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On the vibration problem for an elastic body surrounded by a slightly compressible fluid

ON THE VIBRATION PROBLEM
FOR AN ELASTIC BODY
SURROUNDED BY A SLIGHTLY COMPRESSIBLE FLUID (*)

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Abstract. — *The vibration problem for an elastic bounded body immersed in an ideal fluid with small compressibility (associated with the small parameter ε ↝ 0) is considered in the framework of scattering theory. It is shown that the scattering frequencies converge as ε ↝ 0 to the eigenfrequencies of the selfadjoint problem associated with an incompressible fluid. The asymptotic behaviour of the eigenfrequencies for small ε is given via the implicit function theorem. Some general considerations about the physics of the problem and experimental results are also given.*

Résumé. — *Nous considérons le problème de vibration d'un corps élastique borné plongé dans un fluide idéal de petite compressibilité (associé au petit paramètre ε ↝ 0) dans le cadre de la théorie de la diffusion (scattering). Nous montrons que les fréquences de scattering convergent lorsque ε ↝ 0 vers les fréquences propres du problème autoadjoint correspondant à un fluide incompressible. Le comportement asymptotique des fréquences propres pour ε petit est obtenu via le théorème de fonctions implicites. Nous donnons aussi certains résultats expérimentaux et des considérations générales sur le problème physique.*

I. GENERALITIES ABOUT THE PHYSICAL PROBLEM

It is proposed to investigate the problem of the low-frequency forced vibrations of a structure immersed in an infinite liquid medium.

The main response parameters are :

— the displacement field at any point of the structure,

— the pressure field in the neighbourhood of the structure (the so-called near-field pressure),

— the far-field pressure,

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as functions of the excitation frequency (considered as a continuously increasing parameter).

It should be emphasized that the experimental analysis follows the above scheme by using displacement and pressure devices, the excitation frequency parameter being controlled by the experimentalist through appropriate devices.

If L (resp. Λ) denotes a characteristic length of the structure (resp. the wavelength of the radiated wave), the low-frequency range considered here corresponds to the case $\Lambda \gg L$. Experimental analysis carried on test specimens of revolution immersed in an " infinite " liquid medium with a free surface — for example, ring-stiffened cylinders — have shown that, for low-frequency vibration, the response curves present " resonant peaks " similar to classical slightly damped conservative linear systems analysed theoretically through asymptotic expansions [2]. Moreover, the values of the frequencies corresponding to those peaks were " practically " the same for the three field parameters described above. Therefore, a new set of response parameters is :

— the " resonant frequencies " corresponding to the successive peaks of the response spectrum and the corresponding structural displacement field and liquid near- and far-field pressure.

Let us now discuss the origin of the damping of the fluid-structure coupled system-.

Refering again to the experimental analysis, for example for a test specimen of 1 m height and 0.3 m of radius, for a range of low-frequency between 40 Hz and 120 Hz, the damping effects arising either from the radiation of gravity waves or from the viscosity of the fluid were negligible in comparison with the two other sources of damping, namely the one occuring in the structure dissipation and the one arising from the compressibility of the liquid (radiation). It should be recalled that for *bounded* fluid-structure systems, the system is conservative even if the fluid is compressible [8, 10], which is not the case if the external fluid volume is infinite as a consequence of the radiation phenomena.

It results from the above experimental considerations and from theoretical and numerical investigations (without a free surface in [2], and with a free surface in [9]), that the conservative fluid-structure system associated with the real physical system corresponds to the following hypotheses :

— elastic structure,
— *incompressible* liquid,
— free surface with a Dirichlet boundary condition $\varphi = 0$ (φ denotes the displacement potential of the fluid).

The last condition will not be considered in the present paper. Experimental

investigations bring to the conclusion that — for instance, in the case of a vertical cylinder immersed in a liquid at a 1 meter depth — the results differ slightly from the case where the specimen is completely surrounded by the fluid filling the whole space. We have already said that the case considered here corresponds to $\Lambda \gg L$. By considering the Helmholtz equation in the fluid, it is easy to see that the behaviour of the liquid in the neighbourhood of the structure is governed by the Laplace equation $\Delta \varphi = 0$ (incompressible limit). This fact is verified by the experimental analysis (for the low-frequency behaviour) as very sharp peaks are observed concerning the structural displacement and the very near pressure field.

f Hz	Experimental	Computed (Incompressible liquid)
1	70,5	71,2
2	80,2	81
3	120	121

The following table shows the validity of the incompressible assumption concerning the " eigenfrequencies " of a cylinder immersed in a liquid [9].

On the other hand, it seems that the radiation effect due to the compressibility of the fluid acts at large distances from the structure i.e., on the far-field pressure, in the low frequency range.

