RAIRO. ANALYSE NUMÉRIQUE

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RAIRO. Analyse numérique, tome 17, nº 4 (1983), p. 385-395
<http://www.numdam.org/item?id=M2AN_1983__17_4_385_0>

R.A.I.R.O. Analyse numérique/Numerical Analysis
(vol. 17, n° 4, 1983, p. 385 à 395)

CONVERGENCE OF THE DISCRETE FREE BOUNDARIES FOR FINITE ELEMENT APPROXIMATIONS (*)

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Résumé — *Sur un problème d'obstacle modèle, on établit que la frontière libre discrète, obtenue par éléments finis linéaires par morceaux, converge vers la frontière libre du problème continu, avec un ordre de convergence qui est approximativement la racine carrée de la distance dans L^∞ entre la solution continue et la solution discrète.*

Abstract — *We show, on a model " obstacle problem " that the discrete (piecewise linear) finite element free boundary converges to the free boundary of the continuous problem with a rate which is approximately the square root of the L^∞ distance between the continuous and the discrete solution.*

1. INTRODUCTION

It is well known that a certain number of stationary free boundary problems can be written, directly or after some manipulations, as an elliptic variational inequality. The usual finite element approximation will then, in general, provide a sequence $u_h(x)$ convergent to the exact solution $u(x)$ of the variational inequality as h tends to zero. Hence, from the knowledge of $u_h(x)$ one tries to have information on some " approximate free boundary ". However, the usual estimates on the rate of convergence of $u_h(x)$ to $u(x)$ (in the H^1-norm or in the L^∞-norm) do not yield, by themselves, any estimate on the rate of convergence of the free boundaries.

In the present paper we discuss, for the sake of simplicity, the following " model problem " ·

$$\left.\begin{array}{c} \textit{find } u \in K \textit{ such that} \\ a(u, v-u) \geqslant (f, v-u) \quad \forall v \in K\,, \end{array}\right\} \qquad (1.1)$$

(*) Received in November 1982

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where

$$K = \{ v \mid v \in H^1(\Omega), v \geqslant 0 \text{ a.e. in } \Omega \; v|_{\partial\Omega} = g \}, \tag{1.2}$$

$$a(u, v) = \int_\Omega \nabla u . \nabla v \, dx, \tag{1.3}$$

$$(f, v) = \int_\Omega fv \, dx, \tag{1.4}$$

and where Ω is a bounded domain in $\mathbb{R}^n$, f is an element of $L^2(\Omega)$ bounded from above by a negative constant

$$f(x) \leqslant c(f) < 0 \text{ a.e. in } \Omega, \tag{1.5}$$

and g is a nonnegative function in $H^{1/2}(\partial\Omega) \cap C^0(\partial\Omega)$.

We show that if the finite element approximation of (1.1) verifies the discrete maximum principle then the discrete solutions of (1.1) " leave " the obstacle (zero in our case) with a certain " minimum speed ", showing a behaviour completely similar to the one proved, for the exact solution, by Caffarelli [2]. By means of this " minimum speed " property, added to some regularity assumptions on $u(x)$ and to known L^∞-estimates for the finite element approximation, we are then able to prove quasi-optimal error bounds (in measure and in distance, following different regularity assumptions on $u(x)$) for the approximation of the free boundaries.

2. FINITE ELEMENT APPROXIMATIONS

Assume for the sake of simplicity that Ω is a polyhedron in $\mathbb{R}^n$ and let $\{ \mathcal{T}_h \}_h$ be a family of decompositions of Ω into n-simplexes of diameter $\leqslant h$. We assume that the family $\{ \mathcal{T}_h \}_h$ is *regular* and *quasi-uniform* in the following sense : for any $\mathcal{T}_h$ and for any $S \in \mathcal{T}_h$, let P be a vertex of S, $F_S(P)$ the opposite face in S, and $\pi_S(P)$ the hyperplane containing $F_S(P)$; we set

$$d_S(P) = \text{dist}\,(P, \pi_S(P)) \tag{2.1}$$

and we *assume* :

$$\left.\begin{array}{l} \exists \theta > 0 \quad \text{s.t.} \quad \forall h > 0, \quad \forall S \in \mathcal{T}_h, \quad \forall P \in S, \\ \qquad d_S(P) \geqslant \theta h . \end{array}\right\} \tag{2.2}$$

Remark : The assumption of regularity and quasi-uniformity of the family $\{ \mathcal{T}_h \}_h$ can be written in many different equivalent ways (see for instance [4]). We chose (2.2) for convenience.

