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NORM ESTIMATES FOR A MAXIMAL RIGHT INVERSE OF THE DIVERGENCE OPERATOR IN SPACES OF PIECEWISE POLYNOMIALS (*)

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Abstract. — *In this paper we study the divergence operator acting on continuous piecewise polynomials of degree* $p + 1$, $p \geqslant 3$, *on triangulations of a plane polygonal domain* Ω. *We give a characterization of the range of the divergence operator and the full details of a combinatorial verification of this. As the central result we show that for very general families of meshes it is possible to find a maximal right inverse for the divergence operator with a* $\mathscr{B}(L_2\,;H^1)$ *norm which is bounded independently of the mesh size. The norm of this right inverse grows at most algebraically with p, but it necessarily blows up as a certain measure of singularity of the meshes approaches* 0.

Résumé. — *Dans cet article nous étudions l'opérateur de divergence agissant sur des espaces de fonctions continues, polynômes par morceaux de degré* $p + 1$, $p \geqslant 3$, *sur des triangulations d'un domaine polygonal plan* Ω. *Nous donnons une caractérisation de l'image de l'opérateur divergence et tous les détails d'une preuve combinatoire de ce résultat. Notre résultat principal est de montrer, ensuite, que pour des familles très générales de triangulation, il est possible de trouver un inverse à droite maximal pour l'opérateur divergence, avec une norme* $\mathscr{B}(L_2\,;H^1)$ *bornée indépendamment de la longueur de la maille. La norme de cet inverse à droite croît au plus algébriquement en fonction de p, mais elle explose nécessairement lorsqu'une certaine mesure de la singularité du maillage tend vers* 0.

1. INTRODUCTION

Incompressibility constraints, such as constraints on the divergence of a velocity field or a displacement field, occur in many equations of physical interest, e.g. the Navier-Stokes equations or the equations of elasticity. When analyzing the stability of finite element approximations to these equations a central question concerns the behaviour of the divergence operator, or a

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discrete version thereof, on the corresponding spaces of piecewise polynomials (see for instance [4, 7, 17, 19]). It is well documented that continuous piecewise polynomials of low degree applied directly to the velocity- (or displacement)-formulation are often inadequate, due to the lack of a uniformly bounded right inverse for the divergence operator. This has led various authors to study non-conforming low order elements in connection with mixed formulations. The analysis in this paper points in another direction: our results imply that continuous piecewise polynomials of degree four or higher directly applied to the velocity- (or displacement)-formulation lead to optimal (uniform) convergence rates (for a discussion of this, see [12]).

The paper [18] contains a characterization of the range of the divergence operator on spaces of continuous piecewise polynomials of degree $p + 1$, $p \geqslant 3$, on an arbitrary triangulation (Theorem 2.1 and Remark 2.1); it also gives a proof of the fact that on a fixed triangulation it is always possible to construct a maximal right inverse for the divergence operator, the norm of which grows at most algebraically with p (Theorem 2.1). These results were used to prove that the so-called p-version of the finite element method, when applied directly to the displacement formulation of plane strain elasticity, converges at « almost » optimal rate independently of the value of Poisson's ratio ([19]).

The analysis presented in this paper extends the results of [18] in a rather surprising way — it shows that the aforementioned right inverse ($p \geqslant 3$) has a $\mathscr{B}(L_2, H^1)$ operator norm which is bounded independently of the mesh size of the triangulation. This uniform bound can only hold provided a certain measure of singularity of the meshes is bounded away from zero (*cf.* Exemple 3.1). Available numerical experiments (*cf.* [16]) and recent theoretical results (*cf.* [12]) show that a similar bound does not exist for $p < 3$.

For reasons of exposition we have chosen to express our main result in terms of a bound for the norm of a maximal right inverse for the divergence operator. It is easy to see (*cf.* Section 5) that this is equivalent to a uniform, positive lower bound for the expression

$$\inf_{\phi} \sup_{\underline{V}} \int_{\Omega} \nabla . \underline{V} \phi \, d\underline{x} / | \underline{V} |_{H^1} \| \phi \|_{L_2}$$

as studied by other authors (here $\underline{V}$ varies over the space of piecewise polynomials and ϕ varies over the divergence of this space).

The organization of this paper is as follows: in Section 2 we introduce the necessary notation concerning the triangulations and the polynomial subspaces. It should be emphasized that our triangulations are quite general and only restricted by the assumption of quasiuniformity. Section 3 independently characterizes the range of the divergence operator acting on continuous piecewise polynomials of degree $p + 1$, $p \geqslant 3$. Combinatorial proofs are carried out both with and without boundary conditions; in the latter case the argument is identical to one found in [18] and depends crucially on the formula for the dimension of C^1 piecewise polynomials proven in [10]; in the first case we have to establish a similar formula for C^1 piecewise polynomials that vanish to second order on the boundary (this is done in Section 6). Sections 4 and 5 contain the proof of the main theorem, the existence of a uniformly bounded maximal right inverse. The analysis relies heavily on [18], but an important new element is the localization procedure formulated in Lemmas 4.3, 5.1 and 5.2. The idea behind Lemmas 5.1 and 5.2 is in many ways similar to that underlying the macro-element technique and the corresponding local test for stability found in [3] or [14]. Much of the rest of the proof of the main theorem consists of verifying that the constants in various of the estimates found in [18] scale appropriately with the mesh size.

The attention in this paper is restricted to plane domains; it should be interesting to see if a similar analysis could be carried out in $\mathbb{R}^3$.

2. NOTATION

Throughout this paper Ω denotes a bounded polygonal domain in $\mathbb{R}^2$. $\Sigma_h = \{ \mathcal{T}_i^h \}_{i=1}^{N(h)}$, $0 < h \leqslant 1$, is a family of triangulations of Ω, parametrized by mesh size h. To be more precise : the $\mathcal{T}_i^h$, $1 \leqslant i \leqslant N(h)$, for fixed h, are disjoint triangles with

$$\operatorname{diam} \mathcal{T}_i^h \leqslant h$$

and

$$\bigcup_{i=1}^{N(h)} \overline{\mathcal{T}_i^h} = \overline{\Omega}\,.$$

An edge of a triangle of Σ_h is called an internal edge of Σ_h if its interior lies in Ω (not on $\partial\Omega$). We assume that no vertex of a triangle of Σ_h falls in the interior of an internal edge of Σ_h. This does not prevent boundary edges from having vertices in their interior (as in *fig.* 2 and *fig.* 3). Furthermore we assume that the family Σ_h, $0 < h \leqslant 1$, is *quasiuniform* in the sense that

$$\rho_0\, h \leqslant \rho(\mathcal{T}) \qquad \forall \mathcal{T} \in \Sigma_h\,, \quad 0 < h \leqslant 1\,, \tag{2.1}$$

where $\rho(\mathcal{T})$ denotes the supremum of diameters of discs contained in $\mathcal{T}$, and $0 < \rho_0$. In the rest of this section and all of the next we shall, to simplify notation, omit the subscript h when referring to a fixed triangulation.

If Σ' is an arbitrary collection of triangles from Σ, we then define the corresponding polygonal domain (*)

$$\Omega(\Sigma') = \text{interior}\left(\bigcup_{\mathcal{T}\in\Sigma} \overline{\mathcal{T}} \cap \Omega\right), \tag{2 2}$$

with this notation $\Omega(\Sigma) = \Omega$ For any integer $p \geqslant 0$ and $r = 0$ or 1

$$\mathscr{P}^{[p]\,r}(\Sigma')$$

denotes the set of functions in $C^r(\Omega(\Sigma'))$ that are given by a polynomial of degree $\leqslant p$ on each of the triangles of Σ'

An internal edge of Σ' is an edge whose interior lies in $\Omega(\Sigma')$ (not on $\partial\Omega(\Sigma')$) An internal vertex of Σ' is a vertex that lies in $\Omega(\Sigma')$ (not on $\partial\Omega(\Sigma')$) We shall say that an internal vertex (of Σ') is singular if the edges meeting at this vertex fall on two straight lines (*cf* [10])

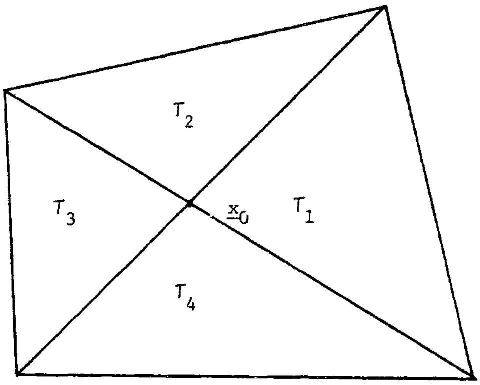

Figure 1. — Singular internal vertex $\underline{x}_0$.

Following [18] we introduce, for $p \geqslant 0$, the space

$$\mathscr{P}^{[p]\,-1}(\Sigma')$$

of functions, ϕ, which are given by a polynomial of degree $\leqslant p$ on each individual triangle (no continuity requirements) and which have the property that

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(*) « $\cap\Omega$ » in the definition of $\Omega(\Sigma)$ matters only when interior $(\overline{\Omega}) \neq \Omega$ e g when Ω is a slit domain

R.1 : at any singular internal vertex of Σ', $\underline{x}_0$,

$$\sum_{i=1}^{4} (-1)^i \phi_i(\underline{x}_0) = 0$$

where $\phi_i(\underline{x}_0) = \phi|_{\mathcal{T}_i}(\underline{x}_0)$ and $\mathcal{T}_1, \ldots, \mathcal{T}_4$ are the triangles meeting at $\underline{x}_0$, numbered consecutively, as shown in figure 1.

An explanation for the requirement (R.1) is most easily given by the following simple observation.

Proposition 2.1 : *For any $\Sigma' \subseteq \Sigma$ and any $p \geqslant 0$,*

$$\nabla.(\mathcal{P}^{[p+1],0}(\Sigma') \times \mathcal{P}^{[p+1],0}(\Sigma')) \subseteq \mathcal{P}^{[p],-1}(\Sigma').$$

The proof of this proposition consists of a straightforward calculation, the details of which are given in [12]. Special cases of this result have been used by other authors, e.g. Mercier [9] and Fix *et al.* [6].

When homogeneous Dirichlet boundary conditions are imposed, a new set of requirements become important. Let

$$\mathring{\mathcal{P}}^{[p],r}(\Sigma'), \quad p \geqslant r+1, \quad r = 0, 1$$

denote the subspace of $\mathcal{P}^{[p],r}(\Sigma')$ consisting of those functions that vanish

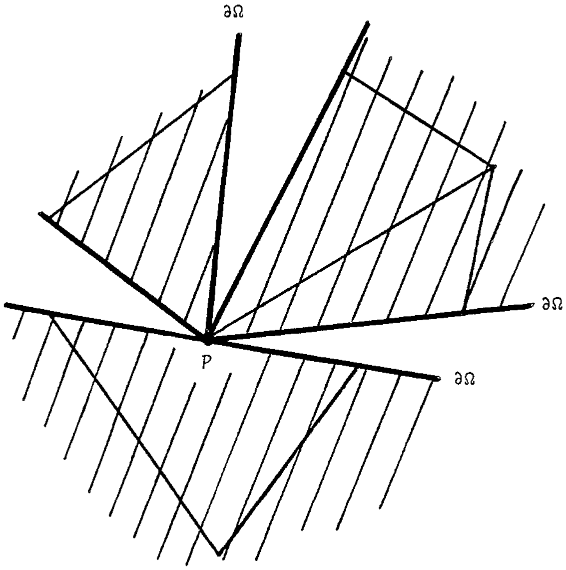

Figure 2. — Point at the boundary which is considered to be two different boundary vertices.

to $r+1$st order on $\partial\Omega(\Sigma')$; that is, functions in $\mathring{\mathscr{P}}^{[p],r}(\Sigma')$ are always zero on $\partial\Omega(\Sigma')$, and in addition, functions in $\mathring{\mathscr{P}}^{[p],1}(\Sigma')$ are required to have a vanishing normal derivative.