The preceeding features of the experimental phenomena, in particular the different behaviour in near- and far-field pressures is explained by the scattering theory (see for instance [7] or [11]) if a slightly compressible fluid is considered.

The behaviour of the scattering functions (which play in some sense the role of eigenfunctions in unbounded domains) in the far field area, i.e. the exponentially large behaviour with respect to the distance r from the structure ([11], sect. 15.7) justify the fact that the damping is predominant far from the structure.

In this paper we give a theoretical study of the scattering frequencies of vibration of an elastic body in a slightly compressible fluid. An asymptotic expansion of the scattering frequencies, based on the implicite function theorem is given as well as explicite formulae for the first order term, suitable for further numerical computation.

Finally we recall that dissipative viscosity effects are not been taken into account here. See [2], [14] for some considerations about that question. We also point out the paper [6] on a related problem.

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II. MATHEMATICAL FORMULATION OF THE PROBLEM

We consider an elastic body occuping the bounded connected domain Ω_s of R^3. The outer region of R^3 (see *fig.* 1), Ω_f is occupied by a compressible fluid with small compressibility. The interface between Ω_s and Ω_f is the smooth surface Σ, and we denote **n** its unit outer normal. In order to introduce some mathematical simplifications, we assume that the body is clamped at some inner surface Γ, but this point is not essential.

We consider throughout the paper a time dependence in $e^{-i\alpha t}$ where α is some (in general complex) constant (perhaps the scattering frequency to be obtained).
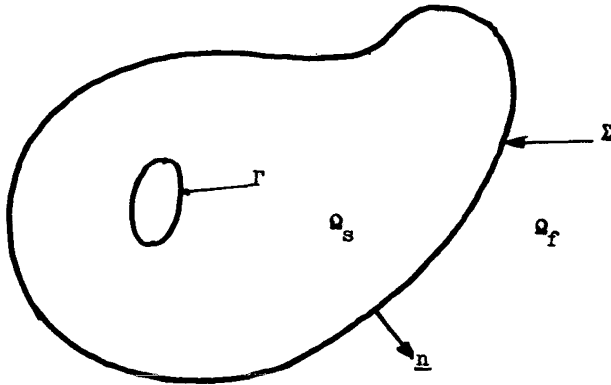

Figure 1.

If $\mathbf{u}(x)$ denotes the displacement vector (in fact it is $\mathbf{u}(x) \exp(-i\alpha t)$), the equation to be satisfied in Ω_s is :

$$- \alpha^2 \rho_s u_i = \frac{\partial \sigma_{ij}}{\partial x_j} \quad \text{in} \quad \Omega_s$$

$$\sigma_{ij} = a_{ijkl} e_{kl}(\mathbf{u}); \quad e_{kl}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \qquad (2.1)$$

where ρ_s is the (constant) density of the solid and a_{ijkl} are its elastic constants, which are assumed to satisfy the standard positivity and symmetry conditions (see for instance [11], sect. 6.1).

In the fluid region, the displacement vector is assumed to be the gradient of a potential φ satisfying the Helmholtz equation

$$- \alpha^2 \varphi - c^2 \Delta\varphi = 0 \quad \text{in} \quad \Omega_f \qquad (2.2)$$

where c is the propagation velocity, which is of the form :

$$c^2 = \frac{1}{\rho_f \, \varepsilon} \tag{2.3}$$

where ρ_f is the density of the fluid and ε is a real small parameter which will tend to zero (this is the mathematical meaning of the small compressibility condition). By introducing the new small parameter η and ω :

$$\eta = \sqrt{\rho_f \, \varepsilon} \, ; \quad \omega = \alpha \eta \tag{2.4}$$

equation (2.2) becomes :

$$- \omega^2 \, \varphi - \Delta \varphi = 0 \quad \text{in} \quad \Omega_f \, . \tag{2.5}$$

The pressure in the fluid is given by :

$$p = \frac{\omega^2}{\varepsilon} \, \varphi = \frac{- \Delta \varphi}{\varepsilon} \, . \tag{2.6}$$

The boundary conditions are :

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma \tag{2.7}$$

$$\mathbf{u} . \mathbf{n} = \frac{\partial \varphi}{\partial n} \quad \text{on} \quad \Sigma \tag{2.8}$$

$$\sigma_{ij} \, n_j = - p n_i = - \frac{\omega^2}{\varepsilon} \, \varphi \tag{2.9}$$

where (2.7) expresses the clamping at Γ, (2.8) is the continuity of the normal component of the displacement on Σ and (2.9) is the continuity of the stress across Σ. It is to be noticed that, as the fluid is inviscid, there may be a discontinuity of the tangential displacement at Σ. Moreover, we shall impose an *outgoing radiation condition* on φ, i.e., φ is a convolution of the outgoing fundamental solution

$$\frac{-1}{4 \pi} \, \frac{e^{i \omega r}}{r} \, ; \quad r = | x - y | \, . \tag{2.10}$$