For any given $\mathcal{T}_h$ let now $P_1, P_2, \dots, P_{N(h)}$ be the vertices of $\mathcal{T}_h$ and assume, to simplify the notations, that the numbering is such that $P_1, P_2, \dots, P_{N_0(h)}$ are the internal vertices while $P_{N_0(h)+1}, \dots, P_{N(h)}$ lie on $\partial\Omega$. We disregard the trivial case assuming $N_0(h) \geqslant 1$. We define now the following finite element sets

$$W_h = \{ v_h \mid v_h \in C^0(\Omega), v_{h|S} \in \mathcal{P}_1 \ \forall S \in \mathcal{T}_h \} \tag{2.3}$$

(with $\mathcal{P}_1$ = polynomials of degree $\leqslant 1$)

$$W_h^g = W_h \cap \{ v \mid v \in C^0(\Omega), v = g \text{ at each vertex of } \partial\Omega \} \tag{2.4}$$

$$K_h = W_h^g \cap \{ v \mid v \geqslant 0 \text{ in } \Omega \} \tag{2.5}$$

and we consider the discrete problem :

$$\left.\begin{array}{c} \textit{find } u_h \in K_h \textit{ such that :} \\ a(u_h, v_h - u_h) \geqslant (f, v_h - u_h) \quad \forall v_h \in K_h . \end{array}\right\} \tag{2.6}$$

We shall now briefly discuss some well known property of the solution u_h of (2.6). To this end we introduce in W_h the canonical basis $\phi_1^h, \phi_2^h, \dots, \phi_{N(h)}^h$ defined by

$$\phi_i^h \in W_h \quad \text{and} \quad \phi_i^h(P_j) = \delta_{ij}, \quad i, j = 1, \dots, N(h), \tag{2.7}$$

and we set

$$\Omega_0^h = \{ x \mid x \in \Omega, u_h(x) = 0 \} \tag{2.8}$$

$$\Omega_+^h = \Omega - \Omega_0^h \quad \text{(note that } \Omega_+^h \text{ is open)} \tag{2.9}$$

$$F_h = (\partial\Omega_+^h) \cap \Omega . \tag{2.10}$$

It is well known (and easy to check !) that if P_i is a node (i.e. a vertex of $\mathcal{T}_h$) then

$$P_i \in \Omega_+^h \Rightarrow a(u_h, \phi_i^h) = (f, \phi_i^h) \tag{2.11}$$

$$P_i \in \Omega - \Omega_+^h \Rightarrow a(u_h, \phi_i^h) \geqslant (f, \phi_i^h) . \tag{2.12}$$

Setting now :

$$U_i = u_h(P_i) \quad i = 1, \dots, N(h), \tag{2.13}$$

$$A_{ij} = \int_\Omega \nabla\phi_i^h \, \nabla\phi_j^h \, dx \quad i, j = 1, \dots, N(h), \tag{2.14}$$

$$f_i = \int_\Omega f\phi_i^h \, dx \quad i = 1, \dots, N(h), \tag{2.15}$$

it is easy to see that (2.6) can be written as : *find* $U_1, \ldots, U_{N(h)}$ *such that* :

$$U_i \geqslant 0\,, \quad i = 1, \ldots, N_0(h)\,; \quad U_i = g(P_i)\,, \quad i = N_0(h) + 1, \ldots, N(h)\,; \tag{2.16}$$

$$\sum_{j=1}^{N_0(h)} A_{ij}\, U_j \geqslant f_i\,, \quad i = 1, \ldots, N_0(h)\,; \tag{2.17}$$

$$\left(\sum_{j=1}^{N_0(h)} A_{ij}\, U_j - f_i \right) U_i = 0\,, \quad i = 1, \ldots, N_0(h)\,. \tag{2.18}$$

3. THE DISCRETE MAXIMUM PRINCIPLE (D.M.P.)

In this section we shall recall some known results on the discrete maximum principle in a form which is convenient for the following section. We shall also discuss some natural properties of the " discrete Laplacian " of a function of type $\sum_{i=1}^{n} x_i^2$.

