M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique

Martine Vullierme-Ledard

Asymptotic study of the vibration problem for an elastic body deeply immersed in an incompressible fluid

M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 19, nº 1 (1985), p. 145-170

<http://www.numdam.org/item?id=M2AN_1985__19_1_145_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE
(vol 19, n° 1, 1985, p 145 à 170)

ASYMPTOTIC STUDY OF THE VIBRATION PROBLEM FOR AN ELASTIC BODY DEEPLY IMMERSED IN AN INCOMPRESSIBLE FLUID (*)

by Martine VULLIERME-LEDARD (1)

Communicated by E SANCHEZ-PALENCIA

Abstract — *We consider the harmonic vibrations of the coupled system composed of an elastic body immersed in an inviscid, incompressible fluid of infinite extend, bounded above by a free surface As the distance from the body to the free surface tends to infinity, we study the asymptotic behaviour of the associated scattering frequencies The results obtained specify in what way, when the body is deeply immersed, the behaviour of the system is close to the one of a conservative vibrating system and energy radiation resulting from the free surface is negligibly small*

Résumé — *Nous considérons les vibrations harmoniques du système couplé, constitué par un solide élastique immergé dans un fluide parfait incompressible occupant un domaine non borné et présentant une surface libre Lorsque la profondeur d'immersion du solide tend vers l'infini, nous étudions le comportement asymptotique des fréquences de scattering associées Les résultats obtenus précisent en quel sens, lorsque le corps est profondément immergé, le comportement du système s'apparente à celui d'un système vibratoire conservatif et l'énergie rayonnée par la surface libre est négligeable*

1. INTRODUCTION

Let us consider in the three dimensional space, the harmonic vibrations of an elastic body surrounded by an inviscid and incompressible fluid of infinite extend (see e g [1]) afterwards, we shall refer to this problem as the unperturbed problem P_0 This coupled system has a behaviour analogous to the one of an elastic body vibrating in the vacuum, that is to say, it has a countable set of real eigenfrequencies and associated eigenmodes

R Ohayon and E Sanchez-Palencia have studied in [2] a perturbation of this problem occurring when one considers a slightly compressible fluid in this case the coupled system has complex eigenfrequencies called scattering

(*) Received on November 1983
(1) E N S T A, Groupe Hydrodynamique Navale, Centre de l'Yvette Chemin de la Hunière, 91120 Palaiseau France

frequencies The authors have derived the asymptotic behaviour of these scattering frequencies as ε (a small parameter associated with the compressibility) vanishes

In this paper, we consider another perturbation of the problem P_0, occurring when the fluid is limited by a free surface located far above the body

We study the asymptotic behaviour of the corresponding scattering frequencies as $\varepsilon \to 0$, where the small parameter ε represents here the inverse of the distance from the body to the free surface More precisely, we show that if α^0 is an eigenfrequency with multiplicity m of the unperturbed problem, then there are m scattering frequencies (assuming as usual that we count each one with its algebraic multiplicity) which converge to α^0 as $\varepsilon \to 0$ Moreover, if α^0 is a simple eigenvalue, the corresponding scattering frequency has an asymptotic expansion in terms of powers of ε in which all coefficients are real

Accordingly, the effect of the free surface is, in a way, to shift each simple eigenfrequency along the real axis From the qualitative point of view, the behaviour of the perturbed system would then be close to the one of a conservative system Our result (proved for simple eigenvalues) can therefore give an explanation to the fact admitted for long, that energy radiation resulting from the free surface is negligibly small when the body is deeply immersed This conclusion is also consistent with the studies performed on the wave equation in an odd number of space dimensions (see e g [3], [4]), where it appears that the imaginary part of the scattering frequencies accounts for the magnitude of energy decay phenomena

It would be of great interest to compare the influence of the free surface with that of the fluid compressibility, which would lead to deal at the same time with the two small parameters Right now, the comparison between our results and those obtained in [2] allows to think that the radiation effect arising from the fluid compressibility is preponderant This fact has already been pointed out by experimental analysis and is readily used in submarine acoustics

In Section 2, we give the set of equations of the whole problem Section 3 is devoted to the study of the exterior problem one has to solve, in order to find the displacement potential in the fluid, assuming the displacements in the solid are known Thus, we can introduce the operators used in Section 4 to define the scattering frequencies and obtain a variational formulation of the whole problem In Section 5, by means of perturbation theory for linear operators, we study the asymptotic behaviour of the scattering frequencies In Section 6, we briefly describe a similar problem, in which a Dirichlet condition at the free surface is perturbed by the introduction of a small term due to gravity effect Finally, in Section 7, we recall some results about the Green function used in Section 3

Notations

All indices i, j, k, h run through 1, 2, 3 ; sums over j, k, l are understood.

n denotes the outer unit normal to surfaces ; it will sometimes have as subscript the symbol denoting the surface under consideration.

ε is a strictly positive real number.

C denotes different constants.

δ_{ij} denotes the Kronecker symbol.

2. EQUATIONS OF THE COUPLED PROBLEM

By means of the three independent quantities ρ_f, g and L, which denote respectively the fluid density, the acceleration of gravity, and a characteristical length of the body (the dimensions of which are assumed to be of the same order), we rescale all physical quantities involved in the problem and we shall then deal only with variables without dimension.

We consider in the $\mathbb{R}^3$ space, $\mathbb{R}^3_- = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < 0 \}$ and $F_\varepsilon = \{ x \in \mathbb{R}^3 \mid x_3 < 1/\varepsilon \}$. The body occupies a bounded connected domain B of $\mathbb{R}^3_-$. In order to prevent rigid motions, we assume that the body is clamped on some inner surface Γ'. The fluid fills the domain Ω_ε complementary to B in F_ε. The interface between B and Ω_ε is a smooth surface Γ and the equation of the mean position FS_ε of the free surface is $x_3 = 1/\varepsilon$.

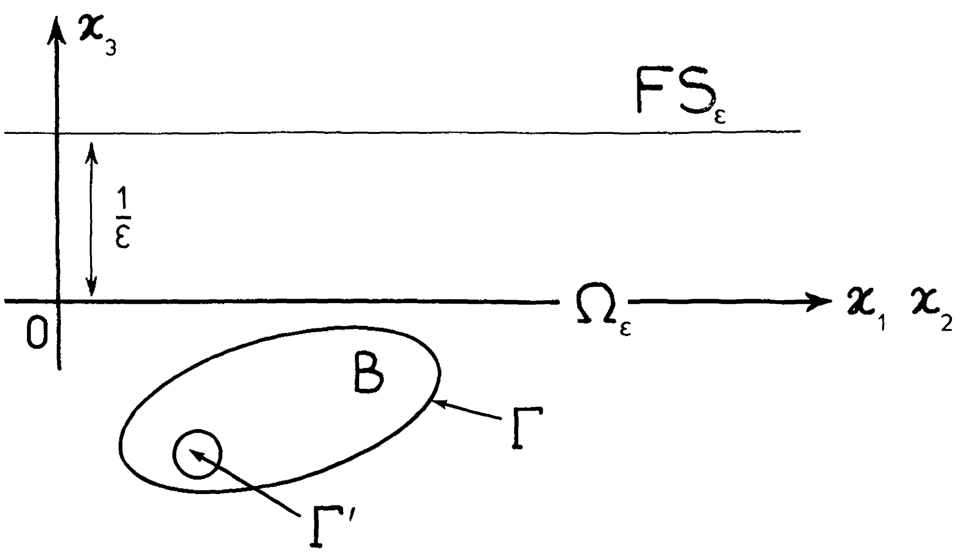

Figure 1.

We assume that the motions are all of small amplitude and study the harmonic vibrations of the system. All variables thus have a time dependence in $e^{-i\alpha t}$, where the pulsation α is first assumed to be real.

The solid is assumed to have a linear elastic behaviour. Thus, if u denotes the displacement vector, the equations to be satisfied are :

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\rho\alpha^2 u_i \quad \text{in } B, \tag{2.1}$$

$$\sigma_{ij} = a_{ijkh}\, e_{kh}(u), \quad e_{kh}(u) = \frac{1}{2}\left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right), \tag{2.2}$$

$$a_{ijkh} = \lambda \delta_{ij}\, \delta_{kh} + \mu(\delta_{ih}\, \delta_{jk} + \delta_{ik}\, \delta_{jh}),$$

where λ and μ are the Lamé constants and $\rho = \rho_s/\rho_f$, ρ_s denoting the solid density.