As we shall see later, we shall consider " finite " values of α (i.e. not tending to infinity as ε or η tend to zero) and according to (2.4) the corresponding values of ω will tend to zero. This enables us solving (2.5), (2.8) with the radiation condition if $\mathbf{u} . \mathbf{n}$ is considered as a given function on Σ. This amounts to solve the exterior Neumann problem for (2.5), with ω in a neighbourhood of the origin, and this is possible because $\omega = 0$ is not a scattering frequency of this problem (see [11], chap. 15, theorem 7.3).

Then, according to the general theory (see [11], sect. 17.2 or remark 2.1 here after), there exists a holomorphic function $T(\omega)$ defined in a neighbourhood of $\omega = 0$ with values in the space of the compact operators from $L^2(\Sigma)$ into itself such that if $\varphi \mid_\Sigma$ is the trace on Σ of the solution of (2.5), (2.8) with the radiation condition, we have :

$$\varphi \mid_\Sigma = T(\omega)\,(\mathbf{u.n}) \mid_\Sigma . \tag{2.11}$$

Remark 2.1 : The general proof of the properties of $T(\omega)$ is here simplified, because $\omega = 0$ is not an eigenvalue of the interior Dirichlet problem in Ω_s, and consequently the solution of the exterior Neumann problem may be searched as a single layer potential on Σ (see [11], sect. 15.5 or [12], sect. 23.1 if necessary); the operator giving the density of the single layer as a function of $(\mathbf{u.n}) \mid_\Sigma$ is compact in $L^2(\Sigma)$ (see [11], chap. 15, Lemma 5.1, part T_{11}) as well as the operator giving φ on Σ as a function of the density (see [11], chap. 15, formula (4.8) and theorem 4.2). ∎

By using (2.11) we may reduce our problem to another one for the displacement $\mathbf{u}$ in Ω_s, satisfying (2.1), (2.7) and (2.9). Then, (2.9) is a Neumann condition whose " given " term φ is, *via* (2.11), an operational function of $\mathbf{u}$.

We define the functional space :

$$V = \{ \mathbf{u} \in \mathbf{H}^1(\Omega_s);\ \ \mathbf{u} \mid_\Gamma = 0 \}$$

endowed with the scalar product

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega_s} a_{ijkl}\, e_{kl}(\mathbf{u})\, e_{ij}(\bar{\mathbf{v}})\, dx \tag{2.12}$$

which defines a norm which is equivalent to the norm induced by $\mathbf{H}^1(\Omega_s)$. Then, solving (2.1), (2.7), (2.9) with (2.11) amounts finding $\mathbf{u} \in V$ such that for any test function $\mathbf{v} \in V$,

$$(\mathbf{u}, \mathbf{v})_V = \alpha^2\, \rho_s \int_{\Omega_s} u_i\, \bar{v}_i\, dx - \alpha^2\, \rho_f \int_\Sigma [T(\omega)\,(u_i\, n_i)]\, (\bar{v}_j\, n_j)\, d\Sigma . \tag{2.13}$$

Now we may define the scattering frequencies α and the associated scattering functions $\mathbf{u}$ according to the general theory (see [11], chap. 15, if necessary) :

DEFINITION 2.1 : *For each real (small) value of* η *the scattering frequencies* $\alpha(\eta)$ *of the problem under consideration are the (complex, in general) values of* α *such that* (2.13) *has a non-zero solution* $\mathbf{u}$ *(which is the corresponding scattering function). Of course we have* $\omega = \eta\alpha$.

Remark 2.2 : The scattering function **u**, as defined here is an element of V, and consequently a function defined on Ω_s. It is clear that it may be uniquely continued to Ω_f by **grad** φ, where φ is defined by (2.5), (2.8) and the radiation condition. ■

In order to write (2.13) as an operator equation we define the operators B_1 and $B_2(\omega)$ such that

$$\rho_s \int_{\Omega_s} u_i \, \bar{v}_i \, dx = (B_1 \, \mathbf{u}, \mathbf{v})_V \quad \forall \mathbf{v} \in V \qquad (2.14)$$

$$- \rho_f \int_\Sigma \left[T(\omega) \, (u_i \, n_i) \right] (\bar{v}_j \, n_j) \, d\Sigma = (B_2(\omega) \, \mathbf{u}, \mathbf{v})_V \quad \forall \mathbf{u}, \mathbf{v} \in V . \qquad (2.15)$$

