in the neighbourhood of the boundaries ([9], [11]). There are at least two main reasons ; firstly we have generally :

$$\boldsymbol{\sigma}^0(x, z).n \not\equiv 0 \quad \text{on} \quad \Gamma_2$$

and the boundary condition on Γ_2 is only satisfied by the mean value of $\boldsymbol{\sigma}^0$, (see (2.9)) ; secondly near the boundaries, there is no reason why the stresses $\boldsymbol{\sigma}^0$ in each period are similar to those of the period beside.

One suggests to introduce, in expansions (2.3) and (2.4), boundary layer terms for which the Z periodicity hypothesis is replaced by an hypothesis of periodicity parallel to the free boundary. The boundary layer stresses are defined on the boundary so that their sum with microscopic stresses $\boldsymbol{\sigma}^0$ (2.7) satisfy the free boundary condition at each point.

3. STATEMENT OF THE BOUNDARY LAYER PROBLEM

We assume that the free boundary Γ_2 is a plane (it is not a restriction : providing that Γ_2 is smooth enough near a point, one can identify it with the tangent plane). One specifies that Γ_2 is generated by the vectors $\vec{e}_1$ and $\vec{e}'_2 = \cos \alpha \, \vec{e}_2 + \sin \alpha \, \vec{e}_3$ with $0 < \alpha \leqslant \pi/2$; it means that Γ_2 is inclinated at angle α to the plane $(\vec{e}_1, \vec{e}_2)$. Then it is convenient to change the set of the axes and to use $R' = (0, \vec{e}_1, \vec{e}'_2, \vec{e}'_3)$; O is the point of Γ_2 which is the subject of the following study and $\vec{e}'_3$ is the unit vector defined by : $\vec{e}'_3 = - \sin \alpha \, \vec{e}_2 + \cos \alpha \, \vec{e}_3$.

One denotes $x' = (x'_1, x'_2, x'_3)$ the coordinates of a point of Ω referring to axes R'.

We assume that the point O is far enough from another boundary. One introduces the following microscopic variables :

$$y_2 = \frac{x_2}{\varepsilon} \qquad y_3 = \frac{x_3}{\varepsilon}. \tag{3.1}$$

We can easily show that :

$$z = \sin \alpha y_2 + \cos \alpha y_3 . \tag{3.2}$$

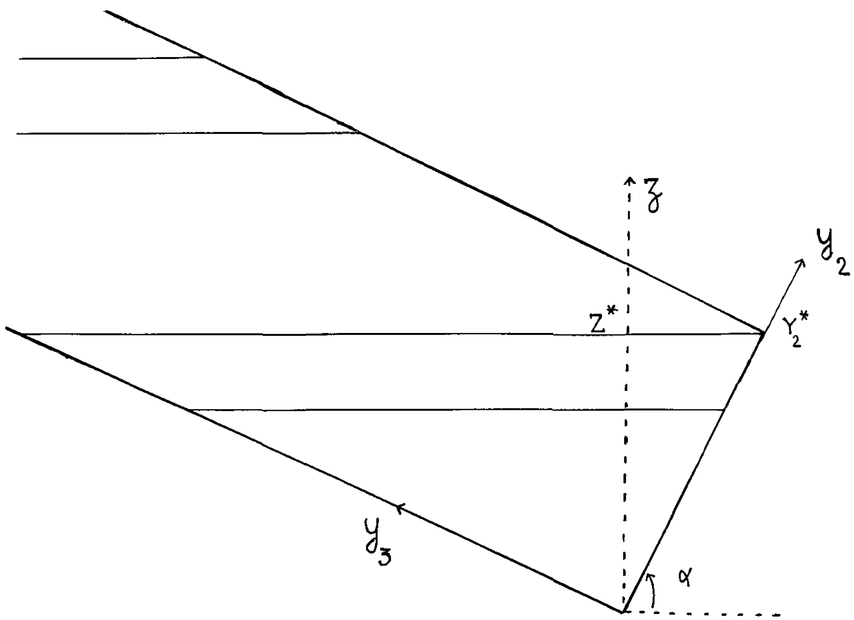

Figure 1. — The semi-infinite strip G.

Let us set :

$$G = Y_2 \times]0, +\infty[\quad (\text{see } \textit{fig. } 1)$$

with $Y_2 =]0, Y_2^*[$ and $Y_2^* = \dfrac{Z^*}{\sin \alpha}$.

We can remark that the domain G will reproduce by Y_2 periodicity and that the elastic coefficients :

$$\tilde{\mathscr{A}}(y_2, y_3) = \mathscr{A}(z = \sin \alpha y_2 + \cos \alpha y_3)$$

are Y_2-periodic, at y_3 fixed.

So that in the neighbourhood of the point O, the structure can be considered Y_2 periodic and then it is natural to try to find the displacement $u^\varepsilon(x)$ and the stresses $\boldsymbol{\sigma}^\varepsilon(x)$ as follows :

$$\left.\begin{aligned} u^\varepsilon(x') &= u^0(x') + \varepsilon[u^1(x', z) + u^{1BL}(x', y_2, y_3)] + \varepsilon^2 \ldots && (3.3)\\ \boldsymbol{\sigma}^\varepsilon(x') &= [\boldsymbol{\sigma}^0(x', z) + \boldsymbol{\sigma}^{0BL}(x', y_2, y_3)] + \varepsilon \ldots && (3.4) \end{aligned}\right\}$$

with (y_2, y_3) and z given by (3.1) and (3.2), where the boundary layer terms : $u^{iBL}(x', y_2, y_3)$ $(i \geqslant 1)$ and $\boldsymbol{\sigma}^{jBL}(x', y_2, y_3)$ $(j \geqslant 0)$ are :

$$\left.\begin{aligned} &\bullet \ \text{defined for } x \in \Gamma_2 \ (\text{i.e. } x' = (x_1', x_2', 0)) \text{ and } (y_2, y_3) \in G \\ &\bullet \ Y_2\text{-periodic in } y_2 \end{aligned}\right\} \quad (3.5)$$

and satisfy :

- $\boldsymbol{\sigma}^{jBL}(x', y_2, y_3)$ converge to zero as $y_3 \mapsto +\infty$ (3.6)
- $\boldsymbol{\sigma}_{p3}^{jBL}(x_1', x_2', 0, y_2, 0) = -\,\boldsymbol{\sigma}_{p3}^{j}(x_1', x_2', 0, y_2, 0)\,,$ (3.7)

$$\forall y_2 \in Y_2\,, \quad (\forall p = 1, 2, 3)$$

and where $u^0(x')$, $u^i(x', z)$ $(i \geqslant 1)$, $\boldsymbol{\sigma}^j(x', z)$ $(j \geqslant 0)$ are the classical terms of expansions (2.3), (2.4) expressed in the system R'.

