RAIRO. ANALYSE NUMÉRIQUE

MICHAL KŘÍŽEK

Conforming equilibrium finite element methods for some elliptic plane problems

RAIRO. Analyse numérique, tome 17, nº 1 (1983), p. 35-65

<http://www.numdam.org/item?id=M2AN_1983__17_1_35_0>

R.A.I.R.O. Analyse numérique/Numerical Analysis
(vol. 17, n° 1, 1983, p. 35 à 65)

CONFORMING EQUILIBRIUM FINITE ELEMENT METHODS FOR SOME ELLIPTIC PLANE PROBLEMS (*)

by Michal KŘÍŽEK (¹)

Communicated by P. G. CIARLET

Abstract. — *An equilibrium finite element method for the Stokes problem and for the dual variational formulations of the steady-state heat conduction problem, linear elasticity and biharmonic problem is described. Using the stream and Airy function, the convergence of such a method is transformed into the well-known convergence results for finite elements. A number of finite element spaces of divergence-free functions is constructed and an easy way of generating basis functions with small supports in these spaces is shown. The approximate solutions of all the above problems can be obtained by solving a system of linear algebraic equations.*

Résumé. — *On s'intéresse à une méthode d'éléments finis « équilibre » pour le problème de Stokes et pour les formulations variationnelles duales du problème stationnaire de la chaleur, du problème de l'élasticité linéaire et du problème biharmonique. En utilisant la fonction de courant et la fonction d'Airy, la convergence de cette méthode se ramène à des résultats connus de la convergence, de la méthode des éléments finis. On construit un certain nombre d'espaces d'éléments finis de fonctions à divergence nulle et on donne un moyen facile de construire des fonctions de base avec des petits supports dans ces espaces. Les solutions approchées des problèmes ci-dessus peuvent être obtenues par la solution de systèmes d'équations algébriquement linéaires.*

The aim of this paper is to investigate finite element subspaces of the spaces of divergence-free functions, which play an important role in some problems of continuum mechanics and incompressible fluids. We shall deal with standard elliptic boundary value problems of the second and fourth order, where the cogradients of the solutions (heat flows, stresses, bending moments) are often more desired than the solutions (temperatures, displacements, deflections). One of the most natural ways of obtaining approximations of cogradient is to use pure equilibrium finite element models based on the principle of complementary energy. However, these models are not as popular as e.g. compatible models, because they require the unpleasant condition div $\alpha = 0$ to be settled.

(*) Received in December 1981.
(¹) Mathematical Institute of the Czechoslovak Academy of Sciences, Žitná 25, 115 67 Prague 1, Czechoslovakia.

Moreover, we shall deal with the Stokes problem, where the incompressibility condition has also this form.

Finite element spaces of functions, whose divergence exists in the sense of distributions, and various degrees of freedom (parameters) of these spaces are given in [6, 7, 15, 17, 24]. However, if we add the equilibrium condition div $\alpha = 0$ we get a constraint among the parameters of each element, i.e., the equilibrium finite element method then consists in minimizing some quadratic functional with linear constraints. These constraints can be removed e.g. by the method of Lagrange multipliers or by the elimination of all independent parameters from the set of dependent parameters etc. — see [6, 7, 10, 16, 22], but this is certainly more complicated than a mere solving of the system of algebraic equations in compatible finite element methods. In this paper we show that the solution of equilibrium finite element models can be obtained also from the system of linear equations with a matrix of a similar size and structure as for compatible models. Solving the system, we obtain the coefficients of the linear combination of basis functions of divergence-free finite element spaces. Let us still remark that the advantage of conforming equilibrium models (used simultaneously with conforming compatible models) is the possibility of a posteriori error estimates and two-sided bounds of energy [7, 11, 20, 21].

1. STREAM AND AIRY FUNCTION

Let $\Omega \subset \mathbb{R}^2$ be a non-empty bounded domain with a Lipschitz boundary $\partial\Omega$, let $\Omega_1, \ldots, \Omega_H$ $(0 \leqslant H < \infty)$ be all bounded components of the set $\mathbb{R}^2 - \overline{\Omega}$ and write

$$\Omega_0 = \Omega \cup \bigcup_{i=1}^{H} \overline{\Omega}_i, \tag{1.1}$$

i.e. $\partial\Omega = \partial\Omega_0 \cup \cdots \cup \partial\Omega_H$. The outward unit normal to $\partial\Omega$ is always denoted by $\nu = (\nu_1, \nu_2)^T$. Throughout the paper let the symbols $L^\infty(\Omega)$, $L^2(\Omega)$, $L^2(\partial\Omega)$, $\mathscr{C}^\infty(\overline{\Omega})$, $\mathscr{D}(\Omega)$, $P_k(\Omega)$, $H^k(\Omega)$, $|\cdot|_{k,\Omega}$, $\|\cdot\|_{k,\Omega}$, $(\cdot,\cdot)_{k,\Omega}$ for $k \geqslant 0$, $H_0^1(\Omega)$, $H_0^2(\Omega)$, ∂_i, ∂_{ij}, ∂_ν, δ_{ij} have the usual meaning (the same as in [4]). All vectors will be column vectors. Since there is no danger of ambiguity, the norm, seminorm, and scalar product of vector or matrix functions, the components of which are from $H^k(\Omega)$, will be denoted equally like in $H^k(\Omega)$ and the subscript Ω will be often omitted, e.g.

$$\| v \|_k = \left(\sum_{i=1}^{n} \| v_i \|_k^2 \right)^{1/2} \quad \text{for} \quad v = (v_1, \ldots, v_n)^T \in (H^k(\Omega))^n .$$

In the usual way we define the operator grad : $H^1(\Omega) \to (L^2(\Omega))^2$ by

$$\operatorname{grad} v = (\partial_1 v, \partial_2 v)^T, \qquad v \in H^1(\Omega).$$

Let $f \in L^2(\Omega)$ be arbitrary. If

$$(q, \operatorname{grad} v)_0 = (f, v)_0 \qquad \forall v \in \mathscr{D}(\Omega) \tag{1.2}$$

holds for some $q \in (L^2(\Omega))^2$, then we say that the divergence of the vector function q exists in the sense of distributions in Ω and define

$$\operatorname{div} q = -f \quad \text{in} \quad L^2(\Omega).$$

Evidently, if $q = (q_1, q_2)^T \in (H^1(\Omega))^2$ then $\operatorname{div} q = \partial_1 q_1 + \partial_2 q_2$.

Further, let us remark (see [8], Theorem I.2.2) that the functional $\gamma_\nu : q \mapsto q^T \nu/\partial\Omega$ defined on $(\mathscr{C}^\infty(\overline{\Omega}))^2$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_ν, from

$$H(\operatorname{div}; \Omega) = \{ q \in (L^2(\Omega))^2 \mid \operatorname{div} q \in L^2(\Omega) \}$$

into $H^{-1/2}(\partial\Omega)$, which is the dual space to the space $H^{1/2}(\partial\Omega)$ of the traces on $\partial\Omega$ of all functions from $H^1(\Omega)$. Now, the Green formula will be of the form

$$(q, \operatorname{grad} v)_0 + (\operatorname{div} q, v)_0 = \langle \gamma_\nu q, \gamma_0 v \rangle_{\partial\Omega} \quad \forall q \in H(\operatorname{div}; \Omega) \quad \forall v \in H^1(\Omega). \tag{1.3}$$

Here $\gamma_0 v$ denotes the trace of v and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ is the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. Especially, if $\gamma_\nu q = q^T \nu/\partial\Omega \in L^2(\partial\Omega)$ then

$$\langle \gamma_\nu q, \gamma_0 v \rangle_{\partial\Omega} = \int_{\partial\Omega} v q^T \nu \, ds \quad \forall v \in H^1(\Omega). \tag{1.4}$$

Now, for any $q \in H(\operatorname{div}; \Omega)$ we can define the functional $\gamma_\nu q \in H^{-1/2}(\partial\Omega_i)$, $i \in \{0, \dots, H\}$ as

$$\langle \gamma_\nu q, \gamma_0 v \rangle_{\partial\Omega_i} = (q, \operatorname{grad} v)_0 + (\operatorname{div} q, v)_0, \quad v \in V_i,$$

where

$$V_i = \{ v \in H^1(\Omega) \mid \gamma_0 v = 0 \quad \text{on} \quad \partial\Omega_j \quad \forall j \in \{0, \dots, H\} - \{i\} \},$$

i.e., $\langle \cdot, \cdot \rangle_{\partial\Omega_i}$ represents the duality between $H^{-1/2}(\partial\Omega_i)$ and $H^{1/2}(\partial\Omega_i)$.

Next, we define the operator curl : $H^1(\Omega) \to (L^2(\Omega))^2$ by

$$\operatorname{curl} w = (\partial_2 w, -\partial_1 w)^T, \quad w \in H^1(\Omega).$$

The following necessary and sufficient condition for the existence of the stream function of divergence-free vector is proved in [8], Theorem I.3.1.

THEOREM 1.1 : *A function* $q \in (L^2(\Omega))^2$ *satisfies*

$$\operatorname{div} q = 0\,, \quad \langle \gamma_\nu\, q, 1 \rangle_{\partial\Omega_i} = 0 \quad \textit{for} \quad i = 0, \dots, H\,, \tag{1.5}$$

iff there exists a stream function w *in* $H^1(\Omega)$ *such that*

$$q = \operatorname{curl} w\,,$$

and this function w *is unique apart from an additive constant.*

Before establishing an analogical necessary and sufficient condition for the existence of the Airy function of a divergence-free tensor, we introduce some further notations. Let

$$(L^2(\Omega))^4_{\text{sym}} = \{ \tau \in (L^2(\Omega))^{2\times 2} \mid \tau = \tau^T \}$$

be the subspace of symmetric tensors from $(L^2(\Omega))^{2\times 2}$ with the scalar product

$$(\tau, \tau')_0 = \left(\sum_{i,j=1}^{2} (\tau_{ij}, \tau'_{ij})_0^2 \right)^{1/2}, \quad \tau, \tau' \in (L^2(\Omega))^{2\times 2}\,.$$

Define the operator $\varepsilon : (H^1(\Omega))^2 \to (L^2(\Omega))^4_{\text{sym}}$ by

$$\varepsilon(v) = \begin{pmatrix} \partial_1 v_1 & , & \frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) \\ \frac{1}{2}(\partial_1 v_2 + \partial_2 v_1)\,, & & \partial_2 v_2 \end{pmatrix}, \; v = (v_1, v_2)^T \in (H^1(\Omega))^2\,. \tag{1.6}$$

Further, let $f \in (L^2(\Omega))^2$ be arbitrary. If

$$(\tau, \varepsilon(v))_0 = (f, v)_0 \quad \forall v \in (\mathscr{D}(\Omega))^2 \tag{1.7}$$

holds for some $\tau \in (L^2(\Omega))^4_{\text{sym}}$, then we say that the divergence of the tensor function τ exists in the sense of distributions in Ω and define

$$\operatorname{Div} \tau = -f \quad \text{in} \quad (L^2(\Omega))^2\,. \tag{1.8}$$

Evidently, for a symmetric $\tau = (\tau_{ij}) \in (H^1(\Omega))^{2\times 2}$ we have

$$\operatorname{Div} \tau = (\partial_1 \tau_{11} + \partial_2 \tau_{12}, \partial_1 \tau_{12} + \partial_2 \tau_{22})^T\,.$$