From now on we shall assume that for every $h > 0$ the decomposition $\mathcal{T}_h$ satisfies the following condition :

For all $S \in \mathcal{T}_h$ *and for all vertex* $P \in S$ *the projection of* P *on the opposite hyperplane* $\pi_S(P)$ *falls in the closure of the opposite face* $F_S(P)$. (3.1)

Remark : In the two dimensional case (3.1) requires that all the angles are $\leqslant \pi/2$.

The following proposition is well known (see for instance [3], [4], [5]).

PROPOSITION 3.1 : *Assume that* $\mathcal{T}_h$ *satisfies* (3.1) *and let* P_i *and* P_j *be two nodes with an n-simplex* S *in common. Then*

$$\int_S \nabla \phi_i^h . \nabla \phi_j^h \, dx \leqslant 0\,. \tag{3.2}$$

It is also well known that from (3.1) one can derive the following additional properties.

THEOREM 3.1 : *Assume that* $\mathcal{T}_h$ *satisfies* (3.1). *Then the " stiffness matrix "* A *defined by* (2.14) *has the following properties* :

$$A_{ii} > 0 \quad i = 1, \ldots, N(h) \tag{3.3}$$

$$A_{ij} \leqslant 0 \quad i, j = 1, \ldots, N(h)\,, \quad i \neq j \tag{3.4}$$

$$\sum_{j=1}^{N(h)} A_{ij} = 0 \quad i = 1, \ldots, N(h)\,. \tag{3.5}$$

THEOREM 3.2 (d.m.p.) : *Assume that* $\mathcal{T}_h$ *satisfies* (3.1) *and let D be a (connected) union of n-simplexes of* $\mathcal{T}_h$. *Let* $w_h(x) \in W_h$ *be such that*

$$\int_D \nabla \phi_i^h \nabla w_h \, dx < 0 \quad \text{if } P_i \text{ is internal to } D ; \tag{3.6}$$

then

$$\max_{x \in \partial D} w_h(x) > \max_{P_j \in D \setminus \partial D} w_h(P_j) . \tag{3.7}$$

Proof : Results of type (3.7) are classical. However we shall sketch the proof for convenience of the reader. Let P_i be a node in $D \setminus \partial D$ and suppose that $w_h(P_j) \leqslant w_h(P_i)$ for all neighbouring nodes P_j. In that case (3.6), (3.4) and (3.5) give

$$0 > \int_D \nabla \phi_i^h \nabla w_h \, dx = \sum_{P_j \in D} \int_D \nabla \phi_i^h (w_h(P_j) \nabla \phi_j^h) \, dx \geqslant$$

$$\geqslant \sum_{P_j \in D} \int_D \nabla \phi_i^h (w_h(P_i) \nabla \phi_j^h) \, dx = w_h(P_i) \sum_{P_j \in D} \int_D \nabla \phi_i^h \nabla \phi_j^h \, dx = 0 , \tag{3.8}$$

which is contradictory. Hence for each internal node P_i there is at least one neighbouring node P_k where $w_h(P_k) > w_h(P_i)$ and the procedure has to stop on ∂D. ■

We end this section with some remarks on the behaviour of the discrete Laplacian of the function

$$\sigma_Q(x) = | x - Q |^2 \tag{3.9}$$

or, rather, of its piecewise linear interpolant $\sigma_Q^I(x)$ defined by

$$\sigma_Q^I(x) \in W_h ; \quad \sigma_Q^I(P_i) = \sigma_Q(P_i) \quad i = 1, 2, \dots, N(h) . \tag{3.10}$$

For this we remark first that for $P \neq Q$ we have

$$\sigma_Q(x) = \sigma_P(x) + l(x) \tag{3.11}$$

with $l(x)$ polinomial of degree $\leqslant 1$.