Following the classical method, we replace asymptotic expansions (3.3), (3.4) into the equilibrium equation and the constitutive relations (2.2); we identify powers of ε, in this way we find for the boundary layer terms at the first orders :

$$\left.\begin{aligned} &\bullet \ \frac{\partial}{\partial y_2}[\boldsymbol{\sigma}_{i2}^{0BL}] + \frac{\partial}{\partial y_3}[\boldsymbol{\sigma}_{i3}^{0BL}] = 0, \quad (\forall i = 1, 2, 3) \\ &\bullet \ \boldsymbol{\sigma}_{ij}^{0BL}(x', y_2, y_3) = \tilde{a}_{ijkh}(y_2, y_3)\, e_{kh}(u^{1BL}(x', y_2, y_3)) \end{aligned}\right\} \quad (3.8)$$

where e denotes the strain tensor in (y_2, y_3) : $e_{ij}(v) = \dfrac{1}{2}\left[\dfrac{\partial v_i}{\partial y_j} + \dfrac{\partial v_j}{\partial y_i}\right]$. Let us set, P^{BL}, the boundary layer problem defined by equations : (3.5), (3.6), (3.7), (3.8).

Remark 3.1 : In expansions (3.3) and (3.4), we have postulated that the terms $u^{iBL}(x', y_2, y_3)$ and $\sigma^{iBL}(x', y_2, y_3)$ are functions of $x' \in \Gamma_2$. In fact, we can consider these terms depending on $x' \in \Omega$, but the following study shows easily that they depend on x_3 only as a function of $\frac{x_3}{\varepsilon}$.

Remark 3.2 : Homogenized boundary conditions (2.9) is a necessary and sufficient condition for the existence of the boundary layer.

4. STATIC FORMULATION OF THE BOUNDARY LAYER PROBLEM

In this section, we choose to consider the boundary layer stresses as the unknowns of the boundary layer problem, which explains why we introduce the Lekhnitskii's stress potentials [8] :

$$\Psi(x', y_2, y_3) \quad \text{and} \quad \Phi(x', y_2, y_3)$$

defined for $x' \in \Gamma_2$ and $(y_2, y_3) \in G$ such that :

$$\left.\begin{aligned} \bullet\ \frac{\partial \Psi}{\partial y_3}(x', y_2, y_3) &= -\sigma_{12}^{0BL}(x', y_2, y_3) \\ \bullet\ \frac{\partial \Psi}{\partial y_2}(x', y_2, y_3) &= \sigma_{13}^{0BL}(x', y_2, y_3) \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \bullet\ \frac{\partial^2 \Phi}{\partial y_2\, \partial y_2}(x', y_2, y_3) &= \sigma_{33}^{0BL}(x', y_2, y_3) \\ \bullet\ \frac{\partial^2 \Phi}{\partial y_3\, \partial y_3}(x', y_2, y_3) &= \sigma_{22}^{0BL}(x', y_2, y_3) \\ \bullet\ \frac{\partial^2 \Phi}{\partial y_2\, \partial y_3}(x', y_2, y_3) &= -\sigma_{23}^{0BL}(x', y_2, y_3) \end{aligned}\right\} \tag{4.2}$$

In this way, the study of the boundary layer problem leads us to consider two scalar elliptic problems, one of them being of the 2nd order and the other of the 4th order.

Remark 4.1 : Given that $e_{11}(u^{1BL}) = 0$, constitutive relations (3.8) allows us to show that :

$$\sigma_{11}^{0BL} = \frac{\nu}{E}(y_2, y_3)\left(\frac{\partial^2 \Phi}{\partial y_2\, \partial y_2} + \frac{\partial^2 \Phi}{\partial y_3\, \partial y_3}\right)$$

where $E(y_2, y_3)$ is Young's modulus and $\nu(y_2, y_3)$ Poisson's ratio of the material.

4.1. Study of the second order problem

The study of the boundary layer problem shows that the function $\Psi(x', y_2, y_3)$ is defined up to the addition of an arbitrary function $\tilde{\Psi}(x')$ and that it can be expressed as follows :

$$\Psi(x', y_2, y_3) = \tau^{rs}(y_2, y_3)\, \varepsilon_{rs}(u^0)\, |_{(x_1', x_2', 0)} + \tilde{\Psi}(x_1', x_2', 0) \qquad (4.4)$$

where $u^0(x')$ is the solution of homogenized problem (2.8), (2.9), and where functions $\tau^{rs}(y_2, y_3)$ satisfy the following problem on G :

$$\begin{cases} \bullet\ \dfrac{\partial}{\partial y_2}\left[\left(\dfrac{1+\nu}{E}\right)\dfrac{\partial \tau^{rs}}{\partial y_2}\right] + \dfrac{\partial}{\partial y_3}\left[\left(\dfrac{1+\nu}{E}\right)\dfrac{\partial \tau^{rs}}{\partial y_3}\right] = 0 & (4.5) \\ \bullet\ \tau^{rs}(y_2, 0) = h^{rs}(y_2) = \displaystyle\int_0^{y_2} b_{13rs}(t, 0)\, dt - \frac{y_2}{Y_2^*}\int_0^{Y_2^*} b_{13rs}(y_2, 0)\, \mathrm{d}y_2 & (4.6) \\ \bullet\ \left(\dfrac{\partial \tau^{rs}}{\partial y_2}, \dfrac{\partial \tau^{rs}}{\partial y_3}\right) \text{ converge to 0 as } y_3 \mapsto +\infty & (4.7) \\ \bullet\ \tau^{rs}(y_2, y_3) \text{ are } Y_2\text{-periodic in } y_2 . & (4.8) \end{cases}$$

Relation : $\dfrac{\partial^2 u_1^{1BL}}{\partial y_2\, \partial y_3} = \dfrac{\partial^2 u_1^{1BL}}{\partial y_3\, \partial y_2}$ and formulas (3.8), (4.1), (4.4) implie (4.5). Equations (4.6) and (4.7) are obtained from boundary conditions (3.6), (3.7). And (4.8) follows from (3.5) and from homogenized boundary condition (2.9).