Now, for τ from the space

$$H(\operatorname{Div}; \Omega) = \{ \tau \in (L^2(\Omega))^4_{\text{sym}} \mid \operatorname{Div} \tau \in (L^2(\Omega))^2 \}$$

we can define the linear functional $\gamma_\nu \tau \in (H^{-1/2}(\Gamma))^2$ by

$$\langle \gamma_\nu \tau, t \rangle_\Gamma = \langle \gamma_\nu q^1, t_1 \rangle_\Gamma + \langle \gamma_\nu q^2, t_2 \rangle_\Gamma, \quad t = (t_1, t_2)^T \in (H^{1/2}(\Gamma))^2,$$

where q^1 and q^2 are the columns of the tensor τ and Γ is either $\partial\Omega$ or $\partial\Omega_i$ for some $i \in \{0, \dots, H\}$. Thus the Green formula will clearly have the form

$$(\tau, \varepsilon(v))_0 + (\operatorname{Div} \tau, v)_0 = \langle \gamma_\nu \tau, \gamma_0 v \rangle_{\partial\Omega} \; \forall \tau \in H(\operatorname{Div}; \Omega) \forall v \in (H^1(\Omega))^2. \tag{1.9}$$

We moreover define the operator $\rho : H^2(\Omega) \to (L^2(\Omega))^4_{\text{sym}}$ by

$$\rho(z) = \begin{pmatrix} \partial_{22} z, & -\partial_{12} z \\ -\partial_{12} z, & \partial_{11} z \end{pmatrix}, \quad z \in H^2(\Omega), \tag{1.10}$$

and the space

$$P = \{ v \in (H^1(\Omega_0))^2 \mid \varepsilon(v) = 0 \}. \tag{1.11}$$

Remark 1.1 : We can ascertain as in [20], Theorem 6.3.2 that $\{ (1, 0)^T, (0, 1)^T, (x_2, -x_1)^T \}$ is the basis of P.

THEOREM 1.2 : *A function* $\tau \in (L^2(\Omega))^4_{\text{sym}}$ *satisfies*

$$\operatorname{Div} \tau = 0, \quad \langle \gamma_\nu \tau, \gamma_0 v \rangle_{\partial\Omega_i} = 0 \quad \forall v \in P, \quad i = 0, \dots, H, \tag{1.12}$$

iff there exists the Airy function z in $H^2(\Omega)$ such that

$$\tau = \rho(z),$$

and this function z is unique apart from a linear function.

Proof : Let $\tau \in (L^2(\Omega))^4_{\text{sym}}$ and let (1.12) be satisfied. We extend τ to $\overset{*}{\tau} = (\overset{*}{\tau}_{ij})$ defined on the simply connected domain Ω_0 so that $\operatorname{Div} \overset{*}{\tau} = 0$ in the whole Ω_0.

Let $i \in \{1, \dots, H\}$ be arbitrary and consider the following Neumann problem of linear elasticity in the component Ω_i.

Find $v^i = (v_1^i, v_2^i)^T$ such that

$$\begin{aligned} \operatorname{Div} \varepsilon(v^i) &= 0 \quad \text{in} \quad (L^2(\Omega_i))^2, \\ \varepsilon(v^i)\, \nu^i &= -\gamma_\nu \tau \quad \text{in} \quad (H^{-1/2}(\partial\Omega_i))^2, \end{aligned} \tag{1.13}$$

where ν^i is the outward unit normal to $\partial\Omega_i$. It is known [20] that this problem has a weak solution v^i in $(H^1(\Omega_i))^2$ since by (1.12) the conditions of total equilibrium (called sometimes also the compatibility conditions) are satisfied. This solution v^i is not unique but $\varepsilon(v^i)$ is already uniquely determined. We put

$$\begin{aligned} \overset{*}{\tau} &= \tau \quad \text{in} \quad \Omega\,, \\ \overset{*}{\tau} &= \varepsilon(v^i) \quad \text{in} \quad \Omega_i\,, \quad i = 1, \dots, H\,. \end{aligned} \tag{1.14}$$

Using (1.1), (1.14), (1.9), and (1.13), we have that

$$(\overset{*}{\tau}, \varepsilon(v))_{0,\Omega_0} = (\tau, \varepsilon(v))_{0,\Omega} + \sum_{i=1}^{H} (\varepsilon(v^i), \varepsilon(v))_{0,\Omega_i} = \langle \gamma_\nu \tau, \gamma_0 v \rangle_{\partial\Omega} +$$

$$+ \sum_{i=1}^{H} \langle \varepsilon(v^i)\, \nu^i, \gamma_0 v \rangle_{\partial\Omega_i} = \sum_{i=0}^{H} \langle \gamma_\nu \tau, \gamma_0 v \rangle_{\partial\Omega_i} - \sum_{i=1}^{H} \langle \gamma_\nu \tau, \gamma_0 v \rangle_{\partial\Omega_i} = 0$$

for all $v \in (\mathscr{D}(\Omega_0))^2$. Thus we see by (1.7) that Div $\overset{*}{\tau} = 0$ in Ω_0. Let $\overset{*}{q}{}^1$ and $\overset{*}{q}{}^2$ be the columns of $\overset{*}{\tau}$. Then

$$\operatorname{div} \overset{*}{q}{}^j = 0 \quad \text{and} \quad \langle \gamma_\nu \overset{*}{q}{}^j, 1 \rangle_{\partial\Omega_0} = 0\,, \qquad j = 1, 2\,.$$

By Theorem 1.1, there exist $\overset{*}{w}_1, \overset{*}{w}_2 \in H^1(\Omega_0)$ such that $\overset{*}{q}{}^j = \operatorname{curl} \overset{*}{w}_j$, $j = 1, 2$. Since $\overset{*}{\tau}_{21} = \overset{*}{\tau}_{12}$ we have $-\partial_1 \overset{*}{w}_1 = \partial_2 \overset{*}{w}_2$. Hence putting $\overset{*}{w} = (\overset{*}{w}_1, \overset{*}{w}_2)^T$, we obtain div $\overset{*}{w} = 0$ in Ω_0 and using (1.3), we obtain $\langle \gamma_\nu \overset{*}{w}, 1 \rangle_{\partial\Omega_0} = 0$. Applying Theorem 1.1 once again we see that there exists $\overset{*}{z} \in H^1(\Omega_0)$ such that curl $\overset{*}{z} = \overset{*}{w}$ in Ω_0. But $\overset{*}{z} \in H^2(\Omega_0)$, as $\partial_1 \overset{*}{z} = -\overset{*}{w}_2$ and $\partial_2 \overset{*}{z} = \overset{*}{w}_1$ are from the space $H^1(\Omega_0)$. Now, clearly $\rho(\overset{*}{z}) = \overset{*}{\tau}$ in Ω_0 and we can set $z = \overset{*}{z}/\Omega$.

Conversely, let $z \in H^2(\Omega)$ and let $\overset{*}{z} \in H^2(\Omega_0)$ be such that $\overset{*}{z}/\Omega = z$ (Calderon's extension — see [19], Theorem 2.3.10). Write $\tau = \rho(z)$, $\overset{*}{\tau} = \rho(\overset{*}{z})$, $\overset{*}{q}{}^1 = \operatorname{curl}(\partial_2 \overset{*}{z})$, $\overset{*}{q}{}^2 = \operatorname{curl}(-\partial_1 \overset{*}{z})$, i.e., $\overset{*}{q}{}^1$ and $\overset{*}{q}{}^2$ are the columns of $\overset{*}{\tau}$. By Theorem 1.1, we see that div $\overset{*}{q}{}^1 = \operatorname{div} \overset{*}{q}{}^2 = 0$. Hence, Div $\overset{*}{\tau} = 0$ in Ω_0 and using (1.9) for any Ω_i, $i = 0, \dots, H$, we arrive at

$$0 = \langle \gamma_\nu \overset{*}{\tau}, \gamma_0 v \rangle_{\partial\Omega_i} = \langle \gamma_\nu \tau, \gamma_0 v \rangle_{\partial\Omega_i} \quad \forall v \in P\,,$$

thus (1.12) holds.

The Airy function of the divergence-free tensor τ is unique apart from a linear function, since if $z^1, z^2 \in H^2(\Omega)$ and $\rho(z^1) = \rho(z^2) = \tau$, then $\rho(z^1 - z^2) = 0$, which yields $z^1 - z^2 \in P_1(\Omega)$. ■

2. CONSTRUCTION OF THE SPACES OF EQUILIBRIUM FINITE ELEMENTS FOR THE STEADY-STATE HEAT CONDUCTION PROBLEM

Classically, the steady-state heat conduction problem can be formulated as follows. Find u such that

$$\begin{aligned} -\operatorname{div}(A \operatorname{grad} u) &= f \quad \text{in} \quad \Omega, \\ u &= \bar{u} \quad \text{on} \quad \Gamma_1, \\ (A \operatorname{grad} u)^T \nu &= g \quad \text{on} \quad \Gamma_2, \end{aligned} \tag{2.1}$$

where Γ_1 and Γ_2 are disjoint and open in $\partial\Omega$,

$$\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 = \partial\Omega,$$

and Γ_0 is a finite set of those and only those points, where one type of the boundary condition changes into another. Further, $A \in (L^\infty(\Omega))^{2\times 2}$, $f \in L^2(\Omega)$, $\bar{u} \in H^1(\Omega)$, $g \in L^2(\Gamma_2)$ are given and A is supposed to be symmetric and uniformly positive definite.

Let us introduce bilinear forms a and b by the formulae

$$\begin{aligned} a(v, v') &= (A \operatorname{grad} v, \operatorname{grad} v')_0, \quad v, v' \in H^1(\Omega), \\ b(q, q') &= (A^{-1} q, q')_0, \quad\quad\quad\quad q, q' \in (L^2(\Omega))^2, \end{aligned} \tag{2.2}$$

where the inverse matrix A^{-1} is also symmetric, uniformly positive definite and from $(L^\infty(\Omega))^{2\times 2}$ (see [24], Lemma IV.2.1).

Let us remark that the primal variational formulation of the problem (2.1) for $\Gamma_1 \neq \emptyset$ consists (see [4, 8, 11]) in minimizing the functional

$$\bar{I}(v) = \frac{1}{2} a(v, v) + a(v, \bar{u}) - (f, v)_0 - \int_{\Gamma_2} gv \, ds \tag{2.3}$$

over the space

$$V = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1 \}. \tag{2.4}$$

For the dual variational formulation let us introduce the set (of statically admissible heat flows)

$$Q(f, g) = \left\{ q \in H(\operatorname{div}; \Omega) \mid (q, \operatorname{grad} v)_0 = (f, v)_0 + \int_{\Gamma_2} gv \, ds \; \forall v \in V \right\} \tag{2.5}$$

and suppose that the compatibility condition

$$(f, 1)_0 + \langle g, 1 \rangle_{\partial\Omega} = 0$$

holds if $\Gamma_1 = \emptyset$. It is known [11] that $Q(f, g)$ is a non-empty affine closed manifold of $(L^2(\Omega))^2$ and using (1.2), (1.3), and (1.4), one can easily see that $q \in Q(f, g)$ iff $\operatorname{div} q = -f$ in Ω and $q^T \nu = g$ on Γ_2. The dual formulation of the problem (2.1) consists in minimizing the functional of complementary energy

$$J(q) = \frac{1}{2} b(q, q) - \langle \gamma_\nu q, \gamma_0 \bar{u} \rangle_{\partial\Omega}$$

over the set $Q(f, g)$. This can be equivalently formulated (see also [11]) as follows.