Remark 2.1 : In this paper we use the very natural convention, that a point on $\partial\Omega(\Sigma')$, which is a *vertex* for k different parts of $\partial\Omega(\Sigma')$, be considered k different boundary vertices. As an example there are two different boundary vertices at the point P in figure 2. A similar convention is applied to edges that lie on " internal " boundaries. These are considered two different boundary edges if they are common to two different triangles of Σ'. Note that, conforming with this convention, our definitions of piecewise polynomial spaces do not impose any continuity conditions at vertices or edges where the boundary intersects itself. □

The vertices (of Σ') that lie on $\partial\Omega(\Sigma')$ are called boundary vertices of Σ'. A vertex on $\partial\Omega(\Sigma')$ is called a singular boundary vertex of Σ' if all the edges of Σ' meeting at this vertex fall on two straight lines. There are four possible configurations for a singular boundary vertex, as shown in figure 3. (The fourth case in figure 3 differs slightly from that in [18] since it also illustrates the possibility of a boundary vertex lying in the interior of a boundary edge.)

We let

$$\tilde{\mathscr{P}}^{[p],-1}(\Sigma')\,,\quad p \geqslant 0$$

denote the subspace of $\mathscr{P}^{[p],-1}(\Sigma')$ consisting of functions, ϕ, which additionally satisfy the following two requirements, that

R.2 : At any singular boundary vertex of Σ', $\underline{x}_0$,

$$\sum_{i=1}^{k} (-1)^i \,\phi_i(\underline{x}_0) = 0\,,$$

where $\phi_i(\underline{x}_0) = \phi\,|_{\mathcal{T}_i}\,(\underline{x}_0)$, and $\mathcal{T}_1, \dots, \mathcal{T}_k$ are the triangles of Σ' meeting at $\underline{x}_0$ (k can be any number from 1 to 4, and the triangles are numbered consecutively as shown in figure 3).

R.3 : For any connected component of $\Omega(\Sigma')$, Ω'',

$$\int_{\Omega''} \phi \, d\underline{x} = 0\,.$$

It is a simple exercise to show that the following holds.

PROPOSITION 2.2 : *For any $\Sigma' \subseteq \Sigma$ and any $p \geqslant 0$,*

$$\nabla.\left(\mathring{\mathscr{P}}^{[p+1],0}(\Sigma') \times \mathring{\mathscr{P}}^{[p+1],0}(\Sigma')\right) \subseteq \tilde{\mathscr{P}}^{[p],-1}(\Sigma')\,.$$

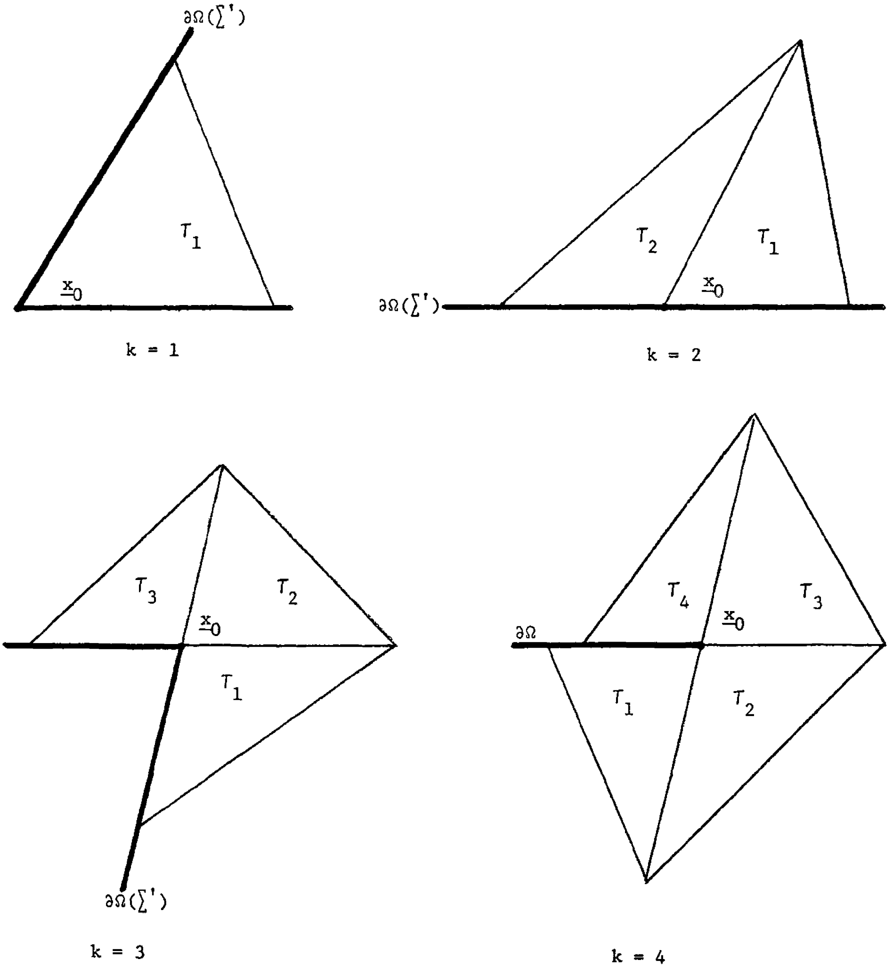

Figure 3. — The four types of singular boundary vertices of Σ'.

Our notation for Sobolev spaces is standard : if $\Omega' \subseteq \Omega$ is a polygonal (sub) domain and k is a nonnegative integer then $H^k(\Omega')$, denotes the set of functions with derivatives of order $\leqslant k$ in $L_2(\Omega')$; the corresponding norm is denoted $\| \cdot \|_{k,\Omega'}$. $\mathring{H}^k(\Omega')$ is the closure of $C_0^\infty(\Omega')$ in $H^k(\Omega')$.

3. CHARACTERIZING THE RANGE OF THE DIVERGENCE OPERATOR

For the analysis of finite element discretizations of equations with a divergence constraint it is important to have precise information about the range of the divergence operator on the finite dimensional subspaces. In general a uniform norm estimate of a right inverse is sufficient to guarantee stability, however, in order to estimate the convergence rate, the algebraic character of the range of the divergence operator has to be known. In the present situation it furthermore turns out that the characterization of the range automatically leads to a necessary condition for the existence of a uniformly bounded right inverse. For ease of notation we omit the subscript h when referring to a fixed triangulation. The following result was proven in [18].

PROPOSITION 3.1 : *For any $\Sigma' \subseteq \Sigma$ and any $p \geqslant 3$ the divergence operator maps*

$$\mathscr{P}^{[p+1],0}(\Sigma') \times \mathscr{P}^{[p+1],0}(\Sigma')$$

onto

$$\mathscr{P}^{[p],-1}(\Sigma').$$

As shown in [18] this result permits a simple combinatorial proof. We give the full details of the combinatorial argument below.

Consider first the case that $\Omega(\Sigma')$ is simply connected. The curl operator

$$\nabla \times \phi = \left(\frac{\partial}{\partial x_2}\phi, -\frac{\partial}{\partial x_1}\phi\right)$$

maps $\mathscr{P}^{[p+2],1}(\Sigma')$ onto the nullspace of the divergence operator

$$\mathscr{N}^{p+1}(\nabla .) \subseteq \mathscr{P}^{[p+1],0}(\Sigma') \times \mathscr{P}^{[p+1],0}(\Sigma').$$

In [10] it is shown, that for $p \geqslant 3$

$$\dim(\mathscr{P}^{[p+2],1}(\Sigma')) = \frac{1}{2}(p+3)(p+4)\,T - (2p+5)\,E_0 + 3\,V_0 + \sigma_0, \quad (3.1)$$

where T is the number of triangles, E_0 is the number of internal edges, V_0 is the number of internal vertices and σ_0 is the number of singular internal vertices, all of the triangulation $\Sigma' \subseteq \Sigma$. Since the nullspace of the curl operator consists of only the constants, it follows from Grassmann's dimension formula that

$$\dim(\mathscr{N}^{p+1}(\nabla .)) = \dim(\mathscr{P}^{[p+2],1}(\Sigma')) - 1. \quad (3.2)$$

If $R^p(\nabla .)$ denotes the range of the divergence operator acting on

$$\mathscr{P}^{[p+1],0}(\Sigma') \times \mathscr{P}^{[p+1],0}(\Sigma'),$$

then the same dimension formula gives that

$$\dim(R^p(\nabla.)) = \dim(\mathscr{P}^{[p+1],0}(\Sigma') \times \mathscr{P}^{[p+1],0}(\Sigma')) - \dim(\mathscr{N}^{p+1}(\nabla.)) . \quad (3.3)$$

The first term in the right hand side of (3.3) is easily found to be

$$(p-1)pT + 2pE + 2V, \quad (3.4)$$

where T is as before, E denotes the total number of edges and V the total number of vertices of the triangulation Σ'. Inserting (3.1), (3.2) and (3.4) into (3.3) one gets

$$\begin{aligned} \dim(R^p(\nabla.)) &= \frac{1}{2}(p^2 - 9p - 12)T + 2(p+1)(E+E_0) - \\ &\quad - 2E + 3E_0 + 2V - 3V_0 - \sigma_0 + 1 \\ &= \frac{1}{2}(p+3)pT + E - V - \sigma_0 + 1, \end{aligned} \quad (3.5)$$

with the second identity based entirely on the relations $V - V_0 = E - E_0$ and $E + E_0 = 3T$. Euler's formula states that

$$T - E + V = 1,$$

and in combination with (3.5) this gives

$$\dim(R^p(\nabla.)) = \frac{1}{2}(p+2)(p+1)T - \sigma_0 . \quad (3.6)$$

The right hand side of (3.6) is exactly the expression for the dimension of $\mathscr{P}^{[p],-1}(\Sigma')$. This observation together with Proposition 2.1 implies that

$$R^p(\nabla.) = \mathscr{P}^{[p],-1}(\Sigma') .$$

If $\Omega(\Sigma')$ is *not* simply connected then we extend any function in $\mathscr{P}^{[p],-1}(\Sigma')$ by piecewise linear functions onto triangles filling the holes of $\Omega(\Sigma')$. This can be done in such a way that the extension is still in $\mathscr{P}^{[p],-1}$ and we may now rely on the previous argument to ensure the existence of a field in $\mathscr{P}^{[p+1],0} \times \mathscr{P}^{[p+1],0}$ with the right divergence. (It has here implicitly been assumed that the holes of $\Omega(\Sigma')$ have boundaries that are not selfintersecting; selfintersecting boundaries can be dealt with by a perturbation argument.)

