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ON A MIXED FINITE ELEMENT METHOD FOR THE STOKES PROBLEM IN $\mathbb{R}^3$ (*)

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Abstract — *We prove an error estimate for a mixed finite element method for solving the Stokes problem on a rectangular domain in $\mathbb{R}^3$ The scheme is based on piecewise trilinear velocities and piecewise constant pressure on a uniform rectangular grid*

Résume — *On etablit une estimation de l'erreur pour une methode d'elements finis mixtes pour le problème de Stokes sur un domaine rectangulaire de $\mathbb{R}^3$ Le schema met en œuvre des vitesses trilineaires par morceaux et une pression constante par morceaux sur un maillage rectangulaire uniforme*

1. INTRODUCTION

One of the simplest ways of discretizing the Stokes equations on a rectangular domain in $\mathbb{R}^n$ is to apply the finite element technique with continuous, piecewise multilinear velocities and piecewise constant pressure on a rectangular grid. The resulting finite difference equations resemble those of the classical Marker — and — Cell method [4]. In two dimensions the method has been used successfully also on irregular meshes, *cf.* [10].

From a theoretical point of view, the above finite element scheme falls into the category of mixed methods, which can be analyzed along the lines of Babuška [1] and Brezzi [2]. The analysis was recently carried out in the two-dimensional case [7]. It was shown that although the method is not uniformly stable in the classical sense of [1, 2], a weaker stability estimate holds which yields optimal convergence rates for the velocities in $H^1(\Omega)$ and $L_2(\Omega)$, provided that the exact solution is sufficiently regular.

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In this paper we analyze the three-dimensional scheme where the velocities are approximated by piecewise trilinear functions The analysis proceeds following closely the lines of [7] In particular, we establish a weak Babuška-Brezzi-type stability estimate for the pressures and combine this with certain superapproximation properties for the velocities As in two dimensions, we are able to prove that the velocities converge with the optimal rate $O(h)$ in $H^1(\Omega)$, if the exact solution is sufficiently smooth We also state the three-dimensional analogues of the L_2-estimates proved in [7] for the velocities and for the pressures smoothed in an appropriate way

Due to the fact that the stability estimate we can prove is weaker than in two dimensions, we end up requiring relatively high regularity on the exact solution, in order to be able to balance the weak stability with superapproximation results Only the case of a regular mesh is considered, a constraint that seems to play an essential role in the analysis

The plan of the paper is as follows In section 2 we state the problem and define its finite element discretization Section 3 is devoted to the error analysis

Throughout the paper we denote by $W^{m,p}(\Omega)$, $\Omega \subset \mathbb{R}^3$, $m \geqslant 0$, $1 \leqslant p < \infty$, the usual Sobolev spaces with the norms

$$\| v \|_{m,p} = \left(\sum_{l=0}^{k} | v |_{l,p}^{p} \right)^{1/p},$$

where $| \cdot |_{l,p}$ denote the seminorms

$$| v |_{l,p} = \left\{ \sum_{i+j+k=l} \int_{\Omega} \left| \frac{\partial^{l} v}{\partial x_1^{i} \, \partial x_2^{j} \, \partial x_3^{k}} \right|^{p} dx_1 \, dx_2 \, dx_3 \right\}^{1/p}$$

Here we omit to indicate the domain with a subindex, since it will be the same throughout the paper For non-integral $s \geqslant 0$, $W^{s,p}(\Omega)$ is defined as usual by interpolation For $p = 2$ we set $H^m(\Omega) = W^{m,2}(\Omega)$, $| \cdot |_m = | \cdot |_{m,2}$ and $\| \cdot \|_m = \| \cdot \|_{m,2}$ As usual, $H_0^1(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in the norm $\| \cdot \|_1$

The same notation will be used for the corresponding (semi) norms in $[W^{m,p}(\Omega)]^3$ The scalar products in $L_2(\Omega)$ or $[L_2(\Omega)]^3$ will be denoted by $(\cdot, \cdot)$

Finally, by C or C_j we denote positive constants, possibly different at different occurrences, which may depend on the domain Ω considered but not on any other parameter to be introduced unless indicated explicitly We also denote by P_k the set of polynomials in three variables of degree at most k

2. THE PROBLEM AND ITS DISCRETIZATION

Let Ω be a rectangular domain in $\mathbb{R}^3 : \Omega = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3,$ $x_i \in (0, a_i), i = 1, 2, 3 \}$. We consider the Stokes problem for an incompressible fluid with viscosity equal to one :

$$\begin{aligned} -\Delta u + \nabla\lambda &= f \quad \text{in} \quad \Omega\,, \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega\,, \\ u &= 0 \quad \text{on} \quad \partial\Omega\,, \\ \int_\Omega \lambda\, dx &= 0\,. \end{aligned} \tag{2.1}$$

Here $u = (u_1, u_2, u_3)$ is the velocity of the fluid and λ is the pressure, which we normalize to have the zero mean value. For simplicity we consider only the homogeneous Dirichlet boundary condition.

