M. I. Cᴏᴍᴏᴅɪ

Approximation of a fourth order variational inequality

APPROXIMATION OF A FOURTH ORDER VARIATIONAL INEQUALITY (*)

by M. I. Comodi (^{1})

Communicated by F. Brezzi

Abstract. — *We study a variational inequality related to a bending plate problem with a boundary unilateral constraint. We approximate the problem by a non-conforming finite element method, and we prove the convergence of the discrete scheme.*

Résumé. — *Nous étudions une inégalité variationnelle relative au problème de la flexion d'une plaque avec une condition aux limites unilatérale. Nous approximons le problème par une méthode d'élément fini non conforme et prouvons la convergence du schéma discret.*

1. INTRODUCTION

The aim of this paper is the study of a fourth order unilateral problem introduced by Duvaut and Lions (« Unilateral Phenomena in the Theory of Flat Plates »; chap. IV; [6]). Precisely : let us consider an elastic plate [14] occupying in its reference configuration a convex open bounded subset Ω of $\mathbb{R}^2$ with boundary $\partial\Omega$, subject to the action of a vertical force f. The unknown of the problem is the displacement u of all points of the plate : we want to minimize the total potential energy of the plate under some constraints on the boundary values of u. In a previous paper [5] we considered the case $u \geqslant 0$ on $\partial\Omega$; now, we seek for u subject to another type of unilateral constraint : normal derivative $\geqslant 0$ on $\partial\Omega$.

Paragraph 2 is devoted to the introduction of the notations used throughout this paper. In paragraph 3, following [6], we study the problem as a variational inequality in a closed convex subset of $H^2(\Omega)$ ([3], [9]). We first prove that

---

the necessary condition $(f, 1) = 0$ (where $(. , .) =$ inner product in $L^2(\Omega)$) is also sufficient for the existence of a solution u (that was left as an open question in [6]); then we examine the problem of the uniqueness of u according to the vanishing of (f, p), $p \in P_1(\Omega)$. If $(f, p) \neq 0$ for at least one $p \in P_1(\Omega)$, then we assume simple conditions on Ω and f so that u becomes unique in the class of functions with zero average. On the other hand if $(f, p) = 0$ for all $p \in P_1(\Omega)$ (as assumed in [6]) then u is unique up to a polynomial p in $P_1(\Omega)$ (as proved in [6]).

Paragraph 4 is devoted to the approximation of the problem by a finite element method ([4], [13], [15]) of non-conforming type. More precisely, we use the method due to Morley ([10], [12]), still better, the modification of it introduced in [2]. We prove the existence of at least one approximate solution, under the only assumption $(f, 1) = 0$, and we remark some analogies between the continuous problem and the discrete one.

In paragraph 5, first of all we give an optimal error bound for the error on the moments $u_{/ij}$, $i, j = 1, 2$, and we show that this estimate also bounds the error done on the displacement if $(f, p) = 0$ for all $p \in P_1(\Omega)$. On the other hand, if $(f, p) \neq 0$ for at least one $p \in P_1(\Omega)$, we prove an error bound $0(h \,|\ln h|^{1/2})$ for the approximate displacement in the piecewise $H^{2,h}(\Omega)$ norm.

2. NOTATIONS AND GREEN'S FORMULAE

Let Ω be an open, bounded, convex subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We denote respectively by $\underline{n} = (n_1, n_2)$ and $\underline{t} = (t_1, t_2) = (-n_2, n_1)$ the unit outward normal and the unit tangent to $\partial\Omega$

— $v_{/i}$, $v_{/n}$, $v_{/t}$ indicate respectively the derivatives of v with respect to the variable x_i, the normal and the tangential derivative.

Let $H^m(\Omega) = W^{m,2}(\Omega)$ be the usual Sobolev space ([1], [11]) consisting of real valued functions defined on Ω which belong to $L^2(\Omega)$ along with their derivatives of order i, $1 \leqslant i \leqslant m$:

— (f, g) denotes the $L^2(\Omega)$ inner product, i.e. $(f, g) = \displaystyle\int_\Omega fg \, d\underline{x}$, $f, g \in L^2(\Omega)$,

$\underline{x} = (x_1, x_2)$

— $| \cdot |_{m,\Omega} \equiv | \cdot |_m$ and $\| \cdot \|_{m,\Omega} \equiv \| \cdot \|_m$ denote respectively the seminorm and the norm on $H^m(\Omega)$.

Throughout this paper we use the convention of repeated indices, we denote by $P_k(\Omega)$ the space of polynomials of degree $\leqslant k$ and by C, or C_i, $i \in N$, a generic positive constant, which may change value at different occurrences. Given a tensor valued function $\underline{\theta} = (\theta_{ij})$, $i, j = 1, 2$, we define :

$$M_n(\underline{\theta}) = -\theta_{ij} \, n_i \, n_j = \text{normal bending moment}$$
$$M_{nt}(\underline{\theta}) = \theta_{ij} \, n_i \, t_j = \text{twisting moment}$$
$$Q_n(\underline{\theta}) = -\theta_{ij/i} \, n_j = \text{normal shear force}$$
$$K_n(\underline{\theta}) = Q_n(\underline{\theta}) - M_{nt}(\underline{\theta})_{/t} = \text{normal Kirchhoff shear force}.$$

Then, given $w \in H^4(\Omega)$, $\underline{\theta} = (\theta_{ij}) = w_{/ij}$, $v \in H^2(\Omega)$, the following Green's formulae hold (note that (2.1) needs only $w \in H^3(\Omega)$)

$$(2.1) \qquad \int_\Omega w_{/ij} \, v_{/ij} \, d\underline{x} = - \int_\Omega w_{/iji} \, v_{/j} \, d\underline{x} + \int_{\partial\Omega} M_{nt}(\underline{\theta}) \, v_{/t} \, ds -$$

$$- \int_{\partial\Omega} M_n(\underline{\theta}) \, v_{/n} \, ds$$

$$(2.2) \qquad - \int_\Omega w_{/iji} \, v_{/j} \, d\underline{x} = \int_\Omega w_{/ijij} \, v \, d\underline{x} + \int_{\partial\Omega} Q_n(\underline{\theta}) \, v \, ds.$$