THEOREM 3.3 : *There exist two positive constants* δ_0, δ_1 *such that for all* $\mathcal{T}_h$ *satisfying* (2.2) *and* (3.1) *and for all* $Q \in \mathbb{R}^n$ *we have* :

$$- \delta_0 \int_\Omega \phi_i^h \, dx \leqslant a(\sigma_Q^I, \phi_i^h) \leqslant - \delta_1 \int_\Omega \phi_i^h \, dx \quad i = 1, \dots, N_0(h) . \tag{3.12}$$

Proof : Let P_i be an internal node; using (3.11) we have

$$\sigma_Q^I(x) = \sigma_{P_i}^I(x) + l(x); \tag{3.13}$$

since

$$a(l(x), \phi_i^h) = \int_\Omega \nabla l . \nabla \phi_i^h \, dx = 0 \tag{3.14}$$

we get

$$a(\sigma_Q^I, \phi_i^h) = a(\sigma_{P_i}^I, \phi_i^h) = \sum_{j=1}^{N(h)} A_{ij} \mid P_i - P_j \mid^2 = \sum_{j \neq i} A_{ij} \mid P_i - P_j \mid^2 . \tag{3.15}$$

Let now h_i' and h_i'' be respectively the minimum and the maximum value of $\mid P_i - P_j \mid$ for P_j adjacent to P_i; from (3.4) and (3.15) we have

$$(h_i'')^2 \sum_{j \neq i} A_{ij} \leqslant a(\sigma_Q^I, \phi_i^h) \leqslant (h_i')^2 \sum_{j \neq i} A_{ij} \tag{3.16}$$

and therefore from (3.16) and (3.5)

$$-(h_i'')^2 A_{ii} \leqslant a(\sigma_Q^I, \phi_i^h) \leqslant -(h_i')^2 A_{ii} . \tag{3.17}$$

It is an easy matter to check that (2.2) implies, for each node P_i,

$$c_0 (h_i')^2 A_{ii} \geqslant \int_\Omega \phi_i^h \, dx \geqslant c_1 (h_i'')^2 A_{ii} \tag{3.18}$$

with c_0, c_1 depending only on θ. Hence (3.12) follows from (3.17), (3.18).

Remark : Formula (3.12) merely expresses the fact that the " Laplacian " of σ_Q^I is bounded and strictly positive, as naturally does $\Delta \sigma_Q(x)$.

4. APPROXIMATION OF THE FREE BOUNDARIES

Let now $u(x)$ be the solution of the continuous problem (1.1); we set

$$\Omega_+ = \{ x \mid x \in \Omega, u(x) > 0 \} , \tag{4.1}$$

$$\Omega_0 = \Omega \setminus \Omega_+ ; \; \Omega_0 = \Omega \setminus \Omega_+ \tag{4.2}$$

$$F = (\partial \Omega_+) \cap \Omega ; \; \tilde{F} = \overline{\partial \Omega_+} \cap \overline{\Omega}_0 , \tag{4.3}$$

$$\Gamma_+ = (\partial \Omega_+) \cap \Gamma . \tag{4.4}$$

Moreover for any set $A \subseteq \overline{\Omega}$ and for any $\varepsilon > 0$ we set

$$\mathscr{S}_\varepsilon(A) = \{ x \mid x \in \overline{\Omega}, \text{dist}(x, A) \leqslant \varepsilon \} \tag{4.5}$$

and for any compact set $K \subset \Omega$

$$F_K = \partial\Omega_+ \cap K\,. \tag{4.6}$$

Let us recall first some results from [2] on the continuous problem (1.1).

THEOREM 4.1 : *Assume that f verifies* (1.5) *and let $u(x)$ be the solution of* (1.1); *then* :

$$\left.\begin{array}{l} \forall K \subset \Omega, \quad \forall x_0 \in \overline{\Omega}_+ \cap K \quad \exists r_0(x_0, K) \quad \exists c_0(K)\,; \quad \forall r < r_0 \\ \sup\limits_{x \in B_r(x_0) \cap K} u(x) \geqslant c_0(K)\, r^2\,; \end{array}\right\} \tag{4.7}$$

moreover, if f is smooth :