The body is clamped on Γ'

$$u_i = 0 \quad \text{on } \Gamma'. \tag{2.3}$$

We assume that the fluid is inviscid, incompressible, and that the flow is irrotational. Consequently, the displacement in the fluid domain is the gradient of a potential Φ which is harmonic

$$\Delta\Phi = 0 \quad \text{in } \Omega_\varepsilon. \tag{2.4}$$

The linearized free surface condition reads (see e.g. [5]) :

$$\frac{\partial \Phi}{\partial x_3} = \alpha^2 \Phi \quad \text{on } FS_\varepsilon. \tag{2.5}$$

The coupling conditions are :

$$\sigma_{ij}\, n_j = -\alpha^2 \Phi n_i \quad \text{on } \Gamma, \tag{2.6}$$

$$\frac{\partial \Phi}{\partial n} = u.n \qquad \text{on } \Gamma, \tag{2.7}$$

where (2.6) expresses the continuity of the normal stress and (2.7) the continuity of the normal component of the displacement across the fluid-structure interface Γ.

To make this system complete, we must specify the asymptotic spatial behaviour of Φ. The fluid is at rest as $x_3 \to -\infty$:

$$\lim_{x_3 \to -\infty} \frac{\partial \Phi}{\partial x_3} = 0, \tag{2.8}$$

and energy radiates toward infinity in the (x_1, x_2) direction, which is expressed by means of the so-called Rellich radiation condition :

$$\lim_{R \to +\infty} \int_{-\infty}^{0} \int_{0}^{2\pi} R \left| \frac{\partial \Phi}{\partial R} - i\alpha^2 \Phi \right|^2 d\theta \, dx_3 = 0 , \tag{2.9}$$

where (R, θ, x_3) denote the cylindrical coordinates.

We are interested in eigenvibrations of the system, that is in values of α such that there exists a non-zero (u, Φ) solution of (2.1)-(2.9). In a standard way (see e.g. [2]), we shall look for such values of the parameter α — called scattering frequencies — in the complex plane. Then, for non real α, we must replace (2.9) by a convenient condition.

This last matter is basically related to the study of the exterior problem (2.4), (2.5), (2.7), set for the fluid potential when in (2.7) $u.n$ is temporarily considered as a datum. In the next section, we shall prove that under certain conditions bearing on the parameters α and ε, this exterior problem has a unique solution $\Phi = R(\alpha, \varepsilon)\,(u.n)$. Thus, we shall eliminate the unknown Φ from the equations.

3. STUDY OF THE EXTERIOR PROBLEM

We are given f in $L^2(\Gamma)$; then, according to the values of the real parameter α, we want to find out whether uniqueness and existence properties hold for the following problem :

$$P_{\alpha,\varepsilon} \begin{cases} \text{Find } \Phi \in H^1_{\text{loc}}(\Omega_\varepsilon) \text{ such that :} & \\ \Delta \Phi = 0 \quad \text{in } \Omega_\varepsilon , & (3.1) \\ \dfrac{\partial \Phi}{\partial x_3} = \alpha^2 \Phi \quad \text{on } FS_\varepsilon , & (3.2) \\ \dfrac{\partial \Phi}{\partial n} = f \quad \text{on } \Gamma , & (3.3) \\ \lim\limits_{x_3 \to -\infty} \dfrac{\partial \Phi}{\partial x_3} = 0 , & (3.4) \\ \Phi \text{ satisfies the Rellich radiation condition .} & (3.5) \end{cases}$$

Maz'ja proved in [6] a uniqueness theorem, but under very restrictive geometrical hypotheses. In the case of finite depth of the fluid, Beale [7] proved that the problem is well-posed except maybe for a discrete set of values of α.

We give hereafter, when the fluid is infinitely deep, a new method to investigate the existence and uniqueness properties of problem $P_{\alpha,\varepsilon}$, derived from the study of an equivalent problem $\tilde{P}_{\alpha,\varepsilon}$ set in a bounded domain. The results obtained constitute an extension of those of [7]. When there is a unique solution Φ, they allow to give a very practical expression of Φ as a function of f. This expression will be the main tool for the variational formulation of the coupled vibration problem.

3.1. The Green function

Let M and P be two points in $\mathbb{R}^3_-$, and δ_M be the Dirac measure at point M. For real α, we denote $G(\alpha, \varepsilon, M, P)$ (and very often $G(\alpha, \varepsilon)$) the Green function of the problem $P_{\alpha,\varepsilon}$. It satisfies (3.4), the radiation condition (3.5) and :

$$\Delta_{PG}(\alpha, \varepsilon, M, P) = \delta_M(P) \qquad \text{in } F_\varepsilon, \tag{3.6}$$

$$\frac{\partial G}{\partial x_3}(\alpha, \varepsilon, M, P) = \alpha^2 G(\alpha, \varepsilon, M, P) \quad \text{on } FS_\varepsilon. \tag{3.7}$$

The main properties of $G(\alpha, \varepsilon)$ are summarized in the following lemma. As the proofs are rather tedious, we only sketch them in the appendix.

LEMMA 3.1 :

(i) *for fixed* ε, $G(\alpha, \varepsilon)$ *has an analytic continuation for* $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$; *this continuation shows a cut along* $\mathbb{R}_-$.

(ii) $G(\alpha, \varepsilon)$ *has an expansion at all orders in terms of powers of* ε :

$$G(\alpha, \varepsilon) = G_0 + \varepsilon G_1 + \varepsilon^2 G_2(\alpha) + \cdots + \varepsilon^p G_p(\alpha) + R_{p+1}(\alpha, \varepsilon) \tag{3.8}$$

where

$$G_0(M, P) = -\frac{1}{4\pi \| MP \|}$$

and $G_1(M, P)$ *does not depend on* α. *The coefficients* $G_i(\alpha, M, P)$ *are holomorphic functions of* α, $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$; *moreover, if* α *belongs to a compact subset of* $\mathbb{C} \setminus \mathbb{R}_-$, *if* M *and* P *belong to compact subsets of* F_ε, *then we have the following estimate for the remainder term :*

$$| R_{p+1}(\alpha, \varepsilon) | \leqslant C\varepsilon^{p+1}.$$

(iii) *if* α *is a strictly positive real number, all coefficients* $G_i(\alpha)$ *are real-valued functions of* M *and* P.

Remark : All these properties are valid in the same conditions for first and second partial derivatives of $G(\alpha, \varepsilon)$ with respect to the coordinates of M or P ;

the analytic continuations for $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$ can be obtained by differentiation of the analytic continuation of $G(\alpha, \varepsilon)$, and the formula (3.8) may be differentiated term by term.

In order to study $P_{\alpha,\varepsilon}$ for real α, we are led to define it for non-real α.

3.2. The problem for Im $\alpha > 0$

DÉFINITION 3.2 : *For* $\operatorname{Im} \alpha > 0$, *we define* $P_{\alpha,\varepsilon}$ *in the following way* :

$$\left\{ \begin{array}{l} \textit{Find } \Phi \textit{ satisfying } (3.1), (3.2), (3.3) \textit{ such that :} \\ \Phi \in \{ \Psi \mid (1+r^2)^{-1/2} \Psi \in L^2(\Omega_\varepsilon), \nabla\Psi \in [L^2(\Omega_\varepsilon)]^3, \Psi |_{FS_\varepsilon} \in L^2(FS_\varepsilon) \} \end{array} \right. \quad (3.9)$$

where r denotes the radial distance in $\mathbb{R}^3$.

LEMMA 3.3 : *If* $\operatorname{Im} \alpha > 0$, $P_{\alpha,\varepsilon}$ *has a unique solution* Φ. *This solution satisfies the integral representation formula* :

$$\Phi = \int_\Gamma \left[\Phi \frac{\partial G}{\partial n}(\alpha, \varepsilon) - \frac{\partial \Phi}{\partial n} G(\alpha, \varepsilon) \right] d\Gamma \quad \text{in } \Omega_\varepsilon . \quad (3.10)$$

This lemma is proved in [8].

Remark : If α is real positive and Φ is a solution of $P_{\alpha,\varepsilon}$, then the representation formula (3.10) is also valid [8]. In both cases, Φ and $G(\alpha, \varepsilon)$ behave the same way asymptotically with respect to the space variables ; the condition (3.9) specifies that behaviour when $\operatorname{Im} \alpha > 0$.

3.3. Problem $\tilde{P}_{\alpha,\varepsilon}$ derived from $P_{\alpha,\varepsilon}$

The technique used in this subsection has been developed by A. Jami and M. Lenoir (see e.g. [9]).