By combining (2.1) and (2.2) and integrating by parts the term $\displaystyle\int_{\partial\Omega} M_{nt}(\underline{\theta}) \, v_{/t} \, ds$, we obtain

$$(2.3) \qquad \int_\Omega w_{/ij} \, v_{/ij} \, d\underline{x} = \int_\Omega w_{/ijij} \, v \, d\underline{x} - \int_{\partial\Omega} M_n(\underline{\theta}) \, v_{/n} \, ds + \int_{\partial\Omega} K_n(\underline{\theta}) \, v \, ds.$$

3. THE CONTINUOUS PROBLEM

Let us consider the closed convex subset of $H^2(\Omega)$

$$(3.1) \qquad \tilde{K} = \{ v \in H^2(\Omega), \, v_{/n} \geqslant 0 \text{ on } \partial\Omega \}.$$

Now, given the continuous bilinear non-negative form

$$(3.2) \qquad a(u, v) = \int_\Omega u_{/ij} \, v_{/ij} \, d\underline{x} \qquad u, v \in H^2(\Omega)$$

and a function $f \in L^2(\Omega)$, we want to solve the following unilateral problem

$$(3.3) \qquad \begin{cases} \text{Find } u \in \tilde{K} \text{ such that} \\ a(u, v - u) \geqslant (f, v - u) \quad \text{for all } v \in \tilde{K}. \end{cases}$$

Since

$$(3.4) \qquad a(v, p) = 0 \quad \text{for all} \quad v \in H^2(\Omega), \quad p \in P_1(\Omega),$$

by taking, in (3.3), $v = u \pm c,\ c \in P_0(\Omega)$, we easily obtain

(3.5) $(f, c) = 0$ for all $c \in P_0(\Omega)$.

The two relations above suggest to seek for a solution of (3.3) belonging to the following closed convex subset of $\tilde{K}$

(3.6) $K = \left\{ v \in \tilde{K},\ \int_\Omega v\,d\underline{x} = 0 \right\}.$

More exactly, we can consider the problem

(3.7) $\begin{cases} \text{Find } u \in K \text{ such that} \\ a(u, v - u) \geqslant (f, v - u) \quad \text{for all} \quad v \in K. \end{cases}$

The choice (3.6) implies

(3.8) $K \cap P_1(\Omega) = 0$

and we shall see, in the following two theorems, that (3.4), (3.5), (3.8) are sufficient conditions for the existence of at least one solution of (3.7).

To this purpose let us introduce the quadratic functional

(3.9) $J(v) = \dfrac{1}{2} a(v, v) - (f, v) \quad v \in H^2(\Omega).$

As K is convex and $(.,.)$ is symmetric and non-negative, we know that (3.7) is equivalent to the following minimum problem

(3.10) $\begin{cases} \text{Find } u \in K \text{ such that} \\ J(u) = \min\limits_{v \in K} J(v). \end{cases}$

We could get the existence of a solution of (3.7) and (3.10) by using [7], but, in view of the approximation of the problem, we prefer to give the two following Theorems. We begin by proving a basic property of the functions of K.

THEOREM 3.1 : *There exists C, independent of v, such that*

(3.11) $\| v \|_2 \leqslant C | v |_2$ *for all* $v \in K$.

Proof : If (3.11) is false, then, for all $n \in N$, there exists v_n such that

(3.12) $v_n \in K$

(3.13) $\| v_n \|_2 > n | v_n |_2$.

The functions

$$(3.14) \quad \tilde{v}_n = \frac{1}{\| v_n \|_2}$$

verify, for all $n \in N$,

$$(3.15) \quad \tilde{v}_n \in K$$

$$(3.16) \quad \| \tilde{v}_n \|_2 = 1$$

$$(3.17) \quad | \tilde{v}_n |_2 < \frac{1}{n}$$

hence, owing to (3.16), there exists a subsequence, which we still denote by $\tilde{v}_n$, weakly convergent in $H^2(\Omega)$ as $n \to + \infty$. Let $\tilde{v}$ be its limit. Because of (3.17), (3.15) and (3.8), we have $\tilde{v} \equiv 0$. But, as $\tilde{v}_n \to 0$ strongly in $H^1(\Omega)$, (3.17) implies $\tilde{v}_n \to 0$ strongly in $H^2(\Omega)$, and this contradicts (3.16). ∎

THEOREM 3.2 : *Problem* (3.10) (*hence problem* (3.7)) *has at least one solution. Moreover, given the set*

$$(3.18) \quad P = \left\{ p \in P_1(\Omega), \int_\Omega p \, d\underline{x} = 0 \right\}$$

and given two solutions u_1 and u_2, then

$$(3.19) \quad u_1 - u_2 = p^0 \qquad p^0 \in P$$

$$(3.20) \quad (f, p^0) = 0 .$$

Proof : By the above theorem and the well known inequality

$$(3.21) \quad 2 \, ab \geqslant - \frac{1}{c^2} a^2 - c^2 \, b^2 \quad \text{for all} \quad a, b, c \in \mathbb{R}, \quad c \neq 0$$

we obtain that the quadratic, lower semicontinuous functional $J(v)$ verifies, for all $v \in K$,

$$(3.22) \quad J(v) = \frac{1}{2} | v |_2^2 - (f, v) \geqslant C_1 \| v \|_2^2 - \| f \|_0 \| v \|_2 \geqslant$$

$$\geqslant \frac{C_1}{2} \| v \|_2^2 - C_2 \| f \|_0^2$$

with C_1, C_2 independent of v. This implies that $J(v) \to + \infty$ as $\| v \|_2 \to + \infty$ and it is bounded from below on the closed convex K. That gives the existence of at least one solution of (3.10).

