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Ewa Demirska

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EXTERNAL APPROXIMATION OF BIFURCATION PROBLEMS (*)

by Ewa Demirska [1]

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Abstract. — *This paper deals with an external approximation of bifurcation problems. It is a continuation of the articles Descloux, Rappaz* [5], [6].

Résumé. — *Le sujet est l'approximation extérieure du problème de la bifurcation. Cet article peut être regardé comme la continuation des articles Descloux, Rappaz* [5], [6].

I. INTRODUCTION

In their two papers [5, 6] Descloux and Rappaz consider the approximation of the solution branches of the nonlinear equation

$$F(x) = 0 \tag{1}$$

by the solution branches of the equations

$$F_h(x_h) = 0 . \tag{2}$$

There X, Y are real Banach spaces ; $F : X \to Y$ is a nonlinear operator approximated by the family of nonlinear operators $F_h : X_h \to Y_h$; $\{ X_h \}_h$, $\{ Y_h \}_h$ are families of finite dimensional subspaces of X and Y respectively. The equation $F(x) = 0$ is considered in the neighbourhood of the point x^* satisfying : $F(x^*) = 0$, $F'(x^*)$ is a Fredholm operator of index 1.

First Descloux and Rappaz prove existence of a solution branch of (2) and its convergence to a solution branch of (1) in the neighbourhood of a regular point x^*. Next the case of a critical point is discussed. It is of special interest because it covers a great many known types of bifurcation points — for example double limit points, simple and double bifurcation points.

(*) Received in January 1985.
[1] Institute of Mathematics, Politechnika Warszawska, Plac Jedności Robotniczej 1, Warsaw, Poland.

In the case of a critical point the authors assume additionally that the operators F_h defined on the finite-dimensional subspaces X_h of the space X — have prolongations $\tilde{F}_h$ onto the whole space X. Then they prove convergence results similar to those obtained for the case of a regular point.

The aim of our work is to release from this assumption of the existence of the prolongations $\tilde{F}_h$ of the operators F_h. This paper, however, remains in strong connection to the articles [5, 6] by Descloux and Rappaz.

Throughout this paper we suppose that the approximating operators F_h operate between some real Banach spaces X_h and Y_h connected with X and Y by restriction operators $r_h : X \to X_h$ and $s_h : Y \to Y_h$. To make it clear, we do not assume that X_h, Y_h are finite-dimensional. However, the case when X_h, Y_h are finite-dimensional is the most interesting from the practical point of view for in practice infinite dimensional problems are usually approximated by the finite-dimensional ones. By our assumptions the theory of an interior and external approximation can be used (see Temam [11] or Aubin [1]).

In Chapter II some preliminaries are given.

At the beginning of Chapter III all the assumptions are precisely formulated. Then by means of Lyapunov-Schmidt method we introduce bifurcation functions f, f_h in such a way that they operate from $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$ ($n =$ codim Range $F'(x^*)$) and possess properties which will justify the application of the results proved in [5, 6]. The main results are formulated in Theorems 1, 3, 4. Theorem 2 dealing with bifurcation equations is a quotation from [6].

In Chapter IV we present an example illustrating the theory of Chapter III.

Similar problems to ours were examined by for example Moore, Spence [7] or Weiss [12]. Moore, Spence [7] were the first to take up the question of an external approximation of bifurcation problems in so general a form, although they dealt only with the case of a regular point. Their work can be put into the framework of ours.

II. PRELIMINARIES

In our work families of approximating operators and spaces will be indexed by a parameter $h \in (0, h_0]$. Where it does not cause misunderstanding, the letter « h » will be omitted. For example instead of denoting an open ball in a normed space X_h — by a symbol $B_{X_h}(x_0^h, \delta)$, we will simply write $B(x_0^h, \delta)$.

Our main tools will be the generalized implicit function theorem and a corollary from it giving an important error estimate in the versions presented by Descloux, Rappaz in [6] pp. 323-324. In [6] these results were applied for a family of operators $G_h : X \times Y \to Z$, each considered in the neighbourhood of a point $(x_0, y_0) \in X \times Y$; where X, Y, Z were Banach spaces. These theorems, however, can be applied in a more general context. And we will apply them for a family of operators $G_h : X_h \times Y_h \to Z_h$, each considered in the neighbourhood of a point $(x_0^h, y_0^h) \in X_h \times Y_h$; where X_h, Y_h, Z_h will be Banach spaces for each h.

At the end we quote a well known result on a uniform convergence :

THEOREM II : *Let X be a Banach space; $D \subset X$ — be a precompact set; $\{X_h\}_{h \leqslant h_0}$ be a family of normed spaces. Let the mappings $f_h : D \to X_h$, $\rho : D \to \mathbb{R}_+$ fulfil the conditions* :

1) $\| f_h(x) \|_h \to \rho(x) \quad \forall x \in D$,
2) $\exists L \geqslant 0 \quad \forall h \leqslant h_0 \quad \forall x, y \in D : \| f_h(x) - f_h(y) \|_h \leqslant L \| x - y \|$.

Then the convergence in 1) *is uniform on the set D.*

III. MAIN RESULTS

Let X, Y be real Banach spaces approximated by the families $\{X_h\}$, $\{Y_h\}$ ~~of real Banach spaces. Let a nonlinear operator $F : X \to Y$ be approximated~~ by nonlinear operators $F_h : X_h \to Y_h$. Let us formulate :

Exact Problem : Find the solution set of the equation :

$$F(x) = 0 \tag{1}$$

in a neighbourhood of a point x^* which is regular or critical.

Approximate Problem : Find the solution set of the equation :

$$F_h(x_h) = 0 \tag{2}$$

in a neighbourhood of a certain point x_h^*. Examine how the solutions of (2) approximate the solutions of (1).

We will deal with these problems assuming :

(A1) The operators F and F_h for $h \leqslant h_0$ are of class C^p; $p \geqslant 2$.
(A2) x^* is regular or critical with codim Range $F'(x) = n \geqslant 0$ (i.e. $F(x^*) = 0$, Range $F'(x^*)$ is a closed subspace of Y, dim Ker $F'(x^*) = n + 1$, codim Range $F'(x^*) = n$).

If we denote :

$$X_1 = \operatorname{Ker} F'(x^*) \qquad Y_2 = \operatorname{Range} F'(x^*), \tag{3}$$

the assumption (A2) implies existence of two closed subspaces $X_2 \subset X$ and $Y_1 \subset Y$ such that $\dim Y_1 = n$ and the following decompositions hold :

$$X = X_1 \oplus X_2 , \tag{4}$$

$$Y = Y_1 \oplus Y_2 . \tag{5}$$

Next we assume that the spaces X and X_h are connected by restriction operators $r_h \in L(X, X_h)$, while the spaces Y and Y_h — by restriction operators $s_h \in L(Y, Y_h)$. Let the following decompositions be true :

$$X_h = X_{1h} \oplus X_{2h} \qquad \overline{X}_{1h} = X_{1h} \qquad \overline{X}_{2h} = X_{2h} \qquad \forall h \leqslant h_0 , \tag{6}$$

$$Y_h = Y_{1h} \oplus Y_{2h} \qquad \overline{Y}_{1h} = Y_{1h} \qquad \overline{Y}_{2h} = Y_{2h} \qquad \forall h \leqslant h_0 . \tag{7}$$

Let us introduce further definitions :

(8) $P : X \to X_2$, $P_h : X_h \to X_{2h}$ are projections associated with the decompositions (4) and (6) respectively (i.e. $P^2 = P$, $PX = X_2$, $(I - P) X = X_1$),

(9) $Q : Y \to Y_2$, $Q_h : Y_h \to Y_{2h}$ are projections associated with the decompositions (5) and (7) respectively,

(10) $\{ x_0, x_1, \dots, x_n \}$ is a basis of $X_1 = \operatorname{Ker} F'(x^*)$,

(11) $\{ y_1, \dots, y_n \}$ is a basis of Y_1.

