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On the existence and the regularity of an initial boundary problem of vorticity equation

ON THE EXISTENCE AND THE REGULARITY
OF AN INITIAL BOUNDARY PROBLEM
OF VORTICITY EQUATION (*) (**)

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Abstract. — *We consider the Euler equation for incompressible fluids in a two dimensional domain depending on the time. In this situation we prove the existence of a weak solution and the existence and uniqueness of a smooth solution when the size of the domain increases fast enough with the time.*

Résumé. — *On considère l'équation d'Euler dans un domaine à deux dimensions variant avec le temps. Dans cette situation nous prouvons l'existence d'une solution faible et l'existence et l'unicité d'une solution forte quand la taille du domaine augmente avec le temps.*

I. INTRODUCTION

This article is devoted to a slightly generalised version of the Euler equation. This equation is used in numerical weather prediction (*cf.* Kuo Pen Yu [2, 5]). The main differences with the classical Euler Equation are the following. We deal with non homogeneous boundary conditions and we assume that the domain depends on the time. We will study the existence of a weak solution and in some cases the existence and uniqueness of a strong solution. The proofs follow for weak and strong solution the methods of Bardos [1], Kato [2], Wolibner [8] and Schaeffer [7]. However the main improvement is the following. With non homogeneous boundary conditions it is in general impossible to prove the regularity of the solution. This is due to the very singular behaviour of the solutions of the transport equation in a bounded domain with change

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of type of boundary condition. This situation may be improved when the domain increases with the time. To ensure the regularity this increase of size of the domain must be large enough with respect to the size of the initial data.

Computational experience have been made for the Euler equation with artificial boundary condition and it has been noticed that the computation becomes stable when the size of the domain increases with the time. The regularity result which we prove here is the continuous version of this observation.

To make the reading easier we focus on the a priori estimates, when these estimates are proven the rest of the work is mere routine and we leave it to the reader. In section II we present the equations and we prove the existence of a weak solution. In section III we prove the regularity result and the uniqueness of the solution.

II. DESCRIPTION OF THE EQUATIONS AND EXISTENCE OF A WEAK SOLUTION

We will denote by G an open set of $\mathbb{R}_x \times \mathbb{R}_t$, and by T a positive number, T is finite but needs not to be small. For $t \in [0, T]$ we introduce the family

$$\Omega(t) = \{ x \mid (x, t) \in G \}$$

and we assume that $\Omega(t)$ is a bounded nonempty simply connected open set of $\mathbb{R}^2$ with smooth boundary.

In two dimensions the usual Euler equation in an open set Ω of $\mathbb{R}^2$ is :

$$\frac{\partial u}{\partial t} + u \nabla u = - \nabla p, \qquad \nabla . u = 0 \tag{1}$$

with the homogeneous boundary condition :

$$u . \nu \mid_{\partial \Omega} = 0. \tag{2}$$

Using the relation $\nabla . u = 0$ one can introduce the stream function Φ and the vorticity ω; we then have :

$$u = \left(\frac{\partial \Phi}{\partial x_2}, - \frac{\partial \Phi}{\partial x_1} \right), \qquad \omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

and therefore $\omega = - \Delta \Phi$.

Now taking the curl of (1) we obtain the classical system :

$$\frac{\partial \omega}{\partial t} + (\nabla \wedge \Phi) . \nabla \omega = 0, \qquad - \Delta \Phi = \omega. \tag{3}$$

If we denote by $\partial/\partial s$ the tangential derivative on the curve $\partial\Omega$, we see that (2) can be written in the form :

$$\frac{\partial\Phi}{\partial s} = 0 \quad \text{or} \quad \Phi\,|_{\partial\Omega} = 0. \tag{4}$$

Now the generalisation of (3), (4) will be the following :

$$\frac{\partial\omega}{\partial t} + \nabla \wedge \Phi.\nabla\omega = 0 \quad \text{in} \quad G_T = \bigcup_{0<t<T} \Omega(t) \tag{5}$$

$$- \Delta\Phi + \lambda^2\,\Phi = \omega + f \quad \text{in} \quad G_T \tag{6}$$

$$\Phi = \Psi(x, t) \quad \text{in} \quad \Gamma_T = \bigcup_{0<t<T} \partial\Omega(t) \tag{7}$$

$$\omega(x, t) = \zeta(x, t) \quad \text{in} \quad (\partial G_T^-) \tag{8}$$

in (6) λ denotes a non negative smooth function, f is also a smooth function. With Φ given on $\partial\Omega(t)$ the solution of (6) and (7) is uniquely determined, when ω is prescribed.