Let C_h^0 be a uniform partitioning of Ω into rectangular subdomains of size $h_1 \times h_2 \times h_3$, i.e.,

$$\begin{aligned} C_h^0 &= \{ K_{ijk} : i = 1, \dots, m_1, j = 1, \dots, m_2, k = 1, \dots, m_3 \}\,, \\ K_{ijk} &= \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : (i-1)\, h_1 < x_1 < ih_1, \\ &\qquad (j-1)\, h_2 < x_2 < jh_2, (k-1)\, h_3 < x_3 < kh_3 \}\,, \end{aligned}$$

where $m_i = a_i/h_i$ are integers. We assume that h_1, h_2 and h_3 depend on the mesh parameter h in such a way that h_i/h is bounded from below and from above by constants independent of h.

Let C_h be a partitioning of Ω obtained by dividing each $K_{ijk} \in C_h^0$ into eight equal 3-rectangles :

$$\begin{aligned} C_h &= \{ \Delta_{ijk} : i = 1, \dots, 2\, m_1, j = 1, \dots, 2\, m_2, k = 1, \dots, 2\, m_3 \}\,, \\ \Delta_{ijk} &= \{ x \in \mathbb{R}^3 ; (i-1)\, h_1/2 < x_1 < ih_1/2, \\ &\qquad (j-1)\, h_2/2 < x_2 < jh_2/2, (k-1)\, h_3/2 < x_3 < kh_3/2 \}\,. \end{aligned}$$

We associate to C_h the following finite element spaces :

$$\begin{aligned} S_h &= \{ v \in H_0^1(\Omega) : v\,|_{\Delta_{ijk}} \quad \text{is trilinear} \quad \forall \Delta_{ijk} \in C_h \} \\ Q_h &= \{ \mu \in L_2(\Omega) : \mu\,|_{\Delta_{ijk}} \quad \text{is constant} \quad \forall \Delta_{ijk} \in C_h \}\,. \end{aligned}$$

Setting $V_h = (S_h)^3$ we can now define a finite element method for the solution of (2 1) as Find $(u_h, \lambda_h) \in V_h \times Q_h$ such that

$$\left.\begin{aligned}(\nabla u_h, \nabla v) - (\lambda_h, \operatorname{div} v) &= (f, v) \quad \forall v \in V_h \\ (\operatorname{div} u_h, \mu) &= 0 \qquad \forall \mu \in Q_h\end{aligned}\right\} \qquad \begin{aligned}&(2\ 2a)\\&(2\ 2b)\end{aligned}$$

This set of equation does not have a unique solution (see section 3 below) To make the solution unique, it is customary to replace (2 2b) by

$$\varepsilon(\lambda_h, \mu) + (\operatorname{div} u_h, \mu) = 0 \quad \forall \mu \in Q_h , \qquad (2\ 2b')$$

where $\varepsilon > 0$ is a small parameter The perturbed system (2 2a)-(2 2b') now has a unique solution, as is easily seen by setting $v = u_h$ $\mu = \lambda_h$ Upon eliminating λ_h from the perturbed system one obtains for u_h the equation

$$(\nabla u_h, \nabla v) + \frac{1}{\varepsilon}(\operatorname{div} u_h, \operatorname{div} v)_* = (f, v) \quad \forall v \in V_h \qquad (2\ 3)$$

where $(.,.)_*$ indicates that the inner product is evaluated by first taking the average of div u_h and div v over each $\Delta_{ijh} \in C_h$ Eg (2 3) may also be regarded as a penalty method where the so-called selective reduced integration (*cf* [8]) is applied

In the analysis below we will only treat the unperturbed scheme (2 2a, b) It is possible to show (see [7] for details) that the results also hold for the scheme (2 2a, b), provided that $\varepsilon \leqslant Ch^2$