Following the method used by Lions [9] for the diffusion equation, which consists in applying a lemma due to Tartar ([9], p. 49-58), after stated, one can easily show the below result :

THEOREM 4.1 : *There exists a real* $\gamma > 0$ *such that the problem* :

$$\begin{cases} \bullet\ (4.5), (4.6), (4.7) \\ \bullet\ \left(e^{\gamma y_3}\dfrac{\partial \tau^{rs}}{\partial y_2}\right) \in L^2(G), \left(e^{\gamma y_3}\dfrac{\partial \tau^{rs}}{\partial y_3}\right) \in L^2(G) \end{cases}$$

admits a unique solution $\tau^{rs} \in V^{rs}$, *where* V^{rs} *is given by* : $V^{rs} = \{ v \in L^2(0, R; V(Y_2))\ \forall \text{ finite } R,\ e^{\gamma y_3}\dfrac{\partial}{\partial y_2} v \in L^2(G),\ e^{\gamma y_3}\dfrac{\partial}{\partial y_3} v \in L^2(G),\ v(y_2, 0) = h^{rs}(y_2) \}$ *with* $V(Y_2) = \{ \Phi \in H^1(Y_2), \Phi$ *takes equal values on the ends of* $Y_2 \}$.

4.2. Study of the fourth order problem

As before, we express function $\Phi(x', y_2, y_3)$, defined up to the addition of arbitrary functions of x' : $\tilde{\Phi}_1(x')$, $\tilde{\Phi}_2(x')$ and $\tilde{\Phi}_3(x')$, as follows :

$$\Phi(x_1', x_2', 0, y_2, y_3) = \varphi^{rs}(y_2, y_3)\, \varepsilon_{rs}(u^0)\,\big|_{(x_1', x_2', 0)} + \\ + \tilde{\Phi}_1(x_1', x_2', 0)\, y_2 + \tilde{\Phi}_2(x_1', x_2', 0)\, y_3 + \tilde{\Phi}_3(x_1', x_2', 0) \quad (4.9)$$

where $u^0(x')$ is the solution of homogenized problem (2.8), (2.9) and where functions $\varphi^{rs}(y_2, y_3)$ satisfy the following problem on G :

- $A[\varphi^{rs}(y_2, y_3)] = 0$ (4.10)
- $\varphi^{rs}(y_2, 0) = g^{rs}(y_2)$
- $\dfrac{\partial \varphi^{rs}}{\partial y_3}(y_2, 0) = k^{rs}(y_2)$ (4.11)
- $\left(\dfrac{\partial^2 \varphi^{rs}}{\partial y_i\, \partial y_j}\right) (i, j = 2, 3)$ converge to 0 as $y_3 \mapsto +\infty$ (4.12)
- $\varphi^{rs}(y_2, y_3)$ are Y_2-periodic in y_2 (4.13)

where A denotes the operator :

$$A = \frac{\partial^2}{\partial y_2\, \partial y_2}\left[\left(\frac{1 - \nu^2}{E}\right) \frac{\partial^2}{\partial y_2\, \partial y_2}\right] + \frac{\partial^2}{\partial y_3\, \partial y_3}\left[\left(\frac{1 - \nu^2}{E}\right) \frac{\partial^2}{\partial y_3\, \partial y_3}\right] \\ + 2 \frac{\partial^2}{\partial y_2\, \partial y_3}\left[\frac{\nu(1 + \nu)}{E} \frac{\partial^2}{\partial y_2\, \partial y_3}\right] - \frac{\partial^2}{\partial y_2\, \partial y_2}\left[\left(\frac{1 + \nu}{E}\right) \frac{\partial^2}{\partial y_3\, \partial y_3}\right] \\ - \frac{\partial^2}{\partial y_3\, \partial y_3}\left[\left(\frac{1 + \nu}{E}\right) \frac{\partial^2}{\partial y_2\, \partial y_2}\right],$$

$k^{rs}(y_2)$ and $g^{rs}(y_2)$ the following functions defined on Y_2 :

- $k^{rs}(y_2) = \displaystyle\int_0^{y_2} b_{23rs}(t, 0)\, dt - y_2 \frac{1}{Y_2^*} \int_0^{Y_2^*} b_{23rs}(t, 0)\, dt$
- $g^{rs}(y_2) = \displaystyle -\int_0^{y_2}\left[\int_0^t b_{33rs}(t_1, 0)\, dt_1\right] dt + \frac{y_2^2}{2\, Y_2^*} \int_0^{Y_2^*} b_{33rs}(y_2, 0)\, dt$
  $\displaystyle + \int_0^{Y_2^*}\left[\int_0^{t_1} b_{33rs}(t_1, 0)\, dt_1\right] dt - \frac{Y_2^*}{2} \int_0^{Y_2^*} b_{33rs}(y_2, 0)\, dy_2$

Equation (4.10) follows from the equations of compatibility and formulas (3.8), (4.2), (4.9) ; by integrating boundary conditions (3.6), (3.7) one obtains equations (4.11) and (4.12). Finally (3.5), (4.2), (4.9) and the boundary homogenized condition (2.9) lead to (4.13).

Remark 4.2 : One can verify that the displacements u_1^{1BL} and (u_2^{1BL}, u_3^{1BL}) associated to $\Psi(x', y_2, y_3)$, solution of (4.5) ... (4.8) and to $\Phi(x', y_2, y_3)$, solution of (4.10) ... (4.13) by the constitutive equation (3.8), (4.1), (4.2), are periodic in y_2.

We are going to present a result for the existence of the functions $\varphi^{rs}(y_2, y_3)$ and the main steps of its proof. This result extends to higher orders equations the results due to Lions [9] for the diffusion equation :

THEOREM 4.2 : *There exists a real $\gamma > 0$ such that the problem :*

$$\begin{cases} \bullet\ (4.10), (4.11), (4.13) \\ \bullet\ \left(e^{\gamma y_3} \dfrac{\partial^2 \varphi^{rs}}{\partial y_i \, \partial y_j}\right) \in L^2(G)\,, \quad (i, j = 2, 3) \end{cases}$$

admits a unique solution $\varphi^{rs} \in W^{rs}$, where W^{rs} is given by :

$$W^{rs} = \left\{ v \in L^2(0, R; W(Y_2)),\ \forall \text{ finite } R,\ e^{\gamma y_3} \frac{\partial^2 v}{\partial y_i \, \partial y_j} \in L^2(G),\ (i, j = 2, 3) \right.$$
$$\left. v(y_2, 0) = g^{rs}(y_2),\ \frac{\partial v}{\partial y_3}(y_2, 0) = k^{rs}(y_2) \right\}$$

with $W(Y_2) = \{ \Phi \in H^2(Y_2),\ \Phi$ and its derivative take equal values at the ends of $Y_2 \}$. (4.14)