Let linear operators $S : \mathbb{R}^{n+1} \to X_1 = (I - P) X$, $S_h : \mathbb{R}^{n+1} \to X_{1h} = (I - P_h) X_h$ map $\sigma =]\sigma_0, \sigma_1, \dots, \sigma_n]^\times \in \mathbb{R}^{n+1}$ into $S\sigma$ and $S_h \sigma$ respectively where :

$$S\sigma = \sum_{i=0}^{n} \sigma_i x_i \qquad S_h \sigma = (I - P_h) r_h S\sigma . \tag{12}$$

Let linear operators $E : \mathbb{R}^n \to Y_1 = (I - Q) Y$, $E_h : \mathbb{R}^n \to Y_{1h} = (I - Q_h) Y_h$ map $\alpha = [\alpha_1, \dots, \alpha_n]^\times \in \mathbb{R}^n$ into $E\alpha$ and $E_h \alpha$ respectively where :

$$E\alpha = \sum_{i=1}^{n} \alpha_i y_i \qquad E_h \alpha = (I - Q_h) s_h E\alpha . \tag{13}$$

Now we are prepared to introduce further assumptions by which the main results of this paper will be proved :

(B1)	$\| x_h^* - r_h x^* \|_h \to 0$
(B2)	$\exists M_1 \geqslant 0 \ \exists \delta > 0 : \forall 0 \leqslant k \leqslant p \ \forall x_h \in B(x_h^*, \delta)$ $$\| F_h^{(k)}(x_h) \|_h \leqslant M_1 .$$
(B3)	$Q_h F_h'(x_h^*) \|_{X_{2h}}$ are isomorphisms of X_{2h} onto Y_{2h} with inverses uniformly bounded
(B4)	$\exists r > 0 : \forall x \in B(x^*, r) \ \forall 0 \leqslant k \leqslant p - 1$
	$\forall \xi_1, \xi_2, \ldots, \xi_k \in X$ – fixed $$\| s_h F^{(k)}(x)(\xi_1, \ldots, \xi_k) - F_h^{(k)}(r_h x)(r_h \xi_1, \ldots, r_h \xi_k) \|_h \to 0 .$$
(B5)	$\exists M_2 \geqslant 0 : \quad \| r_h \|_h \leqslant M_2 \quad \forall h \leqslant h_0$
(B6)	$\exists M_3 \geqslant 0 : \quad \| P_h \|_h \leqslant M_3 \quad \forall h \leqslant h_0$
(B7)	$\forall x \in X \qquad \| (r_h P - P_h r_h) x \|_h \to 0$
(B8)	S_h are isomorphisms of $\mathbb{R}^{n+1}$ onto X_{1h} with inverses uniformly bounded.
(B9)	$\exists M_4 \geqslant 0 : \quad \| s_h \|_h \leqslant M_4 \quad \forall h \leqslant h_0$
(B10)	$\exists M_5 \geqslant 0 : \quad \| Q_h \|_h \leqslant M_5 \quad \forall h \leqslant h_0$
(B11)	$\forall y \in Y \qquad \| (s_h Q - Q_h s_h) y \|_h \to 0$
(B12)	E_h are isomorphisms of $\mathbb{R}^n$ onto Y_{1h} with inverses uniformly bounded.

Remark III.1 :

a) (B2)-(B4) characterize the approximation of the operator F,

(B5)-(B8) — the approximation of the space X and the decomposition $X = X_1 \oplus X_2$, (B9)-(B12) — the approximation of Y and the decomposition $Y = Y_1 \oplus Y_2$.

(B4), (B7), (B11) are called conditions of consistency; (B3) is a condition of stability and it is justified by the fact that $F'(x^*) |_{X_2}$ is an isomorphism of X_2 onto Y_2 (see [5, 6]).

b) There may arise difficulties with the choice of the spaces X_{1h}, X_{2h}, Y_{1h}, Y_{1h}. Sometimes there is some indication that a certain point $x_h^* \in X_h$ is a bifurcation point for the operator F_h. But it is still a question and we want to prove it. Then we suggest that the very natural choice : $X_{1h} = \operatorname{Ker} F_h'(x_h^*)$, $Y_{2h} =$

Range $F_h'(x_h^*)$ be tried at first. So that X_{2h} and Y_{1h} could be defined, we think that for some types of problems spectral projections could be used. In order to check (B3), (B6), (B7), (B10), (B11), one should then use the theory of external approximation of linear eigenvalue problems — see for example Chatelin [2], Descloux, Nassif, Rappaz [4], Regińska [8]. All this will be illustrated in the example in Chapter IV.

c) Assumptions (B8) and (B12) may seem strange. It will turn out later that (B12) will enable us to introduce bifurcation functions $f, f_h : \mathbb{R}^{n+1} \to \mathbb{R}^n$ in a sensible way. (B8) is an equivalent of (B12) for X and its usefulness will be pointed to at the very end of our considerations.

d) The existence and uniform boundedness of E_h^{-1} in (B12) can be concluded from (B9)-(B11), when it is known from other considerations that $\dim Y_{1h} = n$ and :

1) a stable and convergent external approximation $\{ Y, \mathscr{F}_Y, \omega_Y, Y_h, s_h, q_h \}_{h \leqslant h_0}$ is given (for the definition see Temam [11]) or instead of 1) :

or instead of 1) :

2) Norms in Y_1 and Y_h are « matched », i.e. $\| s_h y \|_h \to \| y \|_{nc}$ $\forall y \in Y_1$. The symbol $\| . \|_{nc}$ denotes any norm in Y_1. This norm need not be induced from Y.

For the proof of *d*), see [3]. An analogous result is true for the operators S_h^{-1} from (B8).

Now we will apply the Lyapunov-Schmidt method. Exact Problem (1) is replaced equivalently by a problem of solving the two equations :

$$\begin{cases} QF(x) = 0 & (14') \\ (I - Q)\, F(x) = 0 \quad \text{(see (5), (9))}, & (14'') \end{cases}$$

each one in the neighbourhood of x^*.

Analogously Approximate Problem (2) is replaced, equivalently by a problem of solving the two equations :

$$\begin{cases} Q_h\, F_h(x_h) = 0 & (15') \\ (I - Q_h)\, F_h(x_h) = 0 \quad \text{(see (7), (9))}, & (15'') \end{cases}$$

each one in the neighbourhood of x_h^*.

Relations between the solutions of the infinite-dimensional problems $QF(x) = 0$, $Q_h\, F_h(x_h) = 0$

Let us introduce nonlinear operators $G : \mathbb{R}^{n+1} \times X_2 \to Y_2$, $G_h : \mathbb{R}^{n+1} \times$

$X_{2h} \to Y_{2h}$ such that : $\forall \sigma \in \mathbb{R}^{n+1}$

(16) $\quad G(\sigma, v) = QF(x^* + S\sigma + v) \qquad \forall v \in X_2 \quad$ (see (10), (12)),

(17) $\quad G_h(\sigma, v_h) = Q_h F_h(x_h^* + S_h \sigma + v_h) \qquad \forall v_h \in X_{2h} \quad$ (see (12)).