3 ERROR ANALYSIS

We will first introduce a special orthogonal basis for the space Q_h The basis consists of the functions ξ_{ijkl}, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$, $k = 1, \ldots, m_3$, $l = 1, \ldots, 8$ defined as follows The support of each ξ_{ijkl}, $l = 1, \ldots, 8$, is contained in $K_{ijk} \in C_h^0$, and on each subrectangle $\Delta_{\nu_1\nu_2\nu_3} \subset K_{ijk}$, $\Delta_{\nu_1\nu_2\nu_3} \in C_h$, the functions ξ_{ijkl}, $l = 1, \ldots, 8$, attain the value ± 1 according to the following rule

$$\begin{aligned}
\xi_{ijk1}(x) &= 1 & \xi_{ijk5}(x) &= (-1)^{\nu_2+\nu_3}\\
\xi_{ijk2}(x) &= (-1)^{\nu_1} & \xi_{ijk6}(x) &= (-1)^{\nu_1+\nu_3}\\
\xi_{ijk3}(x) &= (-1)^{\nu_2} & \xi_{ijk7}(x) &= (-1)^{\nu_1+\nu_2}\\
\xi_{ijk4}(x) &= (-1)^{\nu_3} & \xi_{ijk8}(x) &= (-1)^{\nu_1+\nu_2+\nu_3},
\end{aligned}$$

$$\text{if} \quad x \in \Delta_{\nu_1\nu_2\nu_3} \in C_h,\ \Delta_{\nu_1\nu_2\nu_3} \subset K_{ijk} \in C_h^0$$

Any $\mu \in Q_h$ has the unique representation

$$\mu = \sum_{i,j,k,l} \alpha_{ijkl}\, \xi_{ijkl}\,.$$

Here and below we sum i, j, k and l from 1 to m_1, m_2, m_3 and 8, respectively, unless noted otherwise.

We introduce the following subspaces of Q_h :

$$N_h = \{\, \mu \in Q_h : (\mu, \operatorname{div} v) = 0 \quad \forall v \in V_h \,\}$$
$$N_h^{\perp} = \{\, \lambda \in Q_h : (\lambda, \mu) = 0 \qquad \forall \mu \in N_h \,\}\,.$$

One can verify by simple computation that N_h consists of the linear combinations of functions ψ, φ_i, $i = 1, \dots, m_1$, θ_j, $j = 1, \dots, m_2$ and ρ_k, $k = 1, \dots, m_3$, defined as follows :

$$\psi(x) = 1\,, \qquad x \in \Omega\,,$$

$$\varphi_i(x) = \begin{cases} (-1)^{j+k}\,, & x \in \Delta_{ijk} \in C_h \\ 0\,, & \text{otherwise} \end{cases}$$

$$\theta_j(x) = \begin{cases} (-1)^{i+k}\,, & x \in \Delta_{ijk} \in C_h \\ 0 & \text{otherwise} \end{cases}$$

$$\rho_k(x) = \begin{cases} (-1)^{i+j}\,, & x \in \Delta_{ijk} \in C_h \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Taking into account the relation $\sum_i \varphi_i = \sum_j \theta_j = \sum_k \rho_k$, we conclude easily that $\dim(N_h) = 2(m_1 + m_2 + m_3) - 1$.

The space $N_h^{\perp}$ can now be characterized as

$$N_h^{\perp} = \Big\{ \sum_{i,j,k,l} \alpha_{ijkl}\, \xi_{ijkl} : \sum_{ijk} \alpha_{ijk1} = 0\,,$$
$$\sum_{j,k} \alpha_{ijk5} = \sum_{j,k} \alpha_{ijk8} = 0\,, \quad i = 1, \dots, m_1\,,$$
$$\sum_{i,k} \alpha_{ijk6} = \sum_{i,k} \alpha_{ijk8} = 0\,, \quad j = 1, \dots, m_2\,,$$
$$\sum_{i,j} \alpha_{ijk7} = \sum_{i,j} \alpha_{ijk8} = 0\,, \quad k = 1, \dots, m_3 \Big\}\,.$$

Remark : The solution of (2.2) is not unique, since if (u_h, λ_h) is a solution, then so is $(u_h, \lambda_h + \mu)$ for any $\mu \in N_h$. However, if we require that $\lambda_h \in N_h^{\perp}$

then the solution is unique. Note also that if (u_h, λ_h) is the solution of the perturbed problem $(2.2a, b')$, then $\lambda_h \in N_h^{\perp}$. □