In (8) the expression ∂G_T^- denotes the part of the boundary of G_T where the vector fields $(\nabla \wedge \Phi, 1)$ defined in $\mathbb{R}^2 \times \mathbb{R}_t$ points toward the interior of $\dot{G}_T$. More precisely let $(\hat{v}, v_t)$ denotes the unitary outward normal to G_T then ∂G_T^- is defined by the relation :

$$\partial G_T^- = \{ (x, t) \in \partial G_T \mid 0 > v_t + (\nabla \wedge \Phi).\hat{v} \}.$$

Now let $\vec{\alpha} = (\alpha_1, \alpha_2)$ denotes the unitary outward normal to $\Omega(t)$ in $\mathbb{R}^2$; between $\hat{v}_1$, $\hat{v}_2$ and α_1, α_2 we have the relation :

$$\alpha_i = \hat{v}_i/(1 - v_t^2)^{1/2}$$

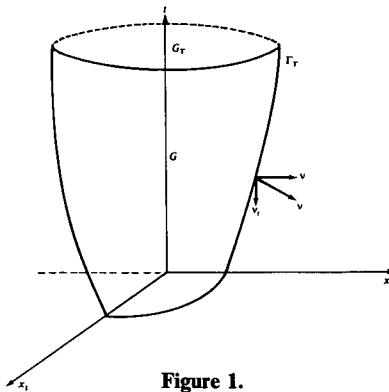**Figure 1.**

and therefore the relation :

$$0 > v_t + (\nabla \wedge \Phi) \, \hat{v} \tag{9}$$

can be written in the equivalent form :

$$0 > v_t(1 - v_t^2)^{1/2} + \alpha_1 \frac{\partial \Phi}{\partial x_2} - \alpha_2 \frac{\partial \Phi}{\partial x_1} =$$

$$= v_t(1 - v_t^2)^{1/2} + \frac{\partial \Phi}{\partial s} = v_t(1 - v_t^2)^{1/2} + \frac{\partial \psi}{\partial s}. \tag{10}$$

In (10), $\partial/\partial s$ denotes the tangential derivative along the curve $\partial\Omega(t)$. As a conse-quence of (10) we see that ∂G_T^- is completely determined by the data : G_T and ψ which is the prescribed value of Φ at the boundary.

It is clear that ∂G_T^- contains the set $\Omega(0) = \{ x \mid (x, 0) \in G \}$ and does not contain $\Omega(T) = \{ x \mid (x, T) \in G \}$.

When Φ and ζ are given, the solution of (5) and (8) is completely deter-mined by a classical transport equation. Indeed, assuming that the vector field $\nabla \wedge \Phi$ is lipschitzian, we can consider the family of curve $x(s) = U_t(s) \, x$ defined by the ordinary differential equation :

$$\dot{x}(s) = (\nabla \wedge \Phi)(x(s), s), \qquad x(t) = x. \tag{11}$$

We denote by $\tau(x, t)$ the first value of s (less that t) for which $x(s)$ reaches the boundary of G_T. Clearly we have :

$$\tau(x, t) \in [0, t]$$

and $U_t(\tau(x, t)) \, x \in \partial G_T^-$.

Then the solution of (5) and (8) is clearly given by :

$$\omega(x, t) = \zeta(U_t(\tau(x, t)) \, x, \tau). \tag{12}$$

We can now state and prove the following :

THÉOREM 1 : *We assume that the data $\lambda(x, t) \geqslant 0$, $f(x, t)$ defined in G_T, $\psi(x, t)$ defined on Γ_T and $\zeta(x, t)$ defined on ∂G_T^- are continuously differentiable*[1] *then there exists a weak solution $(\omega, \Phi) \in L^\infty(G_T)$ of the following system :*

$$\frac{\partial \omega}{\partial t} + (\nabla \wedge \Phi).\nabla \omega = 0 \quad in \quad G_T \tag{13}$$

$$- \Delta\Phi + \lambda^2 \Phi = \omega + f \quad in \quad G_T \tag{14}$$

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[1] We make no attempt for weakening the hypothesis of regularity for the data.