It is obvious that (14′) is equivalent to solving the equation :

(18) $\quad G(\sigma, v) = 0$ in a neighbourhood of $0 \in \mathbb{R}^{n+1} \times X_2$.

If the operator S_h is invertible, then (15′) becomes equivalent to solving the equation :

(19) $\quad G_h(\sigma, v_h) = 0$ in a neighbourhood of $0 \in \mathbb{R}^{n+1} \times X_{2h}$.

Now we will find relations between the solutions of (18) and (19).

THEOREM III.1 : *Let* (A1)-(A2), (B1)-(B7), (B9)-(B11) *hold.*

a) Then there exist constants $h_1, \xi_1, \alpha > 0$, *a unique map*

$$v : B(0, \xi_1) \subset \mathbb{R}^{n+1} \to X_2$$

such that :

(20) $\quad G(\sigma, v(\sigma)) = 0 \qquad \| v(\sigma) \| < \alpha \qquad \forall \sigma \in B(0, \xi_1)$

and for any $h \leqslant h_1$*-a unique map* $v_h : B(0, \xi_1) \subset \mathbb{R}^{n+1} \to X_{2h}$ *such that :*

(21) $\quad G_h(\sigma, v_h(\sigma)) = 0 \qquad \| v_h(\sigma) \|_h < \alpha \qquad \forall \sigma \in B(0, \xi_1)$.

Moreover v, v_n *are of class* C^p *with all the derivatives of orders* 0, 1, ..., p *uniformly bounded with respect to* $\sigma \in B(0, \xi_1)$, $h \leqslant h_1$.

b) For any $k = 0, 1, ..., p - 1$ *and any* $h \leqslant h_1$ *the following estimate is true* (see (10)) :

(22) $\quad \| r_h v^{(k)}(\sigma) - v_h^{(k)}(\sigma) \| \leqslant \text{Const}\, H_h^k(\sigma) \qquad \forall \sigma \in B(0, \xi_1)$,

where

(23) $$H_h^k(\sigma) = \| r_h x^* - x_h^* \| + \sum_{i=0}^{n} \| (P_h r_h - r_h P) x_i \| +$$
$$+ \sum_{l=0}^{k} \left\{ \| (P_h r_h - r_h P) v^{(l)}(\sigma) \| + \left\| (Q_h s_h - s_h Q) \frac{d^l}{d\sigma^l} F(x^* + S\sigma + v(\sigma)) \right\| + \right.$$
$$\left. + \left\| \frac{d^l}{d\sigma^l} [F_h(r_h(x^* + S\sigma + v(\sigma))) - s_h F(x^* + S\sigma + v(\sigma))] \right\| \right\}.$$

Moreover for any $k = 0, 1, \ldots, p - 1$:

$$\sup_{\sigma \in B(0,\xi_1)} H_h^k(\sigma) \to 0 \qquad \sup_{\sigma \in B(0,\xi_1)} \| r_h v^{(k)}(\sigma) - v_h^{(k)}(\sigma) \| \to 0 . \tag{24}$$

c) For any function $\lambda : (-t_0, t_0) \to B(0, \xi_1)$ *which is of class* C^r *and has all its derivatives uniformly bounded;* $t_0 > 0$; $r \leqslant p - 1$, *for any* $0 \leqslant k \leqslant r$ *and any* $h \leqslant h_1$, *the following is true (see (10))* :

$$\left\| r_h \frac{d^k}{dt^k} v(\lambda(t)) - \frac{d^k}{dt^k} v_h(\lambda(t)) \right\| \leqslant \text{Const}\, H_h^k(\lambda, t) \qquad \forall\, | t | < t_0 , \tag{25}$$

where

$$\begin{aligned} H_h^k(\lambda, t) = {} & \| r_h x^* - x_h^* \| + \sum_{i=0}^{n} \| (P_h r_h - r_h P) x_i \| + \\ & + \sum_{l=0}^{k} \left\{ \left\| (P_h r_h - r_h P) \frac{d^l}{dt^l} v(\lambda(t)) \right\| + \left\| (Q_h s_h - s_h Q) \frac{d^l}{dt^l} F(.) \right\| \right. \\ & \left. + \left\| \frac{d^l}{dt^l} [F_h(r_h(.)) - s_h F(.)] \right\| \right\}, \end{aligned} \tag{26}$$

where $x^* + S\lambda(t) + v(\lambda(t))$ *should be inserted into* (.).

Moreover for any $k = 0, 1, \ldots, r \leqslant p - 1$:

$$\sup_{|t|<t_0} H_h^k(\lambda, t) \to 0 \qquad \sup_{|t|<t_0} \left\| r_h \frac{d^k}{dt^k} v(\lambda(t)) - \frac{d^k}{dt^k} v_h(\lambda(t)) \right\| \to 0 . \tag{27}$$

Proof : Part *a)* is an immediate corollary from the generalized implicit function theorem (see Theorem 2.1 in [6], p. 323).

Part *b)* : Since $v : B(0, \xi_1) \to X_2 = PX$, then $v^{(k)}(\sigma) = Pv^{(k)}(\sigma)$ $\forall k$ $\forall \sigma$ and

$$\begin{aligned} & \| r_h v^{(k)}(\sigma) - v_h^{(k)}(\sigma) \| \leqslant \\ & \qquad \leqslant \| (r_h P - P_h r_h) v^{(k)}(\sigma) \| + \| P_h r_h v^{(k)}(\sigma) - v_h^{(k)}(\sigma) \| . \end{aligned} \tag{28}$$

From Part *a)* v is continuous and $v(0) = 0$. Then if $\xi_1 > 0$ is chosen sufficiently small, from the uniform boundedness of r_h and P_h, it will follow that :

$$\forall \sigma \in B(0, \xi_1) \quad \forall h \leqslant h_1 \qquad \| P_h r_h v(\sigma) \| < \alpha \quad \text{so} \quad P_h r_h v(\sigma) \in B_{X_{2h}}(0, \alpha) ,$$

where α is given by Part *a)*. Now we can apply Theorem 2.2 from [6] with $g_h := v_h$, $s_h := P_h r_h v$. The estimate :

$$\| P_h r_h v^{(k)}(\sigma) - v_h^{(k)}(\sigma) \| \leqslant \text{Const} \sum_{l=0}^{k} \left\| \frac{d^l}{d\sigma^l} Q_h F_h(x_h^* + S_h \sigma + P_h r_h v(\sigma)) \right\|$$

after some transformations together with (28) reduces to (22), (23). Hence and from Theorem II we obtain (24).

Part *c*) is proved in the same way as Part *b*).

Remark III.2 : (B8) is not an assumption of Theorem 1. Obtaining complete information about the solutions of $G_h(\sigma, v_h) = 0$ in the neighbourhood of 0, we will not obtain complete information about the solutions of $Q_h F_h(x_h) = 0$ in the neighbourhood of x_h^*, if the operators S_h are not invertible.

Definition and properties of bifurcation functions

Now we will introduce bifurcation functions $f, f_h : B(0, \xi_1) \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ both for Exact and Approximate Problems. We will show that f, f_h are of class C^p with all the derivatives uniformly bounded with respect to $h \leqslant h_2$ and $\sigma \in B(0, \xi_2)$ and that f_h with all its derivatives of orders $0, 1, \ldots, p-1$ converges to f uniformly on a ball $B(0, \xi_2)$.