We will supply Q_h with a special mesh-dependent semi-norm, the meaning of which will be clarified by Lemma 3.1 below. We define

$$| \mu |_h^2 = \sum_{l=1}^{4} \| \mu_l \|_0^2 + h^3 \sum_{l=5}^{8} \sigma(\mu_l)^2 ,$$

$$\mu = \sum_{i,j,k,l} \alpha_{ijkl} \, \xi_{ijkl} ,$$

where

$$\mu_l = \sum_{i,j,k} \alpha_{ijkl} \, \xi_{ijkl} , \quad l = 1, \dots, 8 ,$$

and

$$\sigma(\mu_5)^2 = \sum_{j=1}^{m_2-1} \sum_{i,k} (\alpha_{ijk5} - \alpha_{i,j+1,k5})^2 + \sum_{k=1}^{m_3-1} \sum_{i,j} (\alpha_{ijk5} - \alpha_{ij,k+1,5})^2 ,$$

$$\sigma(\mu_6)^2 = \sum_{i=1}^{m_1-1} \sum_{j,k} (\alpha_{ijk6} - \alpha_{i+1,jk6})^2 + \sum_{k=1}^{m_3-1} \sum_{i,j} (\alpha_{ijk6} - \alpha_{ij,k+1,6})^2 ,$$

$$\alpha(\mu_7)^2 = \sum_{i=1}^{m_1-1} \sum_{j,k} (\alpha_{ijk7} - \alpha_{i+1,jk7})^2 + \sum_{j=1}^{m_2-1} \sum_{i,k} (\alpha_{ijk7} - \alpha_{i,j+1,k7})^2 ,$$

$$\begin{aligned} \sigma(\mu_8)^2 = & \sum_{i=1}^{m_1-1} \sum_{j=1}^{m_2-1} \sum_{k} (\alpha_{ijk8} - \alpha_{i+1,jk8} - \alpha_{i,j+1,k8} + \alpha_{i+1,j+1,k8})^2 + \\ & + \sum_{i=1}^{m_1-1} \sum_{j} \sum_{k=1}^{m_3-1} (\alpha_{ijk8} - \alpha_{i+1,jk8} - \alpha_{ij,k+1,8} + \alpha_{i+1,j,k+1,8})^2 \\ & + \sum_{i} \sum_{j=1}^{m_2-1} \sum_{k=1}^{m_3-1} (\alpha_{ijk8} - \alpha_{i,j+1,k8} - \alpha_{ij,k+1,8} + \alpha_{i,j+1,k+1,8})^2 . \end{aligned}$$

We now prove a stability estimate of Babuška-Brezzi (*cf.* [1, 2]) type.

LEMMA 3.1 : *There are the constants C_1 and C_2 such that*

$$C_1 \, | \mu |_h \geqslant \sup_{v \in V_h} \frac{(\mu, \operatorname{div} v)}{\| v \|_1} \geqslant C_2 \, | \mu |_h$$

for all $\mu \in Q_h$ with $(\mu, 1) = 0$.

In the proof we need the following analogue of Lemma 3.1, obtained by reducing the space Q_h to consist only of functions that are constant on each $K_{ijk} \in C_h^0$.

LEMMA 3.2 : *Let* $\mu_1 = \sum_{ijk} \alpha_{ijk1} \xi_{ijk1}$, *with* $(\mu_1, 1) = 0$. *Then there is a constant* C *such that*

$$\sup_{v \in V_h} \frac{(\mu_1, \operatorname{div} v)}{\| v \|_1} \geqslant C \| \mu_1 \|_0 .$$

Proof : Given μ_1 as in the lemma, there exists (*cf.* [5]) $z \in [H_0^1(\Omega)]^3$ such that $\operatorname{div} z = \mu_1$ in Ω and

$$\| z \|_1 \leqslant C \| \mu_1 \|_0 .$$

We then define $z_h \in V_h$ by requiring

$z_h(P) = w_h(P)$, if P is a vertex or the midpoint or a midpoint of an edge of $K_{ijk} \in C_h^0$,

$$\int_S z_h \, ds = \int_S z \, ds , \quad \text{if } S \text{ is a side of } K_{ijk} \in C_h^0 ,$$

where $w_h \in V_h$ satisfies

$$(\nabla z - \nabla w_h, \nabla v) = 0 \quad \forall v \in V_h .$$

Using the same argument as in [5, pp. 76-77] one can verify that z_h is well defined and that

$$\| z_h \|_1 \leqslant C \| z \|_1 ,$$

$$(\operatorname{div} z_h, \mu_1) = (\operatorname{div} z, \mu_1) .$$

Thus we have

$$\frac{(\mu_1, \operatorname{div} z_h)}{\| z_h \|_1} \geqslant C \frac{(\mu_1, \operatorname{div} z)}{\| z \|_1} \geqslant C \| \mu_1 \|_0 ,$$

which proves the lemma. □

Remark : In the argument of [5] referred to above one assumes that the Laplacian is an isomorphism from $H^2(\Omega) \cap H_0^1(\Omega)$ to $L_2(\Omega)$. This obviously holds in the present case. $\square$

Proof of Lemma 3.1 : Let $\mu = \sum_{ijkl} \alpha_{ijkl}\, \xi_{ijkl} = \sum_{l} \mu_l$ be given with $(\mu, 1) = 0$. We first define the functions $z = (z_1, z_2, z_3) \in V_h$, $w = (w_1, w_2, w_3) \in V_h$ and $g = (g_1, g_2, g_3) \in V_h$ as follows :