LEMMA 2.1 : *The operators B_1, $B_2(\omega)$ are well defined by (2.14) (2.15). They are compact operators from V into V and B_2 is holomorphic of ω in a neighbourhood of $\omega = 0$. (More exactly, $B_2(\omega)$ is a holomorphic function with values in $\mathscr{L}(V, V)$).*

Proof : The compactness of B_1 follows from the compact embedding of H^1 into L^2 because B_1 is in fact continuous from $\mathbf{L}^2(\Omega)$ into V. The compactness of $B_2(\omega)$ follows from analogous considerations and from the trace theorem : in fact, by taking in (2.15) $\mathbf{v} = B_2(\omega) \, \mathbf{u}$ (we omit, as in (2.15) the cumbersome expression of the trace operator) we have :

$$\| B_2(\omega) \, \mathbf{u} \|_V \leqslant C \| T(\omega) \, (u_i \, n_i) \|_{\mathbf{L}^2(\Sigma)} \qquad (2.16)$$

then, if we take a sequence $\mathbf{u}^k$ weakly V-convergent, the corresponding traces $u_i^k \, n_i$ are strongly convergent in $L^2(\Sigma)$ according to (2.16), $B_2(\omega) \, \mathbf{u}^k$ strongly converges in V and $B_2(\omega)$ is compact. As for the holomorphy, it follows from the holomorphy of $T(\omega)$ (see [11]), chap. 11, proposition 4.3 if necessary). ■

As a consequence, (2.13) may be writen :

$$\mathbf{u} = \alpha_2(B_1 + B_2(\omega)) \, \mathbf{u} \; ; \quad \omega \equiv \alpha\eta \qquad (2.17)$$

and we obtain a new equivalent definition of the scattering frequencies.

DÉFINITION 2.2 : *For each real (small) value of η the scattering frequencies $\alpha(\eta)$ are the values of α such that 1 is an eigenvalue of the compact operator $\alpha^2(B_1 + B_2(\eta\alpha))$.*

III. CONTINUITY AND LIMIT VALUE OF THE SCATTERING FREQUENCIES

Before going on the study of the scattering frequencies, let us consider the (formal) limit case $\eta = 0$. According to definition 2.2, the scattering frequencies are the values α such that α^{-2} are eigenvalues of the operator $B_1 + B_2(0)$. The corresponding problem is that of the vibration of an elastic body surrounded by an incompressible fluid; it was studied in [2], section 2. The operator $B_1 + B_2(0)$ is symmetric (and of course compact) in V and zero is not an eigenvalue of it. This leads us the theorem 3.1 here after. Nevertheless, for the sake of completeness, let us prove that $B_2(0)$ is symmetric (as for B_1 this is evident). Let us consider (2.15) with $\omega = 0$. If the elements $\mathbf{u}$ and $\mathbf{v}$ of V are given, we also consider the corresponding potentials φ and ψ which solve (2.5), (2.8) in Ω_f. Note that (2.5) is now the Laplace equation; according to the standard theory of harmonic functions at infinity (see [12], sect. 204 if necessary), φ and ψ vanish at infinity as $|x|^{-1}$ and $\mathbf{grad}\,\varphi$, $\mathbf{grad}\,\psi$ as $|x|^{-2}$. We have :

$$(B_2(0)\,\mathbf{u}, \mathbf{v}) = -\rho_s \int_\Sigma \varphi \frac{\partial\overline{\psi}}{\partial n}\, d\Sigma = -\rho_f \int_\Sigma \varphi \frac{\partial\overline{\psi}}{\partial x_i} n_i\, d\Sigma =$$

$$= \rho_f \int_{\Omega_f} \frac{\partial}{\partial x_i}\left(\varphi \frac{\partial\overline{\psi}}{\partial x_i}\right) dx = \rho_f \int_{\Omega_f} \frac{\partial\varphi}{\partial x_i} \frac{\partial\overline{\psi}}{\partial x_i}\, dx \quad (3.1)$$

where the (not writen) « surface integral of infinity » vanishes because φ grad φ behaves as $|x|^{-3}$ and the surface of the sphere is of order $|x|^2$. The symmetry of $B_2(0)$ follows from (3.1). Moreover, the fact that 0 is not an eigenvalue of $B_1 + B_2(0)$ immediatly follows from (2.14) and (3.1) by taking $\mathbf{u} = \mathbf{v}$ the corresponding eigenvector; we see that $\mathbf{u} = 0$. We consequently have :

THEOREM 3.1 : *The operator $B_1 + B_2(0)$ is compact, symmetric and positive definite on V. Consequently, it possesses eigenvalues which will be denoted by :*

$$\frac{1}{\alpha_1^2} \geqslant \frac{1}{\alpha_2^2} \geqslant \cdots \geqslant \frac{1}{\alpha_j^2} \geqslant \cdots \to 0 \quad (3.2)$$

and the corresponding eigenvectors $\mathbf{v}_1\,\mathbf{v}_2, ...,$ may be taken in such a way that they form an orthonormal basis in V. Correspondingly, for $\varepsilon = \eta = 0$, the scattering frequencies are the real values $\pm\,\alpha_j$, with the associated scattering functions $\mathbf{v}_j$.