Let the mappings v, v_h and the constants $\xi_1, h_1 > 0$ be given by Theorem 1. Let us insert v and v_h into (14″) and (15″) respectively. Let us define functions

$$g : B(0, \xi_1) \to Y_1, \qquad g_h : B(0, \xi_1) \to Y_{1h}$$

by the formulae :

$$g(\sigma) = (I - Q) F(x^* + S\sigma + v(\sigma)) \tag{29}$$

$$g_h(\sigma) = (I - Q_h) F_h(x_h^* + S_h \sigma + v_h(\sigma)) . \tag{30}$$

The fact that $v(0) = 0$, the continuity of v and the uniform discrete convergence of v_h to v (see (24)) make it possible to choose $0 < h_2 \leqslant h_1$ and $0 < \xi_2 \leqslant \xi_1$ such that :

$$x_h^* + S_h \sigma + v_h(\sigma) \in B(x_h^*, \delta) \qquad \forall h \leqslant h_2 \quad \forall \| \sigma \| < \xi_2 , \tag{31}$$

where δ is such as in the assumption (B2). From (31), from (B2) and other assumptions, from the fact that v, v_h are of class C^p with all its derivatives uniformly bounded (see Part *a*) of Theorem 1), it follows that g, g_h are also of class C^p with all the derivatives uniformly bounded, i.e. :

$$\| g^{(k)}(\sigma) \|, \| g_h^{(k)}(\sigma) \| \leqslant \text{Const} \qquad \forall k=0, \ldots, p \quad \forall h \leqslant h_2 \quad \forall \| \sigma \| < \xi_2 . \tag{32}$$

Let us assume (B12) and define *bifurcation functions*

$$f, f_h : B(0, \xi_1) \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$$

by the formulae :

$$f(\sigma) = E^{-1}\, g(\sigma) \tag{33}$$

$$f_h(\sigma) = E_h^{-1}\, g_h(\sigma) \tag{34}$$

(see (11), (13), (29), (30) and then (14″), (15″)).

We will be interested in solving *the bifurcation equations* :

$$f(\sigma) = 0\,, \tag{35}$$

$$f_h(\sigma) = 0\,. \tag{36}$$

Of course f, f_h are of class C^p. Now we will be able to justify the assumption (B12) of the uniform boundedness of the operators E_h^{-1}. Thanks to it and (32) :

$$\| f^{(k)}(\sigma) \|,\ \| f_h^{(k)}(\sigma) \| \leqslant \text{Const} \qquad \forall k = 0, \dots, p \quad \forall h \leqslant h_2 \quad \forall \| \sigma \| < \xi_2\,. \tag{37}$$

From (B12), (13) and the equalities :

$$f_h(\sigma) - f(\sigma) = E_h^{-1}\, g_h(\sigma) - E_h^{-1}[E_h\, E^{-1}\, g(\sigma)] = \\ = E_h^{-1}[g_h(\sigma) - (I - Q_h)\, s_h\, g(\sigma)]\,,$$

we get also :

$$\| f_h^{(k)}(\sigma) - f^{(k)}(\sigma) \| \leqslant \text{Const}\, \| g_h^{(k)}(\sigma) - (I - Q_h)\, s_h\, g^{(k)}(\sigma) \| \leqslant \\ \leqslant \text{Const}\, \| F_h^{(k)}(x_h^* + S_h\, \sigma + v_h(\sigma)) - s_h(I - Q)\, F^{(k)}(x^* + S\sigma + v(\sigma)) \|\,. \tag{38}$$

Making further transformations in (38) and using the estimates (22), (23) given by Theorem 1, we will prove :

$$\| f_h^{(k)}(\sigma) - f^{(k)}(\sigma) \| \leqslant \text{Const}\, H_h^k(\sigma) \\ \forall k = 0, \dots, p-1 \quad \forall h \leqslant h_2 \quad \forall \| \sigma \| < \xi_2\,, \tag{39}$$

where $H_h^k(\sigma)$ is given by (23).

From (24) it will follow that :

$$\sup_{\| \sigma \| < \xi_2} \| f_h^{(k)}(\sigma) - f^{(k)}(\sigma) \| \to 0 \qquad \forall k = 0, 1, \dots, p-1\,. \tag{40}$$

Similarly we will prove that for any function $\lambda : (-\, t_0, t_0) \to B(0, \xi_2)$ which is of class C^r and which has all the derivatives uniformly bounded, where

$t_0 > 0$; $0 \leqslant r \leqslant p - 1$, the following is true :

$$\left\| \frac{d^k}{dt^k} [f_h(\lambda(t)) - f(\lambda(t))] \right\| \leqslant \text{Const } H_h^k(\lambda, t) \tag{41}$$

$$\forall k = 0, \ldots, r \quad \forall h \leqslant h_2 \quad \forall \mid t \mid < t_0 ,$$

where $H_h^k(\lambda, t)$ is given by (26).

Bifurcation equations $f(\sigma) = 0$, $f_h(\sigma) = 0$. Final results

Let us assume that f, f_h are not necessarily bifurcation functions dealt with previously but that they are any functions operating between finite dimensional spaces $\tilde{X}_1$, $\tilde{Y}_1$ such that dim $\tilde{Y}_1 = n$, dim $\tilde{X}_1 = n + 1$; $n > 0$. Let us introduce the following assumptions :

(C1) f, f_h are of class C^p ; $p \geqslant 4$

(C2) $\exists q \in \mathbb{N} : 2 \leqslant q \leqslant p-2, f^{(q)}(0) \neq 0$, while $f^{(k)}(0)=0 \quad \forall k=0, \ldots, q-1$.

(C3) $\exists \sigma_0 \in \tilde{X}_1 : \sigma_0 \neq 0$ and $f^{(q)}(0).\sigma_0^q = 0$

(C4) the relations : $\sigma \in \tilde{X}_1$, $f^{(q)}(0).\sigma_0^{q-1}.\sigma = 0$ imply the existence of $\tau \in \mathbb{R}$ such that $\sigma = \tau\sigma_0$

(C5) $f_h^{(k)}(0) = 0 \qquad \forall k = 0, \ldots, q - 1$

If σ_0 fulfills (C3), then σ_0 is called a *characteristic ray*; if in addition to (C3) the condition (C4) holds, then σ_0 is called a *nondegenerate characteristic ray*.

Let us choose $\psi_0 \in \tilde{X}_1^*$ such that : $\psi_0(\sigma_0) \neq 0$. Let us define the mappings $\mathcal{G}$, $\mathcal{G}_h : \mathbb{R} \times \tilde{X}_1 \to \mathbb{R} \times \tilde{Y}_1$:

$$\mathcal{G}(t, \sigma) = \left(\psi_0(\sigma - \sigma_0), \frac{1}{t^q} f(t\sigma) \right), \tag{42}$$

$$\mathcal{G}_h(t, \sigma) = \left(\psi_0(\sigma - \sigma_0), \frac{1}{t^q} f_h(t\sigma) \right). \tag{43}$$

Then we quote :

THEOREM III.2 : *Let f, $f_h : B(0, \xi_2) \subset \tilde{X}_1 \to \tilde{Y}_1$; ξ_2, $h_2 > 0$; $h \leqslant h_2$. Let f, f_h fulfil* (C1)-(C5) *and posess properties* (37), (40). *Then there exist constants h_3, t_0, $\beta > 0$ and two unique maps σ, $\sigma_h : (-t_0, t_0) \to \tilde{X}_1$ such that :*

$$\mathcal{G}(t, \sigma(t)) = 0 \qquad \| \sigma(t) - \sigma_0 \| < \beta \qquad \forall \mid t \mid < t_0 \tag{44}$$

$$\mathcal{G}_h(t, \sigma_h(t)) = 0 \qquad \| \sigma_h(t) - \sigma_0 \| < \beta \qquad \forall \mid t \mid < t_0 \quad \forall h \leqslant h_3 . \tag{45}$$

The mappings σ, σ_h are of class C^{p-q} with all the derivatives uniformly bonded with respect both to $|t| < t_0$ and $h \leqslant h_3$. Moreover $\mathscr{G}(0, \sigma_0) = 0$ and for $k = 0, \dots, p-q-1$, $h \leqslant h_3$:

$$(46) \quad \sup_{|t|<t_0} \left\| \frac{d^k}{dt^k} [t\sigma_h(t) - t\sigma(t)] \right\| \leqslant \text{Const} \sum_{l=0}^{k+q-1} \sup_{|t|<t_0} \left\| \frac{d^l}{dt^l} f_h(t\sigma(t)) \right\| .$$

Proof : The proof of Theorem 4.2, p. 332 from [6] goes without any changes. Although the assumptions there are formulated otherwise, it does not matter because only (37), (40), (C1)-(C5) are used in the proof.