With homogeneous Dirichlet boundary conditions the corresponding result is :

PROPOSITION 3.2 : *For any $\Sigma' \subseteq \Sigma$ and any $p \geqslant 3$ the divergence operator maps*

$$\mathring{\mathscr{P}}^{[p+1],0}(\Sigma') \times \mathring{\mathscr{P}}^{[p+1],0}(\Sigma')$$

onto

$$\tilde{\mathscr{P}}^{[p],-1}(\Sigma') .$$

The analysis given in [18] verifies this for p sufficiently large by an approximation argument. At the end of Section 4 we show how the result is obtained for general $\Sigma' \subseteq \Sigma$ and $p \geqslant 3$. For completeness we briefly outline the key ingredients of a combinatorial argument. Suppose that $\Omega(\Sigma')$ is simply connected ; in that case

$$\dim (\mathring{\mathscr{P}}^{[p+2],1}(\Sigma')) = \frac{1}{2}p(p-5)\,T + (2p-1)\,E_0 + 3\,V_0 + \sigma\,, \quad (3.7)$$

where σ denotes the total number of singular vertices of the triangulation Σ'. The formula (3.7) is verified in Section 6 by a method based on [10]. We furthermore know that

$$\begin{aligned} &\dim (\mathring{\mathscr{P}}^{[p+1],0}(\Sigma') \times \mathring{\mathscr{P}}^{[p+1],0}(\Sigma')) = (p-1)\,pT + 2\,pE_0 + 2\,V_0 \\ &\dim (\tilde{\mathscr{P}}^{[p],-1}(\Sigma')) = \frac{1}{2}(p+2)\,(p+1)\,T - \sigma - 1\,. \end{aligned} \quad (3.8)$$

The formulae (3.7) and (3.8) in combination with an argument like the preceding may now be applied to prove Proposition 3.2 whenever $\Omega(\Sigma')$ is simply connected. Non-simply connected domains may be treated by a slight variation of this argument (*cf.* Remark 6.1).

Remark 3.1 : Propositions 3.1 and 3.2 remain valid also for $p = 2$, 1 or 0 on any Σ' such that $\Omega(\Sigma')$ is simply connected and the formula (3.1), respectively (3.7), holds. The formula (3.1), which was conjectured by Strang [15], has been verified for certain triangulations Σ' (in decreasing generality, as p decreases) by Morgan and Scott [11]. The formula (3.7), however, fails on the most natural triangulations as soon as $p \leqslant 2$. For a more detailed discussion we refer to [12]. $\square$

The main goal in this paper is to verify the existence of a maximal right inverse for the divergence operator, the norm of which is bounded uniformly in the mesh size and grows at most algebrically with p. It turns out that our proof of this fact does not depend on Propositions 3.1 and 3.2, to the contrary, it provides independent proofs of these. However, these propositions demonstrate the necessity of a certain non-degeneracy condition on the triangulations, if one wants to obtain a uniformly bounded right inverse.

Let $\underline{x}_0$ denote any *non-singular* vertex of Σ' and let θ_i, $1 \leqslant i \leqslant k$, be the angles of the triangles $\mathcal{T}_i$, $1 \leqslant i \leqslant k$, meeting at $\underline{x}_0$ (the triangles are numbered consecutively as before). If $\underline{x}_0$ is an internal vertex we define

$$R(\underline{x}_0) = \max \{ \mid \theta_i + \theta_j - \pi \mid : 1 \leqslant i, j \leqslant k \quad \text{and} \quad i - j = 1 \bmod k \} ;$$

if $\underline{x}_0$ is a boundary vertex, $R(\underline{x}_0)$ is defined in the same way, only deleting the term $\bmod k$; $R(\underline{x}_0)$ thus measures how close $\underline{x}_0$ is to being singular. We furthermore set

$$R(\Sigma') = \min \{ R(\underline{x}_0) : \underline{x}_0 \text{ is a non-singular internal vertex of } \Sigma' \} \quad (3.9)$$

and

$$\overset{\circ}{R}(\Sigma') = \min \{ R(\underline{x}_0) : \underline{x}_0 \text{ is a non-singular vertex of } \Sigma' \} . \quad (3.10)$$

Example 3.1 : Let Σ^δ be the simple triangulation shown in figure 4, with $R(\Sigma^\delta) = \delta$. Let ϕ_δ, δ small, be the piecewise constant that is given by

$$\phi_\delta(\underline{x}) = \begin{cases} 1 \text{ in } \mathcal{T}_4 \\ 0 \text{ otherwise} . \end{cases}$$

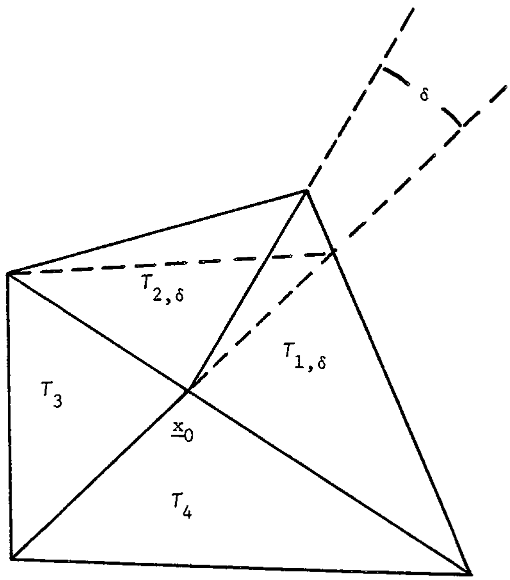

Figure 4. — Σ^δ.

Proposition 3.1 ensures that for $\delta > 0$ there exists

$$\underline{V}_\delta \in \mathscr{P}^{[4],0}(\Sigma^\delta) \times \mathscr{P}^{[4],0}(\Sigma^\delta)$$

with
$$\nabla . \underline{V}_\delta = \phi_\delta ,$$

but $\| \underline{V}_\delta \|_{1,\Omega^\delta}$ *cannot* stay bounded as $\delta \to 0$. If $\| \underline{V}_\delta \|_{1,\Omega^\delta} \leqslant C$, uniformly as $\delta \to 0$, then we could extract a weakly convergent subsequence, which would converge to a field

$$\underline{V}_0 \in \mathscr{P}^{[4],0}(\Sigma^0) \times \mathscr{P}^{[4],0}(\Sigma^0) ,$$

satisfying

$$\nabla . \underline{V}_0 = \begin{cases} 1 \text{ in } \mathscr{T}_4 \\ 0 \text{ otherwise} . \end{cases}$$

This is a contradiction, since $\underline{x}_0$ is a singular internal vertex for Σ^0. □

The previous example shows that it is in general necessary to have (3.9) (or (3.10)) bounded from below in order to establish uniform bounds for the $\mathscr{B}(L_2 ; H^1)$ norm of a maximal right inverse for the divergence operator.

4. LOCAL CONSTRUCTION OF A RIGHT INVERSE FOR ∇.

The first in a series of lemmas is an extension of Lemma 2.6 in [18].

LEMMA 4.1 : *Assume that*

$$R(\Sigma_h) \geqslant \delta > 0 ,$$

where $R(\Sigma_h)$ is the measure of singularity introduced in (3.9), *and δ is independent of h. Let Σ_h' denote any collection of triangles from Σ_h, and let ϕ be any element of $\mathscr{P}^{[p],-1}(\Sigma_h')$. There exists $\underline{V} \in \mathscr{P}^{[3],0}(\Sigma_h') \times \mathscr{P}^{[3],0}(\Sigma_h')$ such that*

$$\phi - \nabla . \underline{V} = 0 \quad \textit{at all vertices of } \Sigma_h' , \tag{4.1a}$$

and

$$\| \underline{V} \|_{1,\Omega(\Sigma_h')} \leqslant C(p+1)^K \| \phi \|_{0,\Omega(\Sigma_h')} \tag{4.1b}$$

with constants C and K that are independent of Σ_h', h, p and ϕ.

Proof : Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two adjacent unit sized triangles, as shown in figure 5.

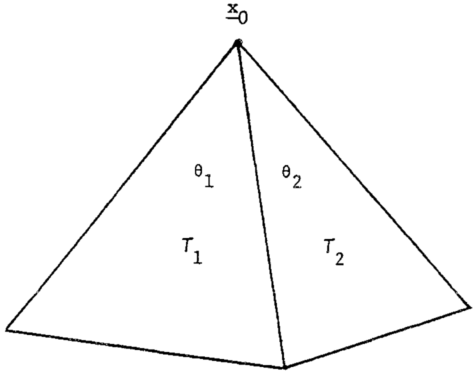

Figure 5. — Two adjacent triangles.

If a is any constant, then it is possible to find a continuous piecewise cubic field $\underline{U}$ on $\overline{\mathcal{T}}_1 \cup \overline{\mathcal{T}}_2$, satisfying

$$\begin{aligned} \nabla . \underline{U} &= a \quad \text{at } \underline{x}_0 \\ \nabla . \underline{U} &= 0 \quad \text{at all other vertices, and} \\ \underline{U} &= 0 \quad \text{on } \partial(\overline{\mathcal{T}}_1 \cup \overline{\mathcal{T}}_2) . \end{aligned} \tag{4.2}$$

From the construction in [18] it follows that

$$\| \underline{U} \|_{1, \overline{\mathcal{T}}_1 \cup \overline{\mathcal{T}}_2} \leqslant C \mid a \mid$$

where C only depends on the minimal angle of $\mathcal{T}_i$, $i = 1, 2$. If furthermore $\theta_1 + \theta_2 \neq \pi$ and a_1, a_2 are any two constants, then one can find a continuous piecewise cubic field $\underline{U}'$ on $\overline{\mathcal{T}}_1 \cup \overline{\mathcal{T}}_2$, such that

$$\begin{aligned} \nabla . \underline{U}' |_{\mathcal{T}_i}(\underline{x}_0) &= a_i \quad \text{for} \quad i = 1, 2 , \\ \nabla . \underline{U}' &= 0 \quad \text{at all other vertices} \\ \underline{U}' &= 0 \quad \text{on} \quad \partial(\overline{\mathcal{T}}_1 \cup \overline{\mathcal{T}}_2) \end{aligned} \tag{4.3}$$

and

$$\| \underline{U}' \|_{1, \overline{\mathcal{T}}_1 \cup \overline{\mathcal{T}}_2} \leqslant C(| a_1 | + | a_2 |) ,$$

where C depends on the minimal angle of $\mathcal{T}_i$, $i = 1, 2$, and $| \theta_1 + \theta_2 - \pi |$.

Let $\underline{x}_0$ be a non-singular internal vertex with N corresponding triangles of unit size, and let a_i, $1 \leqslant i \leqslant N$, be N arbitrary constants. Using (4.2), (4.3) and the same argument as in [18] we obtain a continuous piecewise cubic field $\underline{W}$ with

$$\begin{aligned} \nabla . \underline{W} \,|_{\mathcal{T}_i}(\underline{x}_0) &= a_i \quad \text{for} \quad 1 \leqslant i \leqslant N\,, \\ \nabla . \underline{W} &= 0 \quad \text{at all other vertices} \\ \underline{W} &= 0 \quad \text{on} \quad \partial\left(\bigcup_{i=1}^{N} \overline{\mathcal{T}}_i \right). \end{aligned} \tag{4.4}$$

This field can be estimated by

$$\| \underline{W} \|_{1, \cup \overline{\mathcal{T}}_i} \leqslant C \sum_{i=1}^{N} | a_i |\,, \tag{4.5}$$

where C only depends on the minimal angle of $\mathcal{T}_i$, $1 \leqslant i \leqslant N$, and $R(\underline{x}_0)$ (C blows up when either of these become small). At any singular internal vertex we may similarly find a continuous piecewise cubic field satisfying (4.4), (4.5) provided $\sum_{i=1}^{4} (-1)^i a_i = 0$. The constant C here depends only on the minimal angle. Since we are not imposing any boundary conditions (4.4) (with $\partial(\cup \overline{\mathcal{T}}_i)$ replaced by $\partial(\cup \overline{\mathcal{T}}_i) \cap \Omega$) and (4.5) (with $\cup \overline{\mathcal{T}}_i$ replaced by $\cup \overline{\mathcal{T}}_i \cap \Omega$) can also be satisfied for any boundary vertex and any set of constants a_i, with a constant C that only depends on the minimal angle.