$$\left.\begin{array}{l} \forall K \quad \exists \varepsilon_1(K) \quad \exists c_1(K)\,; \quad \forall \varepsilon < \varepsilon_1(K) \\ \text{meas}\,([\mathscr{S}_\varepsilon(F_K) \cup \{ x \mid 0 < u(x) < \varepsilon^2 \}] \cap K) \leqslant c_1(K)\, \varepsilon\,; \end{array}\right\} \tag{4.8}$$

if finally $F_{\tilde{K}}$ is locally Lipschitz for some $\tilde{K}$ compact, $\tilde{K} \subset \Omega$ then :

$$\left.\begin{array}{l} \forall K \subset\subset \tilde{K} \quad \exists \varepsilon_2(K), c_2(K)\,; \quad \forall \varepsilon < \varepsilon_2(K) \\ \{ x \mid 0 < u(x) < c_2(K)\, \varepsilon^2 \} \cap K \subset \mathscr{S}_\varepsilon(F_{\tilde{K}})\,. \end{array}\right\} \tag{4.9}$$

Remark : Property (4.8) follows from Lemma 1 and Corollary 2 of [2] by a non overlapping covering argument.

In the following we shall prove different results under different regularity assumptions on the solution of the continuous problem. In particular we shall make use, at different levels, of the three following assumptions.

$$\left.\begin{array}{l} \text{A1} : \forall x_0 \in \overline{\Omega}_+ \quad \forall r > 0, \quad \textit{if } B_r(x_0) \cap \Gamma_+ = \varnothing \textit{ then} : \\ \sup\limits_{x \in B_r(x_0) \cap \Omega} u(x) \geqslant \gamma r^2 \\ \textit{with } \gamma \textit{ independent of } x_0 \textit{ and } r. \end{array}\right\} \tag{4.10}$$

$$\left.\begin{array}{l} \text{A2} : \exists \varepsilon_1 > 0 \textit{ and } \gamma_1 > 0 \textit{ such that } \forall \varepsilon < \varepsilon_1 : \\ \text{meas}\,[\mathscr{S}_\varepsilon(F) \cup \{ x \mid 0 < u(x) < \varepsilon^2 \}] \leqslant \gamma_1\, \varepsilon\,. \end{array}\right\} \tag{4.11}$$

$$\left.\begin{array}{l} \text{A3} : \exists \varepsilon_2 > 0 \textit{ and } \gamma_2 > 0 \textit{ such that } \forall \varepsilon < \varepsilon_2 : \\ \{ x \mid 0 < u(x) < \gamma_2\, \varepsilon^2 \} \subset \mathscr{S}_\varepsilon(\tilde{F})\,. \end{array}\right\} \tag{4.12}$$

Remark : Note that (4.10) is immediate if one assumes f and u in $C^0(\Omega)$; (4.11) and (4.12) are also easily proved in many particular cases. We shall now prove that a property of type (4.10) holds for the discrete solution $u_h(x)$ of (2.6).

THEOREM 4.2 : *Let* $u_h(x)$ *be the solution of* (2.6) *and assume* (1.5) *and* (3.1). *There exist two positive constants* γ_0, h_0 *such that : for all* $h \leqslant h_0$, *for all* $\rho \geqslant 2h$ *and for all* $Q \in \Omega_+^h$ *with* $B_\rho(Q) \cap \Gamma_+ = \emptyset$ *we have*

$$\sup_{x \in B_\rho(Q) \cap \Omega} u_h(x) \geqslant \gamma_0 \, \rho^2 . \tag{4.13}$$

Proof : Consider the function

$$w_h(x) = u_h(x) + \frac{c(f)}{2\,\delta_0} \sigma_Q^I(x) \tag{4.14}$$

where σ_Q^I is defined by (3.9), (3.10), $c(f)$ is defined by (1.5) and δ_0 by (3.12). Let D_+ be the connected open region of Ω containing Q and such that $u_h(x) > 0$ in D_+; let D be the biggest union of n-simplexes contained in $\overline{B_\rho(Q)} \cap D_+$. Note that D has at least one internal node. Let P_i be a node internal to D; from (1.5) and (2.11) we have

$$a(u_h, \phi_i^h) = (f, \phi_i^h) \leqslant c(f) \int_\Omega \phi_i^h \, dx \tag{4.15}$$

and from (4.14), (4.15) and (3.12)

$$a(w_h, \phi_i^h) \leqslant c(f) \int_\Omega \phi_i^h \, dx - \delta_0 \frac{c(f)}{2\,\delta_0} \int_\Omega \phi_i^h \, dx = \frac{c(f)}{2} \int_\Omega \phi_i^h \, dx < 0 . \tag{4.16}$$