Given some $\bar{p} \in Q(f, g)$ fixed (for the choice of such $\bar{p}$ see Remark 2.5), find p which minimizes the functional

$$\bar{J}(q) = \frac{1}{2} b(q,q) + b(q, \bar{p}) - \langle \gamma_\nu q, \gamma_0 \bar{u} \rangle_{\partial\Omega} = \frac{1}{2} b(q, q) - l(q) \quad (2.6)$$

over the space

$$Q = Q(0, 0) = \{ q \in H(\operatorname{div}; \Omega) \mid (q, \operatorname{grad} v)_0 = 0 \ \forall v \in V \}. \quad (2.7)$$

The vector $p + \bar{p}$ is considered to be the solution of the dual formulation and to any $\bar{p} \in Q(f, g)$, there exists exactly one p.

Now, let us come to the construction of the spaces of finite elements. For simplicity, we shall suppose from now to the very last that the domain Ω is polygonal. Let $\mathcal{T}_h$ be a set of convex polygons such that the union of all $K \in \mathcal{T}_h$ is $\bar{\Omega}$ and such that two different $K, K' \in \mathcal{T}_h$ either are mutually disjoint or have just one common vertex or an entire side in common (h is the usual mesh size). Such a set $\mathcal{T}_h$ will be called the triangulation of $\bar{\Omega}$ and we shall always assume that $\mathcal{T}_h$ is consistent with Γ_1 and Γ_2, i.e., that the interior of any side of any $K \in \mathcal{T}_h$ is disjoint with Γ_0.

Remark 2.1 : We can identify any triangulation with a connected planar graph, for which the well-known Euler's formula holds (see e.g. [3], Lemma 4.3). This formula has in our case the form

$$S + 1 = N + E + H, \quad (2.8)$$

where N, S, and E are the number of vertices, sides, and elements (convex polygons) of the triangulation, respectively, and H is the number of the holes in Ω (e.g. $S = 44$, $N = 21$, $E = 24$, $H = 0$ for the triangulation in *figure* 1 in the end of Section 2).

Now, let X_h be a finite-dimensional subspace of $H^1(\Omega)$ such that for any $K \in \mathcal{T}_h$ the set

$$P_K = \{ v_h/K \mid v_h \in X_h \}$$

is some space of polynomials and

$$P_K \supseteq P_1(K) \tag{2.9}$$

The conforming primal finite element method consists in minimizing the functional (2.3) over some finite-dimensional space $V_h \subset V$, which can be defined as (see [4])

$$V_h = X_h \cap V . \tag{2.10}$$

To construct the space of finite elements for the conforming dual (equilibrium) finite element method, we first introduce an auxiliary space

$$W_h = X_h \cap W , \tag{2.11}$$

where

$$W = \{ w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_2 \} \tag{2.12}$$

and prove an important Theorem 2.1. We shall denote by curl W the space of the rotations of all the functions from W, i.e.

$$\text{curl } W = \{ q \in (L^2(\Omega))^2 \mid \exists w \in W : q = \text{curl } w \} .$$

We shall often use also the symbols curl $H_0^2(\Omega)$, $\rho(H^2(\Omega))$, $\varepsilon((H^1(\Omega))^2)$ etc., which will have an analogical meaning.

Convention : Up to the end of Section 3, let us assume that Γ_1 and Γ_2 are connected sets. (The general case will be discussed in Section 6.)

THEOREM 2.1 : *It is* $Q = \text{curl } W$, *where* Q *and* W *are defined by* (2.7) *and* (2.12), *respectively.*

Proof : Let $q \in Q$. By (1.2) and then by (1.3) we obtain

$$\text{div } q = 0 , \quad \langle \gamma_\nu q, 1 \rangle_{\partial\Omega} = 0 . \tag{2.13}$$

From the above convention we see that Ω is either simply connected or doubly connected (in which case $\Gamma_1 = \partial\Omega_0$, $\Gamma_2 = \partial\Omega_1$ or $\Gamma_1 = \partial\Omega_1$, $\Gamma_2 = \partial\Omega_0$). Using (2.13) and the fact that $q^T \nu = 0$ on Γ_2, we get that (1.5) holds. Thus there exists a $w \in H^1(\Omega)$ unique apart from an additive constant (this constant will be chosen later) such that $q = \text{curl } w$. Write $s = (\nu_2, -\nu_1)^T$ and let $\Gamma_2 \neq \emptyset$. By the definitions of grad and curl, and by (2.7), (1.3), and (1.4) we see that, for any $v \in V \cap \mathscr{C}^\infty(\overline{\Omega})$, it holds

$$0 = (\text{curl } w, \text{grad } v)_0 = -(\text{curl } v, \text{grad } w)_0 =$$

$$= -\int_{\partial\Omega} w(\text{curl } v)^T \nu \, ds = \int_{\partial\Omega} w(\text{grad } v)^T s \, ds = \int_{\partial\Omega} \frac{\partial v}{\partial s} w \, ds = \int_{\Gamma_2} \frac{\partial v}{\partial s} w \, ds , \tag{2.14}$$

since $v = 0$ on Γ_1. Therefore, for any line segment $\varphi \subset \Gamma_2$,

$$\int_\varphi \frac{\partial \psi}{\partial s} \gamma_0 \, w \, ds = 0 \quad \forall \psi \in \mathscr{D}(\varphi),$$

i.e., the derivative of the function $\gamma_0 \, w/\varphi$ is zero in the sense of distributions. Since the trace $\gamma_0 \, w$ of the function from $H^1(\Omega)$ cannot have a discontinuity of the first order and since Γ_2 is connected, we conclude that w is a constant on Γ_2. Hence, we choose w to be zero on Γ_2. So, $q \in \text{curl } W$.

Conversely, let $w \in W$ and let $v \in V \cap \mathscr{C}^\infty(\overline{\Omega})$. As in (2.14) we can obtain that

$$(\text{curl } w, \text{grad } v)_0 = \int_{\Gamma_2} \frac{\partial v}{\partial s} w \, ds,$$

but the integral vanishes, since $w = 0$ on Γ_2. From (2.7) and from the density of $V \cap \mathscr{C}^\infty(\overline{\Omega})$ in V, we have curl $w \in Q$. ■

Finally, we can define the space of equilibrium finite elements (of heat flows) as

$$Q_h = \text{curl } W_h . \tag{2.15}$$

The finite element approximation of the dual problem consists in finding p_h which minimizes the functional (2.6) over the space Q_h. The vector $\overline{p} + p_h$ is considered to be the solution of this problem. The inclusion $Q_h \subset Q$ desired for the conformity of the finite element method follows from (2.11), (2.15), and Theorem 2.1.

It has been proved in [13] that the dual finite element approximation constructed by the linear triangular elements converges to the solution of the dual problem without any regularity assumptions. In rather another way we prove now an analogical result, even when the space Q_h contains only piecewise constant functions or if $\mathcal{T}_h$ contains e.g. rectangles etc.

THEOREM 2.2: *Let* $\{ W_h \}$ *be a system of finite element subspaces of* W *such that* $\bigcup_h W_h$ *is dense in* W *(with the topology of* $H^1(\Omega)$*). Then*

$$\| p - p_h \|_0 \to 0 \quad \textit{for} \quad h \to 0 .$$

Proof: By Theorem 2.1 there exists $u \in W$ so that $p = \text{curl } u$. Using Cea's Lemma ([4], Theorem 2.4.1) and (2.15), we obtain

$$\frac{1}{C} \| p - p_h \|_0 \leqslant \inf_{q_h \in Q_h} \| p - q_h \|_0 = \inf_{w_h \in W_h} \| \text{curl } u - \text{curl } w_h \|_0 =$$

$$= \inf_{w_h \in W_h} \| \text{grad}(u - w_h) \|_0 \leqslant \inf_{w_h \in W_h} \| u - w_h \|_1 \to 0 \quad \text{for} \quad h \to 0,$$

where $C > 0$ is a constant independent on h. ■

Remark 2.2 : A sufficient condition for the density assumption in Theorem 2.2 is (see e.g. [4]) the regularity of the corresponding family $\{ \mathcal{T}_h \}$ of triangulations, the validity of (2.9) and the existence of one (or a finite number) reference element to which all elements are affine-equivalent.

Remark 2.3 : Let us assume that we can define, for some integer $k \geq 1$ and for all $w \in W \cap H^{k+1}(\Omega)$, an X_h-interpolant $\pi_h w$ in W_h such that

$$\| w - \pi_h w \|_1 \leqslant ch^k \mid w \mid_{k+1} , \tag{2.16}$$

where the constant c is independent of h (see [4], Section 3.2). Then for any $q \in Q \cap (H^k(\Omega))^2$ we can define the interpolant $\Pi_h q \in Q_h$ by

$$\Pi_h q = \operatorname{curl}(\pi_h w) , \tag{2.17}$$

where w corresponds to q by Theorem 2.1, i.e. $q = \operatorname{curl} w$ and $w \in H^{k+1}(\Omega)$ since $\partial_1 w, \partial_2 w \in H^k(\Omega)$. When two different w^1 and w^2 correspond to q (it is clearly only in the case $\Gamma_2 = \emptyset$), then the definition of $\Pi_h q$ remains correct, as by (2.16) we have $\operatorname{curl}(\pi_h(w^1 - w^2)) = 0$. Now, let us suppose that $p \in Q \cap (H^k(\Omega))^2$ and let again $p = \operatorname{curl} u$, $u \in W$. Then $u \in H^{k+1}(\Omega)$ and using Cea's Lemma, (2.16) and (2.17), we arrive at the following a priori error estimate

$$\begin{aligned} \| p - p_h \|_0 &\leqslant C \inf_{q_h \in Q_h} \| p - q_h \|_0 \leqslant C \| p - \Pi_h p \|_0 = \\ &= C \| \operatorname{curl} u - \operatorname{curl}(\pi_h u) \|_0 \leqslant C \| u - \pi_h u \|_1 \leqslant cCh^k \mid u \mid_{k+1} \\ &= cCh^k \mid \operatorname{grad} u \mid_k = cCh^k \mid \operatorname{curl} u \mid_k = cCh^k \mid p \mid_k . \end{aligned}$$

Let us further note that the rate of the convergence of the primal and the corresponding dual approximation need not be always the same, since the smoothness of the solutions of the primal and dual problem can considerably differ (e.g. for composite materials).