Now, let us consider two solutions u_1 and u_2. By adding the following two relations

(3.23) $a(u_1, u_2 - u_1) \geqslant (f, u_2 - u_1)$

(3.24) $a(u_2, u_1 - u_2) \geqslant (f, u_1 - u_2)$

we obtain $| u_1 - u_2 |_2 = 0$, or (3.19). Inserting it in (3.23) and (3.24), we get (3.20).   ∎

Whilst we have solved the problem of the existence of a solution by using only the necessary condition (3.5) (which is also sufficient, by now), the question of the uniqueness of a solution depends strongly on the behaviour of the product (f, p), $p \in P$. Before distinguishing two, quite different, situations, we remark, by a simple application of Green's formulae, that a solution u verifies, at least formally, the following relations

(3.25) $\Delta^2 u = f$ in Ω

(3.26) $u_{/n} \geqslant 0$ on $\partial\Omega$

(3.27) $M_n(\underline{\lambda}) \leqslant 0$ on $\partial\Omega$

(3.28) $u_{/n} M_n(\underline{\lambda}) = 0$ on $\partial\Omega$

(3.29) $K_n(\underline{\lambda}) = 0$ on $\partial\Omega$

where $\underline{\lambda}$ is the following tensor valued function

(3.30) $\underline{\lambda} = (\lambda_{ij}) = (u_{/ij})$ $i, j = 1, 2$.

Now, let us examine the two possible cases that follow.

Case I : The function f verifies

(3.31*a*) $(f, p) = 0$ for all $p \in P$.

That implies, by Green's formulae and (3.25)-(3.30)

(3.31*b*) $\displaystyle\int_{\partial\Omega} M_n(\underline{\lambda}) \, p_{/n} \, ds = (f, p) = 0$ for all $p \in P$.

Case II : There exists at least $\bar{p} \in P$ such that

(3.32*a*) $\displaystyle (f, \bar{p}) = \int_{\partial\Omega} M_n(\underline{\lambda}) \, \bar{p}_{/n} \, ds \neq 0$.

That means :

(3.32b)   there exists only one $p^* \in P$ such that

$$(f, p^*) = \int_{\partial\Omega} M_n(\underline{\lambda}) \, p^*_{/n} \, ds = 0 \, .$$

Case I.

If (3.31) hold, following now [6], we introduce the quotient space

(3.33) $\dot{H} = H/P \, ,$

where

(3.34) $H = \left\{ v \in H^2(\Omega), \int_{\Omega} v \, d\underline{x} = 0 \right\} \, ,$

Let us consider the image $\dot{K}$ of K obtained by applying the canonical mapping

(3.35) $v \to \dot{v}$ of $H \to \dot{H}$.

Since

(3.36) $a(v, w) = a(\dot{v}, \dot{w})$ for all $v \in \dot{v}$, $w \in \dot{w}$,

the following theorem immediately follows.

THEOREM 3.3 : *If* (3.31) *hold, then there exists a unique solution $\dot{u}$ of the following problem*

(3.37) $\begin{cases} Find \ \dot{u} \in \dot{K} \ such \ that \\ a(\dot{u}, \dot{v} - \dot{u}) \geqslant (f, \dot{v} - \dot{u}) \quad for \ all \ \ \dot{v} \in \dot{K} \, . \end{cases}$

Proof : It is an obvious consequence of Theorem 3.2, (3.31a) and of (3.36). ∎
Now, let us deal with Case II.

Case II.

THEOREM 3.4 : *If $\partial\Omega$ does not contain rectilinear portions, then Problems* (3.7) *and* (3.10) *have a unique solution.*

Proof : Clearly, we can begin this proof from the results of Theorem 3.2.
Let us suppose $p^0 \neq 0$, p^0 given by (3.19). As Ω is convex and its boundary does not contain rectilinear parts, only two distinct points s_1, $s_2 \in \partial\Omega$ exist such that

(3.38) $p^0_{/n}(s_1) = p^0_{/n}(s_2) = 0$

and s_1, s_2 divide $\partial\Omega$ into two connected portions $\partial\Omega_1$ and $\partial\Omega_2$ such that

$$(3.39) \quad 0 > p^0_{/n}(s) = (u_1 - u_2)_{/n}(s) \geqslant - u_{2/n}(s) \quad \text{for all} \quad s \in \partial\Omega_1$$

$$(3.40) \quad 0 < p^0_{/n}(s) = (u_1 - u_2)_{/n}(s) \leqslant u_{1/n}(s) \quad \text{for all} \quad s \in \partial\Omega_2 \,.$$

As

$$(3.41) \quad \underline{\lambda}_1 = \underline{\lambda}_2 = \underline{\lambda}$$

where $\underline{\lambda}_1$, $\underline{\lambda}_2$ are the tensor valued functions whose components are the second derivatives of u_1 and u_2 respectively, we obtain, by combining (3.39), (3.40) and (3.28),

$$(3.42) \quad M_n(\underline{\lambda}) = 0 \qquad \text{a.e. on } \partial\Omega$$

which contradicts (3.32a). ∎

The conditions (3.32) also imply the following property of the solution u.

LEMMA 3.5 : *The normal derivative of any solution u of* (3.7) *and* (3.10)*, vanishes at least on a subset of the boundary, which has positive measure.*

Proof : It is obvious, once more by noting that $u_{/n} > 0$ a.e. on $\partial\Omega$, implies $M_n(\underline{\lambda}) = 0$ a.e. on $\partial\Omega$. ∎

In the next paragraph we shall assume that Ω is a convex polygon of $\mathbb{R}^2$. This implies, obviously, the presence of rectilinear portions. Then, in Case II we suppose the following behaviour of f.

ASSUMPTION 3.6 : *Let γ_i, $1 \leqslant i \leqslant N$, be the rectilinear portions of $\partial\Omega$, and let $\tilde{P}$ be the following subset of P*

$$(3.47) \quad \tilde{P} = \{ \tilde{p} \in P, \tilde{p}_{/n} = 0 \text{ on } \gamma_i, \text{ for some } i, 1 \leqslant i \leqslant N \} \,.$$

We assume

$$(3.48) \quad (f, \tilde{p}) \neq 0 \quad \text{for all} \quad \tilde{p} \in \tilde{P} \,.$$

Namely, if γ_i is represented by

$$(3.49) \quad a_i x_1 + b_i x_2 + c_i = 0 \quad 1 \leqslant i \leqslant N$$

we assume

$$(3.50) \quad \int_\Omega f(a_i x_2 - b_i x_1) \, d\underline{x} \neq 0 \quad \text{for all} \quad i, 1 \leqslant i \leqslant N \,. \quad ∎$$

THEOREM 3.7 : *Assumption 3.6 implies the uniqueness of the solution.*

Proof : Notice : p^0 given by (3.19) belongs to $\tilde{P}$ (otherwise $u = (u_1 + u_2)/2$ would contradict Lemma 3.5). Then, owing to the preceding Assumption and (3.20), $p^0 \equiv 0$. ∎

LEMMA 3.8 : *Let G be the subset of $\partial\Omega$ such that*

(3.51) $u_{/n}(s) = 0 \quad on \quad G$.