(i) $\begin{cases} z_1(P) = -h\alpha_{ijk2} & \\ z_2(P) = -h\alpha_{ijk3} & \text{if } P \text{ is the midpoint} \\ z_3(P) = -h\alpha_{ijk4} & \text{of } K_{ijk} \in C_h^0 \end{cases}$

(ii) $w_3(P) = -h(\alpha_{ijk5} - \alpha_{i,j+1,k5})$, or respectively
$w_2(P) = -h(\alpha_{ijk5} - \alpha_{ij,k+1,5})$,
if P is the midpoint of the common side of K_{ijk} and $K_{i,j+1,k} \in C_h^0$, or of K_{ijk} and $K_{ij,k+1} \in C_h^0$,

(iii) $w_3(P) = -h(\alpha_{ijk6} - \alpha_{i+1,j6})$, or respectively
$w_1(P) = -h(\alpha_{ijk6} - \alpha_{ij,k+1,6})$,
if P is the midpoint of the common side of K_{ijk} and $K_{i+1,jk} \in C_h^0$, or of K_{ijk} and $K_{ij,k+1} \in C_h^0$,

(iv) $w_2(P) = -h(\alpha_{ijk7} - \alpha_{i+1,jk7})$, or respectively
$w_1(P) = -h(\alpha_{ijk7} - \alpha_{i,j+1,k7})$,
if P is the midpoint of the common side of K_{ijk} and $K_{i+1,jk} \in C_h^0$, or of K_{ijk} and $K_{i,j+1,k} \in C_h^0$.

(v) $g_3(P) = h(-\alpha_{ijk8} + \alpha_{i+1,jk8} + \alpha_{i,j+1,k8} - \alpha_{i+1,j+1,k8})$, or
$g_2(P) = h(-\alpha_{ijk8} + \alpha_{i+1,jk8} + \alpha_{ij,k+1,8} - \alpha_{i+1,j,k+1,8})$, or
$g_1(P) = h(-\alpha_{ijk8} + \alpha_{i,j+1,k8} + \alpha_{ij,k+1,8} - \alpha_{i,j+1,k+1,8})$,

if, respectively, P is the midpoint of the common edge of
$K_{ijk}, K_{i+1,jk}, K_{i,j+1,k}$ and $K_{i+1,j+1,k} \in C_h^0$, or of
$K_{ijk}, K_{i+1,jk}, K_{ij,k+1}$ and $K_{i+1,j,k+1} \in C_h^0$, or of
$K_{ijk}, K_{i,j+1,k}, K_{ij,k+1}$ and $K_{i,j+1,k+1} \in C_h^0$.

(vi) The remaining degrees of freedom of z, w and g are set equal to zero.

One can easily verify from (i) through (vi) that the following inequalities hold :

$$\| z \|_1 \leqslant C \left\{ \sum_{l=2}^{4} \| \mu_l \|_0^2 \right\}^{1/2},$$

$$\| w \|_1 \leqslant Ch^{3/2} \left\{ \sum_{l=5}^{7} \sigma(\mu_l)^2 \right\}^{1/2},$$

$$\| g \|_1 \leqslant Ch^{3/2} \, \sigma(\mu_8).$$

$$(\mu, \operatorname{div} z) \geqslant C \left(\sum_{l=2}^{4} \| \mu_l \|_0^2 \right),$$

$$\left(\mu_1 + \sum_{l=5}^{8} \mu_l, \operatorname{div} w \right) \geqslant Ch^3 \left(\sum_{l=5}^{7} \sigma(\mu_l)^2 \right),$$

and

$$(\mu_1 + \mu_8, \operatorname{div} g) \geqslant Ch^3 \, \sigma(\mu_8)^2 .$$

We now introduce a fourth function $e = (e_1, e_2, e_3) \in V_h$ which satisfies

$$\| e \|_1 \leqslant C \| \mu_1 \|_0$$

$$(\mu_1, \operatorname{div} e) \geqslant C \| \mu_1 \|_0^2 .$$

Since $(\mu, 1) = (\mu_1, 1) = 0$, the existence of e follows from Lemma 3.2.