We consider in $\mathbb{R}^3_-$ a closed smooth surface Σ enclosing B ; we denote Ω' the domain limited by Γ and Σ, and introduce the problem :

$$\tilde{P}_{\alpha,\varepsilon} \left\{ \begin{array}{ll} \text{Find } \chi \in H^1(\Omega') \text{ such that :} & \\ \Delta\chi = 0 \quad \text{in } \Omega' , & (3.11) \\ \dfrac{\partial \chi}{\partial n} = f \quad \text{on } \Gamma , & (3.12) \\ \chi = \displaystyle\int_\Gamma \left[\chi \frac{\partial G}{\partial n}(\alpha, \varepsilon) - \frac{\partial \chi}{\partial n} G(\alpha, \varepsilon) \right] d\Gamma \quad \text{on } \Sigma . & (3.13) \end{array} \right.$$

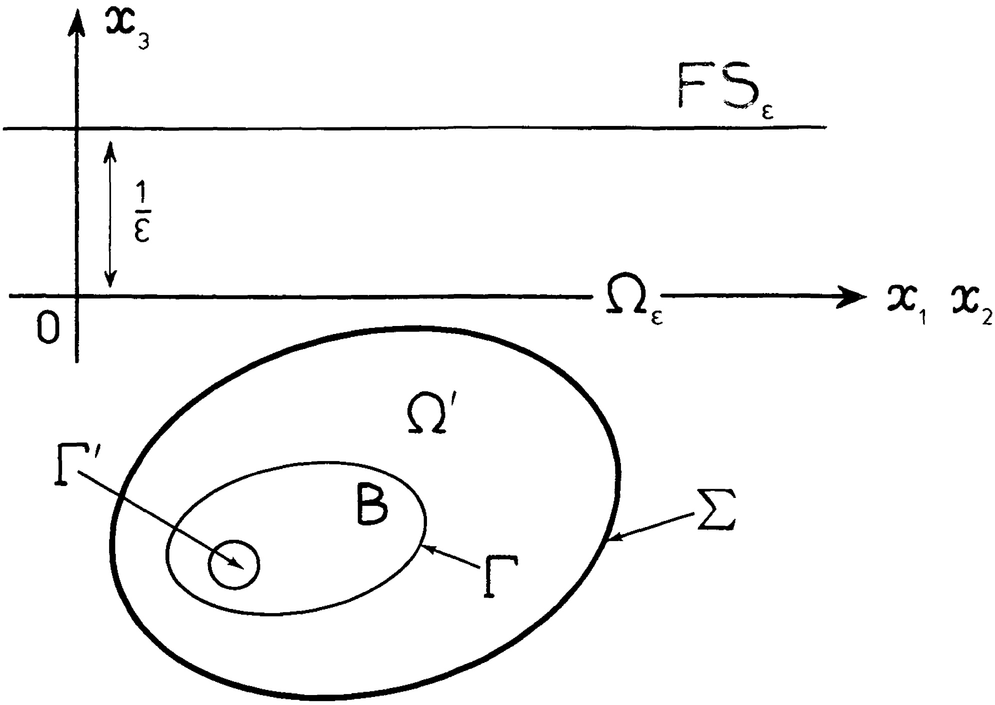

Figure 2.

We show in the following lemma that the problems $P_{\alpha,\varepsilon}$ and $\tilde{P}_{\alpha,\varepsilon}$ are equivalent.

LEMMA 3.4 : *For* $\alpha > 0$ *and for* $\operatorname{Im} \alpha > 0$, *each of the following properties holds for* $P_{\alpha,\varepsilon}$ *if and only if it holds for* $\tilde{P}_{\alpha,\varepsilon}$:

(i) $\Phi = 0$ *is the only solution of the homogeneous problem.*

(ii) $\forall f \in L^2(\Gamma)$, *the problem has at least one solution.*

Moreover, when α *is such that the problems are well-posed, the solution of* $\tilde{P}_{\alpha,\varepsilon}$ *is the restriction of the solution of* $P_{\alpha,\varepsilon}$ *to* Ω'.

Proof : Assuming that $\tilde{P}_{\alpha,\varepsilon}$ has the uniqueness property (i), let Φ be a solution of the homogeneous problem $P_{\alpha,\varepsilon}$. We can write the representation formula (3.10) for Φ, which shows that $\Phi|_{\Omega'}$ is a solution of the homogeneous problem $\tilde{P}_{\alpha,\varepsilon}$. This implies $\Phi|_{\Omega'} = 0$ and then $\Phi = 0$ by analytic continuation.

Conversely, assuming $P_{\alpha,\varepsilon}$ has the uniqueness property, let χ be a solution of the homogeneous problem $\tilde{P}_{\alpha,\varepsilon}$, and let us define :

$$\Psi = \int_{\Gamma} \chi \frac{\partial G}{\partial n}(\alpha, \varepsilon)\, d\Gamma \quad \text{in } \Omega_{\varepsilon}. \tag{3.14}$$

Using the properties (3.6), (3.7) of the Green function we get :

$$\Delta\Psi = 0 \qquad \text{in } \Omega_\varepsilon , \tag{3.15}$$

$$\frac{\partial\Psi}{\partial n} = \alpha^2\,\Psi \quad \text{on } FS_\varepsilon . \tag{3.16}$$

Now, the integral representation formula for χ gives :

$$\chi = \int_{\Gamma\cup\Sigma}\left[\chi\frac{\partial G}{\partial n}(\alpha,\varepsilon) - \frac{\partial\chi}{\partial n}G(\alpha,\varepsilon)\right]d(\Gamma\cup\Sigma) \quad \text{in } \Omega' ,$$

which also reads :

$$\chi = \Psi\,|_{\Omega'} + \eta , \tag{3.17}$$

$$\eta = \int_{\Gamma}\left[\chi\frac{\partial G}{\partial n}(\alpha,\varepsilon) - \frac{\partial\chi}{\partial n}G(\alpha,\varepsilon)\right]d\Sigma . \tag{3.18}$$

It is readily seen that (3.18) defines η in the domain $\Omega'' = \Omega' \cup \overline{B}$ enclosed by Σ and that we have :

$$\Delta\eta = 0 \text{ in } \Omega'' . \tag{3.19}$$

From (3.13), (3.14) and (3.17) we deduce :

$$\eta = 0 \text{ on } \Sigma , \tag{3.20}$$

and (3.19) together with (3.20) implies $\eta = 0$ in Ω''. Thus, $\Psi\,|_{\Omega'} = \chi$ by (3.17), and (3.12) gives :

$$\frac{\partial\Psi}{\partial n} = 0 \text{ on } \Gamma . \tag{3.21}$$

Returning to the definition (3.14), according to the properties of $G(\alpha, \varepsilon)$, we see that for $\alpha > 0$, Ψ satisfies (3.4) and the Rellich radiation condition (3.5), and for $\operatorname{Im}\alpha > 0$, Ψ satisfies (3.9). This, together with (3.15), (3.16) and (3.21) implies that Ψ is a solution of the homogeneous problem $P_{\alpha,\varepsilon}$; thus $\Psi = 0$ and eventually $\chi = 0$.

The equivalence between $P_{\alpha,\varepsilon}$ and $\tilde{P}_{\alpha,\varepsilon}$ with respect to the existence property (ii) follows from analogous considerations.

3.4. Formulation of $\tilde{P}_{\alpha,\varepsilon}$

The space $V = H^1(\Omega')$ is equipped with the usual scalar product :

$$(\Phi, \Psi)_V = \int_{\Omega'} \nabla\Phi \, \overline{\nabla\Psi} \, d\Omega' + \int_{\Omega'} \Phi\overline{\Psi} \, d\Omega'$$

and with the associated norm $\| \Phi \|_V$.

THEOREM 3.5 : *Solving $\tilde{P}_{\alpha,\varepsilon}$ amounts to find $\chi \in V$ such that :*

$$(I + K(\alpha, \varepsilon)) \chi = F(\alpha, \varepsilon) f, \tag{3.22}$$

where $K(\alpha, \varepsilon)$ is a compact operator in V, and $F(\alpha, \varepsilon)$ is an element of $\mathscr{L}(L^2(\Gamma), V)$, defined respectively by :

$$(K(\alpha, \varepsilon) \chi, \Psi)_V = - \int_\Sigma \overline{\Psi} \left[\int_\Gamma \chi \frac{\partial^2 G(\alpha, \varepsilon)}{\partial n_\Gamma \, \partial n_\Sigma} \, d\Gamma \right] d\Sigma - \int_{\Omega'} \chi \overline{\Psi} \, d\Omega', \tag{3.23}$$

$$(F(\alpha, \varepsilon) f, \Psi)_V = \int_\Gamma f \overline{\Psi} \, d\Gamma - \int_\Sigma \overline{\Psi} \left[\int_\Gamma f \frac{\partial G}{\partial n_\Sigma} (\alpha, \varepsilon) \, d\Gamma \right] d\Sigma . \tag{3.24}$$

Remark : The first integral in (3.23) is not singular since the surfaces Γ and Σ involved in the trace operators $\partial/\partial n_\Gamma$ and $\partial/\partial n_\Sigma$ do not intersect.