LEMMA 2.1 : *It is* $\dim W_h = \dim Q_h + 1$ *if* $\Gamma_2 = \emptyset$ *and*

$$\dim W_h = \dim Q_h \quad \textit{if} \quad \Gamma_2 \neq \emptyset .$$

Proof: Obviously, $\dim W_h \geqslant \dim Q_h$ by (2.15). Let $w \in W_h$ be in the kernel of the operator curl, i.e. $\operatorname{curl} w = 0$. Then w must be constant on Ω, since $\partial_1 w = \partial_2 w = 0$ holds on any $K \in \mathcal{T}_h$ and $w \in H^1(\Omega)$. If $\Gamma_2 \neq \emptyset$ then from the definition of W_h it follows $w = 0$ in Ω. ■

LEMMA 2.2 : *Let* $\{ w^i \}_{i=1}^n$ *be a basis in* W_h *and let* $q^i = \operatorname{curl} w^i$, $i = 1, \dots, n$. *Then* $\{ q^i \}_{i=1}^n$ *is a basis in* Q_h *if* $\Gamma_2 \neq \emptyset$. *In the case* $\Gamma_2 = \emptyset$, *let* $\alpha_1, \dots, \alpha_n \in \mathbb{R}^1$ *be such that*

$$\sum_{i=1}^n \alpha_i w^i = 1 \quad \text{in} \quad \Omega . \tag{2.18}$$

Then for any $k \in \{ 1, \dots, n \}$ *for which* $\alpha_k \neq 0$, *the set*

$$\{ q^1, \dots, q^{k-1}, q^{k+1}, \dots, q^n \} \tag{2.19}$$

is a basis in Q_h.

Proof: The case $\Gamma_2 \neq \emptyset$ is evident from Lemma 2.1. Consider the case $\Gamma_2 = \emptyset$. Applying the operator curl to (2.18), we obtain

$$\sum_{i=1}^n \alpha_i q^i = 0 .$$

Thus, for some $\alpha_k \neq 0$,

$$q^k = -\frac{1}{\alpha_k} \sum_{i \neq k} \alpha_i q^i .$$

For an arbitrary $q \in Q_h$ there exist $\beta_1, \dots, \beta_n \in \mathbb{R}^1$ such that

$$q = \sum_{i \neq k} \beta_i q^i + \beta_k q^k = \sum_{i \neq k} \left(\beta_i - \beta_k \frac{\alpha_i}{\alpha_k} \right) q^i .$$

Therefore, the set (2.19) generates the space Q_h and by Lemma 2.1 the number of its vectors is equal to dim Q_h. ∎

Remark 2.4 : Suppose that we have some basis $\{ q^i \}_{i=1}^m$ of Q_h. Evidently the approximation of the dual problem is then equivalent to the system of linear equations

$$\sum_{j=1}^m b(q^i, q^j) y_j = l(q^i) , \qquad i = 1, \dots, m ,$$

and, at the same time, $p_h = \sum_i y_i q_i$. Consequently, by Lemma 2.1, by (2.2) and by the definitions (2.10), (2.11) of the spaces V_h and W_h, we see that the flexibility matrix $\mathcal{B} = (b(q^i, q^j))$ has roughly the same order as the stiffness matrix $\mathcal{A} = (a(v^i, v^j))$ for solving the corresponding approximation of the primal problem on V_h with the basis $\{ v^i \}$, i.e., $\dim V_h \doteq \dim Q_h$. In both extremal cases there we have $\dim V_h < \dim Q_h$ for $\Gamma_2 = \emptyset$ (Dirichlet problem) and $\dim V_h > \dim Q_h$ for $\Gamma_1 = \emptyset$ (Neumann problem). Next, it clearly holds

$$\operatorname{supp} u \supseteq \operatorname{supp} \operatorname{curl} u = \operatorname{supp} \operatorname{grad} u \quad \forall u \in X_h ,$$

where supp denotes a support. Therefore, if we conveniently number the basis functions in V_h and W_h the matrices $\mathscr{A}$ and $\mathscr{B}$ will have roughly the same width of the band. Thus, the demands for assembling $\mathscr{A}$ and $\mathscr{B}$ and solving the corresponding systems will be roughly the same, too. Even in the case of the isotropic and homogeneous material of the body (i.e., the matrix A in (2.1) is the unit one) the majority of the scalar products in $\mathscr{A}$ and $\mathscr{B}$ are identical, since

$$b(q^i, q^j) = (\operatorname{curl} w^i, \operatorname{curl} w^j)_0 = (\operatorname{grad} w^i, \operatorname{grad} w^j)_0 = a(w^i, w^j)$$

for all $w^i, w^j \in V_h \cap W_h$. Let us still emphasize that a simultaneous knowledge of the solutions obtained by the conforming primal and dual finite element method is advantageous for a posteriori error estimates, for two-sided bounds of energy and for utilizing the hypercircle method — see [7, 11, 20].

Remark 2.5 : We shall describe a way of finding some $\bar{p} \in Q(f, g)$ (see (2.6)) in practical cases. Let us define

$$F(x_1, x_2) = \left(- \int_0^{x_1} \bar{f}(\xi, x_2)\, d\xi, 0 \right)^T , \quad (x_1, x_2)^T \in \Omega ,$$

where $\bar{f} = f$ in Ω and $\bar{f} = 0$ in $\mathbb{R}^2 - \Omega$. Let $\Omega' \subset \Omega$ be an arbitrary domain, which has a Lipschitz boundary with the outward unit normal ν', such that $\Gamma_2 \subset \bar{\Omega}'$. Let all the functions occurring below be sufficiently smooth so that the corresponding symbols have the correct sense. We put

$$G = \operatorname{curl} w \quad \text{in} \quad \Omega',$$
$$G = 0 \quad \text{in} \quad \Omega - \Omega',$$

where w is an arbitrary function with the tangential derivative

$$(\operatorname{curl} w)^T \nu = g - F^T \nu \quad \text{on} \quad \Gamma_2 ,$$
$$(\operatorname{curl} w)^T \nu' = 0 \quad \text{on} \quad \partial\Omega' - \partial\Omega ,$$

i.e., w is constant on $\partial\Omega - \partial\Omega'$. Then by (1.3) and (1.4), we get

$$(F + G, \operatorname{grad} v)_{0,\Omega} = (F, \operatorname{grad} v)_{0,\Omega} + (G, \operatorname{grad} v)_{0,\Omega'} = (-\operatorname{div} F, v)_{0,\Omega} +$$

$$+ \int_{\partial\Omega} vF^T \nu \, ds + \int_{\partial\Omega'} vG^T \nu' \, ds = (f, v)_{0,\Omega} + \int_{\Gamma_2} vF^T \nu \, ds$$

$$+ \int_{\Gamma_2} v(g - F^T \nu) \, ds = (f, v)_{0,\Omega} + \int_{\Gamma_2} gv \, ds$$

for all $v \in V$. Thus, with regard to (2.5), we can put $\bar{p} = F + G$.

Numerical example. We have recomputed Example 2 of [7]. On the triangulated domain Ω with $E = 24$ triangles, $S = 44$ sides and $N = 21$ vertices in Fig. 1, the Dirichlet problem for the Laplace equation $\Delta u = \operatorname{div} \operatorname{grad} u = 0$ is considered (we can put here $\bar{p} = 0$). Choosing the space W_h of piecewise linear functions, we get by Lemma 2.1 that $\dim Q_h = \dim W_h - 1 = N - 1 = 20$, i.e., the order of the flexibility matrix is 20, while in [7] 44 " degrees of freedom " for generating the constant heat flow field have been used. But these degrees of freedom are dependent with one constraint on any triangle. For the piecewise linear heat flow field 88 ($= 2\,S$) dependent degrees of freedom have been used, while $\dim Q_h = S + N - 1 = 64$ for W_h composed of quadratic elements.

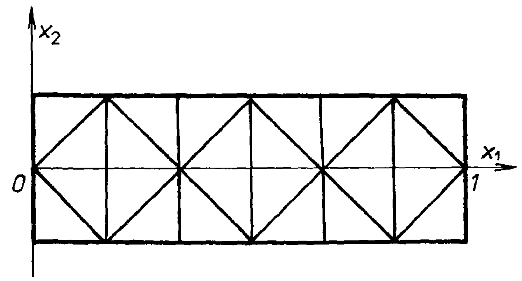

Figure 1.

The Dirichlet boundary condition in [7] was as follows : $\bar{u} = 900\,(^{\circ}\mathrm{C})$ on the axis x_2 and $\bar{u} = 1{,}500\,(^{\circ}\mathrm{C})$ on the remaining part of $\partial\Omega$, i.e. no $\bar{u} \in H^1(\Omega)$ exists with such a trace (it is the case of the non-integrable gradient field). Nevertheless, we have tried to use the linear equilibrium finite element method for computing the heat flow field. The values obtained by the integration of this field from the origin along the axis x_1 are shown in Table 1.

TABLE 1

Temperature on the axis x_1

Linear equilibrium finite element model	$x_1 = 0$	$x_1 = \frac{1}{2}$	$x_1 = 1$
The triangulation in figure 1 with $P = 24$ triangles, dim $Q_h = 64$	900.0	1 488.2	1 499.9
The triangulation refined by midlines, $P = 96$, dim $Q_h = 224$	900.0	1 492.9	1 500.0
Calculation by [7], $P = 24$ (the values have been measured from a graph)	900	1 395	1 460

3. CONSTRUCTION OF THE SPACES OF EQUILIBRIUM FINITE ELEMENTS FOR THE LINEAR ELASTICITY PROBLEM

Given $A = (A_{ijkl})^2_{i,j,k,l=1} \in (L^\infty(\Omega))^{16}$, where $A_{ijkl} = A_{jikl} = A_{klij}$, $f \in (L^2(\Omega))^2$, $\bar{u} \in (H^1(\Omega))^2$, $g \in (L^2(\Gamma_2))^2$, find $u = (u_1, u_2)^T$ so that

$$\begin{aligned} -\operatorname{Div}(\mathbb{A}.\varepsilon(u)) &= f \quad \text{in} \quad \Omega, \\ u &= \bar{u} \quad \text{on} \quad \Gamma_1, \\ (\mathbb{A}.\varepsilon(u))\,\nu &= g \quad \text{on} \quad \Gamma_2, \end{aligned} \tag{3.1}$$

where ε and Div are defined by (1.6) and (1.8), respectively, $\tau = \mathbb{A}.\varepsilon$ is the symmetric stress tensor with the components $\tau_{ij} = \sum_{k,l} A_{ijkl}\,\varepsilon_{kl}$ and we assume that there exists a constant $c > 0$ such that

$$\sum_{i,j,k,l} A_{ijkl}(x)\, e_{ij}\, e_{kl} \geqslant c \sum_{i,j} e_{ij}^2 \quad \forall e = e^T \in \mathbb{R}^{2\times 2} \tag{3.2}$$

holds almost everywhere in Ω.

The dual variational formulation of this classical linear elasticity problem consists (see [12, 20]) in minimizing the functional of complementary energy

$$J(\tau) = \frac{1}{2} b(\tau, \tau) - \langle \gamma_\nu \tau, \gamma_0 \bar{u} \rangle_{\partial\Omega}$$

over the set (of statically admissible stresses)

$$T(f, g) = \left\{ \tau \in H(\operatorname{Div}; \Omega) \mid (\tau, \varepsilon(v))_0 = (f, v)_0 + \int_{\Gamma_2} g^T v \, ds \;\forall v \in \mathscr{V} \right\}$$

Here, b is defined by

$$b(\tau, \tau') = (\mathbb{B}.\tau, \tau')_0 , \quad \tau, \tau' \in (L^2(\Omega))^4_{\text{sym}} ,$$

where $\mathbb{B} = \mathbb{A}^{-1}$ is the fourth order tensor from the linear inverse Hook's law (for details see [20]) and the bilinear form b is $(L^2(\Omega))^4_{\text{sym}}$-elliptic with regard to (3.2). Further,

$$\mathscr{V} = V \times V , \tag{3.3}$$

where V is defined by (2.4), and let the compatibility condition

$$(f, v)_0 + \langle g, \gamma_0 v \rangle_{\partial\Omega} = 0 \quad \forall v \in P$$

hold in the case $\Gamma_1 = \emptyset$, where P is defined by (1.11).