Proof : First step. Introduce functions $\xi^{rs}(y_2, y_3)$, defined on G by :

$$\xi^{rs}(y_2, y_3) = \varphi^{rs}(y_2, y_3) - \tilde{\mathscr{U}}^{rs}(y_2, y_3)$$

where one denotes :

$$\tilde{\mathscr{U}}^{rs}(y_2, y_3) = \chi(y_3)\, \mathscr{U}^{rs}(y_2, y_3)$$

with $\chi \in C^\infty$ (]0, + ∞[), such that :

$$\begin{cases} \bullet\ \chi(y_3) = 0 \quad \text{if} \quad y_3 > r \quad r \text{ fixed} \\ \bullet\ \chi(y_3) = 1 \quad \text{if} \quad 0 < y_3 < r' < r \end{cases}$$

$\mathscr{U}^{rs}(y_2, y_3)$ is the function defined on $Y_2 \times]0, R[$, $\forall$ finite $R > r$ satisfying :

- $A[\mathscr{U}^{rs}(y_2, y_3)] = 0$

$$\left.\begin{array}{l}\bullet\ \hat{u}^{rs}(y_2, R) = \dfrac{\partial \hat{u}^{rs}}{\partial y_3}(y_2, R) = 0 \\ \bullet\ \hat{u}^{rs}(y_2, 0) = g^{rs}(y_2) \\ \bullet\ \dfrac{\partial \hat{u}^{rs}}{\partial y_3}(y_2, 0) = k^{rs}(y_2) \\ \bullet\ \hat{u}^{rs}(y_2, y_3)\ Y_2 \text{ periodic in } y_2\end{array}\right\} \quad (4.15)$$

and functions $\xi^{rs}(y_2, y_3)$ satisfy the following problem :

$$\left\{\begin{array}{ll}\bullet\ A[\xi^{rs}(y_2, y_3)] = l^{rs}(y_2, y_3) & (4.16) \\ \bullet\ \xi^{rs}(y_2, 0) = \dfrac{\partial \xi^{rs}}{\partial y_3}(y_2, 0) = 0 & (4.17) \\ \bullet\ \dfrac{\partial^2 \xi^{rs}}{\partial y_i\, \partial y_j}, \quad (i, j = 2, 3) \text{ converge to } 0 \text{ as } y_3 \mapsto +\infty & \\ \bullet\ \xi^{rs}(y_2, y_3) \text{ are } Y_2\text{-periodic in } y_2 & (4.18)\end{array}\right.$$

where $l^{rs}(y_2, y_3)$ denotes : $l^{rs}(y_2, y_3) = A[\hat{u}^{rs}(y_2, y_3)]$.

One can easily see that problem : (4.10), (4.11), (4.13) and

$$\left(\frac{\partial^2 \varphi^{rs}}{\partial y_i\, \partial y_j}\, e^{\gamma y_3}\right) \in L^2(G)\,, \quad (i, j = 2, 3)$$

is equivalent to the following variational problem. Find ξ^{rs} such that :

$$\left.\begin{array}{l}\xi^{rs} \in W \\ a(\xi^{rs}, v) = L^{rs}(v)\,, \quad \forall v \in W^0\end{array}\right\} \quad (4.19)$$

where we set :

$$W = \left\{ v \in L^2(0, R\,; W(Y_2)) \,\forall \text{ finite } R, \left(e^{\gamma y_3}\, \frac{\partial^2 v}{\partial y_i\, \partial y_j}\right) \in L^2(G), (i, j = 2, 3)\right.$$

$$\left. v(y_2, 0) = \frac{\partial v}{\partial y_3}(y_2, 0) = 0 \right\} \quad (4.20)$$

$$W^0 = \left\{ v \in W, (e^{\gamma y_3}\, v) \in L^2(G), \left(e^{\gamma y_3}\, \frac{\partial v}{\partial y_i}\right) \in L^2(G), (i = 2, 3) \right\} \quad (4.21)$$

with $W(Y_2)$ given by (4.14); where we define $a(u, v)$ for $u \in W$ and $v \in W^0$ by :

$$a(u, v) = \int_G \left(\frac{1 - \nu^2}{E}\right) \frac{\partial^2 u}{\partial y_2\, \partial y_2} \frac{\partial^2 v}{\partial y_2\, \partial y_2}\, e^{2\gamma y_3}\, dy\, +$$

$$+ \int_G \left(\frac{1 - \nu^2}{E}\right) \frac{\partial^2 u}{\partial y_3\, \partial y_3} \frac{\partial^2 (v\, e^{2\gamma y_3})}{\partial y_3\, \partial y_3}\, dy$$

$$- \int_G \frac{\nu(1 + \nu)}{E} \frac{\partial^2 u}{\partial y_2\, \partial y_2} \frac{\partial^2}{\partial y_3\, \partial y_3} (v\, e^{2\gamma y_3})\, dy$$

$$- \int_G \frac{\nu(1 + \nu)}{E} \frac{\partial^2 u}{\partial y_3\, \partial y_3} \frac{\partial^2 v}{\partial y_2\, \partial y_2}\, e^{2\gamma y_3}\, dy$$

$$+ 2 \int_G \left(\frac{1 + \nu}{E}\right) \frac{\partial^2 u}{\partial y_2\, \partial y_3} \frac{\partial}{\partial y_3} \left(\frac{\partial v}{\partial y_3}\, e^{2\gamma y_3}\right) dy \qquad (4.22)$$

and where $L^{rs}(v)$ defined for $v \in W^0$, is given by :

$$L^{rs}(v) = - a(\tilde{u}^{rs}, v)\,. \qquad (4.23)$$

• Second step : We propose to apply the following lemma due to Tartar, which is proved in [9], p. 51-53.

LEMMA 4.3 : *Let V and V^0 be two Hilbert spaces such that : $V^0 \subset V$ with continuous injection, let $a(u, v)$ a continuous bilinear form on $V \times V^0$, M a linear continuous maps V onto V^0 and f an element of V'^0 (dual space of V^0).*

We assume that : there exists $C > 0$ such that :

$$\forall v \in V \quad a(v, Mv) \geqslant C \| v \|_V^2\,. \qquad (4.24)$$

Then, there exists a unique solution of the problem :

$$\begin{cases} u \in W \\ a(u, v) = \langle f, v \rangle \quad \forall v \in V^0 \end{cases}$$

where $\langle\ ,\ \rangle$ denotes the scalar product between V'^0 and V^0.