By rescaling we see that all these versions of (4.4), (4.5) remain valid with a constant that is Ch, where h is the size of the triangles. For each vertex $\underline{x}_0$ of Σ'_h we select a_i, $1 \leqslant i \leqslant N$, to be $\phi \,|_{\mathcal{T}_i^h}(\underline{x}_0)$; the previous construction then leads to

$$\| \underline{W} \|_{1, \cup \overline{\mathcal{T}}_i^h \cap \Omega} \leqslant C(p+1)^K \, \| \phi \|_{0, \cup \overline{\mathcal{T}}_i^h \cap \Omega}$$

for $K > 2$ (*cf.* [18]). Adding the individual $\underline{W}$'s we arrive at a field $\underline{V}$, satisfying (4.1*a*) and (4.1*b*). The constant C is independent of Σ'_h since both $R(\Sigma_h)$ and the minimal angle are bounded away from 0 (the latter because of the quasiuniformity assumption). □

Remark 4.1 : Assume that $R(\Sigma_h) \geqslant \delta > 0$ and that $\phi \in \mathcal{P}^{[p], -1}(\Sigma'_h)$ with $\phi = 0$ at the boundary vertices of Σ'_h. Then it is possible to find

$$\underline{V} \in \mathring{\mathcal{P}}^{[3], 0}(\Sigma'_h) \times \mathring{\mathcal{P}}^{[3], 0}(\Sigma'_h)$$

such that (4.1*a*-*b*) hold. It is crucial that $\phi = 0$ at the vertices on $\partial\Omega(\Sigma'_h)$ provided we want to maintain $R(\Sigma_h)$ as the measure of singularity. If we make the alternate assumption that $\mathring{R}(\Sigma'_h) \geqslant \delta > 0$ then it is possible to find

$$\underline{V} \in \mathring{\mathcal{P}}^{[3], 0}(\Sigma'_h) \times \mathring{\mathcal{P}}^{[3], 0}(\Sigma'_h)$$

satisfying (4.1*a*-*b*) for any $\phi \in \tilde{\mathcal{P}}^{[p],-1}(\Sigma_h')$. These slight variations of Lemma 4.1 follow by a proof very similar to the previous. □

Let $\mathcal{T}_1^h$, $\mathcal{T}_2^h$ be two arbitrary triangles of Σ_h, with a common edge (as in *fig*. 6).

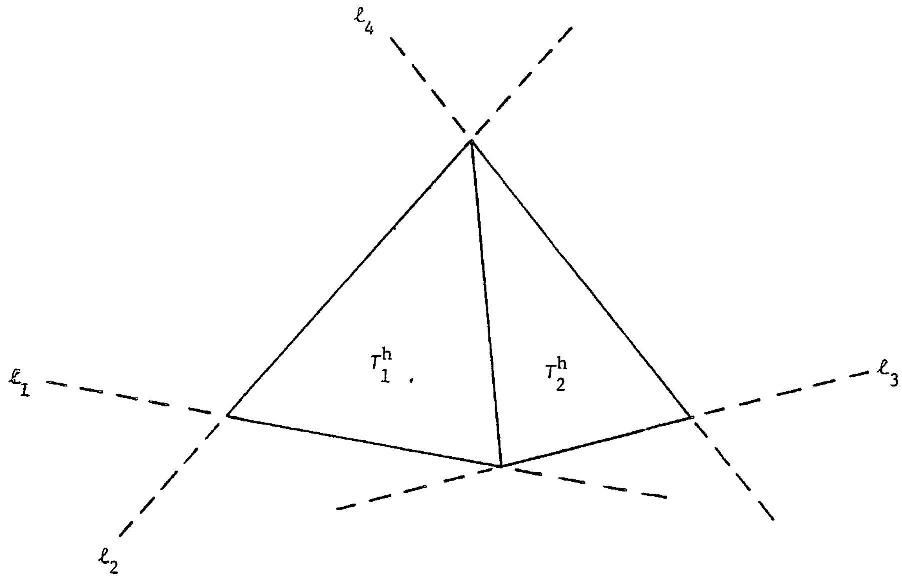

Figure 6.

Denote by

$$l_i(\underline{x}) = \alpha_i x_1 + \beta_i x_2 + \gamma_i = 0, \quad 1 \leqslant i \leqslant 4,$$

the four lines on which the remaining edges lie, and define

$$\psi(\underline{x}) = \begin{cases} l_1^2 \, l_2^2, & \underline{x} \in \mathcal{T}_1^h \\ c l_3^2 \, l_4^2, & \underline{x} \in \mathcal{T}_2^h, \end{cases}$$

where c is chosen such that ψ is continuous in $\overline{\mathcal{T}}_1^h \cup \overline{\mathcal{T}}_2^h$. Let $\underline{n}$ be a normal direction to the common edge, and introduce

$$\underline{W}(\underline{x}) = d\psi(\underline{x})\, \underline{n}.$$

Any such $\underline{W}$ satisfies

$$\int_{\mathcal{T}_1^h} \nabla . \underline{W} \, d\underline{x} = - \int_{\mathcal{T}_2^h} \nabla . \underline{W} \, d\underline{x},$$

and by choosing $d \neq 0$ appropriately we thus obtain

LEMMA 4.2 : *Let $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ be two triangles of Σ_h with a common edge. It is possible to find a continuous field $\underline{W}$ such that*

$$\underline{W} \text{ is given by polynomials of degree } \leqslant 4 \text{ on each of the triangles } \mathcal{T}_i^h, \text{ and } \underline{W} = 0 \text{ on } \partial(\overline{\mathcal{T}}_1^h \cup \overline{\mathcal{T}}_2^h)\,, \tag{4.6a}$$

$$\nabla.\underline{W} = 0 \quad \text{at all vertices of } \mathcal{T}_i^h, \quad i = 1, 2\,, \tag{4.6b}$$

$$-\int_{\mathcal{T}_1^h} \nabla.\underline{W}\, d\underline{x} = \int_{\mathcal{T}_2^h} \nabla.\underline{W}\, d\underline{x} = 1\,, \tag{4.6c}$$

$$\| \underline{W} \|_{1,\overline{\mathcal{T}}_1^h \cup \overline{\mathcal{T}}_2^h} \leqslant D h^{-1} \quad \text{where } D \text{ is independent of } \mathcal{T}_i^h \text{ and } h\,. \tag{4.6d}$$

Note : In the estimate (4.6*d*) we have used the fact that the triangulation Σ_h satisfies a minimal angle condition due to the assumption of quasiuniformity.

DÉFINITION : *A collection of triangles $\Sigma_h' = \{ \mathcal{T}_i^h \}_{i=1}^l$ from Σ_h is called connected if the corresponding polygonal domain $\Omega(\Sigma_h') = \text{interior}\left(\bigcup_{i=1}^l \overline{\mathcal{T}}_i^h \cap \Omega \right)$ is connected.*

LEMMA 4.3 : *Let D be the same constant as in the previous lemma. For any connected collection of triangles $\Sigma_h' = \{ \mathcal{T}_i^h \}_{i=1}^l \subseteq \Sigma_h$ and any set of numbers $\{ b_i \}_{i=1}^l$, with*

$$\sum_{i=1}^{l} b_i = 0\,,$$

one can find $\underline{V} \in \overset{c}{\mathcal{P}}^{[4],0}(\Sigma_h') \times \overset{c}{\mathcal{P}}^{[4],0}(\Sigma_h')$ satisfying

$$\nabla.\underline{V} = 0 \quad \text{at all vertices of } \Sigma_h'\,, \tag{4.7a}$$

$$\int_{\mathcal{T}_i^h} \nabla.\underline{V}\, d\underline{x} = b_i\,, \quad 1 \leqslant i \leqslant l\,, \tag{4.7b}$$

and

$$\| \underline{V} \|_{1,\Omega(\Sigma_h')} \leqslant D l h^{-1} \sum_{i=1}^{l} | b_i |\,. \tag{4.7c}$$

Proof : For $l = 1$ the result follows trivially by choosing $\underline{V}$ identically zero. The proof proceeds by induction. Let $\Sigma_h' = \{ \mathcal{T}_i^h \}_{i=1}^l$ be a connected collection of triangles from Σ_h and let $\{ b_i \}_{i=1}^l$ be a set of numbers, with $\sum_{i=1}^{l} b_i = 0$, $l > 1$. Select $\mathcal{T} \in \Sigma_h'$ so that $\Sigma_h'' = \Sigma_h' \setminus \{ \mathcal{T} \}$ is connected (it is easy to see that this is always possible) ; to simplify notation we shall assume that the numbering of Σ'_h is such that $\mathcal{T} = \mathcal{T}^h_l$ and that $\mathcal{T}^h_l$ and $\mathcal{T}^h_{l-1}$ share a common edge. We define

$$\tilde{b}_i = \begin{cases} b_i, & 1 \leqslant i \leqslant l-2 \\ b_{l-1} + b_l, & i = l-1, \end{cases}$$

and use the induction hypothesis to construct

$$\underline{\tilde{V}} \in \overset{\circ}{\mathscr{P}}{}^{[4],0}(\Sigma''_h) \times \overset{\circ}{\mathscr{P}}{}^{[4],0}(\Sigma''_h)$$

with

$$\nabla . \underline{\tilde{V}} = 0 \quad \text{at all vertices of} \quad \Sigma''_h, \tag{4.8a}$$

$$\int_{\mathcal{T}^h_i} \nabla . \underline{\tilde{V}}\, d\underline{x} = \tilde{b}_i, \quad 1 \leqslant i \leqslant l-1, \tag{4.8b}$$

and

$$\| \underline{\tilde{V}} \|_{1,\Omega(\Sigma''_h)} \leqslant D(l-1)\, h^{-1} \sum_{i=1}^{l-1} | \tilde{b}_i |. \tag{4.8c}$$

Let $\underline{W}$ be the field constructed in Lemma 4.2 corresponding to the triangles $\mathcal{T}^h_{l-1}$ and $\mathcal{T}^h_l$, and set

$$\underline{V} = \underline{\tilde{V}} + b_l\, \underline{W}, \tag{4.9}$$

where $\underline{\tilde{V}}$ and $\underline{W}$ are interpreted to be zero outside Σ''_h and $\overline{\mathcal{T}}^h_{l-1} \cup \overline{\mathcal{T}}^h_l$ respectively. This $\underline{V}$ clearly satisfies (4.7*a*) and (4.7*b*); from (4.9), (4.8*c*) and (4.6*d*) it follows that

$$\| \underline{V} \|_{1,\Omega(\Sigma'_h)} \leqslant D(l-1)\, h^{-1} \sum_{i=1}^{l-1} | \tilde{b}_i | + D h^{-1} | b_l | \leqslant D l h^{-1} \sum_{i=1}^{l} | b_i |, \tag{4.10}$$

(remember that D at all points in this proof is the same constant as in Lemma 4.2). This completes the induction argument. $\square$

Remark 4.2 : Based on (4.7*c*) we immediately conclude that

$$\| \underline{V} \|_{1,\Omega(\Sigma'_h)} \leqslant D l^{3/2}\, h^{-1} \left(\sum_{i=1}^{l} | b_i |^2 \right)^{1/2}; \tag{4.7c'}$$

it is this estimate that shall be used later on. $\square$

A simple rescaling of Lemma 2.5 in [18] leads to the following.

LEMMA 4.4 : *Let $\mathcal{T}^h$ be a single triangle of Σ_h, and let ϕ^p be a polynomial of degree $\leqslant p$ such that $\phi^p = 0$ at the three vertices of $\mathcal{T}^h$ and* $\int_{\mathcal{T}^h} \phi^p\, d\underline{x} = 0$.