Now, we come back to the scattering frequencies for $\eta \geqslant 0$ (see definition 2.2). If we consider the operator $\alpha^2(B_1 + B_2(\eta\alpha))$, it is obviously holo-

morphic in α for fixed η, and jointly continuous in α and η with values in $\mathscr{L}(V, V)$. Moreover, for $\alpha = 0$ the operator

$$[I - \alpha^2(B_1 + B_2(\eta\alpha))]^{-1} \qquad (3.3)$$

is the identity and consequently linear bounded from V to V. Consequently the hypotheses of the classical theorem on continuous dependence on η of the singularities of an operator of the form (3.3) (see [13] or [11], chap. 15, theorem 7.2) applies if we take as D any bounded domain of the α-plane containing the origin. This gives :

THEOREM 3.2 : *The scattering frequencies* $\alpha(\eta)$ *depend continuously on* η *and converge as* $\eta \to 0$ *the values* $\pm \alpha_j$ *of theorem 3.1.*

Remark 3.1 : In the preceding theorem, continuity is understood in the sense that the scattering frequencies for each value η^* are exactly the accumulation points of the scattering frequencies corresponding to the values $\eta \to \eta^*$. But a scattering frequency may split in several ones. On the other hand, no information is given about the corresponding scattering functions. More information is obtained later by using the holomorphic properties.

IV. HOLOMORPHY PROPERTIES OF SCATTERING FREQUENCIES AND FUNCTIONS

In this section we use the fact that the operator $B_1 + B_2(\omega)$ is holomorphic as a function of ω in order to obtain sharper results.

We consider for small η, the scattering frequencies $\alpha(\eta)$ which converge to a given scattering frequency of the unperturbed problem (i.e., one of the values α_j of the theorem 3.1) which we shall denote $\alpha^{(0)}$; we change a little the notation ; the index j is no more written, and $^{(0)}$ expresses « unperturbed value ».

Let us first consider the case where $(\alpha^{(0)})^{-2}$ is a simple eigenvalue of $B_1 + B_2(0)$. According to standard analytic perturbation theory (Kato [3]) the corresponding eigenvalue $\lambda(\omega)$ is a holomorphic function of ω for small ω, and from definition 2.2 we obtain the implicite equation,

$$\frac{1}{\alpha^2} - \lambda(\eta\alpha) = 0 \qquad (4.1)$$

to obtain α as a function of η. The implicit function theorem then gives the existence and local uniqueness of the holomorphic function $\alpha(\eta)$ for small η. Indeed, the derivative of the function in (4.1) with respect to α is :

$$\frac{-2}{\alpha^3} - \lambda'(\eta\alpha)\,\eta \qquad (4.2)$$

is not zero for $\eta = 0$, $\alpha = \alpha^{(0)}$.

Moreover, the corresponding (simple) eigenvalue is a holomorphic function of η with values in V (with an appropriate normalization).

Remark 4.1 : As $B_2'(\omega)$ is defined for complex ω in a eighbourhood of $\omega = 0$, the parameter η may be taken to be complex (with small modulus) and α is then a holomorphic function of η in a neighbourhood of the origin. ∎

We now consider the general case where $(\alpha^{(0)})^{-2}$ is an eigenvalue of $B_1 + B_2(0)$ with multiplicity m. According to the general theory (which is recalled here after) the unperturbed eigenvalue splits for $\omega \neq 0$ into several eigenvalues with total multiplicity m; each eigenvalue is a holomorphic function of a fractional power of ω, and the preceeding equation (4.1) must be modified in a suitable way.