a) We provide W and W^0, defined by (4.20) and (4.21), with the norms :

$$\| u \|_W = \left[\left\| \frac{\partial^2 u}{\partial y_2\, \partial y_2}\, e^{\gamma y_3} \right\|_{L^2(G)}^2 + \right.$$

$$\left. + \left\| \frac{\partial^2 u}{\partial y_2\, \partial y_3}\, e^{\gamma y_3} \right\|_{L^2(G)}^2 + \left\| \frac{\partial^2 u}{\partial y_3\, \partial y_3}\, e^{\gamma y_3} \right\|_{L^2(G)}^2 \right]^{1/2}$$

$$\| v \|_{W^0} = \left[\| v \|_W^2 + \| v \, e^{\gamma y_3} \|_{L^2(G)}^2 + \right.$$

$$\left. + \left\| \frac{\partial v}{\partial y_2} e^{\gamma y_3} \right\|_{L^2(G)}^2 + \left\| \frac{\partial v}{\partial y_3} e^{\gamma y_3} \right\|_{L^2(G)}^2 \right]^{1/2} .$$

Then, they satisfy the hypothesis of lemma 4.3.

b) Under hypothesis (2.1), one can easily verify that $a(u, v)$, defined by (4.22) is a

$$\text{bilinear continuous form on } W \times W^0 . \tag{4.25}$$

Let us introduce operator M as follows :

$$Mu = u - E * \bar{u} \quad \forall u \in W \tag{4.26}$$

where $\bar{u}$ denotes the mean of u :

$$\bar{u}(y_3) = \frac{1}{Y_2^*} \int_0^{Y_2^*} u(y_2, y_3) \, dy_2 ,$$

$*$ is the convolution in t and where $E(y_3)$ is defined for $y_3 > 0$ and given by :

$$E(y_3) = 4 \gamma (1 - \gamma y_3) \, e^{-2\gamma y_3} .$$

Let us extend E by 0 for $y_3 \leqslant 0$ and the convolution for $y_3 \in \mathbb{R}$, one can verify :

$$\frac{d^2}{dy_3^2}(E) + 4 \gamma \frac{dE}{dy_3} + 4 \gamma^2 E = 4 \gamma \delta' + 4 \gamma^2 \delta \tag{4.28}$$

where δ denotes Dirac's function and δ' its derivative.

We remind Poincare-Wirtinger's inequalities [3] : there exist (C_0, C_1, C_2) constants > 0, such that :

$$\left. \begin{aligned} &\forall u \in W \quad \| u - \bar{u} \|_{L^2(Y_2)} \leqslant C_0 \left\| \frac{\partial u}{\partial y_2} \right\|_{L^2(Y_2)} \leqslant C_1 \left\| \frac{\partial^2 u}{\partial y_2 \, \partial y_2} \right\|_{L^2(Y_2)} \\ &\left\| \frac{\partial u}{\partial y_3} - \frac{d\bar{u}}{dy_3} \right\|_{L^2(Y_2)} \leqslant C_2 \left\| \frac{\partial^2 u}{\partial y_2 \, \partial y_3} \right\|_{L^2(Y_2)} . \end{aligned} \right\} \tag{4.29}$$

c) From (4.26) ... (4.29), it follows that : $\forall u \in W \; Mu \in W^0$.

For example, one shows that :

$$\forall u \in W \quad \left(\frac{\partial Mu}{\partial y_3} e^{\gamma y_3} \right) \in L^2(G) ,$$

from (4.26), (4.27), (4.28) it follows :

$$\frac{\partial}{\partial y_3}(Mu)\, e^{\gamma y_3} = \left(\frac{\partial u}{\partial y_3} - \frac{d\bar{u}}{dy_3}\right) e^{\gamma y_3} + \frac{1}{4\,\gamma^2}\left(\frac{dE}{dy_3} * \frac{d^2\bar{u}}{dy_3^2}\right) e^{\gamma y_3} + \\ + \frac{1}{\gamma}\left((E - \delta) * \frac{d\bar{u}}{dy_3}\right) e^{\gamma y_3}.$$

We have $\dfrac{d^2\bar{u}}{dy_3^2} e^{\gamma y_3} \in L^2(G)$ $\forall u \in W$ and by using (4.29), we obtain :

$$\left(\frac{\partial u}{\partial y_3} - \frac{d\bar{u}}{dy_3}\right) e^{\gamma y_3} \in L^2(G) \quad \text{so that} \quad \frac{\partial Mu}{\partial y_3} e^{\gamma y_3} \in L^2(G)\,.$$

d) In the same way, one can verify that M is a continuous operator.

e) One can show that M maps W onto W^0 by using the below remark. Let us set $u = v + (4\,\gamma^2\, y_3 + 4\,\gamma)\, H * \bar{v}$, where $H(y_3)$ is the Heaviside's function, v an element of W^0 and $\bar{v}$ its mean. One can easily see that :

$$u \in W \quad \text{and satisfy :} \quad u - E * \bar{u} = v\,.$$

f) From formulas (4.22), (4.26), (4.28), we express $a(u, Mu)$ $\forall u \in W$ as follows :

$$\begin{aligned} a(u, Mu) = &\int_G \left(\frac{1 - \nu^2}{E}\right)\left[\left(\frac{\partial^2 u}{\partial y_2\, \partial y_2}\right)^2 + \left(\frac{\partial^2 u}{\partial y_3\, \partial y_3}\right)^2\right] e^{2\gamma y_3}\, dy \, + \\ &+ 2\int_G \left(\frac{1 + \nu}{E}\right)\left(\frac{\partial^2 u}{\partial y_2\, \partial y_3}\right)^2 e^{2\gamma y_3}\, dy \\ &+ 4\,\gamma \int_G \left(\frac{1 - \nu^2}{E}\right)\frac{\partial^2 u}{\partial y_3\, \partial y_3}\left(\frac{\partial u}{\partial y_3} - \frac{d\bar{u}}{dy_3}\right) e^{2\gamma y_3}\, dy \\ &+ 4\,\gamma^2 \int_G \left(\frac{1 - \nu^2}{E}\right)(u - \bar{u})\,\frac{\partial^2 u}{\partial y_3\, \partial y_3}\, e^{2\gamma y_3}\, dy \\ &+ 4\,\gamma \int_G \left(\frac{1 + \nu}{E}\right)\frac{\partial^2 u}{\partial y_2\, \partial y_3}\,\frac{\partial u}{\partial y_2}\, e^{2\gamma y_3}\, dy \\ &- 4\,\gamma \int_G \frac{\nu(1 + \nu)}{E}\,\frac{\partial^2 u}{\partial y_2\, \partial y_2}\left(\frac{\partial u}{\partial y_3} - \frac{d\bar{u}}{dy_3}\right) e^{2\gamma y_3}\, dy \\ &- 4\,\gamma^2 \int_G \frac{\nu(1 + \nu)}{E}\,\frac{\partial^2 u}{\partial y_2\, \partial y_2}(u - \bar{u})\, e^{2\gamma y_3}\, dy \\ &- 2\int_G \frac{\nu(1 + \nu)}{E}\,\frac{\partial^2 u}{\partial y_2\, \partial y_2}\,\frac{\partial^2 u}{\partial y_3\, \partial y_3}\, e^{2\gamma y_3}\, dy \end{aligned}$$

From inequalities (4.29), hypothesis (2.1) and the preceding expression of $a(u, Mu)$, we can deduce that (4.24) hold true provided γ is small enough.

g) Lastly, from (4.5), we have : $u^{rs}(y_2, y_3) \in H^2(Y_2 \times]0, R[)$ and then $\tilde{u}^{rs}(y_2, y_3) \in W$; therefore from (4.23) and (4.25), one can easily show that L^{rs} is continuous on W^0.