Proof of theorem 3.5 : (3.23) and (3.24) obviously define two elements of V : $K(\alpha, \varepsilon) \chi$ and $F(\alpha, \varepsilon) f$. Moreover :

$$\| K(\alpha, \varepsilon) \chi \|_V \leqslant C \, \| \chi \|_{L^2(\Gamma)} \left\| \frac{\partial^2 G(\alpha, \varepsilon)}{\partial n_\Gamma \, \partial n_\Sigma} \right\|_{L^2(\Gamma) \times L^2(\Sigma)} + \| \chi \|_{L^2(\Omega')},$$

and the compactness of $K(\alpha, \varepsilon)$ follows from the compactness of the trace operator from $H^1(\Omega')$ into $L^2(\Gamma)$ and from the compact embedding of $H^1(\Omega')$ into $L^2(\Omega')$. Similarly :

$$\| F(\alpha, \varepsilon) f \|_V \leqslant C \, \| f \|_{L^2(\Gamma)} \left(1 + \left\| \frac{\partial G}{\partial n_\Sigma} (\alpha, \varepsilon) \right\|_{L^2(\Gamma) \times L^2(\Sigma)} \right),$$

which proves that $F(\alpha, \varepsilon)$ is a continuous operator from $L^2(\Gamma)$ into V.

Then, if we multiply (3.11) by a test function $\overline{\Psi} \in V$, integrate by parts and use the boundary conditions (3.12) and (3.13), according to the definitions (3.23) and (3.24) we readily obtain :

$$((I + K(\alpha, \varepsilon)) \chi, \Psi)_V = (F(\alpha, \varepsilon) f, \Psi)_V , \quad \forall \Psi \in V ,$$

from which (3.22) immediately follows.

3.5. Consequent results for $P_{\alpha,\varepsilon}$

According to (3.23), and to the properties of the Green function (lemma 3.1 (i)), $K(\alpha, \varepsilon)$ can be defined for $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$; the continued operator depends analytically on α and is again compact. Now, $I + K(\alpha, \varepsilon)$ is invertible if and only if :

$$(I + K(\alpha, \varepsilon))\chi = 0 \tag{3.25}$$

implies $\chi = 0$. Let us verify this last condition for $\operatorname{Im} \alpha > 0$. Because of lemma 3.3, $P_{\alpha,\varepsilon}$ is a well-posed problem; then, using the equivalence proved in lemma 3.4, we deduce that $\tilde{P}_{\alpha,\varepsilon}$ has the uniqueness property (i) and thus (3.25) implies $\chi = 0$.

It follows that $(I + K(\alpha, \varepsilon))^{-1}$ depends meromorphically on $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$ (see e.g. [10], chap. VII, theorem 1.9).

Similarly, $F(\alpha, \varepsilon)$ can be defined for $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$; the continued operator depends analytically on α and is again in $\mathscr{L}(L^2(\Gamma), V)$. We can now deduce the basic result of this section :

THEOREM 3.7 : *For fixed $\varepsilon > 0$, there exists a bounded map $R(\alpha, \varepsilon)$ from $L^2(\Gamma)$ into $H^1_{\mathrm{loc}}(\Omega_\varepsilon)$, meromorphic in $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$, such that for given f in $L^2(\Gamma)$ we have :*

— *if* $\operatorname{Im} \alpha > 0$, *$R(\alpha, \varepsilon)$ is a holomorphic function of α and $R(\alpha, \varepsilon) f$ is the (unique) solution of $P_{\alpha,\varepsilon}$;*

— *if $\alpha \in R^*_+$ is not a pole of $R(\alpha, \varepsilon)$, $P_{\alpha,\varepsilon}$ has a unique solution, namely $R(\alpha, \varepsilon) f$;*

— *if* $\operatorname{Im} \alpha < 0$, *and α is not a pole of $R(\alpha, \varepsilon)$, $R'(\alpha, \varepsilon) f$ defines by means of analytic continuation a solution of* (3.1), (3.2), (3.3).

In all cases, the solution satisfies the integral representation formula (3.10).

Proof : Let γ_0 be the trace operator on Γ, from $H^1(\Omega')$ onto $H^{1/2}(\Gamma)$, we define :

$$T(\alpha, \varepsilon) = \gamma_0 (I + K(\alpha, \varepsilon))^{-1} F(\alpha, \varepsilon) . \tag{3.26}$$

To the Neumann data f for the problem $\tilde{P}_{\alpha,\varepsilon}$, $T(\alpha, \varepsilon)$ associates the trace of the corresponding solution on Γ. This operator belongs to $\mathscr{L}(L^2(\Gamma), H^{1/2}(\Gamma))$ and is a meromorphic function of $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$. Then, setting :

$$R(\alpha, \varepsilon) f = \int_\Gamma \left[(T(\alpha, \varepsilon) f) \frac{\partial G}{\partial n}(\alpha, \varepsilon) - f G(\alpha, \varepsilon) \right] d\Gamma , \tag{3.27}$$

the theorem follows.

Remark : We did not define $P_{\alpha,\varepsilon}$ for $\operatorname{Im} \alpha < 0$, but as a convention we shall say that the solution of $P_{\alpha,\varepsilon}$ is $R(\alpha, \varepsilon) f$, whenever it is defined.

To each value of ε, we can associate a subset M_ε of $\mathbb{C} \setminus \mathbb{R}_-$, such that when α belongs to M_ε, $P_{\alpha,\varepsilon}$ may not have a unique solution. Keeping in mind that our aim is to study the perturbation in the neighbourhood of an eigenfrequency of the unperturbed problem, we can restrict ourselves to α lying in a compact set K. Under that additional hypothesis, we prove in the following theorem that for small ε, M_ε and K do not intersect; this result improves those of theorem 3.7.

THEOREM 3.8 : *Let K be a compact set in $\mathbb{C} \setminus \mathbb{R}_-$, there exists $\varepsilon_0(K) > 0$ such that :*

$$\forall \varepsilon < \varepsilon_0(K), \quad R(\alpha, \varepsilon) \text{ exists for any } \alpha \in K .$$

Proof : We show that the assumption : $\forall k \in \mathbb{N}^*$, $\exists \alpha_k \in K$, $\exists \varepsilon_k < 1/k$, such that α_k is a pole of $R(\alpha, \varepsilon)$ leads to a contradiction. According to (3.26) and (3.27) this amounts to say that α_k is a pole of $(I + K(\alpha, \varepsilon_k))^{-1}$. Thus, there exists for each k, $\Phi_k \in H^1(\Omega')$ solution of the homogeneous problem $\tilde{P}_{\alpha_k,\varepsilon_k}$, and we normalize by requiring $\| \Phi_k \|_V = 1$. We can assume, passing to a subsequence if necessary, that (Φ_k) converges weakly in V and strongly in $L^2(\Omega')$ to a function Φ. Let U be a domain such that $\overline{U} \subset \Omega'$. From elliptic interior regularity we get :

$$\| \Phi_p - \Phi_k \|_{H^2(U)} \leqslant C \| \Phi_p - \Phi_k \|_{L^2(\Omega')}$$

so that (Φ_k) converges to Φ strongly in $H^2(U)$ and $\Delta\Phi = 0$ in U. Finally :

$$\Delta\Phi = 0 \text{ in } \Omega' . \tag{3.28}$$

The variational formulation for $\tilde{P}_{\alpha_k,\varepsilon_k}$ in V gives :

$$\int_{\Omega'} \nabla\Phi_k \nabla\overline{\Psi} \, d\Omega' = 0 , \quad \forall \Psi \in H^1(\Omega') , \ \Psi|_\Sigma = 0 .$$

Passing to the limit $k \to +\infty$, integrating by parts and using (3.28) we get

$$\left\langle \frac{\partial\Phi}{\partial n}, \Psi \right\rangle_{(H^{1/2}(\Gamma))', H^{1/2}(\Gamma)} = 0 , \quad \forall \Psi \in H^1(\Omega') , \quad \Psi|_\Sigma = 0 ,$$

and then

$$\frac{\partial\Phi}{\partial n} = 0 \quad \text{on } \Gamma . \tag{3.29}$$

Next, passing to the limit in the equation (3.13) for Φ_k gives :

$$\Phi = \int_\Gamma \Phi \frac{\partial G_0}{\partial n} d\Gamma \quad \text{on } \Sigma . \tag{3.30}$$

We consider $\check{\Phi}$, defined in $\Omega_0 = \mathbb{R}^3 \setminus \overline{B}$ by the right member of (3.30); $\check{\Phi}$ belongs to $W_0^1(\Omega_0) = \{ \theta \mid (1 + r^2)^{-1/2} \theta \in L^2(\Omega_0), \nabla\theta \in [L^2(\Omega_0)]^3 \}$, satisfies (3.29) and :

$$\Delta\check{\Phi} - 0 .$$

Following Nedelec [11], this implies $\check{\Phi} = 0$. By an argument similar to the proof of lemma 3.4, we show that $\check{\Phi}$ and Φ coincide in Ω', thus $\Phi = 0$.