Then, at least $G_1, G_2 \in G$ exist, with positive measure and not parallel.

Proof : If the Lemma does not hold, there exists $\tilde{p} \in \tilde{P}$ with $\tilde{p}_{/n} = 0$ on G. That means

(3.52) $(f, \tilde{p}) = \displaystyle\int_G M_n(\underline{\lambda}) \, \tilde{p}_{/n} \, ds = 0$,

and this contradicts (3.48). ∎

In what follows, in both Cases (3.32) and (3.31), we assume that u (resp. $\overset{\circ}{u}$) has the following regularity.

ASSUMPTION 3.9 : *The solution u of (3.7) (hence the functions of $\overset{\circ}{u}$, solution of (3.37)) belongs to $H^3(\Omega) \cap W^{2,\infty}(\Omega)$.* ∎

4. THE APPROXIMATE PROBLEM

In order to give an approximation of (3.7), let us assume that Ω is a convex polygon of $\mathbb{R}^2$. We consider a triangulation D_h of Ω with regular triangles T, whose maximum diameter is $\leqslant h$.

We are using a non-conforming finite element method : Morley's method ([12], [10]), modified according to the technique introduced by [2].

To this purpose let us define

— the space

(4.1) $H^{2,h} = \{ v^h \in L^2(\Omega), v^h |_T \in H^2(T) \text{ for all } T \in D_h \}$

with the norm

(4.2) $\| v^h \|_{2,h}^2 = \displaystyle\sum_{T \in D_h} \| v^h \|_{2,T}^2$

and the seminorms

(4.3)　　$| v^h |^2_{i,h} = \sum_{T \in D_h} | v^h |^2_{i,T}$　$i = 1, 2$

— the subspace of $H^{2,h}$

(4.4)　　$\tilde{H}^{2,h} = \left\{ v^h \in H^{2,h}, \int_{\Omega} v^h \, d\underline{x} = 0 \right\}$

— the finite dimensional subspace of $\tilde{H}^{2,h}$

(4.5)　　$V_h = \{ v^h \in \tilde{H}^{2,h}, v^h |_T \in P_2(T)$ for all $T \in D_h$, v^h is continuous at the vertices of the triangles, $v^h_{/n}$ is continuous at the midpoint of each edge $\}$

— the convex closed subset of V_h

(4.6)　　$K_h = \{ v^h \in V_h, v^h_{/n} \geqslant 0$ at the midpoint of each edge belonging to $\partial \Omega \}$

— the continuous bilinear form

(4.7)　　$a_h(v^h, w^h) = \sum_{T \in D_h} \int_T v^h_{/ij} w^h_{/ij} \, d\underline{x}$　$v^h, w^h \in H^{2,h}$.

Now we consider the following problem

(4.8)　　$\begin{cases} \text{Find } u^h \in K_h \text{ such that} \\ a_h(u^h, v^h - u^h) \geqslant (f, v^I - u^I) \quad \text{for all} \quad v^h \in K_h \end{cases}$

where v^I is the piecewise linear interpolate (belonging to $C^0(\overline{\Omega})$) of v^h. The modification introduced by [2] lies in the use of $(v^I - u^I)$ instead of $(v^h - u^h)$.

We shall be able to prove the existence of a solution of (4.8) by using almost the same arguments of the above paragraph, that is : the equivalence of (4.8) to a minimum problem and the equivalence, in K_h, of $| \cdot |_{2,h}$ to $\| \cdot \|_{2,h}$ (theorem 4.1). This allows us to find at least one solution (theorem 4.2).

The minimum problem equivalent to (4.8) is the following

(4.9)　　$\begin{cases} \text{Find } u^h \in K_h \text{ such that} \\ J_h(u^h) = \min_{v^h \in K_h} J_h(v^h) \end{cases}$

where

(4.10) $J_h(v^h) = \dfrac{1}{2} a_h(v^h, v^h) - (f, v^I) \quad v^h \in H^{2,h}$.

By remarking that, analogously with (3.8), we have

(4.11) $K_h \cap P_1(\Omega) = 0$

it is easy to prove the following theorem.

THEOREM 4.1 : *There exists C, independent of v^h, such that*

(4.12) $\| v^h \|_{2,h} \leqslant C \, | \, v^h \, |_{2,h} \quad for \ all \quad v^h \in K_h$.

Proof : By proceeding as in Theorem 3.1, we find a sequence v_n such that, for all $n \in N$,

(4.13) $v_n \in K_h$
(4.14) $\| v_n \|_{2,h} = 1$
(4.15) $v_n \to v$,   as $n \to + \infty$,   in V_h (finite dimensional)
(4.16) $v \in K_h$
(4.17) $| v |_{2,h} = 0$.

The last equality means that $v \in P_1(T)$ for all $T \in D_h$, and since $v \in V_h$ this implies $v \in P$ (*see* (3.18)); then (4.16) and (4.11) imply $v \equiv 0$, which contradicts (4.14).   ∎

THEOREM 4.2 : *Problems (4.8) and (4.9) have at least one solution. If u_1^h and u_2^h are two solutions, then*

(4.18) $(u_1^h - u_2^h) = p \quad p \in P$
(4.19) $(f, p) = 0$.

Proof : Using the technique of Theorem 3.2 and recalling that (e.g. [4])

(4.20) $\| v^I \|_0 \leqslant C(h, \Omega) \| v^h \|_{2,h} \quad for \ all \quad v^h \in V_h$

If v^I is the piecewise linear interpolate of v^h, we can easily check both the existence of at least one solution of (4.8) and (4.9), and the relations

(4.21) $| u_1^h - u_2^h |_{2,h} = 0.$
(4.22) $(f, u_1^I - u_2^I) = 0$

where u_1^h and u_2^h are two solutions of the problems. Formula (4.21) with the continuity of u_i^h and $u_{i/n}^h$, $i = 1, 2$, respectively at the vertices of the triangles and at the midpoints of the edges, implies (4.18); then (4.18) and (4.22) give (4.19).