• Last step : Therefore, we have verified the hypothesis of lemma 4.3; we can apply it to the variational problem (4.9) and it allows us to deduce theorem 4.2.

Remark 4.3 : The calculation of coefficients $b_{ijkh}(y_2, y_3)$ shows that :

$$\tau^{rs}(y_2, y_3) \equiv 0, \quad \text{for} \quad (r, s) \neq (1, 3) \quad \text{and} \quad (r, s) \neq (1, 2)$$

$$\varphi^{rs}(y_2, y_3) \equiv 0, \quad \text{for} \quad (r, s) = (1, 3) \quad \text{and} \quad (r, s) = (1, 2).$$

5. KINEMATIC FORMULATION OF THE BOUNDARY LAYER PROBLEM

We consider now the displacement u^{1BL} as the unknown of the boundary layer problem. We confine ourselves to giving some indications on this formulation, which leads to neighbouring developments to those of the above section.

If we introduce $\eta^{rs}(y_2, y_3)$ by :

• $$\frac{\partial}{\partial y_2}\left[a_{i2kh}\, e_{kh}(\eta^{rs})\right] + \frac{\partial}{\partial y_3}\left[a_{i3kh}\, e_{kh}(\eta^{rs})\right] = 0, \quad \text{in } G \tag{5.1}$$

• $$a_{i3kh}(y_2, 0)\, e_{kh}(\eta^{rs})\big|_{(y_2, 0)} = -\, b_{i3rs}(y_2, 0) + \frac{1}{Y_2^*}\int_0^{Y_2^*} b_{i3rs}(t, 0)\, dt \tag{5.2}$$

• $$\left(\frac{\partial \eta^{rs}}{\partial y_2}, \frac{\partial \eta^{rs}}{\partial y_3}\right) \text{ converge to zero as } y_3 \mapsto +\infty \tag{5.3}$$

• $\eta^{rs}(y_2, y_3)$ are Y_2-periodic in y_2 (5.4)

(which defines η^{rs} up to an additive constant vector), we have :

$$u^{1BL}(x', y_2, y_3) = \eta^{rs}(y_2, y_3)\, \varepsilon_{rs}(u^0)\big|_{(x_1', x_2', 0)} + \tilde{u}^{1BL}(x_1', x_2', 0) \tag{5.5}$$

where $u^0(x')$ is the solution of the homogenized problem.

By introducing quotient spaces and applying lemma 4.3, we prove easily :

THEOREM 5.1 : *There exists a real $\gamma > 0$, such that the problem :*

$$\begin{cases} \bullet\ (5.1), (5.2), (5.4) \\ \bullet \left(e^{\gamma y_3} \dfrac{\partial \eta^{rs}}{\partial y_3}\right) \in [L^2(G)]^3, \quad \left(e^{\gamma y_3} \dfrac{\partial \eta^{rs}}{\partial y_2}\right) \in [L^2(G)]^3 \end{cases}$$

admits a unique solution $\eta^{rs} \in \overset{*}{W} = \dfrac{W}{\mathbb{R}^3}$, *where* $W = \{ v \in [L^2(0, R; W(Y_2)]^3,$

$\forall$ *finite* R, $\left(\dfrac{\partial}{\partial y_2}(v)\, e^{\gamma y_3}\right) \in [L^2(G)]^3, \quad \left(\dfrac{\partial}{\partial y_3}(v)\, e^{\gamma y_3}\right) \in [L^2(G)]^3 \Big\}$

and $W(Y_2) = \{ \Phi \in [H^2(Y_2)]^3, \Phi$ *takes equal values at the ends of* $Y_2 \}$.

Remark 5.1 : One can easily show that : $\eta_1^{rs} = 0$, for $(r, s) \neq (1, 3)$ and $(r, s) \neq (1, 2)$ and $\eta_2^{rs} = \eta_3^{rs} = 0$, for $(r, s) = (1, 3)$ and $(r, s) = (1, 2)$.

6. AN APPROXIMATION OF THE MICRO-STRESSES NEAR THE FREE BOUNDARY

We define a good approximation of the local stresses in the neighbourhood of the free boundary Γ_2, by setting :

- $\sigma^{0*}(x_1', x_2', 0, y_2, y_3) = \sigma^0(x_1', x_2', 0, y_2, y_3) + \sigma^{0BL}(x_1', x_2', 0, y_2, y_3)$. (6.1)

In practice, following the chosen formulation, the calculation of the boundary layer stresses needs to know homogenized displacement u^0, solution of (2.8), (2.9) and to solve problems (4.5) ... (4.8), (4.10) ... (4.13) or (5.1) ... (5.4). Then, it is sufficient to apply the following formulas :

- $\sigma_{ij}^{0*} = c_{ijkh}(y_2, y_3)\, \varepsilon_{kh}(u^0)\,|_{(x_1', x_2', 0)}$ (6.2)

where the not zero coefficients $c_{ijkh}(y_2, y_3)$ are given by :

- for $(r, s) = (1, 2)$ and $(r, s) = (1, 3)$

$$\begin{cases} c_{12rs} = b_{12rs} + \dfrac{\partial \tau^{rs}}{\partial y_3} = b_{12rs} + \tilde{a}_{12kh}\, e_{kh}(\eta^{rs}) \\[2ex] c_{13rs} = b_{13rs} - \dfrac{\partial \tau^{rs}}{\partial y_2} = b_{13rs} + \tilde{a}_{13kh}\, e_{kh}(\eta^{rs}) \end{cases}$$