It is known ([3], chap. VII, sect. 1.3) that the splitting problem is analogous to that in a m-dimensional space, and consequently it amounts to the splitting of the roots of an m-order polynomial (the determinant of the eigenvalue problem for a m-dimensional matrix). (See in this connection [3], chap. II, sect. 1.1, 1.2; additional explanations may be find in [1], chap. IX and [4], vol. II, chap. 5). Then, in a neighbourhood of $\omega = 0$, the m eigenvalues are decomposed into groups of p_j eigenvalues with $\Sigma p_j = m$; each group is a p-cycle (the index j of the cycle is no more written) associated with an irreducible polynomial (with a suitable definition of « irreducibility ») and is formed by p simple eigenvalues in a neighbourhood of the origin $\omega = 0$ (the neighbourhood is « punctured » i.e. the point $\omega = 0$ is removed); these simple eigenvalues are holomorphic functions of ω in this punctered neighbourhood, and they cyclically commute when a simple positive curve is described around $\omega = 0$, this is to say, when ω turns once around $\omega = 0$, the eigenvalue λ_i becomes λ_{i+1} for $i = 1, 2, ..., p - 1$ and λ_p becomes λ_1. Consequently, they are uniform holomorphic functions of the variable $\omega^{1/p}$ in the punctured neighbourhood of the origin and they have Laurent expansions in the vicinity of the origin. Because they are bounded (they converge to the unperturbed eigenvalue) they are in fact holomorphic functions of $\omega^{1/p}$. Moreover, it is easy to see from the preceeding considerations that they are in fact the same holomorphic function of $\omega^{1/p}$ for the p distinct determinations of the p-root. To complete this description, we point out that several p-cycles may in fact coincide, and consequently that the corresponding roots are multiple.

In the problem under consideration, there are some particular features as a consequence of the fact that the unperturbed operator is symmetric, and consequently the unperturbed eigenvalue is semisimple (see [3] chapter II, sect. 2.3). In this case, each p-cycle has a Taylor expansion where the coeffi-

cients of the powers $1, 2, ..., p - 1$ of $\omega^{1/p}$ are zero, that is to say the eigenvectors of the cycle have the form

$$\lambda(\omega) = \lambda^{(0)} + \omega\lambda^{(1)} + \omega\varphi(\omega^{1/p}) \tag{4.3}$$

with φ holomorphic of the variable $\omega^{1/p}$ in the vicinity of the origin. The p eigenvectors are associated with the p determinations of the p-root.

Then in order to find the scattering frequencies $\alpha(\eta)$ we must study the implicit equation (4.1) with (4.3). It is suitable to introduce the new variables

$$\beta = \alpha^{1/p}; \quad \delta = \eta^{1/p} \Rightarrow \omega^{1/p} = \delta\beta \tag{4.4}$$

where δ is taken real and positive then, (4.1), (4.3) becomes :

$$\frac{1}{\beta^{2p}} - \lambda^{(0)} - \delta^p \beta^p \lambda^{(1)} - \delta^p \beta^p \varphi(\delta\beta) = 0 \tag{4.5}$$

which is an implicit equation to obtain $\beta(\delta)$, to be studied in the vicinity of the points

$$\delta = 0, \quad \beta = \alpha(0)^{1/p} = \left(\frac{1}{(\lambda^{(0)})^{1/2}}\right)^{1/p} \tag{4.6}$$

which are p distinct points for the p determinations of the p-root. It is seen as in (4.2) that the derivative with respect to β of the function in (4.5) is not zero at the points (4.6) and consequently β, and then $\alpha = \beta^p$ is a holomorphic function of $\delta = \eta^{1/p}$ in a neighbouhood of the origin (here as in remark 4.1, $\delta = \eta^{1/p}$ may be taken complex). Consequently, in a neighbourhood of $\eta = 0$ the scattering frequencies may be written in the form

$$\alpha(\eta) = \Psi(\eta^{1/p}) \tag{4.7}$$

and of course the scattering functions are the corresponding eigenvectors. If there are two or more identical cycles, the corresponding eigenvalues are multiple and the same is true for the corresponding cycles (4.7). Moreover, from an elementary study of the implicit function it follows that $\alpha(\eta) - \alpha^{(0)}$ is of order η and the differential takes the same value for all eigenvectors of the p-cycle (see explicit computations in sect. 5); consequently, the scattering frequencies have a form analogous to (4.3) :

$$\alpha(\eta) = \alpha^{(0)} + \eta\alpha^{(1)} + \eta\psi(\eta^{1/p}). \tag{4.8}$$

We sum up these results in the

THEOREM 4.1 : *In the vicinity of* $\eta = 0$, *each unperturbed scattering frequency* $\alpha^{(0)}$ *associated with an eigenvalue of* (3.2) *with multiplicity m splits into p-cycles of the form* (4.8), *each one is a holomorphic function of* $\eta^{1/p}$ *in a neighbourhood of the origin and each p-cycle is associated with the corresponding p-cycle of the splitting* (4.3) *of the corresponding eigenvalue as a function of* ω. *As a consequence the multiplicity are the same as in the splitting* (4.3). *The scattering functions are of course the corresponding eigenvalues.*

Remark 4.2 : In the case where the values $\lambda^{(1)}$ in (4.3) are different for the m eigenvalues, the p-cycles are distinct 1-cycles, that is to say, the m eigenvalues (and consequently the m scattering frequencies) are m distinct holomorphic functions taking the same value for $\varepsilon = 0$. This case is studied in detail in section 5.