Returning to the variational formulation of $\tilde{P}_{\alpha_k, \varepsilon_k}$, we have :

$$\int_{\Omega'} | \nabla\Phi_k |^2 \, d\Omega' = \int_\Sigma \frac{\partial \Phi_k}{\partial n} \overline{\Phi}_k \, d\Sigma . \tag{3.31}$$

Now, $(\partial\Phi_k/\partial n_\Sigma)$ is a bounded sequence in $L^2(\Sigma)$, since (Φ_k) converges to zero in $L^2(\Gamma)$ and the representation formula (3.10) yields :

$$\Phi_k = \int_\Gamma \Phi_k \frac{\partial G}{\partial n_\Gamma} (\alpha_k, \varepsilon_k) \, d\Gamma ,$$

$$\frac{\partial \Phi_k}{\partial n_\Sigma} = \int_\Gamma \Phi_k \frac{\partial^2 G}{\partial n_\Sigma \, \partial n_\Gamma} (\alpha_k, \varepsilon_k) \, d\Gamma .$$

Thus, as Φ_k converges to $\Phi = 0$ weakly in $H^1(\Omega')$, the restriction of Φ_k to Σ converges to zero strongly in $L^2(\Sigma)$ and eventually we get from (3.31) :

$$\lim_{k \to +\infty} \int_{\Omega'} | \nabla\Phi_k |^2 \, d\Omega' = 0 ,$$

and then

$$\lim_{k \to +\infty} \| \nabla\Phi_k \|_{H^1(\Omega')} \to 0 ,$$

which contradicts the normalization.

4. FORMULATION OF THE COUPLED PROBLEM

Hypothesis 4.1 : From now on, it will be assumed that α belongs to a compact set K included in $\mathbb{C} \setminus \mathbb{R}_-$ and that ε is sufficiently small to get the assumptions of theorem 3.8 satisfied.

According to the results obtained in Section 3, the problem (2 1)-(2.9) now makes sense when α is complex, we replace equations (2.4), (2.5), (2.7), (2.8) and (2.9) regarding the displacement potential in the fluid by .

$$\Phi = R(\alpha, \varepsilon)(u.n) \quad \text{in } \Omega_\varepsilon .$$

Moreover, if $T(\alpha, \varepsilon)$ is the operator defined in (3.26) :

$$\Phi = T(\alpha, \varepsilon)(u.n) \quad \text{on } \Gamma ,$$

and the whole problem is then .

$$\left.\begin{array}{l} \text{Find } u \in [H^1(B)]^3 \text{ such that :} \\ \dfrac{\partial \sigma_{ij}}{\partial x_j} = -\rho\alpha^2 u_i \quad \text{in } B , \\ u_i = 0 \quad \text{on } \Gamma' , \\ \sigma_{ij} n_j = \alpha^2 (T(\alpha, \varepsilon)(u.n)) n_i \quad \text{on } \Gamma , \end{array}\right\} \tag{4.1}$$

system which only involves the unknowns u and α.

4.1. Scattering frequencies

DEFINITION 4.2 : *A scattering frequency of the problem is a complex number* $\alpha(\varepsilon)$, *such that* (4.1) *admits a non-zero solution* u, u *is called a scattering function associated to* $\alpha(\varepsilon)$.

Note that because of hypothesis 4.1, for a fixed value of ε, we cannot consider the set of all scattering frequencies in $\mathbb{C} \setminus \mathbb{R}_-$.

4.2. Variational formulation

We define the functional space :

$$W = \{ u \in [H^1(B)]^3 \mid u|_{\Gamma'} = 0 \}$$

equipped with the scalar product

$$(u, v)_W = \int_B a_{ijkh}\, e_{ij}(u)\, e_{kh}(\bar{v})\, dB$$

and with the associated norm, which is equivalent to the norm induced by $[H^1(B)]^3$. Then, solving (4.1) amounts to find $u \in W$ such that :

$$\forall v \in W , \quad (u, v)_W = \rho\alpha^2 \int_B u_i \bar{v}_i \, dB - \alpha^2 \int_\Gamma [T(\alpha, \varepsilon)(u.n)] (\bar{v}.n)\, d\Gamma . \tag{4.2}$$

LEMMA 4.3 : *The operators B_1 and $B_2(\alpha, \varepsilon)$ defined by :*

$$(B_1 u, v)_W = \rho \int_B u_i \bar{v}_i \, dB , \tag{4.3}$$

$$(B_2(\alpha, \varepsilon) u, v)_W = - \int_\Gamma [T(\alpha, \varepsilon)(u.n)] (\bar{v}.n) \, d\Gamma , \tag{4.4}$$

are compact operators from W into W.

Proof : The compactness of B_1 follows from the fact that B_1 is continuous from $[L^2(B)]^3$ into W and that $H^1(B)$ has compact embedding into $L^2(B)$.
According to (4.4) :

$$\| B_2(\alpha, \varepsilon) u \|_W \leqslant \| T(\alpha, \varepsilon)(u.n) \|_{L^2(\Gamma)} ,$$

and $T(\alpha, \varepsilon)$ being continuous from $L^2(\Gamma)$ into $L^2(\Gamma)$, the compactness of $B_2(\alpha, \varepsilon)$ follows from the compact embedding of $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$.
Then (4.2) may be written :

$$u = \alpha^2 (B_1 + B_2(\alpha, \varepsilon)) u , \tag{4.5}$$

which leads to the equivalent definition of the scattering frequencies :

DEFINITION 4.4 : *The scattering frequencies are the complex numbers $\alpha(\varepsilon)$ such that 1 is an eigenvalue of the compact operator $\alpha^2(B_1 + B_2(\alpha, \varepsilon))$.*

The following lemma gives the properties of $B_2(\alpha, \varepsilon)$ with respect to the parameters α and ε.

LEMMA 4.5 : *$B_2(\alpha, \varepsilon)$ and $T(\alpha, \varepsilon)$ depend holomorphically on α and have an expansion at all orders in terms of powers of ε, uniformly with respect to α.*

Proof : According to the properties of $G(\alpha, \varepsilon)$ (lemma 3.1 (i) and (ii)), to (3.23) and (3.24) which define $K(\alpha, \varepsilon)$ and $F(\alpha, \varepsilon)$ and to the definition (3.26) of $T(\alpha, \varepsilon)$, we get by composition of asymptotic expansions, an expansion of $T(\alpha, \varepsilon)$ in the form :

$$T(\alpha, \varepsilon) = T_0 + \varepsilon T_1 + \varepsilon^2 T_2(\alpha) + \cdots + \varepsilon^p T_p(\alpha) + R'_{p+1}(\alpha, \varepsilon) , \tag{4.6}$$

the uniformity of the expansion following from the same property for $G(\alpha, \varepsilon)$.
Using (4.4) and (4.6) we get by linearity :

$$B_2(\alpha, \varepsilon) = B_2^0 + \varepsilon B_2^1 + \varepsilon^2 B_2^2(\alpha) + \cdots + \varepsilon^p B_2^p(\alpha) + R''_{p+1}(\alpha, \varepsilon) . \tag{4.7}$$

Remark : Due to a particular feature of the expansion of the Green function, the first two terms in the expansions (4.6) and (4.7) do not depend on α.

4.3. The unperturbed problem P_0

The unperturbed problem is obtained for $\varepsilon = 0$; it is the vibration problem of an elastic body surrounded by an incompressible fluid filling its complementary domain in the whole $\mathbb{R}^3$. According to the definition 4.4, the scattering frequencies are in this case the values of α such that α^{-2} is an eigenvalue of the compact operator $B_1 + B_2^0$.

THEOREM 4.6 . *$B_1 + B_2^0$ is a compact, symmetric, positive definite operator from W into W. Thus, it admits a countable sequence of eigenfrequencies*

$$(\alpha_1^0)^{-2} \geqslant (\alpha_2^0)^{-2} \geqslant \cdots \geqslant (\alpha_j^0)^{-2} \geqslant \cdots \to 0$$

and the corresponding eigenvectors can be chosen so that they form an orthonormal basis in W.

A proof of this theorem is given in [2]. We can see that for the unperturbed problem the scattering frequencies are its eigenfrequencies $\pm \alpha_j^0$.

Definition 4.4 leads us to study how the spectrum of $B_1 + B_2(\alpha, \varepsilon)$ depends on ε, in order to deduce the asymptotic behaviour of the scattering frequencies.

5. ASYMPTOTIC BEHAVIOUR OF THE SCATTERING FREQUENCIES

5.1. Spectrum of $B_1 + B_2(\alpha, \varepsilon)$

Except for some modifications arising from the fact that there is an additional variable α, the results given in this subsection are consequences of the perturbation theory for linear operators, applied to $B_1 + B_2(\alpha, \varepsilon)$ with respect to the parameter ε. An extensive study on this topic may be found in [10] (see chap. 2 especially Section 5, chap. 7 and chap. 8), for that reason lemmas 5.1 and 5.2 will be given without proof.