■

Case I (3.31)

We consider, as we did in the case of the continuous problem, the quotient space

$$(4.23) \quad \dot{H}^{2,h} = \tilde{H}^{2,h}/P,$$

the canonical mapping

$$(4.24) \quad v^h \to \dot{v}^h \quad \text{of} \quad \tilde{H}^{2,h} \to \dot{H}^{2,h}$$

and the images $\dot{V}_h$ of V_h and $\dot{K}_h$ of K_h given by the map (4.24) (of course $\dot{K}_h \subset \dot{V}_h$).

THEOREM 4.3 : *Let U_h be the following closed convex subset of K_h*

$$(4.25) \quad U_h = \{ u^h \in K_h, u^h \text{ solves } (4.7) \}.$$

Given $v^h \in \tilde{H}^{2,h}$ we denote by $\dot{v}^I$ the class of functions obtained by applying (4.24) to the continuous piecewise linear interpolate of v^h.
The problem

$$(4.26) \quad \begin{cases} \text{Find } \dot{u}^h \in \dot{K}_h \text{ such that} \\ a_h(\dot{u}^h, \dot{v}^h - \dot{u}^h) \geqslant (f, \dot{v}^I - \dot{u}^I) \quad \text{for all} \quad \dot{v}^h \in \dot{K}_h \end{cases}$$

has a unique solution. Moreover

$$(4.27) \quad U_h \subset \dot{u}^h.$$

Proof : By the results of Theorem 4.2 and by (3.31).   ■

Now, let us notice that the solutions of (3.7) and the solutions of (4.7) have (in both Cases (3.31) and (3.32)) a very similar behaviour. This will be shown in Theorem 4.6. Before that, let us introduce some notations and recall a result (for the proof see e.g. [10]) which we shall use also in the next paragraph, in order to prove the convergence of the discrete scheme.

$$(4.28) \quad \mathscr{L} \text{ denotes the set of the internal edges } L.$$

$$(4.29) \quad \tilde{\mathscr{L}} \text{ denotes the set of the boundary edges } \tilde{L}.$$

LEMMA 4.4 : *Let* $S(v_{/n}^h)$, $S(v_{/t}^h)$ *be the jumps of* $v_{/n}^h$ *and* $v_{/t}^h$ *at the interelement boundaries. Then*

$$(4.30) \quad \int_L S(v_{/n}^h)\, ds = \int_L S(v_{/t}^h)\, ds = 0 \quad \text{for all} \quad v^h \in V_h, \quad L \in \mathscr{L}. \quad \blacksquare$$

Remark 4.5 : In particular we point out that, given the endpoints m_1, m_2 and the midpoint $\overline{m}$ of an edge L^*, then

$$(4.31) \quad \int_{L^*} v_{/t}^h\, ds = v^h(m_2) - v^h(m_1) \quad \text{for all} \quad v^h \in V_h, \quad L^* \in \mathscr{L} \cup \tilde{\mathscr{L}}$$

$$(4.32) \quad \int_{L^*} v_{/n}^h\, ds = \mu(L^*)\, v_{/n}^h(\overline{m}) \quad \text{for all} \quad v^h \in V_h, \quad L^* \in \mathscr{L} \cup \tilde{\mathscr{L}}. \quad \blacksquare$$

THEOREM 4.6 : *Given* $u^h \in U_h$, *we can consider its bending moment* $M_n(\underline{\lambda}^h)$ *defined at each interelement boundary, with* $\lambda_{ij_1T}^h = u_{/ij_1T}^h$. *Then*

$$(4.33) \quad M_n(\underline{\lambda}^h) \text{ is continuous at the interelement boundaries}$$

$$(4.34) \quad \int_{\tilde{L}} M_n(\underline{\lambda}^h)\, u_{/n}^h\, ds = 0 \quad \text{for all} \quad \tilde{L} \in \tilde{\mathscr{L}}$$

$$(4.35) \quad M_n(\underline{\lambda}^h) \leqslant 0 \quad \text{on} \quad \partial\Omega$$

$$(4.36) \quad a_h(u^h, v^h - u^h) = \sum_{L^* \in \mathscr{L} \cup \tilde{\mathscr{L}}} \int_{L^*} M_{nt}(\underline{\lambda}^h)\, (v^h - u^h)_{/t}\, ds -$$

$$- \sum_{\tilde{L} \in \mathscr{L}} \int_{\tilde{L}} M_n(\underline{\lambda}^h)\, v_{/n}^h\, ds$$

$$(4.37) \quad a_h(u^h, v^h) = (f, v^I) - \sum_{\tilde{L} \in \mathscr{L}} \int_{\tilde{L}} M_n(\underline{\lambda}^h)\, v_{/n}^h\, ds \quad \text{for all} \quad v^h \in K_h$$

$$(4.38) \quad (f, p) = \sum_{\tilde{L} \in \mathscr{L}} \int_{\tilde{L}} M_n(\underline{\lambda}^h)\, p_{/n}\, ds \quad \text{for all} \quad p \in P.$$

Proof : Green's formula gives us

$$(4.39) \quad a_h(u^h, v^h - u^h) = \sum_{L^* \in \mathscr{L} \cup \tilde{\mathscr{L}}} \left(\int_{L^*} M_{nt}(\underline{\lambda}^h)\, (v^h - u^h)_{/t}\, ds - \right.$$

$$\left. - \int_{L^*} M_n(\underline{\lambda}^h)\, (v^h - u^h)_{/n}\, ds \right) \geqslant (f, v^I - u^I) \quad \text{for all} \quad v^h \in V_h.$$

Let us consider some functions of K_h with suitable degrees of freedom. More precisely : let A_i, $1 \leqslant i \leqslant N(h)$, be the vertices of D_h, and m_j, $1 \leqslant j \leqslant M(h)$, the midpoints of the edges of D_h. A function $\bar{v} \in V_h$, is defined as follows

(4.40) $\bar{v} = v - $ (mean value of v)

where v has the following d.o.f.

(4.41) $F_i(v) = $ value of v at the vertex A_i, $1 \leqslant i \leqslant N(h)$

(4.42) $F_{lj}(v) = $ value of $v_{/n}$ at the point m_j, $1 \leqslant j \leqslant M(h)$.