$$\Phi(x, t) = \psi(x, t) \quad in \quad \Gamma_T = \bigcup_{0 < t < T} G_T \qquad (15)$$

$$\omega(x, t) = \zeta(x, t) \quad in \quad \partial G_T^- \qquad (16)$$

where G_T^- denotes the set of points of ∂G_T such that :

$$v_t(1 - v_t^2)^{1/2} + \frac{\partial \Phi}{\partial s}$$

is negative. Furthermore the function Φ satisfies, for any p ($1 \leqslant p < \infty$) the estimate :

$$\sup_t \, \| \Phi(., t) \|_{W^{2, p}(\Omega(t))} \, dt < + \infty . \qquad (17)$$

Remarks : Since (14) and (15) define the classical Dirichlet problem, with ω and f in $L^\infty(G_T)$, Φ and $\dfrac{\partial^2 \Phi}{\partial x_i \, \partial x_j}$ belong to $L^p(G_T)$ for any p ($1 \leqslant p < \infty$) and in particular the value of Φ on Γ_T is it self well defined. On the other hand we say that $(\omega, \Phi) \in L^\infty(G_T)$ is a weak solution if Φ satisfies (14)-(15) and if for any test function $\theta(x, t) \in \mathscr{D}(\overline{G}_T)$ which satisfies :

$$\theta(x, t) = 0 \quad for \quad (x, t) \in \partial G_T^+ = \left\{ (x, t) \in \partial G_T \mid v_t(1 - v_t^2)^{1/2} + \frac{\partial \psi}{\partial s} > 0 \right\}$$

we have :

$$\int\!\!\int_{G_T} \left(\omega \frac{\partial \theta}{\partial t} + \omega.(\nabla \wedge \Phi).\nabla \theta \right) dx \, dt = \int_{\partial G^-} (v_t + \hat{v}.\nabla \wedge \Phi) \, \theta. \omega \, d\sigma_{(x, t)} . \qquad (18)$$

Of course (18) means that in the distribution sense we have :

$$\frac{\partial \omega}{\partial t} + \nabla.((\nabla \wedge \Phi).\omega) = 0$$

and that in some weak sense the restriction of ω to ∂G_T^- is defined and satisfies (16).

We recall that the set $\Sigma(T) = (\Omega(T), T)$ belongs to ∂G_T^+ and that the set $\Sigma(0) = (\Omega(0), 0)$ belongs to ∂G_T^-.

Proof of the theorem 1 : As we have said we give mainly the a priori estimates : we consider the following iterative method : given Φ^n we define ω^{n+1} by the transport equation :

$$\frac{\partial \omega^{n+1}}{\partial t} + (\nabla \wedge \Phi^n) \, \nabla . \omega^{n+1} = 0 \quad \text{in} \quad G_T \tag{19}$$

$$\omega^{n+1}(x, t) = \zeta(x, t) \quad \text{in} \quad \partial G_T^- \tag{20}$$

$\zeta(x, t)$ and ∂G_T^- are independent of n and the solution of (19), (20) is well defined and belongs to $L^\infty(G_T)$, next we define Φ^{n+1} by the equation :

$$- \Delta \Phi^{n+1} + \lambda^2 \, \Phi^{n+1} = f + \omega_n \tag{21}$$

$$\Phi^{n+1} = \Psi \quad \text{on} \quad \Gamma_T . \tag{22}$$

We will have from (19)-(20) the estimate

$$\| \omega^{n+1} \|_{L^\infty(G_T)} \leqslant | \zeta |_{L^\infty(\partial G_T^-)} \tag{23}$$

and for $1 \leqslant p < \infty$

$$\| \Phi^{n+1}(., t) \|_{W^{2,p}(\Omega(t))} \leqslant Cp \left\{ | \zeta |_{L^\infty(\partial G_T^-)} + | f |_{L^\infty(\Omega_T)} \right\} \tag{24}$$

where C denotes a constant depending only of the open set G_T.