Now, let $v = z + \delta w + \delta^2 g + \delta^3 e$, where $\delta \in [0, 1]$ will be chosen below. Then we have

$$\| v \|_1 \leqslant C | \mu |_h , \tag{3.1}$$

and

$$\begin{aligned}(\mu, \operatorname{div} v) \geqslant C \left\{ \delta^3 \| \mu_1 \|_0^2 + \sum_{l=2}^{4} \| \mu_l \|_0^2 + \delta h^3 \sum_{l=5}^{7} \sigma(\mu_l)^2 + \delta^2 h^3 \sigma(\mu_8)^2 \right\} \\ + \delta \sum_{l=2}^{4} (\mu_l, \operatorname{div} w) \\ + \delta^2 \sum_{l=2}^{7} (\mu_l, \operatorname{div} g) \\ + \delta^3 \sum_{l=2}^{8} (\mu_l, \operatorname{div} e) .\end{aligned} \tag{3.2}$$

We will now deed estimates for $| (\mu_l, \operatorname{div} g) |$ and $| (\mu_l, \operatorname{div} e) |$ for $l = 5, ..., 8$. We proceed as follows. For $v = (v_1, v_2, v_3) \in V_h$, let

$$v_{nijk} = v_n(ih_1 /2, jh_2/2, kh_3/2) ,$$

$i = 0, \ldots, 2\,m_1, j = 0, \ldots, 2\,m_2, k = 0, \ldots, 2\,m_3, n = 1, 2, 3$. Then we can write $(\mu_l, \operatorname{div} v)$, $l = 5, \ldots, 8$, $v \in V_h$, explicitly in terms of α_{ijkl} and v_{nijk}. For example, we find by straightforward computation that

$$(\mu_5, \operatorname{div} v) = \sum_{j=1}^{m_2-1} \sum_{i,k} (\alpha_{ijk5} - \alpha_{i,j+1,k5})\, \Delta^3_{ijk}(v_3) + \\ + \sum_{k=1}^{m_3-1} \sum_{i,j} (\alpha_{ijk5} - \alpha_{ij,k+1,5})\, [\Delta^1_{ijk}(v_1) + \Delta^2_{ijk}(v_2)]\,,$$

where

$$\Delta^1_{ijk}(v) = \frac{1}{16} h_2\, h_3 \sum_{\nu=0}^{1} \sum_{\mu=0}^{1} (-1)^{\nu+\mu}\, v_{2i-2\nu,2j-2\mu,2k}\,,$$

and

$$\Delta^2_{ijk}(v) = \frac{1}{16} h_1\, h_3 \sum_{l=0}^{2} c_l (v_{2i-l,2j-2,2k} - 2\, v_{2i-l,2j-1,2k} + v_{2i-l,2j,2k})\,,$$

$$\Delta^3_{ijk}(v) = \frac{1}{16} h_1\, h_2 \sum_{l=0}^{2} c_l (v_{2i-l,2j,2k-2} - 2\, v_{2i-l,2j,2k-1} + v_{2i-l,2j,2k})\,,$$

where $c_0 = 1$, $c_1 = 2$ and $c_2 = 1$.

Similarly, we find that

$$\left(\mu_8, \frac{\partial v_1}{\partial x_1}\right) = \frac{1}{16} h_2\, h_3 \sum_i \sum_{j=1}^{m_2-1} \sum_{k=1}^{m_3-1} \Delta_{ijk}(v_1) \times \\ \times (\alpha_{ijk8} - \alpha_{i,j+1,k8} - \alpha_{ij,k+1,8} + \alpha_{i,j+1,k+1,8})\,,$$

where

$$\Delta_{ijk}(v) = v_{2i-2,2j2k} - 2\, v_{2i-1,2j,2k} + v_{2i,2j,2k}\,.$$

Using these relations and similar expressions for $(\mu_6, \operatorname{div} v)$, $(\mu_7, \operatorname{div} v)$, $(\mu_8, \partial v_2/\partial x_2)$ and $(\mu_8, \partial v_3/\partial x_3)$, and noting that

$$C_1\, |\, v\, |_1^2 \leqslant h \sum_{n=1}^{3} \sum_{i=0}^{2m_1-1} \sum_{j=0}^{2m_2-1} \sum_{k=0}^{2m_3-1} [(v_{nijk} - v_{n,i+1,jk})^2 + \\ + (v_{nijk} - v_{ni,j+1,k})^2 + (v_{nijk} - v_{nij,k+1})^2] \\ \leqslant C_2\, |\, v\, |_1^2\,, \quad v \in V_h\,,$$

we can now easily verify that

$$|\, (\mu_l, \operatorname{div} v)\, | \leqslant C h^{3/2}\, \sigma(\mu_l)\, |\, v\, |_1\,, \quad l = 5, 6, 7\,, \quad v \in V_h\,, \tag{3.3}$$

and

$$| (\mu_8, \operatorname{div} v) | \leqslant Ch^{3/2}\, \sigma(\mu_8)\, | v |_1 , \quad v \in V_h . \tag{3.4}$$