It is known [5, 20] that $T(f, g)$ is a non-empty affine closed manifold of $(L^2(\Omega))^4_{\text{sym}}$, and $\tau \in T(f, g)$ iff $\operatorname{Div} \tau = -f$ in Ω and $\tau\nu = g$ on Γ_2. These equations of equilibrium can be obtained also by (1.7) and (1.9). As in Section 2 or in [12], the dual formulation of (3.1) can be transformed into the following problem.

Given $\bar{\sigma} \in T(f, g)$ fixed, find σ which minimizes the functional

$$\bar{J}(\tau) = \frac{1}{2} b(\tau, \tau) + b(\tau, \bar{\sigma}) - \langle \gamma_\nu \tau, \gamma_0 \bar{u} \rangle_{\partial\Omega} \tag{3.4}$$

over the space

$$T = T(0, 0) = \{ \tau \in H(\operatorname{Div}; \Omega) \mid (\tau, \varepsilon(v))_0 = 0 \quad \forall v \in \mathscr{V} \} . \tag{3.5}$$

The stress tensor $\sigma + \bar{\sigma}$ is considered to be the solution of the dual formulation of (3.1) and, to any $\bar{\sigma} \in T(f, g)$, there exists exactly one solution σ. Using the operator (1.10), we can find some particular solution $\bar{\sigma}$ of the equations of equilibrium as in Remark 2.5.

THEOREM 3.1 : *It is* $T = \rho(Z)$, *where* ρ *is defined by* (1.10) *and*

$$Z = \{ z \in H^2(\Omega) \mid z = \partial_\nu z = 0 \text{ on } \Gamma_2 \} .$$

Proof: Let $\tau = (\tau_{ij}) \in T$. Obviously, its columns denoted by q^1, q^2 are in Q. Therefore, by Theorem 2.1 there exist $w_1, w_2 \in W$ such that $\operatorname{curl} w_j = q^j$, $j = 1, 2$. Since $\tau_{21} = \tau_{12}$, we obtain $-\partial_1 w_1 = \partial_2 w_2$. Thus $w = (w_1, w_2)^T \in Q$ as $w^T \nu = 0$ on Γ_2. Applying Theorem 2.1 once again, we see that there exists $z \in W$ such that $\operatorname{curl} z = w$. But this z belongs to Z since $\partial_1 z, \partial_2 z \in W$. Now, it is easy to show that $\rho(z) = \tau$ (e.g., $\partial_{11} z = -\partial_1 w_2 = \tau_{22}$).

Conversely, let $z \in Z$. Since $z = \partial_\nu z = 0$ on Γ_2 we obtain that $\partial_1 z, \partial_2 z \in W$. Thus, by Theorem 2.1,

$$\operatorname{curl}(\partial_2 z) = (\partial_{22} z, -\partial_{12} z)^T \in Q \quad \text{and} \quad -\operatorname{curl}(\partial_1 z) = (-\partial_{12} z, \partial_{11} z)^T \in Q,$$

i.e. $\rho(z) \in T$. ∎

Remark 3.1 : By this theorem and (3.5), we see that for any $\tau \in (L^2(\Omega))^4_{\text{sym}}$ it holds

$$(\tau, \varepsilon(v))_0 = 0 \quad \forall v \in \mathscr{V} \Leftrightarrow \exists z \in Z : \tau = \rho(z).$$

Therefore, the well-known orthogonal decomposition (see [20]) of the space of symmetric stresses into the closed subspaces of equilibrium and compatible stresses (for the identical tensor $\mathbb{B}$) will be now of the form

$$(L^2(\Omega))^4_{\text{sym}} = \rho(Z) \oplus \varepsilon(\mathscr{V}). \tag{3.6}$$

We shall use this consequence of Theorem 3.1 in Section 4.

Next, let Y_h be a finite-dimensional subspace of $H^2(\Omega)$ such that for any $K \in \mathcal{T}_h$ the set

$$P_K = \{ v_h/K \mid v_h \in Y_h \}$$

is the space of piecewise polynomial functions and $P_K \supseteq P_2(K)$. Due to Theorem 3.1, we can define the subspace of T of equilibrium finite elements of stresses as

$$T_h = \rho(Z_h), \tag{3.7}$$

where $Z_h = Y_h \cap Z$.

The finite element approximation of the dual formulation of (3.1) will now consist in finding a σ_h which minimizes the functional (3.4) over the space T_h.

THEOREM 3.2 : *Let $\{ Z_h \}$ be a system of finite element subspaces of Z such that the set $\bigcup_h Z_h$ is dense in Z (with the topology of $H^2(\Omega)$). Then*

$$\| \sigma - \sigma_h \|_0 \to 0 \quad \textit{for} \quad h \to 0.$$

Proof : By Theorem 3.1 there exists $\bar{z} \in Z$ such that $\sigma = \rho(\bar{z})$. Using Cea's Lemma and (3.7), we obtain

$$\frac{1}{C} \| \sigma - \sigma_h \|_0 \leqslant \inf_{\tau_h \in T_h} \| \sigma - \tau_h \|_0 = \inf_{z_h \in Z_h} \| \rho(\bar{z}) - \rho(z_h) \|_0 = \inf_{z_h \in Z_h} | \bar{z} - z_h |_2 \to 0$$

for $h \to 0$ and $C > 0$ is a constant independent of h. ∎

Remark 3.2 : Suppose that for some $k \geqslant 1$ and for all $z \in Z \cap H^{k+2}(\Omega)$, we can define a Z_h-interpolant $\pi_h z$ in Z_h so that

$$\| z - \pi_h z \|_2 \leqslant ch^k \, | z |_{k+2} , \tag{3.8}$$

where $c > 0$ is a constant independent of h. Then the following a priori error estimate can be derived by an analogical procedure as in Remark 2.3 :

$$\| \sigma - \sigma_h \|_0 \leqslant cCh^k \, | \sigma |_k \quad \text{for} \quad \sigma \in T \cap (H^k(\Omega))^{2 \times 2} , \quad h \to 0 .$$

LEMMA 3.1 : *It is*

$$\dim Z_h = \dim T_h + 3 \ \textit{if} \ \Gamma_2 = \emptyset \quad \textit{and} \quad \dim Z_h = \dim T_h \ \textit{if} \ \Gamma_2 \neq \emptyset .$$

Proof: Clearly, $\dim Z_h \geqslant \dim T_h$ by (3.7). Let $\rho(z) = 0$ for some $z \in Z_h$, i.e., $\partial_{11} z = \partial_{12} z = \partial_{22} z = 0$ on any $K \in \mathcal{T}_h$. Hence, z is linear on any K and since $z \in H^2(\Omega)$, we have $z \in P_1(\Omega)$. As $\dim P_1(\Omega) = 3$, we get the first part of the lemma for $\Gamma_2 = \emptyset$. For $\Gamma_2 \neq \emptyset$ we obtain that $z = 0$ from the condition $z = \partial_\nu z = 0$ on Γ_2. ■

Remark 3.3 : Let $\{ z^i \}$ be a basis in Z_h and let us put $\tau^i = \rho(z^i)$, $i = 1, \ldots, \dim Z_h$. By Lemma 3.1 it is evident that $\{ \tau^i \}$ is a basis in T_h if $\Gamma_2 \neq \emptyset$. Let us further consider the case $\Gamma_2 = \emptyset$. It is an easy exercise in linear algebra to show that, for instance, if $z^i(a_j) = \delta_{ij}$, $i, j \in \{ k, l, m \}$, for some points a_k, a_l, $a_m \in \overline{\Omega}$ not lying in a straight line, then τ^i are basis functions in T_h for

$$i \in \{ 1, \ldots, \dim Z_h \} - \{ k, l, m \} .$$

As in Section 2 we can now transform the problem of finding σ_h into the solution of the system of linear equations. The flexibility matrix of the system will be a band matrix if we select the basis in Z_h properly, since

$$\operatorname{supp} z \supseteq \operatorname{supp} \rho(z) \quad \text{for all} \quad z \in Y_h \subset H^2(\Omega) .$$

Note that the approximation of the primal formulation of (3.1) consists in minimizing some quadratic functional over the space $(V_h)^2$ which is in $(X_h)^2 \subset (H^1(\Omega))^2$. Therefore, the comparison of the orders of the stiffness and flexibility matrices of the primal and dual approximation of (3.1) cannot be correctly performed (like in Section 2). But we shall show an interesting example.

For simplicity, let $\mathcal{T}_h$ contain only triangles and let the whole space Y_h be generated only by the Hsieh-Clough-Tocher element, which is uniquely determined (see [4]) by three degrees of freedom of each vertex and by one degree of freedom of the mid-point of each side. Thus by (2.8) we get that

$$\dim Y_h = 3N + S = 2N + 2S - E - H + 1$$

and by Lemma 3.1 we arrive at

$$\dim \rho(Y_h) = 2N + 2S - E - H - 2. \tag{3.9}$$

Since $\rho(Y_h)$ contains piecewise linear stresses, it is reasonable for comparison to choose the space $(X_h)^2$ with piecewise quadratic displacements on the same triangulation $\mathcal{T}_h$ (see also [1]). For the most common quadratic element which is uniquely determined by degrees of freedom of each vertex and of the mid-point of each side, we get

$$\dim (X_h)^2 = 2N + 2S. \tag{3.10}$$

Using (3.9) and (3.10), a simple consideration leads now to the fact that $\dim T_h < \dim (V_h)^2$ in general. Even it can be shown that we can get also the width of the band of the flexibility matrix less than for the stiffness matrix. Moreover, the displacements of $(V_h)^2$ are quadratic on any $K \in \mathcal{T}_h$, while the stress tensors of T_h are piecewise linear on any K which is composed of three subtriangles. The conforming methods using this composed element are presented in [10, 12].

Some non-conforming methods, where the equilibrium equation or the symmetry of the stress tensor is satisfied only approximately, are described in [15, 25, 26, 27].

4. CONSTRUCTION OF THE SPACES OF EQUILIBRIUM FINITE ELEMENTS FOR THE BIHARMONIC PROBLEM

For simplicity, we shall consider the biharmonic problem only with the Dirichlet boundary condition (the bending problem for a clamped plate). Given $f \in L^2(\Omega)$, find z such that

$$\begin{aligned} \Delta^2 z &= f \quad \text{in} \quad \Omega, \\ z = \partial_\nu z &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.1}$$

The dual variational formulation of this classical problem consists (see [2]) in minimizing the functional

$$J(\mu) = \frac{1}{2}(\mu, \mu)_0$$

over the set (of statically admissible bending moments)

$$M(f) = \{ \mu \in (L^2(\Omega))^4_{\text{sym}} \mid (\mu, \operatorname{hes} z)_0 = (f, z)_0 \quad \forall z \in H^2_0(\Omega) \},$$

where hes : $H^2(\Omega) \to (L^2(\Omega))^4_{\text{sym}}$ is defined in the usual way

$$\operatorname{hes} z = \begin{pmatrix} \partial_{11} z, & \partial_{12} z \\ \partial_{12} z, & \partial_{22} z \end{pmatrix}, \qquad z \in H^2(\Omega).$$

Let us note that by Green's theorem we get

$$\partial_{11}\mu_{11} + 2\,\partial_{12}\mu_{12} + \partial_{22}\mu_{22} = \operatorname{div}(\operatorname{Div}\mu) = f$$

for

$$\mu = (\mu_{ij}) \in M(f) \cap (H^2(\Omega))^4.$$

The previous problem can be equivalently formulated as follows.