- for $(r, s) \neq (1, 2)$ and $(r, s) \neq (1, 3)$

$$\begin{cases} c_{22rs} = b_{22rs} + \dfrac{\partial^2 \varphi^{rs}}{\partial y_3\, \partial y_3} = b_{22rs} + \tilde{a}_{22kh}\, e_{kh}(\eta^{rs}) \\[2ex] c_{33rs} = b_{33rs} + \dfrac{\partial^2 \varphi^{rs}}{\partial y_2\, \partial y_2} = b_{33rs} + \tilde{a}_{33kh}\, e_{kh}(\eta^{rs}) \\[2ex] c_{23rs} = b_{23rs} - \dfrac{\partial^2 \varphi^{rs}}{\partial y_3\, \partial y_2} = b_{23rs} + \tilde{a}_{23kh}\, e_{kh}(\eta^{rs}) \\[2ex] c_{11rs} = b_{11rs} + \dfrac{\nu}{E}(c_{22rs} + c_{33rs} - b_{22rs} - b_{33rs}) \\ \qquad = b_{11rs} + \tilde{a}_{11kh}\, e_{kh}(\eta^{rs}), \end{cases}$$

with $\tau^{rs}(y_2, y_3)$ the solution of (4.5) ... (4.8), $\varphi^{rs}(y_2, y_3)$ the solution of (4.10) ... (4.13) and $\eta^{rs}(y_2, y_3)$ the solution of (5.1) ... (5.4).

Remark 6.1 : In the special case $\alpha = \frac{\pi}{2}$, (it amounts to considering the Y_2 and Z periodicity as equivalent), we have :

$$\sigma_{12}^{0BL} = \sigma_{13}^{0BL} = 0 \quad \text{and therefore} \quad \sigma_{12}^{0*} = \sigma_{12}^{0}, \quad \sigma_{13}^{0*} = \sigma_{13}^{0}.$$

7. SOME NUMERICAL RESULTS AND COMMENTS

In this section, we present the main results of the computations of the local stresses near a free boundary; a more detailed study of the numerical implement of the above method is given in [5]. We consider a simplified structure problem, where the value of angle α (see section 3) is fixed to $\frac{\pi}{4}$; the layers are composed of two isotropic materials with equal thickness and mechanic characteristics are given by :

$$\begin{cases} \bullet\ E_1 = 0.84\ 10^{11}\ \text{Pa} & \nu_1 = 0.22 \\ \bullet\ E_2 = 0.4\ \ 10^{10}\ \text{Pa} & \nu_2 = 0.34\,. \end{cases}$$

The computations, which are presented here, use a finite element method with P_1 Lagrange approximation and are obtained on C.D.C. Cyber 750 and Mini 6-Bull computers.

The first step of the computations consists in solving equilibrium problem P^{ε} (2.2), where the material is considered heterogeneous. This resolution is used as a reference in the comparison of the different stresses $\boldsymbol{\sigma}^0$, $\boldsymbol{\sigma}^{0BL}$ and $(\boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^{0BL})$.

The second step consists in resolving homogenized problem (2.8), (2.9), the micro-stresses are computed from (2.7).

Lastly, the third step consists in solving the boundary layer problem. Since the kinematic formulation (see section 5) is well adapted to the computation codes (Modulef-Inria [10]), we choose to use this formulation. But, there is no difficulty to implement the static formulation. The domain G is truncated in y_3 direction far enough from $y_3 = 0$. In figure 2, we present the mesh of G. We compute the displacements η^{rs} solutions of (5.1) ... (5.4); we obtain the boundary layer stresses by setting :

$$\begin{cases} \bullet\ \sigma_{ij}^{0BL}(x', y_2, y_3) = s_{ij}^{rs}(y_2, y_3)\, \varepsilon_{rs}(u^0)\,|_{(x_1', x_2', 0)} \\ \bullet\ s_{ij}^{rs}(y_2, y_3) = \tilde{a}_{ijkh}(y_2, y_3)\, e_{kh}(\eta^{rs})\,. \end{cases}$$

and the local stresses $\boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^{0BL}$ by formulas (6.1), (6.2).

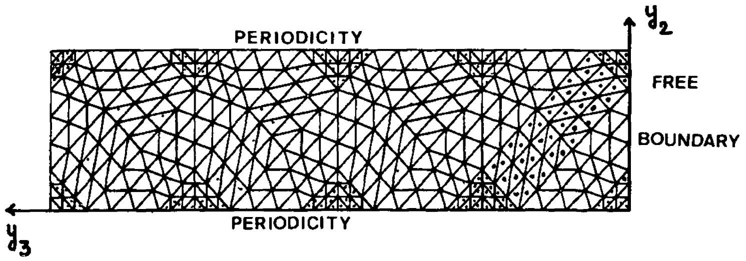

Figure 2. — The triangulation of domain G.
Number of triangles : 504. Number of degree of freedom : 596.
The triangles of the material 2 are distinguished by a point.

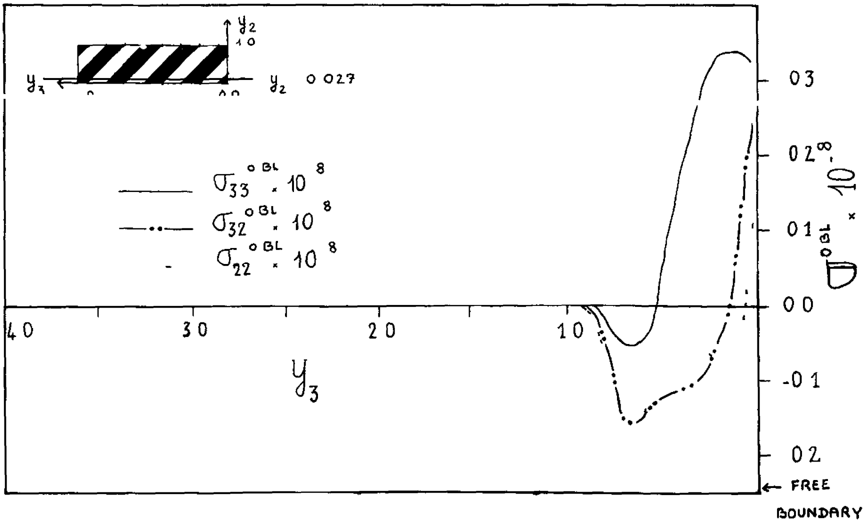

Figure 3. — The boundary layer stresses σ^{OBL} ploted against y_3, at $y_2 = 0.027$ fixed.