We may now apply Theorem 3.2 and see that $w_h(x)$ has its maximum on a node, say P_k, on ∂D. Clearly $w_h(P_k) > 0$, so that $u_h(P_k) > 0$ and hence $P_k \notin \partial D_+$; it follows that

$$\text{dist}\,(P_k, \partial B_\rho(Q)) < h . \tag{4.17}$$

On the other hand, $w_h(P_k) > 0$ also implies

$$u_h(P_k) > \frac{-c(f)}{2\,\delta_0} \sigma_Q^I(P_k) ; \tag{4.18}$$

recall that (3.9), (3.10) give

$$\sigma_Q^I(P_k) = | P_k - Q |^2 \tag{4.19}$$

which combined with (4.17) gives

$$\sigma_Q^I(P_k) > (\rho - h)^2 ; \tag{4.20}$$

hence from (4.18), (4.20) we have for $\rho \geqslant 2h$

$$u_h(P_k) > \frac{-c(f)}{2\,\delta_0}\frac{\rho^2}{4} \tag{4.21}$$

which proves (4.13) with

$$\gamma_0 = -c(f)/8\,\delta_0\,. \quad \blacksquare \tag{4.22}$$

We shall assume from now on that an L^∞ error estimate is known for $u(x) - u_h(x)$ of the type :

$$\left.\begin{array}{ll} \text{(i)} & u = \textit{solution of}\ (1.1) \\ \text{(ii)} & u_h = \textit{solution of}\ (2.6) \\ \text{(iii)} & \| u - u_h \|_{L_\infty(\Omega)} \leqslant \eta^2(h) \\ \text{(iv)} & \lim\limits_{h \to 0} \eta(h) = 0 \\ \text{(v)} & h^{-1}\,\eta(h) \geqslant \sqrt{2\,\gamma_0}\ \textit{for } h \textit{ small enough} \end{array}\right\} \tag{4.23}$$

with, *here and in the following*, γ_0 *given by* (4.23). Estimates of type (4.23) are well known in the litterature; see for instance [1], [6], [7], [8].

The following lemma will be used in the estimate of the rate of convergence of the free boundaries.

LEMMA 4.1 : *Assume* (1.5), (3.1) *and* (4.23). *There exists an* $h_0 > 0$ *such that for all positive* $h \leqslant h_0$, *for all* $Q \in \Omega$, *and for all* $r > 0$ *with* $r \geqslant 2h$, $B_r(Q) \cap \Gamma_+ = \varnothing$ *and* $\eta^2(h) \leqslant \gamma_0 r^2$ *we have* :

$$u \not\equiv 0 \quad \text{in} \quad B_r(Q) \cap \Omega \Rightarrow u_h(Q) = 0\,. \tag{4.24}$$

Proof : Assume that $u \equiv 0$ in $B_r(Q)$ and suppose that $u_h(Q) > 0$. Apply now Theorem 4.2 to get

$$\sup_{x \in B_r(Q) \cap \Omega} u_h(x) \geqslant \gamma_0\, r^2 > \eta^2(h) \tag{4.25}$$

which contradicts (4.23) (iii).

Remark : Using (4.10) instead of (4.13) one gets

$$u_h \equiv 0 \quad \text{in} \quad B_r(Q) \cap \Omega \Rightarrow u(Q) = 0 \tag{4.26}$$

for all $r > 0$ such that $B_r(Q) \cap \Gamma_+ = \varnothing$ and $\eta^2(h) < \gamma r^2$.