Now from the family (ω^n, Φ^n) one can extract a subfamily converging in $(L^\infty(\Omega_T))$ with the weak star topology, the fact that $(\nabla \wedge \Phi^n) (., t)$ is uniformly bounded in $W^{1,p}(\Omega(t))$ and the equation (19) are used to obtain the limit $(\nabla \wedge \Phi) . \nabla \omega$ (in the sense of distribution) in the non linear term $(\nabla \wedge \Phi^n) . \nabla \omega^{n+1}$.

III. CONSTRUCTION OF A SMOOTH SOLUTION

In this section we will assume that the domain increases fast enough compared with the size of the data and we will show that with this hypothesis the method of Wolibner [8] can be adapted. Wolibner has noticed that from the estimate curl $\omega \in L^\infty(G_T)$ one cannot deduce an estimate on the sup norm of $\nabla_x u$. Therefore the first step is to prove, using an analysis of the pair dispersion, that ω is bounded in some Holder space $C^{0,\alpha}$. From this bound one deduces that $\nabla_x u$ remains bounded in the same space. Then it is easy to prove that the solution will be as regular as the initial data.

To make a precise statement, we will assume that $\Gamma = \bigcup_{0 \leqslant t \leqslant T} \partial \Omega(t)$ is defined by a smooth C^2 function $\Gamma(x, t)$:

$$G_T = \left\{ (x, t) \mid 0 \leqslant t \leqslant T, \quad \Gamma(x, t) < 0 \right\}$$

$$\Gamma = \left\{ (x, t) \mid 0 \leqslant t \leqslant T, \quad \Gamma(x, t) = 0 \right\} \tag{25}$$

and for the transport equation we will prove the following.

LEMMA 1 : *Let* $G_T = \bigcup\limits_{0 \leqslant t < T} \Omega(t) = \{ (x, t) \mid 0 \leqslant t < T, \Gamma(x, t) < 0 \}$ *be an open set of* $\mathbb{R}_x^2 \times \mathbb{R}_t$. *We assume that the open sets* $\Omega(t)$ *are simply connected and that their boundary* $\partial\Omega(t)$ *is smooth. We denote by* u *a smooth vector field defined in* $\overline{G}_T$. *We assume that* $\nabla . u \equiv 0$ *in* G_T *and we assume that there exists a strictly positive constant* γ *with the following properties*

$$\frac{\partial\Gamma}{\partial t}(x, t) < -\gamma \quad and \quad (\gamma - |\nabla_x\Gamma(x, t)| \, |u|_\infty) \geqslant \delta > 0 \qquad (26)$$

for any point $(x, t) \in \mathbb{R}_x^2 \times \mathbb{R}_t$.

Then for $\zeta \in C^1(\partial G_T^-)$ *the solution of the transport equation*

$$\frac{\partial\omega}{\partial t} + u.\nabla\omega = 0, \qquad \omega \mid_{\partial G_T^-} = \zeta \qquad (27)$$

satisfies for any pairs $(x, t) \in G_T$ *and* $(y, t) \in G_T$ *the estimates :*

$$|\omega(x, t)| \leqslant |\zeta|_{L^\infty(\partial G_T^-)} \qquad (28)$$

and

$$|\omega(x, t) - \omega(y, t)|/|x - y|^{e^{-Kt}} \leqslant C |\zeta|_{C^1(\partial G_T^-)}. \qquad (29)$$

In the relation (28) *the constant* K *denotes the sup norm of* $\nabla \wedge u$ *in* G_T. C *depends on the sup norme of* $\nabla \wedge u$ *in* G_T, *of the open set itself and of the value of* $u.\hat{v}$ *in* $C^{1,\alpha}(\Gamma)$. *But it is independent of the other properties of the vector field* u.

Proof : We will use the fact that ω is constant along the caracteristics of the vector field u. Since we assume that u is smooth we can extend it in $\mathbb{R}_x^2 \times [0, T]$ by a smooth vector field still denoted $u(x, t)$. We may assume that u and $\nabla u = (\partial u_i / \partial x_j)$ are uniformly bounded in $\mathbb{R}_x^2 \times [0, T]$. As it is done in the introduction we denote by $U_t(s) x$ the value at the time s of the solution of the ordinary differential equation

$$\dot{x}(s) = u(x(s), s), \qquad x(t) = x.$$

From the transport equation we deduce that we have

$$\frac{d}{ds} \omega(U_t(s) s, s) = 0 \qquad (30)$$

and therefore that $\omega(x, t)$ is given by the relation

$$\omega(x, t) = \zeta(U_t(\tau(x, t)) x) \qquad (31)$$

where $\tau(x, t)$ denotes the largest time, in the intervalle $[0, t]$ where the curve $s \to U_t(s) x$ reaches the boundary of G_T. Now the relation (28) is a direct conse-

quence of (31). To prove the relation (29) we will adapt the method of Wolibner [8].