In figure 3, we have represented the boundary layer stresses as function of y_3 for a fixed value of y_2 in the neighbourhood of a point of the free boundary It appears, as planned in section 4 1, that these stresses decrease exponentially when $y_3 \mapsto +\infty$ The figures 4, 5, 6 allow us to compare the local stresses $\sigma^0 + \sigma^{OBL}$, the micro-stresses σ^0 and the real stresses in the neighbourhood of a free boundary It is obvious that the stresses σ^0 and $\sigma^0 + \sigma^{OBL}$ are good approximations of σ^ε, far enough from the free boundary, but near

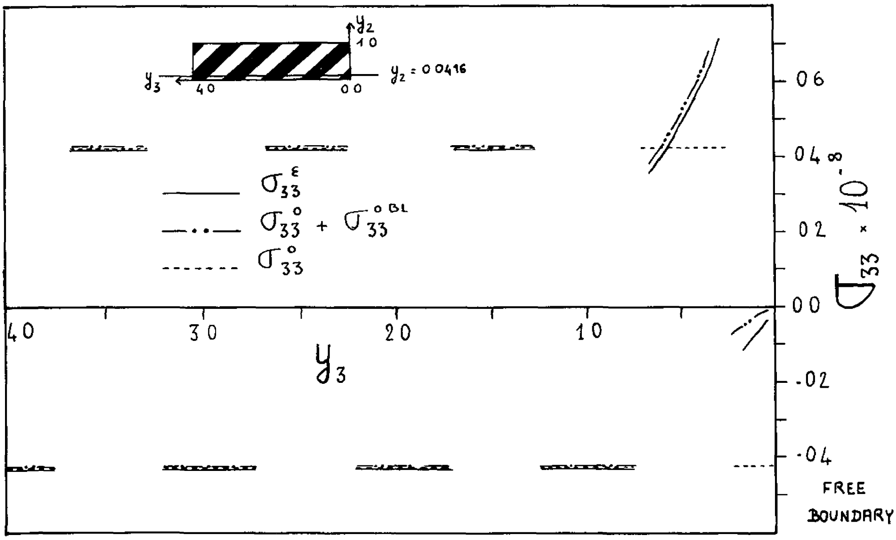

Figure 4. — The (3, 3) components of the real stresses σ^{ε}, of the micro-stresses σ^0 and of the boundary layer stresses σ^{OBL} added to σ^0, ploted against y_3, at $y_2 = 0.0416$ fixed.

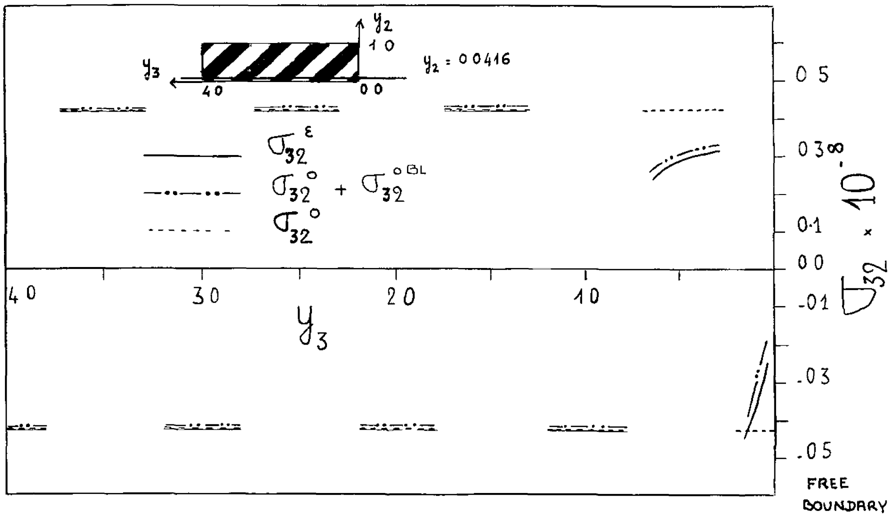

Figure 5. — The (2, 3) components of σ^{ε}, σ^0 and ($\sigma^0 + \sigma^{OBL}$) ploted against y_3, at $y_2 = 0.0416$ fixed.

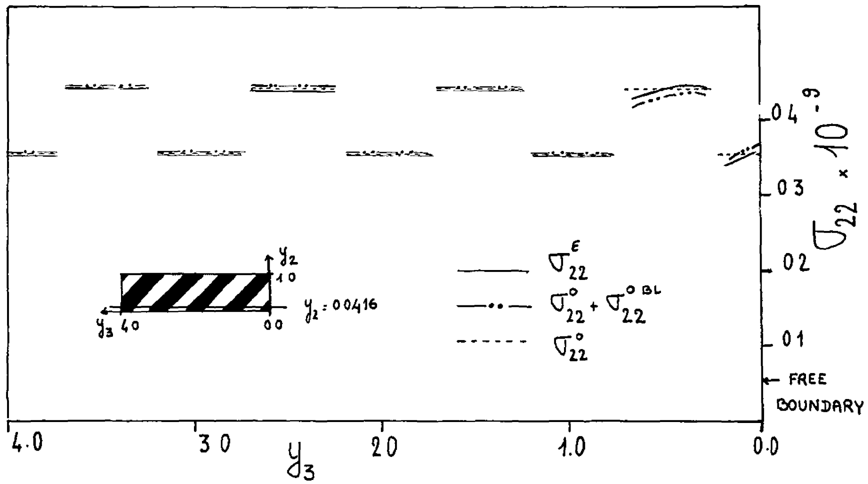

Figure 6. — The (2, 2) components of σ^{ε}, σ^0 and ($\sigma^0 + \sigma^{OBL}$) ploted against y_3, at $y_2 = 0.0416$ fixed.

the free boundary, one can see that $\boldsymbol{\sigma}^0$ do not coincide with $\boldsymbol{\sigma}^{\varepsilon}$. In addition, one can remark that the free boundary condition is correctly satisfied by sum $\boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^{0BL}$, when it is not the case of the only stresses $\boldsymbol{\sigma}^0$.

Remark 7.1 : The question of the possible presence of stress singularities in the intersection of the free boundary and an interface between any two elastic materials is very discussed (see for example [4], [12], [14], [15]). In some cases, it seems that the stresses are singular. By constructing an adapted mesh, our method, described the stress field near the boundary, should be allow the numerical studies of the singularities presence.

Conclusion : This method, which gives an approximation of the local stresses in the neighbourhood of the free boundary, can be considered as a complement of the classical homogenization theory. The mathematical study of the boundary layer problem leads to extend to a system and a higher order equation the results obtained by Lions for the diffusion equation. We remark that the numerical implement of this method is easy and inexpensive and that the computations of the local stresses near a free boundary confirme the correction due to the boundary layer terms. Finally, let us point out that this method can be applied to other materials, such as the stratified orthotropic materials, the materials renforced by unidirectional fibers and other boundaries.

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