There exists a field $\underline{V}^{p+1}$ of polynomials of degree $\leqslant p+1$ satisfying

$$\underline{V}^{p+1} = 0 \quad on \quad \partial\mathcal{C}^h \tag{4.11a}$$

$$\nabla . \underline{V}^{p+1} = \phi^p \tag{4.11b}$$

$$\| \underline{V}^{p+1} \|_{1,\mathcal{C}^h} \leqslant C(p+1)^K \| \phi^p \|_{0,\mathcal{C}^h} \tag{4.11c}$$

with constants C and K that are independent of $\mathcal{C}^h$, h, p and ϕ^p.

In this lemma we have again used the fact that Σ_h satisfies a minimal angle condition.

Lemmas 4.1 through 4.4 give rise to a local construction of a right inverse for the divergence operator. We give the details of this construction with particular boundary conditions; this result shall prove useful in our proof of Theorem 5.1.

PROPOSITION 4.1 : *Assume that*

$$R(\Sigma_h) \geqslant \delta > 0 ,$$

where $R(\Sigma_h)$ is the measure of singularity introduced in (3.9), and δ is independent of h. Let $\Sigma'_h = \{ \mathcal{C}^h_i \}^l_{i=1}$ denote any collection of triangles from Σ_h, and let ϕ be any element of $\mathcal{P}^{[p],-1}(\Sigma'_h)$, $p \geqslant 3$, that vanishes at all boundary vertices of Σ'_h. Assume that

$$\int_{\Omega''} \phi \, d\underline{x} = 0 ,$$

for any connected component Ω'' of $\Omega(\Sigma'_h)$. There exists $\underline{V} \in \overset{c}{\mathcal{P}}^{[p+1],0}(\Sigma'_h) \times \overset{c}{\mathcal{P}}^{[p+1],0}(\Sigma'_h)$ such that

$$\nabla . \underline{V} = \phi \quad in \quad \Omega(\Sigma'_h) , \tag{4.12a}$$

and

$$\| \underline{V} \|_{1,\Omega(\Sigma'_h)} \leqslant C p^K \, l^{3/2} \, \| \phi \|_{0,\Omega(\Sigma'_h)} , \tag{4.12b}$$

with constants C and K that are independent of Σ'_h, h, p and ϕ.

Proof : We shall without loss of generality restrict our attention to the case that $\Omega(\Sigma'_h)$ has only one connected component. Lemma 4.1 in combination with Remark 4.1 shows how to construct $\underline{V}_1 \in \overset{c}{\mathcal{P}}^{[3],0}(\Sigma'_h) \times \overset{c}{\mathcal{P}}^{[3],0}(\Sigma'_h)$ with

$$\phi - \nabla . \underline{V}_1 = 0 \quad \text{at all vertices of } \Sigma'_h .$$

Lemma 4.3 applied with

$$b_i = \int_{\mathcal{T}_i^h} (\phi - \nabla.\underline{V}_1)\, d\underline{x}\,, \quad 1 \leqslant i \leqslant l\,,$$

yields $\underline{V}_2 \in \overset{\circ}{\mathscr{P}}{}^{[4],0}(\Sigma_h') \times \overset{\circ}{\mathscr{P}}{}^{[4],0}(\Sigma_h')$ such that

$$\phi - \nabla.(\underline{V}_1 + \underline{V}_2) = 0 \quad \text{at all vertices of } \Sigma_h'\,,$$

and

$$\int_{\mathcal{T}} (\phi - \nabla.(\underline{V}_1 + \underline{V}_2))\, d\underline{x} = 0 \quad \text{for any} \quad \mathcal{T} \in \Sigma_h'\,.$$

The problem is now completely localized, and applying Lemma 4.4 triangle by triangle we find $\underline{V}_3 \in \overset{\circ}{\mathscr{P}}{}^{[p+1],0}(\Sigma_h') \times \overset{\circ}{\mathscr{P}}{}^{[p+1],0}(\Sigma_h')$, satisfying

$$\phi - \nabla.(\underline{V}_1 + \underline{V}_2) = \nabla.\underline{V}_3\,,$$

i.e., the field

$$\underline{V} = \underline{V}_1 + \underline{V}_2 + \underline{V}_3 \in \overset{\circ}{\mathscr{P}}{}^{[p+1],0}(\Sigma_h') \times \overset{\circ}{\mathscr{P}}{}^{[p+1],0}(\Sigma_h')$$

has the desired property (4.12*a*). It follows directly from Lemmas 4.1 and 4.4 that

$$\| \underline{V}_1 \|_{1,\Omega(\Sigma_h')} \leqslant Cp^K \| \phi \|_{0,\Omega(\Sigma_h')}\,,$$

and (4.13)

$$\| \underline{V}_3 \|_{1,\Omega(\Sigma_h')} \leqslant Cp^K \left(\| \phi \|_{0,\Omega(\Sigma_h')} + \sum_{j=1}^{2} \| \underline{V}_j \|_{1,\Omega(\Sigma_h')} \right).$$

Since

$$| b_i | = \left| \int_{\mathcal{T}_i^h} (\phi - \nabla.\underline{V}_1)\, d\underline{x} \right| \leqslant Ch(\| \phi \|_{0,\mathcal{T}_i^h} + \| \underline{V}_1 \|_{1,\mathcal{T}_i^h})\,,$$

the estimate (4.7*c*′) shows

$$\| \underline{V}_2 \|_{0,\Omega(\Sigma_h')} \leqslant Cl^{3/2}(\| \phi \|_{0,\Omega(\Sigma_h')} + \| \underline{V}_1 \|_{1,\Omega(\Sigma_h')})\,. \qquad (4.14)$$

A combination of (4.13) and (4.14) yields the estimate (4.12*b*) for $\underline{V}$. □

The previous argument, with minor changes, provides proofs of both Proposition 3.1 and Proposition 3.2. Note, however, that for the estimate (4.12*b*) to be valid for $\phi \in \tilde{\mathscr{P}}^{[p],-1}(\Sigma_h')$ and corresponding

$$\underline{V} \in \overset{\circ}{\mathscr{P}}{}^{[p+1],0}(\Sigma_h') \times \overset{\circ}{\mathscr{P}}{}^{[p+1],0}(\Sigma_h')$$

we have to require that $\overset{\circ}{R}(\Sigma_h') \geqslant \delta > 0$, independent of Σ_h' and h (this latter is the reason we use Proposition 4.1 and not the corresponding version of Proposition 3.2 in our proof of Theorem 5.1). If Σ_h' is taken to be all of Σ_h, then $l \sim 0(h^{-2})$, and the estimate (4.12b) reads

$$\| \underline{V} \|_{1,\Omega} \leqslant Cp^K h^{-3} \| \phi \|_{0,\Omega} ,$$

i.e., the local construction does not immediately give a bound for a right inverse which is uniform in h.

5. THE MAIN THEOREM

As announced earlier the main focus of this paper is to estimate the norm of a right inverse for the divergence operator. Our estimate is the central part of the following theorem.

THEOREM 5.1 : *Let Σ_h, $0 < h \leqslant 1$, be a quasiuniform family of triangulations of the polygonal domain Ω, and let p be an integer $\geqslant 3$. Assume that*

$$R(\Sigma_h) \geqslant \delta > 0 , \quad \delta \textit{ independent of } h ,$$

where $R(\Sigma_h)$ is the measure of singularity introduced in (3.9). *Then*

$$\nabla . (\mathscr{P}^{[p+1],0}(\Sigma_h) \times \mathscr{P}^{[p+1],0}(\Sigma_h)) = \mathscr{P}^{[p],-1}(\Sigma_h) ,$$

and there exists a linear operator

$$\mathscr{L}_p^h : \mathscr{P}^{[p],-1}(\Sigma_h) \to \mathscr{P}^{[p+1],0}(\Sigma_h) \times \mathscr{P}^{[p+1],0}(\Sigma_h)$$

such that

$$\nabla . (\mathscr{L}_p^h \phi) = \phi \qquad \forall \phi \in \mathscr{P}^{[p],-1}(\Sigma_h) , \tag{5.1a}$$

$$\| \mathscr{L}_p^h \phi \|_{1,\Omega} \leqslant Cp^K \| \phi \|_{0,\Omega} \tag{5.1b}$$

with constants C and K that are independent of h, p and ϕ.

Note : The first part of Theorem 5.1 is simply a restatement of Proposition 3.1. Also note that the assumption $R(\Sigma_h) \geqslant \delta > 0$ does not rule out the presence of singular vertices, it merely prevents the nonsingular vertices from becoming too close to singular.

Since

$$\nabla . (\mathscr{P}^{[p+1],0}(\Sigma_h) \times \mathscr{P}^{[p+1],0}(\Sigma_h)) \subseteq \mathscr{P}^{[p],-1}(\Sigma_h)$$

it is well known that the statements of Theorem 5.1 are equivalent to the so-called inf-sup condition ($c = C^{-1}$)

$$\forall \phi \in \mathscr{P}^{[p],-1}(\Sigma_h)$$

$$\sup_{\underline{V}} \frac{\int_\Omega \nabla . \underline{V} \phi \, d\underline{x}}{\| \underline{V} \|_{1,\Omega}} \geqslant c p^{-K} \| \phi \|_{0,\Omega} \tag{5.2}$$

with the supremum taken over $\underline{V} \in [\mathscr{P}^{[p+1],0}(\Sigma_h) \times \mathscr{P}^{[p+1],0}(\Sigma_h)] \setminus \{ 0 \}$ (*cf.* [2]). We shall make use of this fact in the case $p = 3$ of our proof. The proof of Theorem 5.1 relies heavily on the analysis of [18], but an added new element is the localization procedure which has certain similarities to the macro-element concept found in [3], [14]; however, our triangulations are quite arbitrary, except for the assumption of quasiuniformity.

LEMMA 5.1 : *There exists a constant C such that for any given positive integer k and h sufficiently small (how small depends on k) it is possible to partition Σ_h into a disjoint union of connected collections $\Sigma_h^{(m)}$, $1 \leqslant m \leqslant M(k, h)$ with*

$$\textit{each collection } \Sigma_h^{(m)} \textit{ containing at most } Ck \textit{ triangles} \tag{5.3a}$$

$$\textit{each } \Omega_h^{(m)} = \textit{interior}\left(\bigcup_{\mathscr{T} \in \Sigma_h^{(m)}} \overline{\mathscr{T}} \cap \Omega \right) \textit{containing a disc of radius} \sqrt{k} h \,. \tag{5.3b}$$

Proof : Let $\underline{x}_h^{(m)}$, $1 \leqslant m \leqslant M(k, h)$, be those vertices of a uniform lattice, with sidelength $2(\sqrt{k} + 1) h$, that lie in Ω and lie at least a distance $\sqrt{k}\, h$ away from $\partial\Omega$. Let $D_h^{(m)}$ denote the open disc of radius $\sqrt{k}\, h$, centered at $\underline{x}_h^{(m)}$. All triangles of Σ_h that intersect $D_h^{(m)}$ will be assigned to the collection $\Sigma_h^{(m)}$, thus ensuring that (5.3*b*) is satisfied. At this point the collections $\Sigma_h^{(m)}$ are connected, mutually disjoint and each contains at most Ck triangles. It is now easy to distribute the remaining triangles of Σ_h among the $\Sigma_h^{(m)}$, in such a way that their individual connectivity is preserved, and they still satisfy (5.3*a*) (possibly with a larger constant C). $\square$