Given $\bar{\lambda} = (\bar{\lambda}_{ij}) \in M(f)$ fixed (e.g., it is possible to choose $\bar{\lambda}_{11} = \bar{\lambda}_{12} = 0$ in Ω and

$$\bar{\lambda}_{22}(x_1, x_2) = \int_0^{x_2} \int_0^{\eta} \bar{f}(x_1, \xi)\, d\xi\, d\eta, \quad (x_1, x_2)^T \in \Omega,$$

where $\bar{f} = f$ in Ω and $\bar{f} = 0$ in $\mathbb{R}^2 - \Omega$), find λ which minimizes the functional

$$\bar{J}(\mu) = \frac{1}{2}(\mu, \mu)_0 + (\mu, \bar{\lambda})_0 \tag{4.2}$$

over the space

$$M = M(0) = \{ \mu \in (L^2(\Omega))^4_{\text{sym}} \mid (\mu, \operatorname{hes} z)_0 = 0 \quad \forall z \in H^2_0(\Omega) \}. \tag{4.3}$$

Further, we define the operator $\omega : (H^1(\Omega))^2 \to (L^2(\Omega))^4_{\text{sym}}$ by

$$\omega(v) = \begin{pmatrix} \partial_2 v_2 & , & -\frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) \\ -\frac{1}{2}(\partial_1 v_2 + \partial_2 v_1), & & \partial_1 v_1 \end{pmatrix}, \; v = (v_1, v_2)^T \in (H^1(\Omega))^2.$$

THEOREM 4.1 : *If Ω is simply connected, then*

$$M = \omega((H^1(\Omega))^2).$$

Proof: Let $\mu = (\mu_{ij}) \in M$ be arbitrary and let us put

$$\overset{*}{\mu} = \begin{pmatrix} \mu_{22}, & -\mu_{12} \\ -\mu_{12}, & \mu_{11} \end{pmatrix}.$$

Then using (4.3), we get

$$(\overset{*}{\mu}, \rho(z))_0 = (\mu, \text{hes } z)_0 = 0 \quad \forall z \in H_0^2(\Omega). \tag{4.4}$$

By (3.6) we have for a connected $\Gamma_1 = \partial\Omega$ that

$$(L^2(\Omega))^4_{\text{sym}} = \rho(H_0^2(\Omega)) \oplus \varepsilon((H^1(\Omega))^2). \tag{4.5}$$

Hence, by (4.4) there exists $v \in (H^1(\Omega))^2$ so that $\overset{*}{\mu} = \varepsilon(v)$. Thus $\mu = \omega(v)$ and $\mu \in \omega((H^1(\Omega))^2)$.

Conversely, let $v \in (H^1(\Omega))^2$. Then by (4.5) we obtain

$$(\omega(v), \text{hes } z)_0 = (\varepsilon(v), \rho(z))_0 = 0 \quad \forall z \in H_0^2(\Omega).$$

Therefore, (4.3) yields $\omega(v) \in M$. ∎

Now, for a simply connected domain Ω we can introduce the space of an equilibrium finite elements of bending moments as

$$M_h = \omega(\mathscr{V}_h),$$

where $\mathscr{V}_h$ is some space of finite elements in $(H^1(\Omega))^2$. Let us emphasize that if we choose the space $\mathscr{V}_h$ composed of piecewise linear functions, then M_h will contain only piecewise constant functions (see also [14]), while any conforming primal finite element method for the problem (4.1) demands more complicated functions from $Z_h \subset H_0^2(\Omega)$.

The finite element approximation of the dual problem will consist in finding λ_h which minimizes (4.2) over $M_h \subset M$.

For any $\mu \in M$, $\mu_h \in M_h$ and to them corresponding $v \in (H^1(\Omega))^2$, $v_h \in \mathscr{V}_h$ we have

$$\| \mu - \mu_h \|_0 = \| \omega(v) - \omega(v_h) \|_0 = \| \varepsilon(v - v_h) \|_0 \leqslant \| v - v_h \|_1 .$$

Thus by Cea's Lemma we can get analogous convergence results as in Section 2 and 3, i.e., if $\bigcup_h \mathscr{V}_h$ is dense in $(H^1(\Omega))^2$, then λ_h converge to λ without any regularity assumptions on λ and there is a constant C independent of h such that

$$\| \lambda - \lambda_h \|_0 \leqslant Ch^k \, | \lambda |_k \quad \text{for} \quad \lambda \in M \cap (H^k(\Omega))^{2 \times 2}, \quad k \geqslant 1,$$

and for $\mathscr{V}_h \subset (H^1(\Omega))^2$ constructed by elements of sufficiently high order with respect to k.

The next table shows which class of elements [4] is to be used for the conforming primal and dual finite element method.

TABLE 2

Classes of elements for conforming methods

	Primal method	Dual method
Steady-state heat conduction problem	$\mathscr{C}^0$	$\mathscr{C}^0$
Linear elasticity problem	$[\mathscr{C}^0]^2$	$\mathscr{C}^1$
Biharmonic problem	$\mathscr{C}^1$	$[\mathscr{C}^0]^2$

Let us further remark that due to Theorems 2.1, 3.1 and 4.1, we get for a simply connected Ω the relations

$$\begin{aligned}(L^2(\Omega))^2 &= \operatorname{curl} H^1(\Omega) \oplus \operatorname{grad} H_0^1(\Omega) = \operatorname{curl} H_0^1(\Omega) \oplus \operatorname{grad} H^1(\Omega)\,,\\ (L^2(\Omega))^4_{\text{sym}} &= \rho(H^2(\Omega)) \oplus \varepsilon((H_0^1(\Omega))^2) = \rho(H_0^2(\Omega)) \oplus \varepsilon((H^1(\Omega))^2)\\ &= \operatorname{hes} H^2(\Omega) \oplus \omega((H_0^1(\Omega))^2) = \operatorname{hes} H_0^2(\Omega) \oplus \omega((H^1(\Omega))^2)\,.\end{aligned}$$

It will be obvious from Section 6 that the above assumption of simple connectivity of Ω is necessary.

5. CONSTRUCTION OF THE SPACES OF EQUILIBRIUM FINITE ELEMENTS FOR THE STOKES PROBLEM

The homogeneous stationary Stokes problem of the motion of an incompressible viscous fluid in Ω is classically formulated in the following way (see e.g. [4, 6, 8, 9, 18]).

Given $f \in (L^2(\Omega))^2$ and a constant $\eta > 0$, find the velocity $v = (v_1, v_2)^T$ and the preasure p such that

$$\begin{aligned} -\eta\,\Delta v + \operatorname{grad} p &= f \quad \text{in} \quad \Omega\,,\\ \operatorname{div} v &= 0 \quad \text{in} \quad \Omega\,, \qquad\qquad (5.1)\\ v &= 0 \quad \text{on} \quad \partial\Omega\,,\end{aligned}$$

where $\Delta v = (\Delta v_1, \Delta v_2)^T$.

We shall not be concerned with the way of finding p (for this see [4, 8, 9]). The variational solution v of the problem (5.1) can be obtained [6] by minimizing the functional

$$J(u) = \frac{\eta}{2}(\operatorname{Grad} u, \operatorname{Grad} u)_0 - (f, u)_0$$

over the space

$$U = \{ u \in (H_0^1(\Omega))^2 \mid \operatorname{div} u = 0 \},$$

where the operator Grad : $(H^1(\Omega))^2 \to (L^2(\Omega))^{2 \times 2}$ is defined by

$$\operatorname{Grad} u = \begin{pmatrix} \partial_1 u_1, & \partial_1 u_2 \\ \partial_2 u_1, & \partial_2 u_2 \end{pmatrix}, \quad u = (u_1, u_2)^T \in (H^1(\Omega))^2 .$$

As a simple consequence of Theorem 2.1 we get the following assertion (see also [18]).

THEOREM 5.1 : *If Ω is simply connected, then*

$$U = \operatorname{curl} H_0^2(\Omega) .$$

Proof: Let $u \in U$. Since $U \subset Q$ for $\Gamma_2 = \partial\Omega$, there exists $z \in H_0^1(\Omega)$ so that $(\partial_2 z, -\partial_1 z)^T = u$. Thus $z \in H_0^2(\Omega)$ as $u \in (H_0^1(\Omega))^2$.

Conversely, if $z \in H_0^2(\Omega)$, then curl $z \in (H_0^1(\Omega))^2$ and div curl $z = 0$ in Ω, i.e. curl $z \in U$. ■

Now, for a simply connected Ω we can define the space of equilibrium finite elements of velocities for the Stokes (or Navier-Stokes) problem as

$$U_h = \operatorname{curl} Z_h ,$$

where Z_h is some space of finite elements in $H_0^2(\Omega)$, i.e., velocities from U_h are continuous and exactly divergence-free (see also [23]). The finite element approximation of the above variational problem will consist in finding v_h which minimizes J over the space $U_h \subset U$. Choosing the basis $\{ z^i \}$ in Z_h, we see (as in Section 2) that $\{ \operatorname{curl} z^i \}$ is the basis in U_h. Hence, v_h can again be obtained by solving a system of linear algebraic equations.

For any $u \in U$ and $u_h \in U_h$ and to them corresponding $z \in H_0^2(\Omega)$ and $z_h \in Z_h$, respectively, we have

$$\| u - u_h \|_1 = \| \operatorname{curl} z - \operatorname{curl} z_h \|_1 \leqslant \| z - z_h \|_2 .$$

Hence, as in the preceding sections we can get the convergence of v_h to the solution $v \in U$ in the norm $\|\cdot\|_1$, if $\bigcup_h Z_h$ is dense in $H_0^2(\Omega)$ (see [4], Theorem 6.1.7). Further we can obtain the a priori error estimate

$$\| v - v_h \|_1 \leqslant Ch^k \, | v |_{k+1} \quad \text{for} \quad v \in U \cap (H^{k+1}(\Omega))^2 \,, \quad k \geqslant 1 \,,$$

and for $Z_h \subset H_0^2(\Omega)$ generated by elements of sufficiently high order with respect to k (see (3.8)).

6. APPENDIX

In this last Section, we shall investigate the problem of Section 2 without any assumption on the connectivity of Γ_1 and Γ_2. Let us recall that the connectivity of Γ_1, Γ_2 guaranteed the existence of the stream function of any $q \in Q$ since (1.5) was fulfilled — see the proof of Theorem 2.1. But, in general, we evidently need not get the existence of the (one-valued !) stream function of $q \in Q$ defined on a multiply connected domain Ω. Before proving a theorem analogical to Theorem 2.1 for the general case, we formulate an auxiliary lemma.