Now we choose $\bar{v}_1$, $\bar{v}_2 \in V_h$ with

(4.43) $F_{lj}(v_1) = 1 + F_{lj}(u_h)$ at a point $m_{\tilde{j}} \in \Omega$

(4.44) $F_{lj}(v_2) = -1 + F_{lj}(u^h)$ at the same $m_{\tilde{j}}$

(4.45) $F_{lj}(v_1) = F_{lj}(v_2) = F_{lj}(u^h)$ $1 \leqslant j \leqslant M(h)$, $j \neq \tilde{j}$

(4.46) $F_i(v_1) = F_i(v_2) = F_i(u^h)$ $1 \leqslant i \leqslant N(h)$.

By inserting $\bar{v}_1$, $\bar{v}_2$ in (4.39) and using Remark 4.5 we obtain, with n_1 and n_2 outward normals to the edge $L_{\tilde{j}}$ containing $m_{\tilde{j}}$

(4.47) $\displaystyle\int_{L_{\tilde{j}}} (M_{n_1}(\underline{\lambda}^h) - M_{n_2}(\underline{\lambda}^h)) \, ds = 0$

hence (4.33).

   Next, $\bar{v}_1$ and $\bar{v}_2 \in K_h$, with

(4.48) $F_{lj}(v_1) = 0$ at a point $m_{\tilde{j}} \in \partial\Omega$

(4.49) $F_{lj}(v_2) = 2 F_{lj}(u^h)$ at the same point

and once more (4.45), (4.46) give (4.34). On the other hand (4.35) is obtained by taking $\bar{v} \in K_h$ with the same values of all d.o.f. as u^h, except for one $m_{\tilde{j}} \in \partial\Omega$, where

(4.50) $F_{lj}(v) = 1 + F_{lj}(u^h)$.

Formula (4.36) is an obvious application of the above results. Now, in order to obtain (4.37), let us notice that

(4.51) $a_h(u^h, v^h) = (f, v^l)$ for all $v^h \in V_h$, $v_h^h = 0$ on $\partial\Omega$.

Then, because of (4.36), (4.51) and Green's formula, the decomposition

(4.52) $v = \bar{v}_1 + \bar{v}_2$ for all $v \in K_h$,

where

(4.53) $F_i(v_1) = F_i(v)$ at all $A_i, 1 \leqslant i \leqslant N(h)$

(4.54) $F_{/j}(v_1) = F_{/j}(v)$ at all $m_j \in \Omega$

(4.55) $F_{/j}(v_2) = F_{/j}(v)$ at all $m_j \in \partial\Omega$

immediately gives (4.37), from which (4.38) follows.   ∎

Case II (3.32).

Assumption 3.6, Theorem 4.2 and the above theorem lead to the following result, easy to prove by the same techniques of Lemma 3.5, Theorem 3.7 and Lemma 3.8.

THEOREM 4.7 : *In Case* II, *Problem* (4.7) *has a unique solution, whose normal derivative vanishes on at least two not parallel edges belonging to $\partial\Omega$.*   ∎

5. ERROR ESTIMATES

In this paragraph we shall estimate the convergence of the discrete scheme in two steps. The first step will be the following : in Theorem 5.2 we shall bound the error on the moments, which can be done without distinguishing between Case I (3.31) and Case II (3.32). In Theorem 5.3 we shall see that, if we are dealing with the problem in Case I, such estimate also measures the error done on the displacement. In the second step we consider Case II, and we bound $\| u - u^h \|_{2,h}$ mainly by considering the boundary conditions. The real constants C or C_i, $i \in N$, which appear in this paragraph, are independent of h.

First of all let us give the following result in approximation theory.

LEMMA 5.1 : *Given a function v defined and continuously differentiable on all $T \in D_h$, we define as follows its interpolate v_M into V_h.*

(5.1) $v_M = \tilde{v}_M - (\text{mean value of } \tilde{v}_M)$

where

(5.2) $\tilde{v}_M = v$ *at all vertices*

(5.3) $\tilde{v}_{M/n}$ *at the midpoints of each edge $=$ mean value of $v_{/n}$ on the same edge .*

Then we have

(5.4) $| v - v_M |_{m,h} \leqslant Ch^{3-m} | v |_3$ $1 \leqslant m \leqslant 3$.

Proof : In [8] we can find the result

$$(5.5) \qquad | v - \tilde{v}_M |_{m,h} \leqslant Ch^{3-m} | v |_3 \quad 0 \leqslant m \leqslant 3$$

and (5.5) and (5.1) immediately give (5.4). ∎

Finally we can bound the error in the seminorm $| \cdot |_{2,h}$.

THEOREM 5.2 : *Let u be a solution of (3.7) and u^h a solution of (4.8). If the number of points of $\partial \Omega$ where the constraint changes from binding to nonbinding is finite, then*

$$(5.6) \qquad | u - u^h |_{2,h} = 0(h) \, .$$

Proof : Let u_M be the interpolate of u into V_h, which clearly belongs to K_h, and let $\underline{\lambda}$ be given by (3.30).

We define

$$(5.7) \qquad \chi = u_M - u^h$$

$$(5.8) \qquad \chi^I = \text{piecewise linear interpolate of } \chi$$

$$(5.9) \qquad \overline{M}_{nt} = \text{mean value of } M_{nt}(\underline{\lambda}) \text{ on an edge of the triangulation}$$

$$(5.10) \qquad \overline{M}_{n} = \text{mean value of } M_{n}(\underline{\lambda}) \text{ on an edge of the triangulation} \, .$$

We have

$$(5.11) \qquad | \chi |^2_{2,h} = a_h(\chi, \chi) = a_h(u_M - u, \chi) + a_h(u - u^h, \chi) \leqslant$$
$$\leqslant | u_M - u |_{2,h} | \chi |_{2,h} + a_h(u, \chi) - (f, \chi^I)$$
$$= | u_M - u |_{2,h} | \chi |_{2,h} + E_h - (f, \chi^I) \, .$$