Applying (3.3) and (3.4) together with the above estimates for $\| w \|_1$, $\| g \|_1$ and $\| e \|_1$ in (3.2) we find that

$$\begin{aligned}
(\mu, \operatorname{div} v) \geqslant\ & C \left\{ \delta^3 \| \mu_1 \|_0^2 + \sum_{l=2}^{4} \| \mu_l \|_0^2 + \delta h^3 \sum_{l=5}^{7} \sigma(\mu_l)^2 + \delta^2 h^3 \sigma(\mu_8)^2 \right\} - \\
& - C_1\, \delta h^{3/2} \left\{ \sum_{l=2}^{4} \| \mu_l \|_0^2 \right\}^{1/2} \left\{ \sum_{l=5}^{7} \sigma(\mu_l)^2 \right\}^{1/2} \\
& - C_1\, \delta^2 h^{3/2} \left\{ \sum_{l=2}^{4} \| \mu_l \|_0^2 + h^3 \sum_{l=5}^{7} \sigma(\mu_l)^2 \right\}^{1/2} \sigma(\mu_8) \\
& - C_1\, \delta^3 \left\{ \sum_{l=2}^{4} \| \mu_l \|_0^2 + h^3 \sum_{l=5}^{8} \sigma(\mu_l)^2 \right\}^{1/2} \| \mu_1 \|_0 \\
\geqslant\ & (C - C_2\, \delta) \left\{ \delta^3 \| \mu_1 \|_0^2 + \sum_{l=2}^{4} \| \mu_l \|_0^2 + \delta h^3 \sum_{l=5}^{7} \sigma(\mu_l)^2 + \right. \\
& \left. + \delta^2 h^3 \sigma(\mu_8)^2 \right\} .
\end{aligned}$$

Choosing now $\delta = \min \left\{ 1, \dfrac{C}{2\, C_2} \right\}$, we have

$$(\mu, \operatorname{div} v) \geqslant C\, | \mu |_h^2 .$$

Together with (3.1), this proves the asserted lower bound for $| \mu |_h$. To finally prove the upper bound we only need to note that, by (3.3) and (3.4),

$$| (\mu, \operatorname{div} v) | \leqslant C\, | \mu |_h\, | v |_1 , \quad \mu \in Q_h , v \in V_h .$$

Thus, Lemma 3.1 is proved. □

We note that, by the definition of $N_h^\perp$, $| \cdot |_h$ is a norm in $N_h^\perp$. We establish next a lower bound for this norm in terms of h and the usual L_p norms.

LEMMA 3.3 : *If* $\mu \in N_h^\perp$, *then*

$$| \mu |_h \geqslant C \left(\sum_{l=1}^{4} \| \mu_l \|_0 + h \sum_{l=5}^{7} \| \mu_l \|_0 + h^{5/2} \| \mu_8 \|_{0,6} \right) .$$

Proof : Let $\mu = \sum_{i,j,k,l} \alpha_{ijkl}\, \xi_{ijkl} \in N_h^{\perp}$ be given. We recall from the definition of $N_h^{\perp}$ that $\sum_{j,k} \alpha_{ijk,5} = \sum_{i,k} \alpha_{ijk,6} = \sum_{i,j} \alpha_{ijk,7} = 0$. From these relations we conclude, e.g., that

$$h^3\,\sigma(\mu_5)^2 = h^3 \sum_i \left\{ \sum_{j=1}^{m_2-1} \sum_k (\alpha_{ijk,5} - \alpha_{i,j+1,k5})^2 + \right.$$

$$\left. + \sum_j \sum_{k=1}^{m_3-1} (\alpha_{ijk5} - \alpha_{ij,k+1,5})^2 \right\}$$

$$\geqslant Ch^5 \sum_{i,j,k} (\alpha_{ijk5})^2 \geqslant C_1\, h^2 \,\| \mu_5 \|_0^2$$

Here we used discrete Poincare's and Sobolev's inequalities to conclude that if $\sum_{jk} \alpha_{jk} = 0$, then

$$\sum_{j=1}^{m_2-1} \sum_k (\alpha_{jk} - \alpha_{j+1,k})^2 + \sum_j \sum_{k=1}^{m_3-1} (\alpha_{jk} - \alpha_{j\,k+1})^2 \geqslant Ch^2 \sum_{j,k} \alpha_{jk}^2$$

(*cf.* [7] for the details of the argument) Since similar estimates obviously hold for $\sigma(\mu_6)$ and $\sigma(\mu_7)$, we conclude that

$$| \mu |_h \geqslant C\left(\sum_{l=1}^{4} \| \mu_l \|_0 + h \sum_{l=5}^{7} \| \mu_l \|_0 \right). \tag{3 5}$$