Let us note [3] that a graph is said to be a tree if it is connected and has no circuits. Every tree with m vertices has exactly $m - 1$ edges. Marking with $0, \ldots, m - 1$ the vertices and with $1, \ldots, m - 1$ the edges of a tree in some way, we can associate this tree with the $m \times (m - 1)$ incidence matrix $B = (B_{ij})$ whose rows and columns correspond to the vertices and edges, respectively, and whose entry $B_{ij} \neq 0$ iff the jth edge is incident with the ith vertex. For the present, we do not determine the magnitude of $B_{ij} \neq 0$, but we only assume that

$$\sum_{i=0}^{m-1} B_{ij} = 0 \,, \qquad j = 1, \ldots, m - 1 \,, \tag{6.1}$$

(this sum contains exactly two nonzero entries, as any edge is incident with two vertices).

LEMMA 6.1 : *Let an $m \times (m - 1)$ incidence matrix $B = (B_{ij})$ of a tree satisfy (6.1) and let $b = (b_0, \ldots, b_{m-1})^T \in \mathbb{R}^m$ satisfy the condition $b_0 + \cdots + b_{m-1} = 0$. Then the system $Bx = b$ has exactly one solution $x = (x_1, \ldots, x_{m-1})^T$.*

Proof : Using the fact [3] that any tree has at least two vertices of degree one (such a vertex is incident with just one edge), we can easily establish by induction on m that B has the maximum rank. Thus its columns are basis vectors in the space $\{ \xi = (\xi_0, \ldots, \xi_{m-1})^T \in \mathbb{R}^m \mid \xi_0 + \cdots + \xi_{m-1} = 0 \}$. ■

THEOREM 6.1 : *Let m be the number of all the components* $\partial\Omega_i$ $(i \in \{0, \dots, H\})$ *of* $\partial\Omega$ *for which* $\partial\Omega_i \cap \Gamma_1 \neq \emptyset$. *Let n be the number of the components of* Γ_2. *Then there exist functions* $\alpha^1, \dots, \alpha^{m-1}$, $\beta^1, \dots, \beta^{n-1} \in (L_2(\Omega))^2$ – curl W *such that*

$$Q = \mathscr{L}(\text{curl } W \cup \{\alpha^1, \dots, \alpha^{m-1}, \beta^1, \dots, \beta^{n-1}\}),$$

where Q and W are defined by (2.7) *and* (2.12), *respectively, and* $\mathscr{L}$ *denotes the linear hull.*

Proof : The inclusion curl $W \subset Q$ can be proved in the exactly same way as in the proof of Theorem 2.1. Now we proceed to the construction of the functions β^j. If $n \geqslant 2$, then we put

$$\beta^j = \text{curl } \overline{w}^j, \qquad j = 1, \dots, n-1, \tag{6.2}$$

where $\overline{w}^j \in H^1(\Omega)$ are arbitrary functions satisfying

$$\overline{w}^j = \delta_{ij} \quad \text{on} \quad \Gamma_2^i, \quad i = 0, \dots, n-1, \quad j = 1, \dots, n-1. \tag{6.3}$$

Here $\Gamma_2^0, \dots, \Gamma_2^{n-1}$ are the components of Γ_2 and by the definition of Γ_0, the distances among these components are positive. As in (2.14), we can obtain that

$$(\text{curl } \overline{w}^j, \text{grad } v)_0 = \int_{\Gamma_2} \frac{\partial v}{\partial s} \overline{w}^j \, ds \quad \forall v \in V \cap \mathscr{C}^\infty(\overline{\Omega}), \quad j = 1, \dots, n-1.$$

Therefore, by (6.2) and (6.3) we get

$$(\beta^j, \text{grad } v)_0 = \int_{\Gamma_2^j} \frac{\partial v}{\partial s} \, ds = 0 \quad \forall v \in V \cap \mathscr{C}^\infty(\overline{\Omega}), \quad j = 1, \dots, n-1, \tag{6.4}$$

since either Γ_2^j is a closed curve, or $v = 0$ at the end points of $\overline{\Gamma_2^j}$ $(v/\Gamma_0 = 0)$. From (2.7) and from the density of $V \cap \mathscr{C}^\infty(\overline{\Omega})$ in V (Γ_1, Γ_2 have a finite number of components), we get that $\beta^j \in Q$ and (6.3) yields $\beta^j \notin$ curl W.

If $m \geqslant 2$ we shall define the functions $\alpha^1, \dots, \alpha^{m-1}$, but first of all we shall construct their supports $S_1, \dots, S_{m-1}$. Let

$$\partial\Omega_i \cap \Gamma_1 \neq \emptyset \quad \text{for} \quad i = 0, \dots, m-1 \tag{6.5}$$

(otherwise we change the notation of the components of $\partial\Omega$). Let G be an arbitrary tree with m vertices and let us mark with $0, \dots, m-1$ its vertices and with $1, \dots, m-1$ its edges in some way. Let $j \in \{1, \dots, m-1\}$ be fixed and let the end points of the jth edge be the kth and lth vertex of G. We associate the support S_j with this jth edge in the following way.

The set S_j will be an arbitrary simply connected closed domain in $\overline{\Omega}$ with a Lipschitz boundary such that the sets

$$\partial S_j^1 = \partial S_j \cap \partial \Omega_k , \; \partial S_j^3 = \partial S_j \cap \partial \Omega_l \tag{6.6}$$

are connected and contained in $\overline{\Gamma}_1$ and

$$S_j \cap \partial \Omega_i = \varnothing \quad \forall i \in \{ 0, \dots, H \} - \{ k, l \} , \tag{6.7}$$

(see *fig.* 2, 3). Now, let $\tilde{w}^j \in H^1(\Omega)$ be an arbitrary function such that

$$\begin{aligned} \tilde{w}^j &= 1 \quad \text{on } \partial S_j^2 , \\ \tilde{w}^j &= 0 \quad \text{on } \partial S_j^4 , \end{aligned} \tag{6.8}$$

where ∂S_j^2 and ∂S_j^4 are the components of the set ∂S_j-$(\partial S_j^1 \cup \partial S_j^3)$. Let us put, for $j \in \{ 1, \dots, m - 1 \}$,

$$\begin{aligned} \alpha^j &= \operatorname{curl} \tilde{w}^j \quad \text{on} \quad S_j , \\ \alpha^j &= 0 \qquad\qquad \text{on} \quad \Omega - S_j . \end{aligned} \tag{6.9}$$

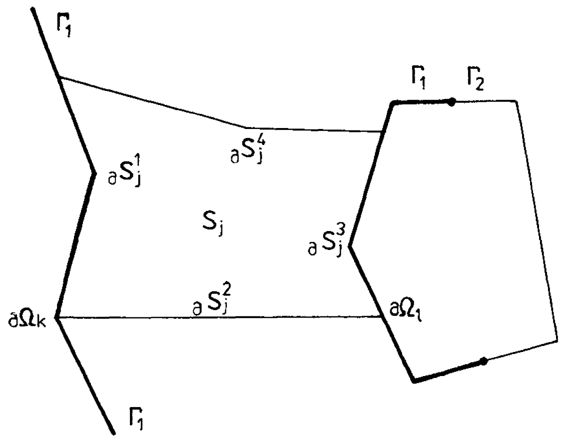

Figure 2.

Denoting by $\nu^j = (\nu_1^j, \nu_2^j)^T$ the outward unit normal to S_j and setting $s^j = (\nu_2^j, - \nu_1^j)^T$ on ∂S_j, we get by (6.9), (1.3), (1.4), (6.6), (6.7) and (6.8) that

$$(\alpha^j, \operatorname{grad} v)_{0,\Omega} = (\operatorname{curl} \tilde{w}^j, \operatorname{grad} v)_{0,S_j} = -(\operatorname{grad} \tilde{w}^j, \operatorname{curl} v)_{0,S_j} =$$

$$= -\int_{\partial S_j} \tilde{w}^j (\operatorname{curl} v)^T \nu^j \, ds = \int_{\partial S_j} \tilde{w}^j (\operatorname{grad} v)^T s^j \, ds = \int_{\partial S_j^2} \frac{\partial v}{\partial s^j} \, ds = 0$$

$$\forall v \in V \cap \mathscr{C}^\infty(\overline{\Omega}) \qquad (6.10)$$

as $v = 0$ on $\Gamma_1 \cap \partial S_j$. This implies that $\alpha^j \in Q$. Further, since $\operatorname{div} \alpha^j = 0$, we get by (1.3), (6.7) and (6.9) that

$$0 = \langle \gamma_\nu \alpha^j, 1 \rangle_{\partial\Omega} = \langle \gamma_\nu \alpha^j, 1 \rangle_{\partial\Omega_k} + \langle \gamma_\nu \alpha^j, 1 \rangle_{\partial\Omega_l} \,. \qquad (6.11)$$

Using (6.8) and (6.9), we can easily ascertain that the absolute value of both the last terms is equal to 1, while by Theorem 1.1

$$\langle \gamma_\nu \operatorname{curl} w, 1 \rangle_{\partial\Omega_i} = 0 \quad \forall w \in W \,, \quad i = 0, \dots, H \,.$$

Hence, $\alpha^j \notin \operatorname{curl} W$.

Conversely, let $q \in Q$ be arbitrary and let for a moment $m \geqslant 2$. Using (1.3) and (6.5), we have

$$0 = \langle \gamma_\nu q, 1 \rangle_{\partial\Omega} = \sum_{i=0}^{H} \langle \gamma_\nu q, 1 \rangle_{\partial\Omega_i} = \sum_{i=0}^{m-1} \langle \gamma_\nu q, 1 \rangle_{\partial\Omega_i} \,, \qquad (6.12)$$

since $\operatorname{div} q = 0$ and $q^T \nu = 0$ on Γ_2. By Lemma 6.1, (6.11) and (6.12) we see that the system

$$\sum_{j=1}^{m-1} \langle \gamma_\nu \alpha^j, 1 \rangle_{\partial\Omega_i} x_j = \langle \gamma_\nu q, 1 \rangle_{\partial\Omega_i} \,, \quad i = 0, \dots, m-1 \,, \qquad (6.13)$$

has the solution $x = (x_1, \dots, x_{m-1})^T$. Let us put

$$\bar{q} = q - \sum_{j=1}^{m-1} x_j \alpha^j \,, \qquad (6.14)$$

and suppose the sum to be zero for $m < 2$. Since $\operatorname{div} \alpha^j = 0$ we get by (6.14) and (6.13) that $\bar{q}$ satisfies

$$\operatorname{div} \bar{q} = 0 \,, \quad \langle \gamma_\nu \bar{q}, 1 \rangle_{\partial\Omega_i} = 0 \quad \text{for} \quad i = 0, \dots, H \,.$$

Now, by Theorem 1.1 there exists $\bar{w} \in H^1(\Omega)$ such that $\bar{q} = \operatorname{curl} \bar{w}$. We can find out by the same manner as in the proof of Theorem 2.1 that $\bar{w}$ is constant on any component Γ_2^i, $i = 0, \dots, n-1$, of Γ_2. If $\Gamma_2 \neq \varnothing$, then let $\bar{w}$ be chosen in such a way that $\overline{w} = 0$ on Γ_2^0. For $n \geqslant 2$ there exist, by (6.3), the coefficients $y_1, \ldots, y_{n-1} \in \mathbb{R}^1$ and $w \in W$ such that

$$\overline{w} = w + \sum_{j=1}^{n-1} y_j \overline{w}^j$$

and the sum is zero for $n < 2$. Hence,

$$\overline{q} = \operatorname{curl} w + \sum_{j=1}^{n-1} y_j \operatorname{curl} \overline{w}^j$$

and by (6.14) and (6.2) we have

$$q = \operatorname{curl} w + \sum_{j=1}^{m-1} x_j \alpha^j + \sum_{j=1}^{n-1} y_j \beta^j . \quad \blacksquare \tag{6.15}$$

Remark 6.1 : Since α^j, β^j are not in curl W and, by (6.10) and (6.4), they are perpendicular to grad V, we get

$$(L^2(\Omega))^2 = \operatorname{grad} V \oplus \operatorname{curl} W \oplus R ,$$

where $R = Q \ominus \operatorname{curl} W$ is non-empty if $m \geqslant 2$ or $n \geqslant 2$. Clearly, the functions from curl $W \oplus R$ are divergence-free, while the functions from grad $V \oplus R$ are rotation-free (rot $q = \partial_1 q_2 - \partial_2 q_1 = 0$ in the sense of distributions).