Remark 5.1 : Based on Lemma 5.1 we may immediately conclude that for h sufficiently small (how small depends on k) it is possible to partition Σ_h into a disjoint union of connected collections $\Sigma_h^{(m)}$, $1 \leqslant m \leqslant M(k, h)$ satisfying

$$\text{each collection } \Sigma_h^{(m)} \text{ contains at most } k \text{ triangles}\,, \tag{5.4a}$$

$$\text{each } \Omega_h^{(m)} = \text{interior}\left(\bigcup_{\mathscr{T} \in \Sigma_h^{(m)}} \overline{\mathscr{T}} \cap \Omega \right) \text{contains a disc of radius } c\sqrt{k}\, h \,. \tag{5.4b}$$

The constant $c > 0$ is independent of k and h. $\square$

LEMMA 5.2 : *Let k be a sufficiently large positive integer. For h sufficiently small (how small depends on k), let $\Sigma_h^{(m)}$, $1 \leqslant m \leqslant M(k, h)$, be the partition of Σ_h introduced in Remark 5.1. For any $1 \leqslant m \leqslant M(k, h)$ and any constant b, one can then find $\phi^{(m)} \in \overset{\circ}{\mathscr{P}}^{[1],0}(\Sigma_h^{(m)})$ such that*

$$\int_{\Omega_h^{(m)}} \phi^{(m)}\, d\underline{x} = b$$

and

$$\| \phi^{(m)} \|_{0,\Omega_h^{(m)}} \leqslant C(\sqrt{k}\, h)^{-1} \, | b | ,$$

$$\| \phi^{(m)} \|_{1,\Omega_h^{(m)}} \leqslant C(\sqrt{k}\, h)^{-1} \, (1 + (\sqrt{k}\, h)^{-1}) \, | b | .$$

Proof : From (5.4*b*) we know that there exists $\underline{z} \in \Omega_h^{(m)}$ such that

$$D_{\underline{z}}(c\sqrt{k}\, h) \subseteq \Omega_h^{(m)}$$

where $D_{\underline{z}}(r)$ is the open disc of radius r centered at $\underline{z}$. Selecting $\underline{z}$ to be the origin and dilating by $(c\sqrt{k}\, h)^{-1}$ we obtain

$$D_{\underline{0}}(1) \subseteq \Omega^{(m)} ,$$

where $\Omega^{(m)}$ is the translated, dilated image of $\Omega_h^{(m)}$. Let $\Sigma^{(m)}$ be the triangulation of $\Omega^{(m)}$ corresponding to $\Sigma_h^{(m)}$. The triangles of $\Sigma^{(m)}$ all have diameter $\leqslant (c\sqrt{k})^{-1}$; if k is sufficiently large, more specifically if $(c\sqrt{k})^{-1} < 1/2$, it then follows that there is a vertex of $\Sigma^{(m)}$ which lies inside $D_{\underline{0}}(1/2)$. Let $0 \leqslant \Psi \leqslant 1$ be a C^1-function with $\Psi = 1$ on $D_{\underline{0}}(1/2)$ and $\Psi = 0$ outside $D_{\underline{0}}(1)$, and let $\psi \in \mathscr{P}^{[1],0}(\Sigma^{(m)})$ be the function which interpolates Ψ at the vertices of $\Sigma^{(m)}$. If we define

$$\tilde{\psi} = \psi \Big/ \int_{\Omega^{(m)}} \psi \, d\underline{x} ,$$

then

$$\tilde{\psi} = 0 \text{ on the boundary of } \Omega^{(m)} , \tag{5.5a}$$

$$\int_{\Omega^{(m)}} \tilde{\psi} \, d\underline{x} = 1, \tag{5.5b}$$

and

$$\| \tilde{\psi} \|_{1,\Omega^{(m)}} \leqslant C . \tag{5.5c}$$

The function

$$\phi^{(m)}(\underline{x}) = b(c\sqrt{k}\, h)^{-2} \, \tilde{\psi}\left[\frac{\underline{x} - \underline{z}}{c\sqrt{k}\, h}\right] .$$

is an element of $\overset{\circ}{\mathscr{P}}^{[1],0}(\Sigma_h^{(m)})$ that satisfies the requirements in this lemma. □

We are now ready for the

Proof of Theorem 5.1 : Consider the case $p = 3$; we shall verify that if h is sufficiently small then for any $\phi \in \mathscr{P}^{[3],-1}(\Sigma_h)$ there exists $\underline{W} \in \mathscr{P}^{[4],0}(\Sigma_h) \times \mathscr{P}^{[4],0}(\Sigma_h)$ satisfying

$$\| \phi - \nabla . \underline{W} \|_{0,\Omega} \leqslant \frac{1}{2} \| \phi \|_{0,\Omega} \tag{5.6a}$$

and

$$\| \underline{W} \|_{1,\Omega} \leqslant C \| \phi \|_{0,\Omega} . \tag{5.6b}$$

It follows immediately from (5.6*a*-*b*) that

$$\sup_{\underline{V}} \frac{\int_\Omega \nabla . \underline{V} \phi \, d\underline{x}}{\| \underline{V} \|_{1,\Omega}} \geqslant \frac{\int_\Omega \nabla . \underline{W} \phi \, d\underline{x}}{\| \underline{W} \|_{1,\Omega}} =$$

$$= \frac{\int_\Omega \phi^2 \, d\underline{x} - \int_\Omega (\phi - \nabla . \underline{W}) \, \phi \, d\underline{x}}{\| \underline{W} \|_{1,\Omega}} \geqslant \frac{1}{2} \frac{\int_\Omega \phi^2 \, d\underline{x}}{\| \underline{W} \|_{1,\Omega}} \geqslant c \| \phi \|_{0,\Omega} ,$$

i.e., the inequality in (5.2) holds for $p = 3$. According to the comments made earlier this proves the theorem, in the case $p = 3$, for h sufficiently small. For $p = 3$ and large h the theorem follows directly from the constructive proof of Proposition 3.1, discussed at the end of Section 4.

The construction of $\underline{W}$ proceeds in several steps.

Step 1 : Using Lemma 4.1, with $\Sigma_h' = \Sigma_h$ and $p = 3$ one finds $\underline{V}_1 \in \mathscr{P}^{[3],0}(\Sigma_h) \times \mathscr{P}^{[3],0}(\Sigma_h)$ such that

$$\phi - \nabla . \underline{V}_1 = 0 \quad \text{at all vertices of } \Sigma_h , \tag{5.7a}$$

$$\| \underline{V}_1 \|_{1,\Omega} \leqslant C \| \phi \|_{0,\Omega} . \tag{5.7b}$$

Step 2 : Let (for k sufficiently large and h sufficiently small) $\{ \Sigma_h^{(m)} \}_{m=1}^{M(k,h)}$ be the disjoint partition of Σ_h introduced in Remark 5.1. Let $\tilde{\phi}^{(m)} \in \mathring{\mathscr{P}}^{[1],0}(\Sigma_h^{(m)})$ be the function constructed in Lemma 5.2 corresponding to

$$b = \int_{\Omega_h^{(m)}} (\phi - \nabla . \underline{V}_1) \, d\underline{x} ,$$

and define

$$\tilde{\phi}(\underline{x}) = \tilde{\phi}^{(m)}(\underline{x}) \quad \text{for} \quad \underline{x} \in \Omega_h^{(m)} , \quad 1 \leqslant m \leqslant M(k, h) .$$

It follows from Lemma 5.2 and (5.7*a-b*) that

$$\phi - \nabla . \underline{V}_1 - \tilde{\phi} = 0 \quad \text{at all vertices on the boundaries of } \Omega_h^{(m)}, \quad 1 \leqslant m \leqslant M(k, h). \tag{5.8a}$$

$$\int_{\Omega_h^{(m)}} (\phi - \nabla . \underline{V}_1 - \tilde{\phi})\, d\underline{x} = 0, \quad 1 \leqslant m \leqslant M(k, h), \tag{5.8b}$$

$$\| \tilde{\phi} \|_{0,\Omega} \leqslant C \| \phi \|_{0,\Omega}, \quad \text{and} \quad \| \tilde{\phi} \|_{1,\Omega} \leqslant C(1 + (\sqrt{k}\, h)^{-1}) \| \phi \|_{0,\Omega}. \tag{5.8c}$$

Step 3 : For each $1 \leqslant m \leqslant M(k, h)$ we apply Proposition 4.1 with $\Sigma_h' = \Sigma_h^{(m)}$ to the function $\phi - \nabla . \underline{V}_1 - \tilde{\phi}$ (this is possible due to (5.8*a-b*)); by composition of the individual solutions we get

$$\underline{V}_2 \in \mathring{\mathscr{P}}^{[4],0}(\Sigma_h) \times \mathring{\mathscr{P}}^{[4],0}(\Sigma_h),$$

satisfying

$$\nabla . \underline{V}_2 = \phi - \nabla . \underline{V}_1 - \tilde{\phi} \quad \text{in } \Omega, \tag{5.9a}$$

and

$$\| \underline{V}_2 \|_{1,\Omega} \leqslant C k^{3/2} \| \phi \|_{0,\Omega}. \tag{5.9b}$$

Step 4 : Finally we shall construct a field $\underline{V}_3 \in \mathscr{P}^{[4],0}(\Sigma_h) \times \mathscr{P}^{[4],0}(\Sigma_h)$ such that

$$\| \tilde{\phi} - \nabla . \underline{V}_3 \|_{0,\Omega} \leqslant C_0(h + \sqrt{k}^{-1}) \| \phi \|_{0,\Omega}, \tag{5.10a}$$

and

$$\| \underline{V}_3 \|_{1,\Omega} \leqslant C \| \phi \|_{0,\Omega}. \tag{5.10b}$$

In combination with (5.7*b*) and (5.9*a-b*) this leads to

$$\| \phi - \nabla . \underline{W} \|_{0,\Omega} \leqslant C_0(h + \sqrt{k}^{-1}) \| \phi \|_{0,\Omega}, \tag{5.11a}$$

and

$$\| \underline{W} \|_{1,\Omega} \leqslant C k^{3/2} \| \phi \|_{0,\Omega}, \tag{5.11b}$$

where $\underline{W} = \sum_{j=1}^{3} \underline{V}_j \in \mathscr{P}^{[4],0}(\Sigma_h) \times \mathscr{P}^{[4],0}(\Sigma_h)$. By choosing k sufficiently large we may arrange that

$$C_0(h + \sqrt{k}^{-1}) < 1/2,$$

for all h sufficiently small, and (5.11*a-b*) therefore verifies the existence of a field $\underline{W}$ with the properties (5.6*a-b*).

The construction of $\underline{V}_3$ is based on an approximation argument. Let Φ be a function satisfying

$$\Delta \Phi = \tilde{\phi} \quad \text{in } \Omega , \tag{5.12}$$

with

$$\| \Phi \|_{2,\Omega} \leqslant C \| \tilde{\phi} \|_{0,\Omega} \tag{5.13a}$$

and

$$\| \Phi \|_{3,\Omega} \leqslant C \| \tilde{\phi} \|_{1,\Omega} , \tag{5.13b}$$

(note that we *do not* specify any boundary condition on $\partial\Omega$, and this is what makes it possible to obtain (5.13*a*-*b*), although $\partial\Omega$ is not smooth).