Green's formula yields

$$(5.12) \qquad E_h - (f, \chi^I) = \sum_{T \in D_h} \left(- \int_T \lambda_{ij/i} \, \chi_{/j} \, dx + \int_{\partial T} M_{nt}(\underline{\lambda}) \, \chi_{/t} \, ds - \right.$$

$$\left. - \int_{\partial T} M_n(\underline{\lambda}) \, \chi_{/n} \, ds \right) + \sum_{T \in D_h} \int_T \lambda_{ij/i} \, \chi^I_{/j} \, dx + \sum_{\tilde{L} \in \mathscr{L}} \int_{\tilde{L}} Q_n(\underline{\lambda}) \chi^I \, ds =$$

$$= \sum_{T \in D_h} \left(\int_T \lambda_{ij/i}(\chi^I - \chi)_{/j} \, d\underline{x} + \int_{\partial T} M_{nt}(\underline{\lambda}) \, \chi_{/t} \, ds - \right.$$

$$\left. - \int_{\partial T} M_n(\underline{\lambda}) \, \chi_{/n} \, ds \right) + \sum_{\tilde{L} \in \mathscr{L}} \left(\int_{\tilde{L}} Q_n(\underline{\lambda}) \, (\chi^I - \chi) \, ds + \int_{\tilde{L}} Q_n(\underline{\lambda}) \, \chi \, ds \right) .$$

Owing to Lemma 4.4, we can write, following [2],

$$\sum_{T \in D_h} \left(\int_{\partial T} M_{nt}(\underline{\lambda}) \, \chi_{/t} \, ds - \int_{\partial T} M_n(\underline{\lambda}) \, \chi_{/n} \, ds \right) =$$

(5.13)
$$= \sum_{L \in \mathscr{L}} \int_L (M_{nt}(\underline{\lambda}) - \overline{M}_{nt}) \, \chi_{/t} \, ds$$

$$+ \sum_{L^* \in \mathscr{P} \cup \mathscr{L}} \int_{L^*} (M_n(\underline{\lambda}) - \overline{M}_n) \, \chi_{/n} \, ds$$

$$+ \sum_{L \in \mathscr{P}} \left(\int_{\tilde{L}} M_{nt}(\underline{\lambda}) \, \chi_{/t} \, ds - \int_{\tilde{L}} \overline{M}_n \, \chi_{/n} \, ds \right).$$

Let us write the partial estimate which we obtain by combining Lemma 5.1, (5.11)-(5.13) and by recalling that $K_n(\underline{\lambda})$ vanishes on $\partial\Omega$.

(5.14) $\quad | \chi |^2_{2,h} \leqslant Ch \lceil u \rfloor_3 | \chi |_{2,h} + \| Q_n(\underline{\lambda}) \|_{-1/2,\partial\Omega} \| \chi - \chi^I \|_{1/2,\partial\Omega} -$

$$- \sum_{\tilde{L} \in \mathscr{P}} \int_{\tilde{L}} \overline{M}_n \, \chi_{/n} \, ds.$$

Classical results of interpolation [11] give

(5.15) $\quad \| \chi - \chi^I \|_{1/2,\partial\Omega} \leqslant \| \chi - \chi^I \|^{1/2}_{0,\partial\Omega} \| \chi - \chi^I \|^{1/2}_{1,\partial\Omega}$

and by using Bramble-Hilbert's techniques we obtain

(5.16) $\quad \| \chi - \chi^I \|_{0,\partial\Omega} \leqslant Ch^{3/2} | \chi |_{2,h}$

(5.17) $\quad \| \chi - \chi^I \|_{1,\partial\Omega} \leqslant Ch^{1/2} | \chi |_{2,h} .$

There remains to bound the last integral of (5.14). By (3.28) and (5.3) we have, as $\int_{\tilde{L}} \overline{M}_n \, u^h_{/n} \, ds \leqslant 0$ for all $\tilde{L} \in \mathscr{P}$,

(5.18) $\quad \sum_{\tilde{L} \in \mathscr{P}} \int_{\tilde{L}} \overline{M}_n(u^h - u_M)_{/n} \, ds \leqslant \sum_{\tilde{L} \in \mathscr{P}} \int_{\tilde{L}} (M_n(\underline{\lambda}) - \overline{M}_n) \, u_{/n} \, ds .$

Remark that, in the right hand side of (5.18), the only terms $\neq 0$ are only the integrals on the edges of $\partial\Omega$ which contain at least a point where the constraint changes from binding to non binding; then, as the number of such points is finite, and, by the regularity assumption, $u \in W^{2,\infty}(\Omega)$, we get

(5.19) $\quad \sum_{\tilde{L} \in \mathscr{P}} \left(\int_{\tilde{L}} (M_n(\underline{\lambda}) - \overline{M}_n) \, u_{/n} \, ds = 0(h^2) . \right.$

By combining (5.14)-(5.19) we obtain

$$(5.20) \quad | \chi |_{2,h} = 0(h)$$

from which, by the triangle inequality, the desired estimate follows.   ∎

Case I (3.31)

THEOREM 5.3 : *We assume that* (3.31) *hold, and we consider the solution* $\dot{u}$ *of* (3.10) *and the solution* $\dot{u}^h$ *of* (4.26). *Then*

$$(5.21) \quad \| \dot{u} - \dot{u}^h \|_{2,h} = 0(h) .$$

Proof : It is immediate if we consider that, in $\dot{H}^{2,h}$, $\| \cdot \|_{2,h} := | \cdot |_{2,h}$, and apply the above theorem.   ∎

Case II (3.32)

Now we want to estimate the error $u - u^h$ in Case II. First of all we decompose the difference $(u_M - u^h)$ (resp. the interpolate of u into V_h and the solution of the discrete problem) as follows

$$(5.22) \quad u_M - u^h = \overline{u}^h + \alpha_h \overline{p} + \beta_h p^* \quad \alpha_h \in \mathbb{R} , \quad \beta_h \in \mathbb{R}$$

where

$$(5.23) \quad \overline{u}^h \in V_h , \quad \int_\Omega \overline{u}^h_{/i} \, d\underline{x} = 0 \quad i = 1, 2$$

$$(5.24) \quad p^* \in P \text{ is given by (3.32b), i.e. } (f, p^*) = 0$$

$$(5.25) \quad \overline{p} \in P \text{ is chosen once and for all, such that } \overline{p} \neq p^* .$$