V. EXPLICIT COMPUTATION

In order to obtain the first term $\alpha^{(1)}$ of the expansion of the scattering frequency it suffices to differenciate (4.1) with (4.3) at $\eta = 0$; this gives :

$$\alpha^{(1)} = \frac{-1}{2} (\alpha^{(0)})^4 \lambda^{(1)} \equiv \frac{-\lambda^{(1)}}{2(\lambda^{(0)})^2} \tag{5.1}$$

and the problem reduces to the computation of $\lambda^{(1)}$. In order to perform explicit calculations, we admit that :

HYPOTHESIS : *The eigenvalues are holomorphic in* ω, *i.e., in the framework of* (4.3) *and remark* 4.2, $\lambda^{(1)}$ *takes m distinct values.*

We shall see that this hypothesis amounts to the fact that the eigenvalues of a certain matrix are distinct. If $B_2(\omega)$ has an expansion

$$B_2(\omega) = B_2(0) + \omega B_2'(0) + \cdots \tag{5.2}$$

we expand the eigenvalue problem under the form

$$[B_1 + B_2(\omega)] \mathbf{u}(\omega) = \lambda(\omega) \mathbf{u}(\omega) \tag{5.3}$$

$$\mathbf{u}(\omega) = \mathbf{u}_j^{(0)} + \omega \mathbf{u}_j^{(0)} + \cdots \tag{5.4}$$

$$\tilde{\ }\ \lambda(\omega) = \lambda^{(0)} + \omega \lambda_j^{(1)} + \cdots \tag{5.5}$$

where $j = 1, 2, ..., m$ is the index for each eigenvalue splitting from the

unperturbed eigenvalue $\lambda^{(0)}$ (with multiplicity m). The expansion of (5.2) gives at the two first orders :

$$(B_1 + B_2(0) - \lambda^{(0)}) \mathbf{u}_j^{(0)} = 0 \tag{5.6}$$

$$(B_1 + B_2(0) - \lambda^{(0)}) \mathbf{u}_j^{(1)} = - (B_2'(0) - \lambda_j^{(1)}) \mathbf{u}_j^0 . \tag{5.7}$$

Relation (5.6) only expresses that $\mathbf{u}_j^{(0)}$ ($j = 1, ..., m$) are eigenvectors of the unperturbed problem. Let us decompose the space V under the form

$$V = V_0 \oplus V_0^\perp \tag{5.8}$$

where V_0 is the m-dimensional eigenspace associated with $\lambda^{(0)}$ and $V_0^\perp$ is its orthogonal complement. If we consider (5.7) as an equation for the unknowns $\mathbf{u}_j^{(1)}$ ($j = 1, ..., m$), the Fredholm alternative shows that the compatibility condition for the existence of a solution is

$$0 = ((B_2'(0) - \lambda_j^{(0)}) \mathbf{u}_j, \mathbf{v})_V = 0 \quad \forall \mathbf{v} \in V_0 \tag{5.9}$$

and by choosing a basis in V_0, (5.9) is an eigenvalue problem for a $m \times m$ matrix in V_0. We shall admit that this problem has m distinct eigenvalues. They are the $\lambda_j^{(0)}$ we were looking for. The study of this problem gives us $\mathbf{u}_j^{(0)}$ and the following terms $\mathbf{u}_j^{(1)}$ may be computed from (5.7) by standard calculations.

Next, the operator $B_2'(0)$ may be defined from (2.15) by differenciating with respect to ω, if $T'(0)$ is known.

In order to study this operator we solve the exterior Neumann problem for small ω by a single-layer potential with density v (see remark 2.1)

$$\varphi(x, \omega) = \int_{\Sigma_y} v(y, \omega) \, \psi^+(r, \omega) \, d\Sigma_y \tag{5.10}$$

$$\psi^+(r, \omega) = \frac{-1}{4\pi} \frac{e^{i\omega r}}{r}; \quad r = |x - y| \tag{5.11}$$

and the integral equation for v is (see [11], chap. 15, (4.10), (4.11) and (5.5) if necessary) :

$$v(x) = 2 \int_{\Sigma_y} v(y) \frac{\partial \psi^+}{\partial n_x} d\Sigma_y + 2 f = S(\omega) v + 2 f \tag{5.12}$$

$$\frac{\partial \psi^+}{\partial n_x} \equiv - \frac{1}{4\pi} \frac{d}{dr} \left(\frac{e^{i\omega r}}{r} \right) \cos (r, n_x) \tag{5.13}$$