Let $\underline{V}_3 \in \mathscr{P}^{[1],0}(\Sigma_h) \times \mathscr{P}^{[1],0}(\Sigma_h)$ be an approximation to $\nabla\Phi$ in the sense that

$$\| \nabla\Phi - \underline{V}_3 \|_{1,\Omega} \leqslant Ch \| \Phi \|_{3,\Omega} \tag{5.14a}$$

and

$$\| \underline{V}_3 \|_{1,\Omega} \leqslant C \| \Phi \|_{2,\Omega} ; \tag{5.14b}$$

(5.12) and the estimates (5.13*b*), (5.14*a*) then lead to

$$\begin{aligned} \| \tilde{\phi} - \nabla . \underline{V}_3 \|_{0,\Omega} &= \| \nabla . (\nabla\Phi - \underline{V}_3) \|_{0,\Omega} \\ &\leqslant Ch \| \Phi \|_{3,\Omega} \\ &\leqslant Ch \| \tilde{\phi} \|_{1,\Omega} , \end{aligned}$$

so that by virtue of (5.8*c*)

$$\| \tilde{\phi} - \nabla . \underline{V}_3 \|_{0,\Omega} \leqslant C_0 (h + \sqrt{k^{-1}}) \| \phi \|_{0,\Omega} .$$

The remaining inequality (5.10*b*) follows immediately from (5.8*c*), (5.13*a*) and (5.14*b*).

This completes the proof of Theorem 5.1 in the case $p = 3$.

Let p be an arbitrary integer $\geqslant 4$. Given $\phi \in \mathscr{P}^{[p],-1}(\Sigma_h)$ it is possible on each triangle $\mathscr{T}^h$ of Σ_h to find a quadratic $q_{\mathscr{T}^h}$ with

$$q_{\mathscr{T}^h} = \phi \quad \text{at the three vertices of } \mathscr{T}^h$$

$$\int_{\mathscr{T}^h} q_{\mathscr{T}^h} \, d\underline{x} = \int_{\mathscr{T}^h} \phi \, d\underline{x} ,$$

and

$$\| q_{\mathscr{T}^h} \|_{0,\mathscr{T}^h} \leqslant Ch \sup_{\underline{x} \in \mathscr{T}^h} | \phi(\underline{x}) | \leqslant Cp^{K'} \| \phi \|_{0,\mathscr{T}^h} \tag{5.15}$$

for $K' > 2$,

(in (5.15) we have used the Sobolev Imbedding Lemma and a Bernstein-type inequality, *cf.* [18]). Define

$$q(\underline{x}) = q_{\mathcal{T}^h}(\underline{x}) \quad \text{for} \quad \underline{x} \in \mathcal{T}^h, \qquad \mathcal{T}^h \in \Sigma_h,$$

then $q \in \mathcal{P}^{[2],-1}(\Sigma_h) \subseteq \mathcal{P}^{[3],-1}(\Sigma_h)$, since $\phi \in \mathcal{P}^{[p],-1}(\Sigma_h)$. From (5.15) we conclude that

$$\| q \|_{0,\Omega} \leqslant C p^{K'} \| \phi \|_{0,\Omega} . \tag{5.16}$$

and

$$\| \phi - q \|_{0,\Omega} \leqslant C p^{K'} \| \phi \|_{0,\Omega} . \tag{5.17}$$

Due to our method of construction

$$\phi - q = 0 \quad \text{at all vertices of } \Sigma_h ,$$

and

$$\int_{\mathcal{T}^h} (\phi - q)\, d\underline{x} = 0 \quad \text{on all triangles of } \Sigma_h .$$

We may now apply Lemma 4.4 separately on each triangle, and by piecing together we get

$$\underline{V}_1 \in \mathcal{P}^{[p+1],0}(\Sigma_h) \times \mathcal{P}^{[p+1],0}(\Sigma_h) ,$$

with

$$\nabla . \underline{V}_1 = \phi - q \quad \text{in } \Omega , \tag{5.18a}$$

and

$$\begin{aligned} \| \underline{V}_1 \|_{1,\Omega} &\leqslant C p^{K} \| \phi - q \|_{0,\Omega} \\ &\leqslant C p^{K+K'} \| \phi \|_{0,\Omega} . \end{aligned} \tag{5.18b}$$

Since $q \in \mathcal{P}^{[3],-1}(\Sigma_h)$ we may use this theorem in the case $p = 3$ (which has already been verified) to find

$$\underline{V}_2 \in \mathcal{P}^{[4],0}(\Sigma_h) \times \mathcal{P}^{[4],0}(\Sigma_h) ,$$

such that

$$\nabla . \underline{V}_2 = q \quad \text{in } \Omega \tag{5.19a}$$

and

$$\begin{aligned} \| \underline{V}_2 \|_{1,\Omega} &\leqslant C \| q \|_{0,\Omega} \\ &\leqslant C p^{K'} \| \phi \|_{0,\Omega} ; \end{aligned} \tag{5.19b}$$

in the last inequality we used (5.16). Defining

$$\underline{V} = \underline{V}_1 + \underline{V}_2 ,$$

the theorem follows directly from (5.18*a*-*b*) and (5.19*a*-*b*) in the case $p \geqslant 4$. This concludes our proof. $\square$

The proof presented above immediately carries over to the case of homogeneous Dirichlet boundary conditions, except for the construction of Φ and $\underline{V}_3$. We need an additional result concerning the invertibility of the divergence operator with homogeneous boundary conditions. The following lemma is proven in [1]; the method of proof relies heavily on the characterization of trace spaces for function spaces on polygonal domains, as found in [8]. Sobolev spaces $H^s(\Omega)$, $0 \leqslant s$, with noninteger indices are defined by interpolation; $\| \cdot \|_{s,\Omega}$ denotes the norm on $H^s(\Omega)$.

LEMMA 5.3 : *Assume that all internal angles at corners of the domain Ω are less than 2π. Let $0 < s < 1$ be fixed and suppose that $\phi \in H^s(\Omega)$, with*

$$\int_{\Omega'} \phi \, d\underline{x} = 0$$

on all connected components Ω' of Ω. Then there exists $\underline{U} \in H^{s+1}(\Omega)$ such that

$$\nabla . \underline{U} = \phi \quad \text{in } \Omega ,$$

$$\underline{U} = 0 \quad \text{on } \partial\Omega ,$$

and

$$\| \underline{U} \|_{s+1,\Omega} \leqslant C \| \phi \|_{s,\Omega} ,$$

$$\| \underline{U} \|_{1,\Omega} \leqslant C \| \phi \|_{0,\Omega} ,$$

with C independent of ϕ.

Let $\tilde{\phi}$ be as introduced in step 2 of the proof of Theorem 5.1; $\tilde{\phi}$ clearly lies in $H^{1/2}(\Omega)$, and it has integral zero on each connected component of Ω. Let $\underline{U}_3 \in H^{3/2}(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)$ be the field, corresponding to $\tilde{\phi}$, which is defined by Lemma 5.3. If $\underline{V}_3 \in \overset{\circ}{\mathscr{P}}{}^{[1],0}(\Sigma_h) \times \overset{\circ}{\mathscr{P}}{}^{[1],0}(\Sigma_h)$ is an approximation to $\underline{U}_3$ in the sense that

$$\| \underline{U}_3 - \underline{V}_3 \|_{1,\Omega} \leqslant Ch^{1/2} \| \underline{U}_3 \|_{3/2,\Omega}$$

and

$$\| \underline{V}_3 \|_{1,\Omega} \leqslant C \| \underline{U}_3 \|_{1,\Omega} ,$$

then

$$\begin{aligned} \| \tilde{\phi} - \nabla . \underline{V}_3 \|_{0,\Omega} &= \| \nabla . (\underline{U}_3 - \underline{V}_3) \|_{0,\Omega} \\ &\leqslant \| \underline{U}_3 - \underline{V}_3 \|_{1,\Omega} \\ &\leqslant Ch^{1/2} \| \underline{U}_3 \|_{3/2,\Omega} \\ &\leqslant Ch^{1/2} \| \tilde{\phi} \|_{1/2,\Omega} \end{aligned} \tag{5.20a}$$

and

$$\| \underline{V}_3 \|_{1,\Omega} \leqslant C \| \tilde{\phi} \|_{0,\Omega} . \tag{5.20b}$$

Due to (5.20*a*), (5.8*c*) and " logarithmic convexity " of the Sobolev norms it follows that

$$\| \tilde{\phi} - \nabla . \underline{V}_3 \|_{0,\Omega} \leqslant C(h + \sqrt{k^{-1}})^{1/2} \| \phi \|_{0,\Omega} ;$$

from (5.20*b*) and (5.8*c*) it follows that

$$\| \underline{V}_3 \|_{1,\Omega} \leqslant C \| \phi \|_{0,\Omega} .$$

By choosing k sufficiently large we thus obtain a field $\underline{V}_3$ which, for h sufficiently small, has the same properties

$$\| \tilde{\phi} - \nabla . \underline{V}_3 \|_{0,\Omega} \leqslant \frac{1}{2} \| \phi \|_{0,\Omega}$$

and

$$\| \underline{V}_3 \|_{1,\Omega} \leqslant C \| \phi \|_{0,\Omega}$$

as the field constructed in step 4 of the previous proof. $\underline{V}_3$ furthermore vanishes on $\partial\Omega$ and hence it may be used in a construction of a field with homogeneous Dirichlet boundary conditions. The rest of the proof proceeds as before, completing our verification of :

THEOREM 5.2 : *Assume that all internal angles at corners of the polygonal domain Ω are less than 2π. Let Σ_h, $0 < h \leqslant 1$ be a quasiuniform family of triangulations of Ω, and let p be an integer $\geqslant 3$. Assume that*

$$\overset{c}{R}(\Sigma_h) \geqslant \delta > 0 , \quad \delta \text{ independent of } h ,$$

where $\overset{c}{R}(\Sigma_h)$ is the measure of singularity introduced in (3.10). *Then*

$$\nabla . (\overset{\circ}{\mathscr{P}}^{[p+1],0}(\Sigma_h) \times \overset{\circ}{\mathscr{P}}^{[p+1],0}(\Sigma_h)) = \tilde{\mathscr{P}}^{[p],-1}(\Sigma_h) ,$$

and there exists a linear operator

$$\mathscr{L}_p^h : \tilde{\mathscr{P}}^{[p],-1}(\Sigma_h) \to \overset{\circ}{\mathscr{P}}^{[p+1],0}(\Sigma_h) \times \overset{\circ}{\mathscr{P}}^{[p+1],0}(\Sigma_h)$$

such that

$$\nabla . (\mathscr{L}_p^h \phi) = \phi \qquad \forall \phi \in \tilde{\mathscr{P}}^{[p],-1}(\Sigma_h) \tag{5.31a}$$

$$\| \mathscr{L}_p^h \phi \|_{1,\Omega} \leqslant C p^K \| \phi \|_{0,\Omega} \tag{5.31b}$$

with constants C and K that are independent of h, p and ϕ.