Let us come back to the situation where f, $f_h : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are bifurcation functions defined by (33), (34). Theorems 1 and 2 together with the estimate (41) will allow us to state :

THEOREM III.3 : *Let* (A1)-(A2), (B1)-(B7), (B9)-(B12) *and* (C1)-(C5) *be fulfilled. Let the mappings $x(.)$, $x_h(.)$ be defined by the formulae :*

$$(47) \quad x(t) = x^* + S(t\sigma(t)) + v(t\sigma(t)) \qquad |t| < t_0 ,$$

$$(48) \quad x_h(t) = x_h^* + S_h(t\sigma_h(t)) + v_h(t\sigma_h(t)) \qquad |t| < t_0 \quad h \leqslant h_3 ,$$

where the mappings σ, σ_h, the numbers β, h_3, $t_0 > 0$ are given by Theorem 2, while the mappings v, v_h — by Theorem 1. Then x, x_h are of class C^{p-q} with all the derivatives uniformly bounded with respect to t and h,

$$(49) \qquad F(x(t)) = 0 \qquad \forall\, |t| < t_0$$

$$(50) \qquad F_h(x_h(t)) = 0 \qquad \forall\, |t| < t_0 \quad \forall h \leqslant h_3 ,$$

$$(51) \qquad x(0) = x^* \qquad x'(0) = S\sigma_0 .$$

Moreover x_h with all its derivatives of orders $0, \dots, p-q-1$ converge to x discreetly and uniformly on the interval $|t| < t_0$. The speed of this convergence is characterized by the estimate :

$$(52) \qquad \sup_{|t|<t_0} \| r_h x^{(k)}(t) - x_h^{(k)}(t) \| \leqslant \text{Const} \sup_{|t|<t_0} H_h^{k+q-1}(\lambda, t) ,$$

where $\lambda(t) = t\sigma(t)$; $H_h^{k+q-1}(\lambda, t)$ is defined by (26).

Proof : We have introduced bifurcation functions f, f_h in such a way that they fulfil (37) and (40), so Theorem 2 is applicable. We have :

$$\begin{aligned} r_h x(t) - x_h(t) &= [r_h x^* - x_h^*] + [(r_h S - S_h)\, t\sigma(t)] + \\ &+ [S_h(t\sigma(t) - t\sigma_h(t))] + [r_h v(t\sigma(t)) - v_h(t\sigma(t))] \\ &+ [v_h(t\sigma(t)) - v_h(t\sigma_h(t))] = [r_h x^* - x_h^*] + W_h^1(t) + W_h^2(t) + W_h^3(t) + W_h^4(t) . \end{aligned}$$

Minding that :

$$\left\| \frac{d^k}{dt^k} W_h^1(t) \right\| = \left\| (r_h S - S_h) \frac{d^k}{dt^k} (t\sigma(t)) \right\| \leqslant \text{Const} \sum_{i=0}^{n} \| (r_h P - P_h r_h) x_i \|$$

(see (10), (12)),

$$\sup_{|t| < t_0} \left\| \frac{d^k}{dt^k} W_h^2(t) \right\| \leqslant \text{Const} \sup_{|t| < t_0} H_h^{k+q-1}(t\sigma(t), t)$$

from (41), (46),

$$\left\| \frac{d^k}{dt^k} W_h^3(t) \right\| \leqslant \text{Const}\, H_h^k(t\sigma(t), t)$$

from the estimate (25) of Theorem 1,

$$\left\| \frac{d^k}{dt^k} W_h^4(t) \right\| \leqslant \text{Const} \sum_{l=0}^{k} \left\| \frac{d^l}{dt^l} [t\sigma(t) - t\sigma_h(t)] \right\|$$

from the uniform boundedness of the derivatives of v_h, the estimate (52) becomes obvious.

Remark III.3 :

a) In (C2) : the condition $q \geqslant 2$ is automatically satisfied, since it follows from Theorem 1 that $v(0) = 0$, $v'(0) = 0$, $f(0) = 0$, $f'(0) = 0$. (C1) is also satisfied.

b) (C5) is rarely fulfilled. One occasion when it holds is the so called « primary bifurcation ». In most cases, however, (C5) does not hold. Then the existence and the uniform discrete convergence of $x_h(.)$ to $x(.)$ can be shown not on the whole interval $|t| < t_0$, but only on its part $(-t_0, -\delta_h/\varepsilon) \cup (\delta_h/\varepsilon, t_0)$, where $\varepsilon > 0$ is a certain constant while

$$\delta_h = \max_{0 \leqslant k \leqslant q-1} \| f_h^{(k)}(0) \|^{1/q-k} . \tag{53}$$

In the case when (C5) does not hold, the same properties (37), (40) allow us to repeat (without any changes) the proof of Theorem 4.4 and some of the estimates in the proof of Theorem 4.5 from Descloux, Rappaz [5], pp. 39-49. In the end the following estimate is obtained :

$$\sup_{\delta_h/\varepsilon < |t| < t_0} \| r_h x(t) - x_h(t) \| \leqslant \text{Const} \left\{ \delta_h + \sup_{|t| < t_0} H_h^{q-1}(\lambda, t) \right\} , \tag{54}$$

where $\lambda(t) = t\sigma(t)$; δ_h is given by (53) ; $H_h^{q-1}(\lambda, t)$ – by (26).

Now we will proceed to characterize *the behaviour of all the solutions* of Exact and Approximate Problems. At first we will deal with bifurcation equations. Let (A1)-(A2), (B1)-(B12) hold. Let the bifurcation functions f, f_h have properties (C2), (C5). Let all the characteristic rays of f (i.e. vectors satisfying (C3)) be nondegenerate (i.e. (C4) holds in addition to (C3)). Then if $\sum$ denotes the set of all the characteristic rays with norm 1, it is easy to show that $\sum$ is finite, say $\sum = \{ \sigma_1, \ldots, \sigma_m \}$, $\| \sigma_i \| = 1$. By Theorem 2 applied m-times, there exist numbers h_{3i}, t_i, $\beta_i > 0$ such that to each σ_i corresponds :

— an implicit function $\sigma_i : (-t_i, t_i) \to \mathbb{R}^{n+1}$ for the operator $\mathcal{G}$ defined by (42),

— for any $h \leqslant h_{3i}$ — an implicit function $\sigma_{ih} : (-t_i, t_i) \to \mathbb{R}^{n+1}$ for the operator $\mathcal{G}_h$ defined by (43).