We write now

$$(5.26) \quad u - u^h = u - u_M + \overline{u}^h + \alpha_h \overline{p} + \beta_h p^* = \xi + \alpha_h \overline{p} + \beta_h p^*$$

where, by Theorem 5.2, (5.22), (5.23),

$$(5.27) \quad \| \xi \|_{2,h} = 0(h) .$$

We shall get our last goal, $\alpha_h = 0(h)$ and $\beta_h = 0(h)$, by considering that u (resp. u^h) minimizes the functional $J(.)$ (resp. $J_h(.)$) given by (3.9) (resp. (4.9)) and by recalling the following result :

$$(5.28) \quad | v^h |_{m,\infty} \leqslant C | \ln h |^{1/2} (| v^h |_m + | v^h |_{m+1})$$

for all $v^h \in H^{m+1}(\Omega) \cap W^{m,\infty}(\Omega)$, $v^h_{|T} \in P_q(T)$ for all T, $q \geqslant m + 1$, (5.28) is proved, for instance, in [5] following arguments of [4] and [8].

In our case, with $v^h = \xi$, $m = 1$, we have

(5.29) $|\xi|_{1,\infty} \leqslant C |\ln h|^{1/2} \| \xi \|_{2,h} = 0(h |\ln h|^{1/2})$.

THEOREM 5.5 : *Let u and u^h be the solutions respectively of* (3.7) *and* (4.8). *Then*

(5.30) $\| u - u_h \|_{2,h} = 0(h |\ln h|^{1/2})$.

Proof : By (5.26) we have

(5.31) $u^h + \alpha_h \overline{p} + \beta_h p^* + \xi = w^h + \xi = u$

with $(f, p^*) = 0$ and $w^h \in V^h$. Hence, by (5.24),

(5.32) $J_h(u^h) + \alpha_h(f, \overline{p}) = J_h(w^h) = J_h(u - \xi) =$

$$= \frac{1}{2} a(u, u) + \frac{1}{2} a_h(\xi, \xi) - a_h(u, \xi) - (f, u^I - \xi^I)$$

$$= J(u) + \frac{1}{2} a_h(\xi, \xi) - a_h(u, \xi) + (f, \xi^I) + (f, u - u^I).$$

Since

(5.33) $J(u) - J_h(u^h) = -\frac{1}{2} a(u, u) + \frac{1}{2} a_h(u^h, u^h) = \frac{1}{2} a(u + u^h, u^h - u)$

(5.27), (5.32) and (5.33) give us

(5.34) $\alpha_h = 0(h)$.

Owing to (5.34) the equality (5.31) becomes

(5.35) $u - \beta_h p^* = u^h + \theta$

where

(5.36) $\| \theta \|_{2,h} = 0(h)$.

Now let us notice that Lemma 3.8 and $(f, p^*) = \displaystyle\int_G M_n(\lambda) p^*_{/n} \, ds = 0$, imply

the existence of s_1 and $s_2 \in \partial\Omega$, such that

(5.37) $s_1 \in G \quad p^*_{/n}(s_1) < 0$

(5.38) $s_2 \in G \quad p^*_{/n}(s_2) > 0$.

and we can choose s_1, s_2 to be midpoints of boundary edges.

We define

(5 39) $\quad g_1(\beta) = + \beta p_{/n}^*(s_1)$

(5 40) $\quad g_2(\beta) = - \beta p_{/n}^*(s_2)$,

and we consider (5 35), (5 36)

If $\beta_h < 0$ we get, using (5 29),

(5 41) $\quad 0 < g_1(\beta_h) = + \beta_h \, p_{/n}^*(s_1) = - u_{/n}^h(s_1) - \theta_{/n}(s_1) \leqslant 0(h \mid \ln h \mid^{1/2})$

Analogously, if $\beta_h > 0$, we obtain

(5 42) $\quad 0 > - \beta_h \, p_{/n}^*(s_2) \geqslant 0(h \mid \ln h \mid^{1/2})$

Hence from (5 41), (5 42) we get that, in any case,

(5 43) $\quad \mid \beta_h \mid \leqslant 0(h \mid \ln h \mid^{1/2})$

and (5 30) follows from (5 35), (5 36), (5 43) ∎

REFERENCES

[1] R ADAMS, *Sobolev Spaces* New York Academic Press 1975

[2] D N ARNOLD, F BREZZI, *Mixed and nonconforming finite element methods implementation, postprocessing and error estimates* To appear on R A I R O

[3] C BAIOCCHI, A CAPELO, *Disequazioni variazionali e quasivariazionali Applicazioni a problemi di frontiera libera*, Vol 1 Bologna Pitagora Editrice 1978

[4] P G CIARLET, *The Finite Element Method for Elliptic Problems* Amsterdam North Holland Publishing Co 1978

[5] M I COMODI, *Approximation of a plate bending problem with a boundary unilateral constraint* Numer Math 47, 435-458 (1985)

[6] G DUVAUT, J L LIONS, *Inequalities in Mechanics and Physics* Grund Math Wiss, 219 Berlin Springer-Verlag 1976

[7] G FICHERA, *Boundary value problems of Elasticity with unilateral constraints* Handbuch der Physik, Band VI a/2 Springer, Berlin (391-424) 1972

[8] C JOHNSON, *A convergence estimate for an approximation of a parabolic variational inequality* SIAM J Numer Anal, 13, 599-606 (1976)

[9] D KINDERLEHRER, G STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications* New York Academic Press 1980

[10] P LASCAUX, P LESAINT, *Some nonconforming finite elements for the plate bending problem* R A I R O Anal Num 8, 9-53 (1975)

[11] J L LIONS, E MAGENES, *Problemes aux limites non Homogenes et Applications*, TOME I PARIS DUNOD 1968

[12] L S D MORLEY, *The triangular equilibrium element in the solution of plate bending problems* Aero Quart 19, 149-169

[13] G STRANG, G FIX, *An Analysis of the Finite Element Method* Englewood Cliffs, New Jersey Prentice Hall 1973

[14] S TIMOSCHENKO, *Theory of Plates and Shells* New York McGraw-Hill 1959

[15] O C ZIENKIEWICZ, *The Finite Element Method in Engineering Science* London McGraw-Hill 1971