Henceforth, let us suppose that the functions $\tilde{w}^j, \overline{w}^j$ from the proof of Theorem 6.1 will be always in X_h (otherwise the following definition would then be dependent upon the choice of α^j, β^j). We define the space of finite elements (of heat flows) as

$$Q_h = \mathscr{L}(\operatorname{curl} W_h \cup \{ \alpha^1, \ldots, \alpha^{m-1}, \beta^1, \ldots, \beta^{n-1} \}),$$

where $W_h = X_h \cap W$. The definition is independent of a particular choice of α^j, β^j, since any other $\hat{\alpha}^j$, $\hat{\beta}^j$ ($\{ \hat{\alpha}^j \}$ can correspond e.g. to a quite different tree) can be expressed as in (6.15) by a linear combination of the functions curl w and $\alpha^1, \ldots, \beta^{n-1}$, where w is now in W_h, as $\tilde{w}^j, \overline{w}^j \in X_h$.

LEMMA 6.2 : *Let* $\{ q^j \}_{j=1}^r$ *be basis in* curl W_h. *Then* $\{ q^j \} \cup \{ \alpha^j \} \cup \{ \beta^j \}$ *is the basis in* Q_h.

Proof : Let there exist $c_j, \tilde{c}_j, \overline{c}_j \in \mathbb{R}^1$ such that

$$\sum_{j=1}^{r} c_j q^j + \sum_{j=1}^{m-1} \tilde{c}_j \alpha^j + \sum_{j=1}^{n-1} \overline{c}_j \beta^j = 0 . \tag{6.16}$$

If $m \geqslant 2$ then by (6.16) and Theorem 1.1 we have

$$\sum_{j=1}^{m-1} \langle \gamma_\nu \alpha^j, 1 \rangle_{\partial\Omega_i} \tilde{c}_j = 0 , \qquad i = 0, ..., m - 1 ,$$

since any q^j, β^j have a stream function. This system has, by Lemma 6.1, exactly one solution $\tilde{c}_j = 0$, $j = 1, ..., m - 1$.

Let $n \geqslant 2$ (the assertion of the lemma for $n < 2$ is evident now) and let $w^j \in W$ be such that $q^j = \operatorname{curl} w^j$. Then (6.2) and (6.16) imply that $\sum_{j=1}^{r} c_j w^j + \sum_{j=1}^{n-1} \bar{c}_j \bar{w}^j$ is a constant. This constant is zero, as $w^j = \bar{w}^j = 0$ on Γ_2^0, and by (6.3) we see that $\bar{c}_j = 0$ for $j = 1, ..., n - 1$. Thus also $c_j = 0$ for $j = 1, ..., r$. ■

Remark 6.2 : For the numerical realization, it is natural to take α^j, β^j so that their supports are as small as possible — see the shaded parts of the triangulation in *fig.* 3 (with X_h consisting of e.g. bilinear elements). To get a suitable

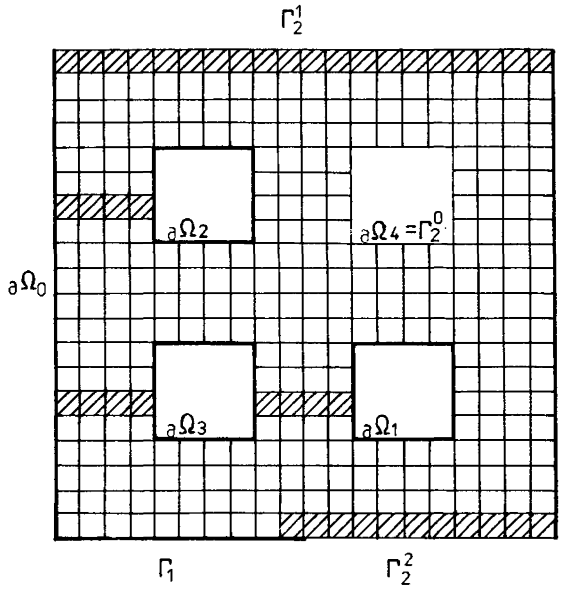

Figure 3.

form of the flexibility matrix, it is moreover necessary to deal with the problems how to choose the tree corresponding to $\partial\Omega_0, ..., \partial\Omega_{m-1}$, which component of Γ_2 to denote by Γ_2^0, in which sequence to mark with numbers basis functions of Q_h, etc.

Remark 6.3 : Let $\{ W_h \}$ be a system of finite element spaces such that $\bigcup_h W_h$ is dense in W. For simplicity, let Q_h be defined by the same functions α^j, β^j as in the expression of Q. Then for p_h which minimizes (2.6) over Q_h and for

$$p = \operatorname{curl} w + \sum_j \tilde{c}_j \alpha^j + \sum_j \bar{c}_j \beta^j , \quad w \in W ,$$

we have by Cea's Lemma that

$$\frac{1}{C} \| p - p_h \|_0 \leqslant \inf_{q_h \in Q_h} \| p - q_h \|_0 =$$

$$= \inf_{w_h \in W_h} \inf_{\tilde{d}_j, \bar{d}_j \in \mathbb{R}^1} \left\| \operatorname{curl} (w - w_h) + \sum_{j=1}^{m-1} (\tilde{c}_j - \tilde{d}_j) \alpha^j + \sum_{j=1}^{n-1} (\bar{c}_j - \bar{d}_j) \beta^j \right\|_0$$

$$= \inf_{w_h \in W_h} \| \operatorname{grad} (w - w_h) \|_0 \to 0 \quad \text{as} \quad h \to 0 .$$

Remark 6.4 : For the Stokes problem, it can be proved by an analogical procedure as in the proof of Theorem 6.1 that

$$U = \mathscr{L}(\operatorname{curl} H_0^2(\Omega) \cup \{ \beta^1, \ldots, \beta^H \}) ,$$

where $\beta^j = \operatorname{curl} \overline{w}^j$, $\overline{w}^j \in H^2(\Omega)$, $\partial_\nu \overline{w}^j = 0$ on $\partial\Omega$ and $\overline{w}^j = \delta_{ij}$ on $\partial\Omega_i$ for $j = 1, \ldots, H$, $i = 0, \ldots, H$. Thus the supports of β^j can have e.g. a circular shape around any hole.

For the linear elasticity problem, we can obtain, due to Theorem 1.2, the results analogical to that of this section. But this is supposed to be the subject of next paper.

REFERENCES

[1] D. J. ALLMAN, *On compatible and equilibrium models with linear stresses for stretching of elastic plates,* in [27].

[2] F. BREZZI, *Non-standard finite elements for fourth order elliptic problems,* in [27].

[3] G. R. BUSACKER and T. L. SAATY, *Finite graphs and network : an introduction with applications,* McGraw-Hill, New York, London, Sydney, 1965.

[4] P. G. CIARLET, *The finite element method for elliptic problems,* North-Holland publishing company, Amsterdam, New York, Oxford, 1978.

[5] G. DUVAUT and J. L. LIONS, *Inequalities in mechanics and physics,* Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[6] M. FORTIN, *Approximation des fonctions à divergence nulle par la méthode des éléments finis,* Proceedings of the Third International Conference on Numerical Methods in Fluid Mechanics, Paris, 1972, vol. 1, 99-103.

[7] B. M. Fraeijs de Veubeke and M. Hogge, *Dual analysis for heat conduction problems by finite elements*, Internat. J. Numer. Methods Energ. 5 (1972), 65-82.

[8] V. Girault and P. A. Raviart, *Finite element approximation of the Navier-Stokes equation*, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[9] R. Glowinski and O. Pironneau, *On numerical methods for the Stokes problem*, in [27].

[10] B. J. Hartz and V. B. Watwood, *An equilibrium stress field model for finite element solution of two-dimensional electrostatic problems*, Internat. J. Solids and Structures 4 (1968), 857-873.

[11] J. Haslinger and I. Hlaváček, *Convergence of a finite element method based on the dual variational formulation*, Apl. Mat. 21 (1976), 43-65.

[12] I. Hlaváček, *Convergence of an equilibrium finite element model for plane elastostatics*, Apl. Mat. 24 (1979), 427-457.

[13] I. Hlaváček, *The density of solenoidal functions and the convergence of a dual finite element method*, Apl. Mat. 25 (1980), 39-55.

[14] C. Johnson, *On the convergence of a mixed finite element method for plate bending problems*, Numer. Math. 21 (1973), 43-62.

[15] C. Johnson and B. Mercier, *Some equilibrium finite element methods for two-~~dimensional elasticity problems~~*, ~~Numer. Math. 30 (1978), 103-116.~~

[16] D. W. Kelly, *Bounds on discretization error by special reduced integration of the Lagrange family of finite elements*, Internat. J. Numer. Methods Energ. 15 (1980), 1489-1506.

[17] M. Křížek, *An equilibrium finite element method in three-dimensional elasticity*, Apl. Mat. 27 (1982), 46-75.

[18] B. Mercier, *Topics in finite element solution of elliptic problems*, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[19] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967.

[20] J. Nečas and I. Hlaváček, *Mathematical theory of elastic and elasto-plastic bodies : an introduction*, Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York, 1981.

[21] A. R. S. Ponter, *The application of dual minimum theorems to the finite element solution of potential problems with special reference to seepage*, Internat. J. Numer. Methods Energ. 4 (1972), 85-93.

[22] G. Sander, *Application of the dual analysis principle*, Proceedings of the IUTAM Symposium on High Speed Computing of Elastic Structures, Congrès et Colloques de l'Université de Liège (1971), 167-207.

[23] R. Temam, *Navier-Stokes equations*, North-Holland publishing company, Amsterdam, New York, Oxford, 1977.

[24] J. M. Thomas, *Sur l'analyse numérique des méthodes d'éléments finis hybrides et mixtes*, Thesis, Université Paris VI, 1977.

[25] J. M. Thomas and M. Amara, *Approximation par éléments finis équilibre du système de l'élasticité linéaire*, C.R. Acad. Sc. Paris, t. 286 (1978), 1147-1150.

[26] J. M. Thomas and M. Amara, *Equilibrium finite elements for the linear elastic problem*, Numer. Math. 33 (1979), 367-383.

[27] Energy methods in finite element analysis, John Wiley & Sons Ltd., Chichester, New York, Brisbane, Toronto, 1979.