Since the open sets $\Omega(t)$ are simply connected, we write, using the relation $\nabla . u = 0$ the following equations :

$$- \Delta \Phi = \nabla \wedge u \quad \text{in} \quad \Omega(t) \quad 0 < t < T \tag{32}$$

$$\partial \Phi / \partial s \mid_{\partial \Omega(t)} = u . \vec{\alpha} \tag{33}$$

$$u = \nabla \wedge \Phi \quad \text{in} \quad G_T. \tag{34}$$

As in the introduction $\vec{\alpha} = (\alpha_1, \alpha_2)$ denotes the components of the outward unitary normal to $\partial \Omega(t)$ in $\mathbb{R}^2$.

It is known (cf. Wolibner [8] or Ladysenskaia and Uraltceva [6]) that there is no uniform bound of the gradient of u in term of the uniform norm of $\nabla \wedge u$ and of the norme of $u . \alpha$ in $C^1(\partial \Omega(t))$. On the other hand assuming (for sake of simplicity) that $u . \alpha$ is bounded in $C^2(\partial \Omega(t))$ one can prove (Wolibner [8], Kato [2]) the following estimate :

$$\mid u(x, t) - u(y, t) \mid \leqslant CK \mid x - y \mid \sup \left(1, \operatorname{Log} \frac{1}{\mid x - y \mid} \right) \tag{35}$$

where C denotes a constant depending only on the open set $\Omega(t)$, and the boundary condition $u . \vec{\alpha}$ in $C^2(\partial \Omega(t))$ and K denotes the norm of $\nabla \wedge u$ in $L^\infty(\Omega(t))$. From the relation (35) one deduces, for two solutions $U_t(s) x$ and $U_t(s) y$ of the differential equation (29), the classical pair dispersion relation :

$$\mid U_t(s) x - U_t(s) y \mid \leqslant C \mid x - y \mid^{e^{-CK(t-s)}} \quad (^1). \tag{36}$$

For the two points (x, t) and (y, t) appearing in the relation (29) we will denote by τ and η the numbers $\tau = \tau(x, t), \eta = \tau(y, t)$ and by a and b the points $U_t(\tau) x$ and $U_t(\eta) y$. We will consider three differents cases.

(i) $a \in \Sigma(0)$ and $b \in \Sigma(0)$
(ii) $a \in \Gamma$ and $b \in \Gamma$
(iii) $a \in \Gamma$ and $b \in \Sigma(0)$.

In the first case we have $\tau = \eta = 0$ and from the relation (36) we deduce the formula :

$$\mid \omega(x, t) - \omega(y, t) \mid / \mid x - y \mid^{e^{-CKt}} \leqslant C \mid \omega(a, 0) - \omega(b, 0) \mid / \mid a - b \mid$$
$$\leqslant C \mid \zeta \mid_{C^1(\partial G_T^-)}. \tag{37}$$

(^{1}) For the rest of the proof C will denote any constant depending on the open set G_T, on $u . \alpha$ and on the norm of u in $L^\infty(G_T)$ but of nothing else K denotes the norm of $\nabla \wedge u$ in $L^\infty(G_T)$.

In the second case we use the relation (26) and we write :

$$0 = \Gamma(a, \tau) - \Gamma(b, \eta) = \int_0^1 \frac{d}{d\sigma} (\Gamma(\sigma a + (1 - \sigma) b, \sigma\tau + (1 - \sigma) \eta) \, d\sigma$$

$$= \int_0^1 \nabla_x \Gamma(\sigma a + (1 - \sigma) b, \sigma\tau + (1 - \sigma) \eta) (a - b) \, d\sigma$$

$$+ \int_0^1 \frac{\partial\Gamma}{\partial t} (\sigma a + (1 - \sigma) b, \sigma\tau + (1 - \sigma) \eta) (\tau - \eta) \, d\sigma \qquad (38)$$

to obtain, with $- \gamma > \partial\Gamma/\partial t$, the relation :

$$\gamma \, | \, \eta - \tau \, | \leqslant | \, \nabla_x \Gamma \, |_\infty \, | \, a - b \, | . \qquad (39)$$