Let $(\alpha^0)^{-2}$ be one of the eigenvalues of the unperturbed operator $B_1 + B_2^0$, with multiplicity m. Let γ be a positively oriented curve, plotted in the λ-complex plane (λ being the spectral variable for $B_1 + B_2^0$), enclosing $(\alpha^0)^{-2}$ but no other eigenvalues. We have (see [10] chap. 2, theorem 1 5 and theorem 5 4) :

LEMMA 5.1 . *For sufficiently small ε, the curve γ is included in the resolvent set of $B_1 + B_2(\alpha, \varepsilon)$; we can then define the operator .*

$$P(\alpha, \varepsilon) = -\frac{1}{2\, i\pi} \int_\gamma [B_1 + B_2(\alpha, \varepsilon) - \lambda]^{-1}\, d\lambda\,.$$

It is a projection operator, equal to the sum of the eigenprojections for all the eigenvalues of $B_1 + B_2(\alpha, \varepsilon)$ *lying inside* γ. $P(\alpha, \varepsilon)$ *depends holomorphically on* α, *and has an expansion at all orders in terms of powers of* ε, *uniformly with respect to* α. *In particular :*

$$P(\alpha, \varepsilon) = P^0 + 0(\varepsilon),$$

where

$$P^0 = -\frac{1}{2\,i\pi}\int_\gamma [B_1 + B_2^0 - \lambda]^{-1}\,d\lambda$$

is the eigenprojection for the eigenvalue $(\alpha^0)^{-2}$ *of* $B_1 + B_2^0$.

Hereafter, we shall assume ε is small enough to get the results of lemma 5.1. It follows that the space $M(\alpha, \varepsilon) = P(\alpha, \varepsilon)\,W$ is isomorphic to the eigenspace $M^0 = P^0\,W$ (in particular it has the same dimension m). Therefore, $B_1 + B_2(\alpha, \varepsilon)$ has exactly m eigenvalues lying inside γ, which are the eigenvalues of $B_1 + B_2(\alpha, \varepsilon)$ in $M(\alpha, \varepsilon)$.

In order to set this eigenvalue problem in a space which does not depend on the parameters, we use the transformation function (see [10] chap. 2, Section 4.2) :

LEMMA 5.2 : *There exists a function* $U(\alpha, \varepsilon)$ *with values in* $\mathscr{L}(W, W)$, *called transformation function for* $P(\alpha, \varepsilon)$ *with the following properties :*

— *the inverse* $U(\alpha, \varepsilon)^{-1}$ *exists; both* $U(\alpha, \varepsilon)$ *and* $U(\alpha, \varepsilon)^{-1}$ *depend holomorphically on* α, *and have an expansion at all orders in terms of powers of* ε, *uniformly with respect to* α;

— $U(\alpha, \varepsilon)\,P^0\,U(\alpha, \varepsilon)^{-1} = P(\alpha, \varepsilon)$.

Thus, finding the eigenvalues of $B_1 + B_2(\alpha, \varepsilon)$ in the neighbourhood of $\lambda = (\alpha^0)^{-2}$ reduces finding the eigenvalues of the operator

$$P^0\,U(\alpha, \varepsilon)^{-1}\,(B_1 + B_2(\alpha, \varepsilon))\,U(\alpha, \varepsilon)\,P^0 \tag{5.1}$$

acting in the fixed subspace M^0.

5.2. Continuity of the scattering frequencies

If we denote in the same way the operator (5.1) and its matrix in any given basis of M^0, the scattering frequencies solve the implicit equation :

$$\det\,[P^0\,U(\alpha, \varepsilon)^{-1}(B_1 + B_2(\alpha, \varepsilon))\,U(\alpha, \varepsilon)\,P^0 - \alpha^{-2}] = 0\,, \tag{5.2}$$

which has a single zero α^0 of order m when $\varepsilon = 0$.

We now choose for compact K (see hypothesis 4.1) a compact neighbourhood of α^0 in $\mathbb{C} \setminus \mathbb{R}_-$, which does not contain any other α_j^0 of theorem 4.6. The left hand-side of (5.2) is thus holomorphic in α in the neighbourhood of α^0, and continuous at $\varepsilon = 0$, uniformly with respect to α. Then, the Rouché theorem (see e.g. [12]) applies to the equation (5.2) and yields :

THEOREM 5.3 : *If α^0 is one of the eigenvalues of the unperturbed problem with multiplicity m, there are exactly m scattering frequencies $\alpha(\varepsilon)$ (counting each one with its algebraic multiplicity) which converge to α^0 as ε converges to zero.*

5.3. Study of the case $m = 1$

When α^0 is a simple eigenvalue. the operator (5.1) reduces to a similarity in the one-dimensional space M^0. Again, from the perturbation theory in a finite dimensional space ([10] chap. 2, theorem 5.4) we have :

LEMMA 5.4 : *For ε sufficiently small, $\lambda(\alpha, \varepsilon)$, the eigenvalue of $B_1 + B_2(\alpha, \varepsilon)$ in $M(\alpha, \varepsilon)$, has an expansion in terms of powers of ε, uniform with respect to α :*

$$\lambda(\alpha, \varepsilon) = (\alpha^0)^{-2} + \varepsilon\lambda^{(1)} + \varepsilon^2 \lambda^{(2)}(\alpha) + \cdots + \varepsilon^p \lambda^{(p)}(\alpha) + 0(\varepsilon^{p+1}) . \quad (5.3)$$

The only scattering frequency of theorem 5.3 may thus be obtained by solving the implicit equation :

$$\alpha^{-2} = \lambda(\alpha, \varepsilon) . \quad (5.4)$$

Let us choose K as we did in Section 5.2, and define $\lambda(\alpha, \varepsilon)$ for negative values of ε by symmetry. Let :

$$\eta(\alpha, \varepsilon) = \lambda(\alpha, \varepsilon) - \alpha^{-2} ,$$

we have :

$$\eta(\alpha^0, 0) = 0 .$$

LEMMA 5.5 : *The partial derivative $\partial\eta/\partial\alpha$ exists and is continuous in a neighbourhood of $(\alpha^0, 0)$.*

Proof : In a neighbourhood of $(\alpha^0, 0)$, $\lambda(\alpha, \varepsilon)$ is the only eigenvalue of $B_1 + B_2(\alpha, \varepsilon)$ and thus depends holomorphically on α (see [10] chap. 2, Section 1.2); consequently $\partial\eta/\partial\alpha$ exists. Moreover, by means of the theorem of Morera (see e.g. [12]), we show that the coefficients $\lambda^{(p)}(\alpha)$ in the expansion (5.3) are holomorphic functions of α :

Let T be a triangle included in the domain D of holomorphy of $\lambda(\alpha, \varepsilon)$ and let us consider :

$$f_\varepsilon(\alpha) = \frac{\lambda(\alpha, \varepsilon) - (\alpha^0)^{-2} - \varepsilon\lambda^{(1)}}{\varepsilon^2}.$$

f_ε is holomorphic in α, $\alpha \in D$, and

$$\lim_{\varepsilon \to 0} f_\varepsilon(\alpha) = \lambda^{(2)}(\alpha),$$

$$\int_T f_\varepsilon(\alpha)\, dT = 0,$$

$$| f_\varepsilon(\alpha) | \leqslant C, \quad \text{where } C \text{ only depends on } D.$$

We can therefore apply the dominated convergence theorem and deduce that :

$$\lim_{\varepsilon \to 0} \int_T f_\varepsilon(\alpha)\, dT = 0 = \int_T \lim_{\varepsilon \to 0} f_\varepsilon(\alpha)\, dT = \int_T \lambda^{(2)}(\alpha)\, dT,$$

which proves that $\lambda^{(2)}(\alpha)$ is holomorphic in D. A recurrence argument gives the holomorphy for the other coefficients.

The expansion (5.3) can then be differentiated term by term with respect to the variable α, and eventually $\partial\eta/\partial\alpha$ is continuous in a neighbourhood of $(\alpha, \varepsilon) = (\alpha^0, 0)$.

The derivative with respect to α at $(\alpha^0, 0)$ is :

$$\frac{\partial\eta}{\partial\alpha}(\alpha^0, 0) = \frac{2}{(\alpha^0)^3} \neq 0,$$

consequently, the implicit function theorem can be used to derive the following :

THEOREM 5.5 : *The scattering frequency* $\alpha(\varepsilon)$ *has an asymptotic expansion at all orders in terms of powers of* ε *in the form* :

$$\alpha(\varepsilon) = \alpha^0 + \varepsilon\alpha^1 + \varepsilon^2\alpha^2 + \cdots + \varepsilon^p\alpha^p + 0(\varepsilon^{p+1}). \tag{5.5}$$

We shall conclude with the main result of this section :

THEOREM 5.6 : *All coefficients* α^p *in the expansion* (5.5) *of* $\alpha(\varepsilon)$ *are real.*

Proof : Taking the partial derivative with respect to ε in (5.4) and using (5.3), we get for $\varepsilon = 0$:

$$\alpha^1 = -\frac{1}{2}(\alpha^0)^3\lambda^{(1)}.$$

By a recurrence argument, we can show that all coefficients α^p are linear combinations of $\lambda^{(q)}(\alpha^0)$ with $q \leqslant p$. Then again ([10] chap. 2. Section 2 2) $\lambda^{(n)}(\alpha^0)$ is the eigenfrequency of the operator .

$$P^0\, \tilde{B}_n(\alpha)\, P^0 = - \sum_{p=1}^{\infty} (-1)^p \sum_{\substack{\nu_1 + \cdots + \nu_p = n \\ k_2 + \cdots + k_p = p-1}} P^0\, B_2^{\nu_1}(\alpha^0)\, S^{k_2} \ldots S^{k_p}\, B_2^{\nu_p}(\alpha^0)\, P^0\,,$$

where S is the reduced resolvent of $B_1 + B_2^0$.