Remark 5.3 : Theorem 5.1 and 5.2 may directly be used to show that minimization of the displacement energy of two dimensional plane strain linear elasticity over the space of continuous piecewise polynomials of degree $p + 1$, $p \geqslant 3$, is an accurate numerical approach. On a quasiuniform family of triangulations (with $R(\Sigma_h)$ or $\overset{\circ}{R}(\Sigma_h)$ bounded away from 0) it leads to approximate solutions that converge at optimal rate in h and at arbitrarily close to optimal rate in p, uniformly with respect to Poisson's ratio (*cf.* [12], [19]). Theorem 5.1 and 5.2 thus disprove a conjecture made by the second author in Remark 3.2 of [19]; it was conjectured, based on numerical evidence, that the h-convergence rates would never be optimal, uniformly in Poisson's ratio. However, the numerical experiments referred to (*cf.* [16]) were all for polynomials of degree $p + 1$, $p < 3$, i.e., exactly the case the theorems here do not cover, and they are not characteristic of the behaviour for $p \geqslant 3$. □

6. A BASIS FOR THE DIVERGENCE FREE SPACE

In many applications, it is of interest to work directly with the null-space of the divergence operator acting on $\mathcal{P}^{[p+1],0} \times \mathcal{P}^{[p+1],0}$ (or $(\overset{\circ}{\mathcal{P}}{}^{[p+1],0} \times \overset{\circ}{\mathcal{P}}{}^{[p+1],0})$). As observed in Section 3 the curl operator maps $\mathcal{P}^{[p+2],1}(\Sigma')$ (respectively $\overset{\circ}{\mathcal{P}}{}^{[p+2],1}(\Sigma')$) onto this nullspace (provided $\Omega(\Sigma')$ is simply connected). Thus a basis for the nullspace can be obtained from one for $\mathcal{P}^{[p+2],1}$ (or $\overset{\circ}{\mathcal{P}}{}^{[p+2],1}$). A basis for $\mathcal{P}^{[p+2],1}$ was given in [10]. We shall extend slightly that work here to construct a basis for $\overset{\circ}{\mathcal{P}}{}^{[p+2],1}$. Our method of proof is to verify the dimension formula

$$\dim(\overset{\circ}{\mathcal{P}}{}^{[p+2],1}(\Sigma')) = \frac{1}{2}p(p-5)\,T + (2p-1)\,E_0 + 3\,V_0 + \sigma\,, \quad (6.1)$$

and in the process exhibit this many linearly independent functions in $\overset{\circ}{\mathcal{P}}{}^{[p+2],1}$ (these functions form a subset of the basis given in [10]); T here denotes the number of triangles of Σ', E_0, V_0 denotes the number of internal edges and internal vertices of Σ' respectively and σ is the total number of singular vertices of Σ'. The dimension formula (6.1) follows directly from Proposition 3.2, a proof of which has already been given in Section 4. The verification presented in this section is totally independent of the previous two sections, and indeed it forms the basis for an alternate proof of Proposition 3.2 (as outlined in Section 3). The polygonal domain $\Omega(\Sigma')$ is assumed to be simply connected.

The operator $\nabla.$ maps the space

$$\overset{\circ}{\mathcal{P}}{}^{[p+1],0}(\Sigma') \times \overset{\circ}{\mathcal{P}}{}^{[p+1],0}(\Sigma')$$

into

$$\overset{\circ}{\mathcal{P}}{}^{[p],-1}(\Sigma')\,.$$

The nullspace of $\nabla\cdot$ is isomorphic to

$$\overset{\circ}{\mathscr{P}}{}^{[p+2]\,1}(\Sigma'),$$

and it thus follows that

$$\dim\left(\overset{\circ}{\mathscr{P}}{}^{[p+1]\,0}\times\overset{\circ}{\mathscr{P}}{}^{[p+1]\,0}\right)-\dim\left(\overset{\circ}{\mathscr{P}}{}^{[p+2]\,1}\right)\leqslant\dim\left(\tilde{\mathscr{P}}{}^{[p]\,-1}\right)\qquad(6\ 2)$$

The first and the last of the dimensions in this inequality have already been computed to be $(p-1)\,pT+2\,pE_0+2\,V_0$ and $\frac{1}{2}(p+2)(p+1)\,T-\sigma-1$ respectively, i e, based on (6 2) we get

$$\dim\left(\overset{\circ}{\mathscr{P}}{}^{[p+2]\,1}(\Sigma')\right)\geqslant\frac{1}{2}p(p-5)\,T+2\,pE_0+2\,V_0+\sigma-T+1$$

Since $T-E+V=1$ and $V-V_0=E-E_0$, this implies

$$\dim\left(\overset{\circ}{\mathscr{P}}{}^{[p+2]\,1}(\Sigma')\right)\geqslant\frac{1}{2}p(p-5)\,T+(2\,p-1)\,E_0+3\,V_0+\sigma\qquad(6\ 3)$$

The inequality (6 3) proves half of the identity (6 1), and it thus remains to verify that

$$\dim\left(\overset{\circ}{\mathscr{P}}{}^{[p+2]\,1}(\Sigma')\right)\leqslant\frac{1}{2}p(p-5)\,T+(2\,p-1)\,E_0+3\,V_0+\sigma\qquad(6\ 4)$$

In [10] it is shown that

$$\dim\left(\mathscr{P}^{[p+2]\,1}(\Sigma')\right)=\frac{1}{2}(p+3)(p+4)\,T-(2\,p+5)\,E_0+3\,V_0+\sigma_0\,,\qquad(6\ 5)$$

through the construction of a purely local basis for this space Among the corresponding nodal values are

(*a*) the value and x_1 and x_2 derivatives at each vertex,

(*b*) the value at each of $p-3$ distinct points in the interior of each edge,

(*c*) the (edge) normal derivative at each of $p-2$ distinct points in the interior of each edge

The remaining nodal values are more complicated to describe, but for vertices on the boundary of $\Omega(\Sigma')$ they do include

(*d*) one cross derivative (i e for each vertex on the boundary, select adjacent edges e_1 and e_2 meeting there and take the e_1, e_2 cross derivative at that vertex),

(*e*) the second edge derivative for all the edges meeting there

For functions in $\mathring{\mathscr{P}}^{[p+2],1}(\Sigma_h')$ the nodal values in (*a*)-(*c*) corresponding to vertices and edges on the boundary of Σ_h' must vanish; by a simple count we get that

$$3(V - V_0) + (2p - 5)(E - E_0) \tag{6.6}$$

nodal values must vanish. The second derivatives along the boundary edges at vertices on the boundary (*e*) must also vanish, and give rise to 2 vanishing nodal values per vertex. Finally, if we pick e_1 or e_2 in (*d*) to be one of the boundary edges, it is clear that this produces one additional nodal value that must vanish for functions in $\mathring{\mathscr{P}}^{[p+2],1}(\Sigma')$. In combination with (6.6) we get a total of

$$3(V-V_0)+(2p-5)(E-E_0)+3(V-V_0)=2p(E-E_0)+(V-V_0) \tag{6.7}$$

vanishing nodal values. Using (6.5), (6.7) and the fact that $E + E_0 = 3T$ and $E - E_0 = V - V_0$ we thus obtain

$$\dim(\mathring{\mathscr{P}}^{[p+2],-1}(\Sigma')) \leqslant$$
$$\leqslant \frac{1}{2}p(p-5)T+(2p-1)E_0+3V_0+\sigma+((V-V_0)-(\sigma-\sigma_0)). \tag{6.8}$$

The right hand side of (6.8) is exactly as desired in (6.4) except for the additional term $(V - V_0) - (\sigma - \sigma_0)$; this term is always nonnegative and it equals the number of *nonsingular boundary vertices*. In order to verify (6.4) it therefore suffices to find one nontrivial linear constraint, for the nodal values corresponding to each nonsingular boundary vertex, which must be satisfied by functions in $\mathring{\mathscr{P}}^{[p+2],1}(\Sigma')$; a constraint, that is, which is not already counted in (6.7).

Let $\underline{x}_0$ be a boundary vertex and let the triangles $\mathcal{T}_i$, angles θ_i and edges e_i meeting at this vertex be numbered consecutively as shown in figure 7.

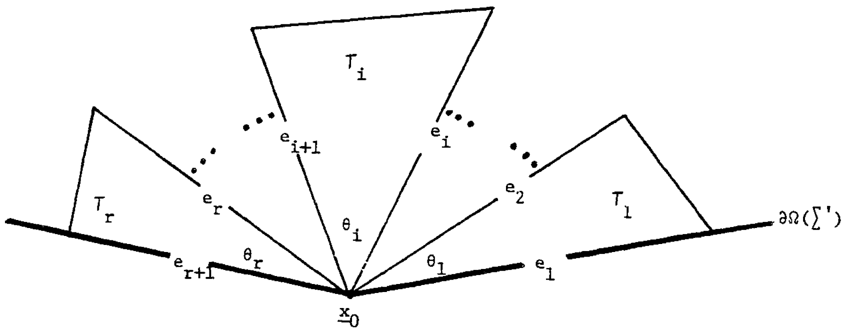

Figure 7. — Boundary vertex.

Let ∂_e denote the directional derivative in the direction parallel to the edge e. There is a simple relationship among all the cross derivatives of $\phi \in \mathscr{P}^{[p+2],1}(\Sigma')$, namely,

$$\sec \theta_i \, \partial_{e_i} \partial_{e_{i+1}}(\phi|_{\mathcal{T}_i})(\underline{x}_0) = -\sec \theta_{i-1} \, \partial_{e_{i-1}} \partial_{e_i}(\phi|_{\mathcal{T}_{i-1}})(\underline{x}_0) + \\ + (\cot \theta_i + \cot \theta_{i-1}) \, \partial^2_{e_i}(\phi|_{\mathcal{T}_{i-1}})(\underline{x}_0), \quad (6.9)$$

$2 \leqslant i \leqslant r$ (see [10] and also [5]). Summation of (6.9) with alternating signs yields

$$\sum_{i=2}^{r} (-1)^i (\cot \theta_i + \cot \theta_{i-1}) \, \partial^2_{e_i}(\phi|_{\mathcal{T}_{i-1}})(\underline{x}_0) = \\ = \sec \theta_1 \, \partial_{e_1} \partial_{e_2}(\phi|_{\mathcal{T}_1})(\underline{x}_0) + (-1)^r \sec \theta_r \, \partial_{e_r} \partial_{e_{r+1}}(\phi|_{\mathcal{T}_r})(\underline{x}_0).$$

For $\phi \in \mathring{\mathscr{P}}^{[p+2],1}(\Sigma'_h)$ both $\partial_{e_1} \partial_{e_2}(\phi|_{\mathcal{T}_1})$ and $\partial_{e_r} \partial_{e_{r+1}}(\phi|_{\mathcal{T}_r})$ must vanish at $\underline{x}_0$, and we thus arrive at the constraint

$$\sum_{i=2}^{r} (-1)^i (\cot \theta_i + \cot \theta_{i-1}) \, \partial^2_{e_i}(\phi|_{\mathcal{T}_{i-1}})(\underline{x}_0) = 0. \quad (6.10)$$

At any nonsingular boundary vertex, r is at least 2 and $\cot \theta_i + \cot \theta_{i-1} \neq 0$ for some i, so that (6.10) represents a nontrivial linear constraint among the second edge derivatives, which is not counted in (6.7); this completes the proof of the identity (6.1). At the nonsingular boundary vertices the expression

$$\sum_{i=2}^{r} (-1)^i (\cot \theta_i + \cot \theta_{i-1}) \, \partial^2_{e_i}(\phi|_{\mathcal{T}_{i-1}})(\underline{x}_0)$$

can be used as a nodal value for $\mathscr{P}^{[p+2],1}$ in place of one of the second edge derivatives (one, for which $\cot \theta_i + \cot \theta_{i-1} \neq 0$). Using these nodal variables we obtain a basis for $\mathring{\mathscr{P}}^{[p+2],1}(\Sigma')$ directly from the basis for $\mathscr{P}^{[p+2],1}(\Sigma')$ by deleting members corresponding to the aforementioned

$$2p(E - E_0) + 2(V - V_0) - (\sigma - \sigma_0)$$

vanishing nodal values.

Remark 6.1 : In the case Ω is not simply connected, one finds that the nullspace of $\nabla\cdot$ (for fields that vanish on $\partial\Omega$) is the curl of the subspace in $\mathscr{P}^{[p+2],1}$ consisting of functions that are constant on each component of $\partial\Omega$, and whose normal derivatives vanish on $\partial\Omega$. This space has a natural basis, and its dimension exceeds (6.1) exactly by the number of components of $\partial\Omega$. Using the corresponding Euler's formula we can thus extend our combinatorial proof of Proposition 3.2 to domains that are not simply connected. □

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