Since $y(s) = U_t(s) \, y$ is a solution of the differential equation $\dot{y} = u(y, s)$ we have :

$$| \, a - b \, | = | \, U_t(\tau(x, t)) \, x - U_t(\tau(t, t)) \, y \, | \leqslant | \, U_t(\tau(x, t) \, x - U_t(\tau(x, t)) \, y \, | +$$

$$+ | \, U_t(\tau(x, t)) \, y - U_t(\tau(y, t)) \, y \, |$$

$$\leqslant | \, U_t(\tau(x, t) \, x - U_t(\tau(x, t)) \, y \, | + | \, \tau - \eta \, | \, | \, u \, |_\infty . \qquad (40)$$

With (40), (39) and the estimate (36) we obtain finally the relation :

$$| \, \tau - \eta \, | \leqslant C(| \, \nabla_x \Gamma \, |_\infty/\gamma)(1 - | \, \nabla_x \Gamma \, |_\infty \, | \, u \, |_\infty/\gamma)^{-1} . \, | \, x - y \, |^{e^{-CKt}} \qquad (41)$$

or with a change of definition of the constant C

$$| \, \tau - \eta \, | \leqslant C \, | \, x - y \, |^{e^{-CKt}} . \qquad (42)$$

An other use of the mean value theorem gives :

$$| \, U_t(\tau(x, t)) \, x - U_t(\tau(y, t)) \, y \, | \leqslant | \, U_t(\tau(x, t)) \, x - U_t(\tau(x, t)) \, y \, |$$

$$+ | \, U_t(\tau(x, t) \, y - U_t(\tau(y, t)) \, y \, |$$

$$\leqslant C \, | \, x - y \, |^{e^{-CKt}} + | \, \tau - \eta \, | \, | \, u \, |_\infty$$

$$\leqslant C \, | \, x - y \, |^{e^{-CKt}} . \qquad (43)$$

Therefore in the second case we have :

$$| \, \omega(x, t) - \omega(y, t) \, |/| \, x - y \, |^{e^{-Kt}} = | \, \omega(a, \tau) - \omega(b, \eta) \, |/| \, x - y \, |^{e^{-Kt}} \leqslant$$

$$\leqslant C \, | \, \omega(a, \tau) - \omega(b, \eta) \, |/|(a, \tau) - (b, \eta) \, | \leqslant C \, | \, \zeta \, |_{C^1(\partial G_T^-)} . \qquad (44)$$

Finally for the third case we use the relations $\Gamma(a, \tau) = 0$, $\Gamma(b, 0) < 0$ and we write :

$$0 \leqslant \Gamma(a, \tau) - \Gamma(b, 0) = \int_0^1 \frac{\partial \Gamma}{\partial t}(\sigma a + (1 - \sigma) b) \tau \, d\sigma +$$

$$+ \int_0^1 \nabla_x \Gamma(\sigma a + (1 - \sigma) b)(a - b) \, d\sigma. \quad (45)$$

From (45) we deduce the relation

$$\gamma \tau \leqslant \int_0^1 | \nabla_x \Gamma |_\infty | (a - b) | \, d\sigma \qquad (46)$$

and the rest of the proof is similar to the case (ii).

Now we can state and prove the main :

THEOREM 2 : *Assume that the lateral boundary* Γ *of* G_T *is defined by a smooth function* $\Gamma(x, t)$ *satisfying the relation*

$$\frac{\partial \Gamma}{\partial t}(x, t) \leqslant - \gamma < 0 \qquad \forall (x, t) \in \mathbb{R}^2 \times [0, T] \qquad (47)$$

and assume that the data $\zeta \in C^1(\partial G_T^-)$, $f \in C^1(G_T)$ *and* $\psi \in C^2(\partial G_T^-)$ *are small enough in the following sense :*

$$| \zeta |_{L^\infty(\partial G_T^-)} + | f |_{L^\infty(G_T)} + | \psi |_{L^\infty(\partial G_T^-)} < \varepsilon \qquad (48)$$

where ε *denotes a suitable constant.*

Then the problem :