From property (iii) of lemma 3.1, it follows that the subspace of W consisting of functions with real values is left invariant by the operators $B_2^m(\alpha^0)$. According to theorem 4.6, S, P^0 and thus $P^0\, \tilde{B}_n(\alpha^0)\, P^0$ have the same property; as $B_1 + B_2^0$ is selfadjoint we can choose a real-valued function u which generates the one-dimensional eigenspace M^0. Finally we deduce that :

$$\lambda^{(n)}(\alpha^0) = \frac{(P^0\, \tilde{B}_n(\alpha^0)\, P^0\, u, u)_W}{(u, u)_W}$$

belongs to $\mathbb{R}$, and we are done with the proof.

Remark When α^0 is not a simple eigenvalue, we know that the eigenvalues $\lambda_p(\alpha, \varepsilon)$, $p = 1, \ldots, m$, can be expanded at least up to the order one ([10] chap. 2, p. 65). This should then allow to expand also the corresponding scattering frequencies up to the order one. Unfortunatly, since we cannot assert the corresponding fonction η has a partial derivative with respect to α in the neighbourhood of $(\alpha^0, 0)$. we were not able to apply an implicit function theorem.

6. INTRODUCTION OF A SMALL GRAVITY TERM

To complete that study, we shall briefly sketch another type of perturbation, related this time to the free surface condition itself.

The elastic body still occupies a bounded domain B in $\mathbb{R}^3_-$; we assume that the incompressible fluid occupies Ω, the complementary domain of B in $\mathbb{R}^3_-$. We make exactly the same physical and geometrical assumptions as for the problem described in Section 1. Let L be a characteristic length of the body, A a Lamé constant, characteristic of the elastic properties of the solid, then $T = L(\rho_s/A)^{1/2}$ is a characteristic time for the vibrations of the solid. We assume that $\rho_s\, Lg/A$ is a small parameter ε. The non-dimensional equations of the problem are in this case .

$$\frac{\partial \sigma_{ij}}{\partial x_j} = - \alpha^2\, u_i \quad \text{in } B\,, \tag{6.1}$$

$$\sigma_{ij} = a_{ijkh}\, e_{kh}(u)\,; \quad e_{kh}(u) = \frac{1}{2}\left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right), \tag{6.2}$$

$$u_i = 0 \quad \text{on } \Gamma', \tag{6.3}$$

$$\Delta\Phi = 0 \quad \text{in } \Omega, \tag{6.4}$$

$$\frac{\partial \Phi}{\partial x_3} = \frac{\alpha^2}{\varepsilon}\Phi \quad \text{on} \quad \{ x_3 = 0 \}, \tag{6.5}$$

$$\sigma_{ij}\, n_j = -\frac{\alpha^2}{\rho}\Phi n_i \quad \text{on } \Gamma, \tag{6.6}$$

$$\frac{\partial \Phi}{\partial n} = u.n \quad \text{on } \Gamma, \tag{6.7}$$

$$\lim_{x_3 \to -\infty} \frac{\partial \Phi}{\partial x_3} = 0, \tag{6.8}$$

Φ satisfies the radiation condition. (6.9)

If we make the changes resulting from the fact that ε appears here, not in the free surface equation, but in the free surface *condition*, the results of section 3 are still valid. The Green function (see Appendix) undergoes a few changes but has exactly the same dependence on the parameters α and ε as that mentioned in Section 3.

The limit problem ($\varepsilon \to 0$) corresponds to the vibrations of the coupled system when there is no gravity; its equations are (6.1)-(6.9) where (6.5) is replaced by :

$$\Phi = 0 \quad \text{on} \quad \{ x_3 = 0 \}.$$

It is a selfadjoint problem which has a countable set of eigenfrequencies (see theorem 4.6).

From the physical point of view, this perturbation corresponds to the introduction of a small gravity term represented by the parameter ε.

As far as the asymptotic behaviour of the scattering frequencies of this system is concerned, the results obtained and the conclusions drawn are consequently similar to those of Sections 4 and 5.

7. APPENDIX : THE GREEN FUNCTION

7.1. Algebraic expression

We give here two expressions of the Green function; more details, particularly about its derivation can be found in [8]. Other expressions of this function are given by John [13].

Let M and P be two points in $\mathbb{R}^3_-$ with respective coordinates (x_1^M, x_2^M, x_3^M), (x_1^P, x_2^P, x_3^P), and M_ε the symmetric of M with respect to the plane $\{x_3 = 1/\varepsilon\}$ with coordinates $(x_1^M, x_2^M, -x_3^M + 2/\varepsilon)$ With the notations of Section 3 1, the Green function may be written

$$G(M, P, \alpha, \varepsilon) = G_0(M, P) + G_0(M_\varepsilon, P) + H(M, P, \alpha, \varepsilon), \qquad (7\ 1)$$

with the following expressions for $H(\alpha, \varepsilon)$

— if $\alpha \in \mathbb{R}_+^*$

$$H(M, P, \alpha, \varepsilon) = -\alpha^2 Pv\left[\int_0^{+\infty} \frac{\exp\left[2\pi t\left(x_3^M + x_3^P - \frac{2}{\varepsilon}\right)\right] J_0(2\pi tR)}{2\pi t - \alpha^2}\, dt\right] - \\ - \frac{i\alpha^2}{2}\exp\left[\alpha^2\left(x_3^M + x_3^P - \frac{2}{\varepsilon}\right)\right] J_0(\alpha^2 R), \qquad (7\ 2)$$

where $R = [(x_1^M - x_1^P)^2 + (x_2^M - x_2^P)^2]^{1/2}$ and Pv is Cauchy's principal value of the integral

$H(M, P, \alpha, \varepsilon)$ can also be written in the following way, which will be very useful in the sequel to derive the dependence of $G(\alpha, \varepsilon)$ on ε

$$H(M, P, \alpha, \varepsilon) = -\frac{\alpha^2}{\pi^2}\,\mathrm{Re}\int_0^{\pi/2} e^{\alpha^2 q} E_1(\alpha^2 q)\, d\theta + \\ + \frac{\alpha^2}{2}\exp\left[\alpha^2\left(x_3^M + x_3^P - \frac{2}{\varepsilon}\right)\right][H_0(\alpha^2 R) - iJ_0(\alpha^2 R)], \qquad (7\ 3)$$

with

$$q = x_3^M + x_3^P - \frac{2}{\varepsilon} + iR\cos\theta$$

H_0 is the Struve function of order zero, J_0 the Bessel function of order zero and E_1 is the complex exponential integral function

— if $\mathrm{Im}\,\alpha > 0$

$$H(M, P, \alpha, \varepsilon) = -\alpha^2\int_0^{+\infty} \frac{\exp\left[2\pi t\left(x_3^M + x_3^P - \frac{2}{\varepsilon}\right)\right] J_0(2\pi tR)}{2\pi t - \alpha^2}\, dt \qquad (7\ 4)$$

This expression provides the continuity of $G(\alpha, \varepsilon)$ and of its partial derivatives with respect to the coordinates of M or P, up to the second order, at each point $\alpha \in \mathbb{R}_+^*$

More precisely let (λ_m) be a sequence of complex numbers, with strictly positive imaginary part, tending to α; then if M and P respectively belong to K_M and K_P, compact sets of $\mathbb{R}^3_-$ which do not intersect :

$$\| \partial^\nu G(\lambda_m, \varepsilon) - \partial^\nu G(\alpha, \varepsilon) \|_{L^\infty(K_M \times K_P)} \to 0 ,$$

when $m \to +\infty$, if $|\nu| \leqslant 2$.