LEMMA III.1 : *There exist numbers* ξ^*, $h^* > 0$ *such that* :

$$(55) \quad A = \{ \sigma \in B(0, \xi^*) \subset \mathbb{R}^{n+1} : f(\sigma) = 0 \} \subset \bigcup_{i=1}^{m} \{ t\sigma_i(t) : | t | < t_i \}$$

and for any $h \leqslant h^*$:

$$(56) \qquad A_h = \{ \sigma \in B(0, \xi^*) : f_h(\sigma) = 0 \} \subset \bigcup_{i=1}^{m} \{ t\sigma_{ih}(t) : | t | < t_i \} .$$

Proof : For $i = 1, \ldots, m$ we define the cones :

$$(57) \qquad C_i = \{ \sigma \in \mathbb{R}^{n+1} : \| \psi_i(\sigma_i) \sigma - \psi_i(\sigma) \sigma_i \| < \beta_i | \psi_i(\sigma) | \} ,$$

where ψ_i have been introduced in (42)-(43). There are no characteristic rays of f in the closed set $D = \mathbb{R}^{n+1} - \bigcup_{i=1}^{m} C_i$. Hence and from the compactness of the sphere in $\mathbb{R}^{n+1}$ we conclude that $a := \frac{1}{q\,!} \inf_{\sigma \in D, \| \sigma \| = 1} \| f^{(q)}(0).\sigma^q \| > 0$. By (40) we get that if h^* is sufficiently small also :

$$(58) \qquad a_h := \frac{1}{q\,!} \inf_{\sigma \in D, \| \sigma \| = 1} \| f_h^q(0).\sigma^q \| > \frac{a}{2} > 0 \qquad \forall h \leqslant h^* .$$

Now we will show that in the set $B(0, \xi^*) \cap D$ there are no solutions of the equations $f(\sigma) = 0$ and $f_h(\sigma) = 0$ for any $h \leqslant h^*$ except $\sigma = 0$ provided that h^*, $\xi^* > 0$ are sufficiently small. Let $h \leqslant h^*$, $\sigma \in D \cap B(0, \xi^*) - \{ 0 \}$ be fixed but such that $f_h(\sigma) = 0$.

Taking (C2) and (C5) into account we obtain by Taylor's expansion :

$$f_h(\sigma) = \frac{1}{q\,!} \| \sigma \|^q f_h^{(q)}(0) \left(\frac{\sigma}{\| \sigma \|} \right)^q + R_h(\sigma) \qquad \| R_h(\sigma) \| \leqslant \frac{N}{(q+1)\,!} \| \sigma \|^{q+1}$$

where N is a constant bounding the derivatives of f_h (see (37)). Hence and from (58) :

$$\| f_h(\sigma) \| \geqslant \left(\frac{a}{2} - \frac{N}{(q+1)\,!} \| \sigma \| \right) \| \sigma \|^q \geqslant \frac{a}{4} \| \sigma \|^q > 0 ,$$

if $\xi^* \geqslant \| \sigma \|$ is sufficiently small. The same is true for f.

We have proved that (see (55), (56)) :

$$A \subset \bigcup_{i=1}^{m} \{ C_i \cap B(0, \xi^*) : f(\sigma) = 0 \} \qquad A_h \subset \bigcup_{i=1}^{m} \{ C_i \cap B(0, \xi^*) : f_h(\sigma) = 0 \} .$$

Now we will show that :

$$\{ \sigma \in C_i \cap B(0, \xi^*) : f(\sigma) = 0 \} \subset \{ t\sigma_i(t) : | t | < t_i \} ,$$
$$\{ \sigma \in C_i \cap B(0, \xi^*) : f_h(\sigma) = 0 \} \subset \{ t\sigma_{ih}(t) : | t | < t_i \} \qquad \forall h \leqslant h^* .$$

Let $h \leqslant h^*$, $\sigma \in C_i \cap B(0, \xi^*)$ be fixed but such that $f_h(\sigma) = 0$. The same procedure may be repeated for f. In the definition (57) of C_i there is a sharp inequality. Therefore if we define $t := \dfrac{\psi_i(\sigma)}{\psi_i(\sigma_i)}$, then $t \neq 0$. For $\lambda := \dfrac{1}{t} \sigma$ we have : $\psi_i(\lambda - \sigma_i) = 0$ and due to (57) : $\| \lambda - \sigma_i \| < \beta_i$. If $\xi^* \geqslant \| \sigma \|$ is small enough then $| t | < t_i$. Taking into account that : $f_h(t\lambda) = 0$, $\psi_i(\lambda - \sigma_i) = 0$, $\| \lambda - \sigma_i \| < \beta_i$, $| t | < t_i$ and $h \leqslant h^* \leqslant h_{3i}$, we conclude from Theorem 2 — from the uniqueness statement — that $\lambda = \sigma_{ih}(t)$, $\sigma = t\sigma_{ih}(t)$.

Now let $\xi_1, h_1, \alpha > 0$ be given by Theorem 1 and let us diminish ξ^*, h^* from Lemma 1 so that : $h^* \leqslant h_1$, $\xi^* \leqslant \xi_1$. Then :

LEMMA III.2 : *There exist positive constant* $\gamma > 0$ *such that* :

$$(59) \quad \{ x \in X : F(x) = 0 \wedge \| x - x^* \| < \gamma \} \subset \\ \subset \{ x^* + S\sigma + v(\sigma) : f(\sigma) = 0 \wedge \| \sigma \| < \xi^* \}$$

and for any $h \leqslant h^*$:

$$(60) \quad \{ x_h \in X_h : F_h(x_h) = 0 \wedge \| x_h - x_h^* \| < \gamma \} \subset \\ \subset \{ x_h^* + S_h \sigma + v_h(\sigma) : f_h(\sigma) = 0 \wedge \| \sigma \| < \xi^* \} .$$

Proof : We choose $\gamma > 0$ in such a way that :

$$(61) \qquad \| P_h \| \gamma < \alpha \qquad \| S_h^{-1}(I - P_h) \| \gamma < \xi^* \qquad \forall h \leqslant h^* .$$

Since P_h, S_h^{-1} (see (B8)) are uniformly bounded such a choice γ of is possible. Let $h \leqslant h^*$, $x_h \in X_h$ be fixed but such that $F_h(x_h) = 0$, $\| x_h - x_h^* \| < \gamma$. Denoting $z_h := x_h - x_h^*$ and minding that $S_h : \mathbb{R}^{n+1} \to X_{1h} = (I - P_h) X_h$ are isomorphisms, we may write :

$$x_h = x_h^* + S_h \sigma_h + v_{2h}, \quad \text{where} \quad \sigma_h = S_h^{-1}(I - P_h) z_h, \qquad v_{2h} = P_h z_h .$$

From (61), and the fact that $\| z_h \| < \gamma$, it follows that :

$$\| \sigma_h \| < \xi^* \leqslant \xi_1, \| v_{2h} \| < \alpha .$$

From the uniqueness guarateed by Theorem 1, Part *a*) we obtain : $v_{2h} = v_h(\sigma_h)$, $x_h = x_h^* + S_h \sigma_h + v_h(\sigma_h)$. Since $f_h(\sigma_h) = E_h^{-1}(I - Q_h) F_h(x_h)$, $F_h(x_h) = 0$, then also $f_h(\sigma_h) = 0$ and (60) is proved. The same is true for (59).

Remark III.4 : If we assumed in (B8) only invertibility of S_h and did not assume their uniform boundedness, then (60) could be proved with γ replaced by $\gamma_h > 0$. However, the case : $\gamma_h \to 0$ could not be then excluded.