$$\frac{\partial \omega}{\partial t} + (\nabla \wedge \Phi).\nabla \omega = 0 \quad in \quad G_T$$

$$- \Delta \Phi + \lambda^2 \Phi = \omega + f \; in \; G_T \; (\lambda \; being \; real \; and \; smooth)$$

$$\Phi = \psi(x, t) \quad in \quad \Gamma_T = \bigcup_{0 \leqslant t \leqslant T} \partial \Omega(t)$$

$$\omega(x, t) = \zeta(x, t) \quad in \quad \partial G_T^-$$

has a unique solution which satisfies the additional regularity property :

$$| \omega(., t) |_{C^{0, \alpha}(\Omega(t))} + | \Phi(., t) |_{C^{2, \alpha}(\Omega(t))} \leqslant C. \qquad (49)$$

In (49) α is taken small enough and C is a constant depending only on the C^2 norm of ψ and of the $W^{1,\infty}$ norm of ζ and f.

Proof : We will only show that the Holder estimate is uniformly preserved by the iterative scheme defined by :

$$- \Delta\Phi^{n+1} + \lambda^2 \Phi^{n+1} = \omega^n + f \quad \text{in} \quad G_T \tag{50}$$

$$\Phi^{n+1} = \psi \quad \text{in} \quad \Gamma_T = \bigcup_{0 \leqslant t \leqslant T} \partial\Omega(t) \tag{51}$$

$$\frac{\partial\omega^{n+1}}{\partial t} + (\nabla \wedge \Phi^{n+1}).\nabla\omega^{n+1} = 0 \quad \text{in} \quad G_T \tag{52}$$

$$\omega^{n+1} = \zeta \quad \text{in} \quad \partial G_T^- . \tag{53}$$

Now if ω^n is uniformly bounded in $L_\infty (G_T)$ by $K = |\zeta|_{L^\infty(\partial G_T^-)}$, the solution of (50) and (51) Φ^{n+1} is bounded in $W^{2,p}(\Omega(t))$ by Cp where C denotes a constant independent of u; the vector field $u^{n+1} = \nabla \wedge \Phi^{n+1}$ satisfies the relations

$$\nabla.u^{n+1} = 0, \quad \nabla \wedge u^{n+1} = - \Delta\Phi^{n+1} = \omega^n + f - \lambda^2 \Phi^{n+1}$$

and

$$u^{n+1}.\vec{\alpha} = \frac{\partial\psi}{\partial s} \quad \text{on} \quad \Gamma .$$

Therefore $\nabla \wedge u^{n+1}$ is uniformly bounded in $L^\infty(G_T)$ by a constant K_1 independent of n. Using the classical Sobolev theorem we have $u^{n+1} = \nabla \wedge \Phi^{n+1}$ uniformly bounded in $L^\infty(G_T)$ by a constant D which depends only on $|f|_{L^\infty (G_T^-)}$, $|\omega^n|_{L^\infty (G_T^-)} = |\zeta|_{L^\infty(\partial G_T^-)}$ and of $|\psi|_{C^1(\Gamma)}$. Therefore if the constant γ which appears in (26) is large enough, compared to D, we can apply the Lemma 1 for the equations (52), (53), ω^{n+1} is uniformly bounded in

$$L^\infty(G_T) \quad \text{by} \quad K = |\zeta|_{L^\infty(\partial G_T^-)}$$

and the quotients :

$$|\omega^{n+1}(x, t) - \omega^{n+1}(y, t)|/|x - y|^{e^{-\kappa t}}$$

are bounded by $E|\zeta|_{C^1(\partial G_T^-)}$ where E denotes a constant independent of (x, t), (y, t) and of n. The proof of the estimate is complete.

The rest of the proof is simple routine is left to the reader. In particular with the estimate on $\omega^n(., t)$ in $C^{0,\alpha}$ (with $\alpha = e^{-K_1 T}$) one deduces from (50) and (51) that the vector field $u^{n+1}(., t)$ is uniformly bounded in $C^{1,\alpha}(\Omega(t))$ then it is easy to prove that the solution is as smooth as the initial data.

Finally we notice that the proof of uniqueness given by Kato in [2] can be adapted in the present case for the class of solutions which satisfy the hypothesis of the Theorem 2.

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