The proof of this continuity property is based on the well-known equality :

$$\lim_{\varepsilon \to 0^+} \frac{1}{x + i\varepsilon} = Pv\frac{1}{x} - i\pi\delta \quad \text{in} \quad \mathscr{D}'(\mathbb{R}) .$$

7.2. Analytic continuation on α

In the same way, the definition of $G(\alpha, \varepsilon)$ for $\operatorname{Im} \alpha < 0$ is chosen to provide its continuity with respect to α, at each point of R^*_+. To that end, we notice that :

$$\lim_{\varepsilon \to 0^-} \frac{1}{x + i\varepsilon} = Pv\frac{1}{x} + i\pi\delta \quad \text{in} \quad \mathscr{D}'(\mathbb{R}) ,$$

which leads to define $H(\alpha, \varepsilon)$ in (7.1) by

$$H(M, P, \alpha, \varepsilon) = -\alpha^2 \int_0^{+\infty} \frac{\exp\left[2\pi t\left(x_3^M + x_3^P - \frac{2}{\varepsilon}\right)\right] J_0(2\pi t R)}{2\pi t - \alpha^2} dt - \\ - i\alpha^2 \exp\left[\alpha^2\left(x_3^M + x_3^P - \frac{2}{\varepsilon}\right)\right] J_0(\alpha^2 R) . \quad (7.5)$$

Rather simple computations based upon the differentiation under the integral symbol, with respect to the parameter α, of the different expressions of $H(M, P, \alpha, \varepsilon)$, yield the following result : the Green function $G(\alpha, \varepsilon)$ defined for $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$ depends holomorphically on α, and shows a cut along $\mathbb{R}_-$. More precisely, for $\alpha \in \mathbb{R}^*_-$:

$$\text{if} \quad (\lambda_m) \to \alpha , \quad \text{with} \quad \operatorname{Im} \lambda_m > 0 ,$$
$$(\mu_p) \to \alpha , \quad \text{with} \quad \operatorname{Im} \mu_p < 0 ,$$

and if we set :

$$D(M, P, \alpha, \varepsilon) = 2i\alpha^2 \exp\left[\alpha^2\left(x_3^M + x_3^P - \frac{2}{\varepsilon}\right)\right] J_0(\alpha^2 R) ,$$

$$\| G(\lambda_m, \varepsilon) - G(\mu_p, \varepsilon) - D(\alpha, \varepsilon) \|_{L^\infty(K_M \times K_P)} \to 0$$

as $m, p \to +\infty$.

Remark : This cut still remains if we choose $\beta = \alpha^2$ as a new variable, it seems therefore to be a specific property of the Green function of this problem (one may notice the analogy with the two dimensional Helmholtz problem, where the Green function $\frac{i}{2} H_0^{(1)}(kR)$ also shows a cut along $\mathbb{R}_-$).

7.3. Asymptotic expansion with respect to ε

We expand separately each term in (7.1).

$$G_0(M_\varepsilon, P) = -\frac{1}{4\pi}\left[(x_1^M - x_1^P)^2 + (x_2^M - x_2^P)^2 + \left(x_3^M + x_3^P - \frac{2}{\varepsilon}\right)^2\right]^{-1/2}$$

has an obvious representation for ε sufficiently small as a power series in ε. We get for the first terms :

$$G_0(M_\varepsilon, P) = -\frac{\varepsilon}{8\pi} - \frac{\varepsilon^2}{16\pi}(x_3^P + x_3^M) + \cdots \tag{7.6}$$

In order to expand $H(M, P, \alpha, \varepsilon)$, we must distinguish the three cases :

$$\alpha \in \mathbb{R}_+^* , \quad \operatorname{Im} \alpha > 0 , \quad \operatorname{Im} \alpha < 0 .$$

$$-\alpha \in \mathbb{R}_+^* .$$

We use formula (7.3). An expansion of the integral is obtained by means of the expansion of $e^z E_1(z)$ in the neighbourhood of $z = \infty$ (see e.g. [14]), in which we replace z by $\alpha^2 q$. We verify that this expansion is valid, uniformly for $\theta \in [0, \pi/2]$ which allows us to perform the integration term by term and get :

$$I = -\frac{\alpha^2}{\pi^2}\int_0^{\pi/2} e^{\alpha^2 q} E_1(\alpha^2 q)\, d\theta =$$

$$= \frac{\alpha^2}{\pi^2}\sum_{k \geq 1}\sum_{p \geq 0}\frac{\varepsilon^{k+p}}{2^{k+p}\alpha^{2k}} k \frac{(k+p-1)!}{p!} \operatorname{Re}\int_0^{\pi/2}(x_3^M + x_3^P + iR\cos\theta)^p\, d\theta .$$

For a given power of ε, we have a finite sum of terms in k and p. The dependence of the coefficients on α is polynomial in $1/\alpha$, so that they are holomorphic on α, $\alpha \in \mathbb{C} \setminus \mathbb{R}_-$. The dependence on the coordinates x_j^P, x_k^M is polynomial too ; it follows that if α belongs to a compact set of $\mathbb{C} \setminus \mathbb{R}_-$, if M and P belong to compact sets of $\mathbb{R}^3_-$, then the remainder term of order p is smaller than $C\varepsilon^{p+1}$, where C only depends on the compact sets involved. This remark is also valid for the expansion (7.6). We get for the first terms :

$$I = \frac{\varepsilon}{4\pi} + \frac{1}{4\pi}\left(\frac{1}{\alpha^2} + \frac{(x_3^M + x_3^P)}{2}\right)\varepsilon^2 + \cdots \tag{7.7}$$

The second term in (7.3) contains the multiplicative coefficient $e^{-2\alpha^2,\varepsilon}$, and is negligible in front of any power of ε ; therefore, whatever large the order of the expansion is, this second term always appears in the remainder. Theorem 5.6 is based on that particular feature of the Green function.

Adding up the expansions of $G_0(M_\varepsilon, P)$ and I, we obtain the one of $G(\alpha, \varepsilon)$; when α is real, (7.6) and (7.7) give for the first terms :

$$G(\alpha, \varepsilon) = G_0(M, P) + \frac{\varepsilon}{8\pi} + \frac{\varepsilon^2}{4\pi}\left(\frac{x_3^M + x_3^P}{4} + \frac{1}{\alpha^2}\right) + \cdots$$

$$-\operatorname{Im}\alpha > 0 \quad \text{or} \quad \operatorname{Im}\alpha < 0\,.$$

We are only left with the expansion of the integral appearing in formula (7.4), because the complementary term in (7.5) contains the multiplicative coefficient $e^{-2\alpha^2\varepsilon}$. Simple calculations show that the expansion obtained is the same as that obtained for $\alpha \in \mathbb{R}^*_+$.

Remark : The Green function of Section 6 is the elementary solution of the Laplacian in $\mathbb{R}^3_-$, satisfying the free surface condition (6.5) and the radiation condition. Its expression is given by (7.1) where M_ε is replaced by M', the symmetric of M with respect to the plane $\{ x_3 = 0 \}$, with coordinates $(x_1^M, x_2^M, -x_3^M)$. Consequently, $x_3^M + x_3^P - 2/\varepsilon$ is replaced by $x_3^P + x_3^M$ and α^2 is replaced by α^2/ε.

Acknowledgments : The author wishes to express her gratitude to professor E. Sanchez-Palencia for helpful discussions and comments about this paper.

REFERENCES

[1] S. CERNEAU et E. SANCHEZ-PALENCIA, *Sur les vibrations libres des corps élastiques plongés dans des fluides,* J. Mecan. 15, 1976, pp. 399-425.

[2] R. OHAYON and E. SANCHEZ-PALENCIA, *On the vibration problem for an elastic body surrounded by a slightly compressible fluid,* R.A.I.R.O. Anal. Num. 17, 1983, pp. 311-326.

[3] P. LAX and R. PHILLIPS, *Scattering theory,* Acad. Press, New York, 1967.

[4] J. RALSTON, *Scattering theory,* Cours de l'école d'été d'analyse numérique INRIA, Sept. 1981.

[5] D. EUVRARD, A. JAMI, C. MORICE and Y. OUSSET, *Calcul numérique des oscillations engendrées par la houle,* J. Mecan. 16, 1977, pp. 289-326.

[6] MAZ'JA, Travaux du séminaire Sobolev, Novosibirsk, 1977

[7] J THOMAS BEALE, *Eigenfunctions expansions for objects floating in an open sea,* Comm Pure Appl Math 30, 1977, pp 283-313.

[8] M. LENOIR and D. MARTIN, *An application of the principle of limiting absorption to the motion of floating bodies,* J. Math. Anal. Appl. 79, 1981, pp. 370-383.

[9] M. LENOIR and A. JAMI, *A variationnal formulation for exterior problems in linear hydrodynamics,* Comp. Meth. Appl. Mech Eng. 16, 1978, pp. 341-359.

[10] T. KATO, *Perturbation theory for linear operators,* Springer, Berlin, 1966

[11] J. C. NEDELEC, *Approximation des équations intégrales en mécanique et en physique,* Cours de l'école d'été d'analyse numérique EDF-IRIA-CEA, June 1977.

[12] W. RUDIN, *Real and Complex Analysis,* McGraw-Hill Book Company, New York, 1966

[13] F. JOHN, *On the motions of floating bodies,* II, Comm Pure Appl. Math 3, 1950, pp 45-101.

[14] M. ABRAMOVITZ and I. STEGUN, *Handbook of mathematical functions,* Dover Publications, Inc., New York, 1970.