From Lemma 1 and 2 and Theorem 3 we have :

THEOREM III.4 : *Let* (A1)-(A2), (B1)-(B12) *hold. Let the bifurcation functions f, f_h fulfil* (C2) *and* (C5). *We also assume that all the characteristic rays of f are nondegenerate. Then there exist an integer m and positive constants h^*, γ, $t_1, \ldots, t_m > 0$ such that :*

$$(62) \qquad \{ x \in X : F(x) = 0 \wedge \| x - x^* \| < \gamma \} \subset \bigcup_{i=1}^{m} \{ x_i(t) : | t | < t_i \}$$

and for any $h \leqslant h^*$:

$$(63) \quad \{ x_h \in X_h : F_h(x_h) = 0 \wedge \| x_h - x_h^* \| < \gamma \} \subset \bigcup_{i=1}^{m} \{ x_{ih}(t) : | t | < t_i \} .$$

The branches x_i and x_{ih} are of classe C^{p-q}; furthermore for any $i = 1, \ldots, m$ the function x_{ih} with all its derivatives of orders $k = 0, \ldots, p - q - 1$ converge uniformly and discreetly to the relevant derivatives of x_i; the speed of this convergence and parametrization of x_i, x_{ih} have been characterized in Theorem 3.

IV. EXAMPLE

Let us define a form $a : H_0^1(0, 1) \times H_0^1(0, 1) \to \mathbb{R}$:

$$a(u, v) = \int_0^1 [u'(x)\, v'(x) + b(x)\, u(x)\, v(x)]\, dx\,, \tag{1}$$

where

$$b \in C^1[0, 1]\,, \qquad b \geqslant \alpha_0 > 0\,, \qquad \alpha_0 \text{ is a constant}. \tag{2}$$

Let us also denote :

$$(u, v) = \int_0^1 u(x)\, v(x)\, dx \qquad \forall u, v \in H_0^1\,.$$

We will be interested in finding solutions $(\lambda, u) \in \mathbb{R} \times H_0^1$ of the equation :

$$a(u, v) = \lambda(u^p + u, v) \qquad \forall v \in H_0^1 \qquad 2 \leqslant p \in \mathbb{N} \tag{3}$$

in a neighbourhood of a point $(\lambda_0, 0) \in \mathbb{R} \times H_0^1$, where $\lambda_0 \neq 0$ is a simple eigenvalue of the problem : $a(u, v) = \lambda(u, v)$ $\forall v \in H_0^1$. By the Lax-Milgram theorem there exists an operator $T \in L(H_0^1)$ such that : $a(Tu, v) = (u, v)$ $\forall u, v \in H_0^1$. So (3) becomes equivalent to :

$$u = \lambda T(u^p + u) \qquad \lambda \in \mathbb{R} \qquad u \in H_0^1\,. \tag{4}$$

From the assumptions about λ_0 we get the existence of an eigenvector $\varphi \neq 0$ such that : $\varphi = \lambda_0 T\varphi$. Let us define as in Chapter III :

$$\left\{\begin{array}{l} Y = H_0^1(0, 1) \qquad X = \mathbb{R} \times Y \\ F(\lambda, u) = u - \lambda T(u^p + u) \qquad x^* = (\lambda_0, 0) \\ Y_2 = \text{Range } F'(x^*) = \text{Range } (I - \lambda_0 T) \quad Y_1 = \text{span}\{\varphi\} \quad y_0 = \varphi \\ X_1 = \text{Ker } F'(x^*) = \mathbb{R} \times Y_1 \quad X_2 = \{0\} \times Y_2 \quad x_0 = (1, 0) \quad x_1 = (0, \varphi)\,. \end{array}\right. \tag{5}$$

Taking into account the following relations (in which f denotes the bifurcation function for the operator F) :

$$DF(\lambda, u)(\mu, v) = v - \lambda T(pu^{p-1}\, v + v) - \mu T(u^p + u)\,,$$

$$D^2 F(\lambda, u)(\mu_1, v_1)(\mu_2, v_2) = -\lambda T(p(p-1)\, u^{p-2}\, v_1\, v_2) - \\ - \mu_1 T(pu^{p-1}\, v_2 + v_2) - \mu_2 T(pu^{p-1}\, v_1 + v_1),$$

$$\frac{\partial^2 f}{\partial \sigma_1^2}(0,0) = E^{-1}(I-Q)\,F''(\lambda_0,0)\,x_0^2 = E^{-1}(I-Q)\,F''(\lambda_0,0)\,(1,0)^2 = 0\,,$$

$$\frac{\partial^2 f}{\partial \sigma_1\,\partial \sigma_2}(0,0) = E^{-1}(I-Q)\,F''(\lambda_0,0)\,x_0\,x_1$$

$$= E^{-1}(I-Q)\,F''(\lambda_0,0)\,(1,0)\,(0,\varphi) = E^{-1}(I-Q)\,(-T\varphi) =$$

$$= E^{-1}(I-Q)\left(-\frac{1}{\lambda_0}\varphi\right) = E^{-1}\left(-\frac{1}{\lambda_0}\varphi\right) = -\frac{1}{\lambda_0},$$

$$\left[\frac{\partial^2 f}{\partial \sigma_1^2}\cdot\frac{\partial^2 f}{\partial \sigma_2^2} - \left(\frac{\partial^2 f}{\partial \sigma_1\,\partial \sigma_2}\right)^2\right](0,0) < 0\,,$$

we see that $(\lambda_0, 0)$ is a simple bifurcation point of F.

Our next step will be defining the approximate problem. To this end let us at first define the external approximation $\{ Y, \mathscr{F}_Y, \omega, Y_h, s_h, q_h \}_h$ of the space Y as it has been done in Regińska [9, 10] :

(6) $$\mathscr{F}_Y = L^2 \times H_0^1 \qquad \omega u = (u, u) \qquad \forall u \in H_0^1$$

(7) $$h = \frac{1}{n+1} \qquad Y_h = \mathbb{R}^n \qquad \| u_h \|_h^2 = | u_h |_h^2 + | \nabla_h u_h |_h^2\,,$$

where

$$| u_h |_h^2 = h \sum_{i=1}^{n} (u_h^i)^2 \qquad \nabla_h u_h = ((u_h^{i+1} - u_h^i)/h)_{i=1}^n$$

$u_h^{n+1} = 0$ for every $u_h = (u_h^i)_{i=1}^n \in \mathbb{R}^n$

(8) $$s_h u = (u(ih))_{i=1}^n \qquad q_h u_h = (q_h^0 u_h, q_h^1 u_h)\,,$$

where

(9) $$q_h^0 u_h = \sum_{i=1}^{n} u_h^i \,\chi\!\left(\frac{x}{h} - i\right) \qquad q_h^1 u_h = \sum_{i=1}^{n} u_h^i \,\pi\!\left(\frac{x}{h} - i\right);$$

χ is a characteristic function of the interval $(0, 1)$ and $\pi(.)$ is a hat function : $\pi(x) = -|x| + 1$ for $|x| \leqslant 1$, $\pi(x) = 0$ for $|x| > 1$. It may be shown that :

(10) $$\| s_h \| \leqslant \text{Const} \qquad \| q_h \| \leqslant \text{Const}$$

(11) $$q_h s_h u \to \omega u \qquad \forall u \in H_0^1\,.$$

Now we will extend the form a to a form $\bar{a} : \mathscr{F}_Y \times \mathscr{F}_Y \to \mathbb{R}$ and introduce forms $a_h : Y_h \times Y_h \to \mathbb{R}$ in the following way :