where the operator $S(\omega)$ defined in (5.12) is holomorphic in ω in a neighbourhood of the origin with values in $\mathscr{L}_{comp}(L^2(\Sigma))$, compact. It is easily seen from the series expansion of $e^{i\omega r}$ (see [5]) that the operator $S(\omega)$ writes for small ω :

$$S(\omega) = S^{(0)} + \omega^2 S^{(2)} + O(\omega^3) \tag{5.14}$$

where

$$S^{(2)} v(x) = \frac{1}{2\pi} \int_{\Sigma_y} \frac{\partial r}{\partial n_x} v(y) \, d\Sigma_y \tag{5.15}$$

and the solution of (5.12) is given by :

$$v(\omega) = [I - S^{(0)} \pm \omega^2 S^{(2)} + O(\omega^3))]^{-1} \, 2 \, f$$

and by using the Neumann series we see that the expansion of v is of the form :

$$v(\omega) = v^{(0)} + \omega^2 v^{(2)} + O(\omega^3). \tag{5.16}$$

By expansion of (5.10) in powers of ω we see that the value of φ on Σ (which is of course the direct value of (5.10)) is given by

$$\varphi(\omega)\,|_\Sigma = \frac{+1}{2\pi} \int_{\Sigma_y} (v^{(0)} + O(\omega^2)) \left(\frac{1}{r} + i\omega + O(\omega^2) \right) d\Sigma_y$$

and it follows that

$$\frac{\partial \varphi}{\partial \omega}\bigg|_{\omega=0} (x) = \frac{+i}{2\pi} \int_{\Sigma_y} v^{(0)}(y) \, d\Sigma_y$$

and consequently :

$$T'(0) f = \frac{+i}{2\pi} \int_{\Sigma_y} v^{(0)}(y) \, d\Sigma_y \tag{5.17}$$

where f is the given value of the normal derivative

$$\frac{\partial \varphi}{\partial n} = \mathbf{u}.\mathbf{n} \tag{5.18}$$

and $v^{(0)}$ is the density of the simple layer solving the exterior Neumann problem for $\omega = 0$ and the given values (5.18).

On this basis, the behaviour of the scattering function at large distance is easily obtained. From (5.10), we see that the displacement potential φ is constructed with the elementary solution $\psi^+(r, \omega)$ given by (5.11); for large r

its amplitude is associated with the real part of the exponent : as $\alpha^{(0)}$ is real we have :

$$\text{Re} \{ i\omega r \} = \text{Re} \{ ri\eta(\alpha^{(0)} + \eta\alpha^{(1)} + \cdots) \} = - \eta^2 r(\text{Im } \alpha^{(1)}) \Rightarrow$$

$$\left| \frac{e^{i\omega r}}{r} \right| \sim \frac{\eta^2}{R} e^{-R \text{Im} \alpha^{(1)}} ; \quad R = \eta^2 r \qquad (5.19)$$

where the exponent is positive according to the general properties of the scattering frequencies ([11], sect. 17.7). We see that R is the characteristic coordinate for the far field. Nevertheless, it is to be noticed that the corresponding behaviour is only used in the inner region of a wave front ([11], sect. 15.8) associated with the initial values of the problem. As the sound velocity in the fluid is of order η^{-1}, the far field region is only attained a time of order η^{-1} after the initial moment (see *fig.* 2).

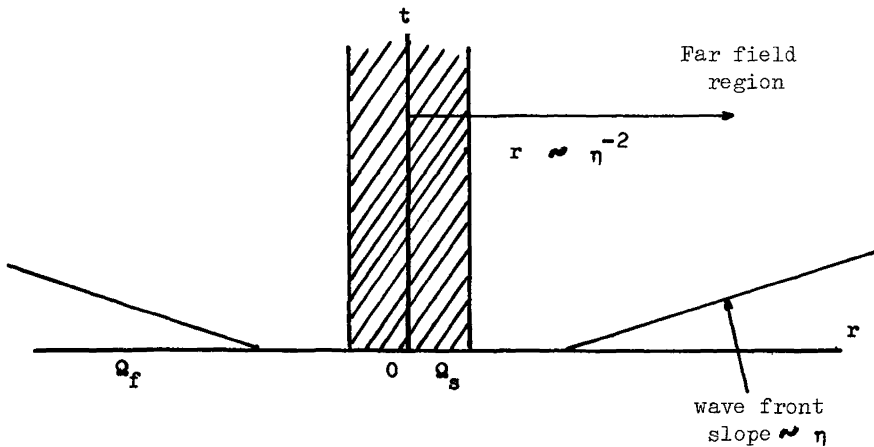
Figure 2.

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