LEMMA 4.2 : *Assume* (1.5), (3.1) *and* (4.23), *and set*

$$\varepsilon_1(h) = \eta(h)\sqrt{2/\gamma_0}. \tag{4.27}$$

There exists an $h_0 > 0$ *such that for all positive* $h < h_0$ *we have* :

$$\Omega_0^h \supseteq \Omega_0 \setminus \mathcal{S}_{\varepsilon_1(h)}(F). \tag{4.28}$$

Proof : Let $Q \in \Omega_0 \setminus \mathcal{S}_{\varepsilon_1(h)}(F)$. Clearly $u \equiv 0$ in $B_{\varepsilon_1(h)}(Q) \cap \Omega$ and $B_{\varepsilon_1(h)}(Q) \cap \Gamma_+ = \emptyset$ From (4.23) (v) and (4.27) we get $\varepsilon_1(h) \geqslant 2h$ for h small enough. Finally (4.27) implies $\eta^2(h) < \gamma_0\, \varepsilon_1^2(h)$; hence we are allowed to use Lemma 4.1 with $r = \varepsilon_1(h)$ and get $u_h(Q) = 0$.

The following theorem gives an estimate for the measure of the symmetric difference $\Omega_+ \div \Omega_+^h$ under the regularity assumption (4.11).

THEOREM 4.3 : *Assume* (1.5), (3.1), (4.11) *and* (4.23). *Then there exists an* $h_0 > 0$ *and a constant* $C_1 > 0$ *such that for all positive* $h \leqslant h_0$

$$\text{meas}\,(\Omega_+ \div \Omega_+^h) \leqslant C_1\, \eta(h). \tag{4.29}$$

Proof : Lemma 4.2 ensures that $\Omega_h^+ \setminus \Omega^+ = \Omega_0 \setminus \Omega_0^h \subseteq \mathcal{S}_{\varepsilon_1(h)}(F)$. On the other hand for $x \in \Omega^+ \setminus \Omega_h^+$ we have $u_h(x) = 0$ and (4.23) (iii) implies $0 < u(x) < \eta^2(h)$. Hence :

$$\Omega_+ \div \Omega_+^h \subseteq \mathcal{S}_{\varepsilon_1(h)}(F) \cup \{\, x \mid 0 < u(x) < \eta^2(h) \,\} \tag{4.30}$$

and (4.11) gives (4.29). ■

This gives already some kind of estimate on the " distance " between Ω_+ and Ω_+^h. In order to have better informations we need the stronger assumption (4.12).

THEOREM 4.4 : *Assume* (1.5), (3.1), (4.12) *and* (4.23). *There exists an* $h_0 > 0$ *and a constant* $C_2 > 0$ *such that for all* $h \leqslant h_0$ *we have* :

$$F_h \subseteq \mathcal{S}_{C_2\eta(h)}(\tilde{F}), \tag{4.31}$$

(*that is, the free boundary of the discrete problem lies in an* η*-neighbourhood of the free boundary of the continuous problem*).

Proof : Let γ_2 be the constant appearing in (4.12); set

$$\varepsilon_2(h) = \eta(h)\sqrt{2/\gamma_2} \tag{4.32}$$

and let $Q \in \Omega_+ \setminus \mathcal{S}_{\varepsilon_2(h)}(\tilde{F})$. Assumption (4.12) joined to (4.32) gives

$$u(Q) \geqslant \gamma_2\, \varepsilon_2^2(h) > \eta^2(h) \tag{4.33}$$

and from (4 23) (iii) and (4 33) we get $u_h(Q) > 0$ Hence

$$\Omega_+ \setminus \mathscr{S}_{\varepsilon_2(h)}(\tilde{F}) \subseteq \Omega_+^h \tag{4 34}$$

which added to Lemma 4 2 completes the proof

Remark We could obviously work in $K \subset\subset \Omega$ instead of Ω and get, using (4 7)-(4 9), *interior* estimates of the type

$$\text{meas}\,\{(\Omega_+ - \Omega_+^h) \cap K\} \leqslant C_1(K)\,\eta\,, \tag{4 35}$$

$$F_h \cap K \subseteq \mathscr{S}_{C_2(K)\eta}(F_K) \tag{4 36}$$

Obviously, a priori, the constants $C_1(K)$ and $C_2(K)$ depends on K It should be noted, however, that (4 35), (4 36) hold under very general assumptions

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