(12) $\bar{a}(\bar{u}, \bar{v}) = \int_0^1 [u_1' v_1' + b u_0 v_0] \qquad \forall \bar{u} = (u_0, u_1), \quad \forall \bar{v} = (v_0, v_1) \in \mathscr{F}_Y$

(13) $$a_h(u_h, v_h) = \bar{a}(q_h u_h, q_h v_h) \qquad \forall u_h, v_h \in Y_h .$$

We will be interested in finding solutions $(\lambda, u_h) \in \mathbb{R} \times Y_h$ of the equation :

(14) $$a_h(u_h, v_h) = \lambda(u_h^p + u_h, v_h)_h \qquad \forall v_h \in Y_h ,$$

where

$$(u_h, v_h)_h = h \cdot \sum_{i=1}^{n} u_h^i v_h^i \qquad u_h^p = ((u_h^i)^p)_{i=1}^n .$$

The assumptions (2) imply the continuity and the coerciveness of the form $\bar{a}$. Hence, from (13), (10) and the fact that $s_h q_h^1 u_h = u_h \ \forall u_h \in Y_h$, it follows that the forms a_h are uniformly coercive and uniformly continuous. By Lax-Milgram theorem there exist operators $T_h \in L(Y_h)$ such that $a_h(T_h u_h, v_h) = (u_h, v_h)_h \ \forall u_h, v_h \in Y_h$ and

(15) $$\| T_h \| \leqslant \text{Const} .$$

The approximate problem (14) becomes equivalent to :

(16) $$u_h = \lambda T_h(u_h^p + u_h) \qquad \lambda \in \mathbb{R} \qquad u_h \in Y_h = \mathbb{R}^n .$$

Making use of the general results from [8], Regińska proves in [9] :

(P1) $\| (T_h s_h - s_h T) v \| \to 0 \ \ \forall v \in H_0^1$

(P2) If $\mu_0 = \dfrac{1}{\lambda_0}$ is an isolated simple eigenvalue of T and $B(\mu_0, \delta) \cap \sigma\{T\} = \{\mu_0\}$, $\delta > 0$, then for h sufficiently small : $B(\mu_0, \delta) \cap \sigma\{T_h\} = \{\mu_h\}$. Moreover the algebraic multiplicity of μ_h is also 1 and $\mu_h \to \mu_0$.

(P3) If $\Gamma \subset \rho(T)$ is a compact set, then for h small enough : $\Gamma \subset \rho(T_h)$ and $\| (T_h - \lambda)^{-1} \| \leqslant M$, where M is independent both of h and $\lambda \in \Gamma$.

(P4) If $\Gamma = \left\{ \mu : | \mu - \mu_0 | = \dfrac{\delta}{2} \right\}$ and R, R_h are spectral projections :

(17) $$R = -\frac{1}{2\pi i} \int_\Gamma (T - \lambda)^{-1} \, d\mu \qquad R_h = -\frac{1}{2\pi i} \int_\Gamma (T_h - \lambda)^{-1} \, d\mu ,$$

then : $\| (R_h s_h - s_h R) v \| \to 0 \ \forall v \in H_0^1$.

From (17) and (P3) it follows immediately that :

$$\| R_h \| \leqslant \text{Const}. \tag{18}$$

Let us introduce further definitions as in Chapter III (see (5)) :

$$\left\{ \begin{array}{lll} X_h = \mathbb{R} \times Y_h & r_h(\lambda, u) = (\lambda, s_h u) & \\ F_h(\lambda, u_h) = u_h - \lambda T_h(u_h^p + u_h) & x_h^* = (\lambda_h, 0) & \lambda_h = \dfrac{1}{\mu_h} \\ Q_h = I - R_h & P_h(\lambda, u_h) = (0, Q_h u_h) & \forall u_h \in Y_h . \end{array} \right. \tag{19}$$

By these definitions :

$$Q_h F_h'(x_h^*) = \frac{1}{\mu_h}(\mu_h - T_h) \qquad X_{2h} = \{0\} \times Y_{2h} \tag{20}$$

$$Y_{2h} = Q_h Y_h = \text{Range } F_h'(x_h^*) \qquad X_{1h} = (I - P_h) X_h = \text{Ker } F_h'(x_h^*) .$$

Coming back for a while to (5) we notice that also $Q = I - R$, $P(\lambda, u) = (0, Qu)$ $\forall u \in H_0^1$.

Then using (P1)-(P4), (15), (18) and (20) we check easily that all the assumptions of Theorem III.4 are fulfilled. For example :

— (B8), (B12) follow from (10), (11) and Remark III.1, *d*).

— (B3) follows from (20), (P2), (P3) and the formulae :

$$[Q_h F_h'(x_h^*)|_{X_{2h}}]^{-1} = (0, -\mu_h[(T_h - \mu_h)|_{Y_{2h}}]^{-1}) =$$
$$= \left(0, \mu_h \left(\frac{1}{2\pi i} \int_\Gamma \frac{(T_h - \mu)^{-1}}{(\mu_h - \mu)} d\mu \,\bigg|_{Y_{2h}} \right)\right).$$

— (C5) follows immediately from (20).

— The set of all the characteristic rays of the bifurcation function f of the norm 1 consists of exactly 2 elements and they are nondegenerate — since $(\lambda_0, 0)$ is a simple bifurcation point of F.

It follows from Theorem III.4 that there exist a constant $\gamma > 0$ such that the set of all the solutions of (4) contained in the ball $B((\lambda_0, 0), \gamma) \subset \mathbb{R} \times H_0^1$ consists of exactly two solution branches $x_1(.)$, $x_2(.)$ which turn out to be of class C^∞. The set of all the solutions of (16) contained in the ball $B_h((\lambda_h, 0), \gamma) \subset \mathbb{R} \times \mathbb{R}^n$ consists of exactly two solution branches $x_{1h}(.)$, $x_{2h}(.)$ which are of class C^∞. The solution branches $x_{1h}(.)$, $x_{2h}(.)$ with all their derivatives converge uniformly and discreetly to the relevant derivatives of the solution branches $x_1(.)$, $x_2(.)$.

Remark IV.1 : Let us consider a more general case when :

— the form a corresponds to a self-adjoint differential operator of the order $2\,m$, $m \geqslant 1$,

— the external approximation of $Y := H_0^m(0, 1)$ is the generalization of the approximation (6)-(9) of H_0^1 (look for the partial piece-wise-polynomial approximation of H_0^m in Aubin [1] p. 338), — $\mu_0 = \dfrac{1}{\lambda_0}$ is of finite multiplicity not necessarily 1.

If μ_0 does not split into more than 1 eigenvalue of the approximate problem (a restrictive assumption !), then it follows from Regińska [8, 9] that (B1)-(B12) and (C5) are fulfilled. Thus the conclusions of Theorem III.4 hold also in this case.

If, however, μ_0 splits into $\mu_h^1, \ldots, \mu_h^k$ and we set $x_h^* := \left(\dfrac{1}{\mu_h^1}, 0\right)$, then the choice suggested in Remark III.1, *b*) is not good since then $\dim Y_{1h} < \dim Y_1$, $\dim X_{1h} < \dim X_1$. The choice $Y_{1h} = (I - Q_h)\, Y_h$, $Y_{2h} = Q_h\, Y_h$, $X_{1h} = (I - P_h)\, X_h$, $Y_h = P_h\, X_h$, where the projections Q_h, P_h are defined by means of spectral projections in exactly the same manner as in (17), (19) — renders that (B1)-(B12) are fulfilled, (C5) is not. Thus only the conclusions of Remark III.3*b*) hold.

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