To obtain a bound for the component $\mu_8 = \sum_{i,j,k} \alpha_{ijk8}\, \xi_{ijk8}$, let k be fixed, $1 \leqslant k \leqslant m_3 - 1$, and define

$$\beta_{ij} = \alpha_{ijk8} - \alpha_{i+1,jk8} - \alpha_{ij,k+1,8} + \alpha_{i+1,j,k+1,8}\,,$$

$$\gamma_{ij} = \alpha_{ijk8} - \alpha_{i,j+1,k8} - \alpha_{ij,k+1,8} + \alpha_{i,j+1,k+1,8}\,,$$

$$\delta_{ij} = \alpha_{ijk8} - \alpha_{ij,k+1,8}\,.$$

Then we easily find that

$$\delta_{ij} = \delta_{1,1} - \sum_{l=1}^{i-1} \beta_{l1} - \sum_{l=1}^{j-1} \gamma_{il}\,. \tag{3.6}$$

Recalling that $\sum_{ij} \alpha_{ijk8} = 0$ for $k = 1, .., m_3$ (since $\mu \in N_h^{\perp}$), we have in particular that $\sum_{i,j} \delta_{ij} = 0$ Using this we may solve for $\delta_{1,1}$ in (3 6) to obtain

$$\delta_{1,1} = \sum_{i=1}^{m_1-1} c_i \beta_{i1} + \sum_i \sum_{j=1}^{m_2-1} d_{ij} \gamma_{ij},$$

where the coefficients satisfy

$$| c_j | \leqslant C$$
$$| d_{ij} | \leqslant Ch .$$

Substituting this back to (3 6) we obtain

$$h \sum_{ij} \delta_{ij}^2 \leqslant Ch^{-2} \left(\sum_{i=1}^{m_1-1} \sum_j \beta_{ij}^2 + \sum_i \sum_{j=1}^{m_2-1} \gamma_{ij}^2 \right). \qquad (3\ 7)$$

Repeating this argument for all k and for permuted indices, and summing up the resulting inequalities (3 7), we find that

$$\sigma(\mu_8) \geqslant Ch \mid \mu_8 \mid_{1\,h}, \qquad (3\ 8)$$

where

$$| \mu_8 |_{1\,h}^2 = h \left\{ \sum_{i=1}^{m_1-1} \sum_{j,k} (\alpha_{ijk8} - \alpha_{i+1,jk8})^2 + \right.$$
$$\left. + \sum_{j=1}^{m_2-1} \sum_{i,k} (\alpha_{ijk8} - \alpha_{i,j+1,k8})^2 + \sum_{k=1}^{m_3-1} \sum_{i\,j} (\alpha_{ijk8} - \alpha_{ij,k+1,8})^2 \right\}.$$

To finally get a lower bound for $| \mu_8 |_{1,h}$, we construct a function $\varphi \in H^1(\Omega)$ satisfying

$$C_1 \mid \mu_8 \mid_{1,h} \leqslant \mid \varphi \mid_1 \leqslant C_2 \mid \mu_8 \mid_{1,h},$$
$$C_1 \| \mu_8 \|_{0,p} \leqslant \| \varphi \|_{0,p} \leqslant C_2 \| \mu_8 \|_{0,p}, \quad 1 \leqslant p < \infty$$

and

$$\int_\Omega \varphi \, dx = h_1 h_2 h_3 \sum_{i,j,k} \alpha_{ijk8} = 0 .$$

The function φ is found, e.g. as follows. Consider another rectangular subdivision C_h^1 of Ω, the interior nodes of which are located at the midpoints of $K_{ijk} \in C_h^0$. Then define φ to be the continuous piecewise trilinear function on C_h^1, which satisfies $\varphi(x) = \alpha_{ijk8}$ if x is a node of C_h^1 such that $x \in \overline{K}_{ijk}$, $K_{ijk} \in C_h^0$. It is then easy to see that the above relations hold, and so, using Poincare's and Sobolev's inequalities, we find that

$$\begin{aligned} | \mu_8 |_{1,h} \geqslant C | \varphi |_1 \geqslant C_1 \| \varphi \|_1 \geqslant C_2 \| \varphi \|_{0,6} \\ \geqslant C_3 \| \mu_8 \|_{0,6} . \end{aligned}$$

Combining this with (3.8) and recalling the definition of $| \mu |_h$, we obtain

$$| \mu |_h \geqslant h^{3/2} \sigma(\mu_8) \geqslant Ch^{5/2} \| \mu_8 \|_{0,6} .$$

Together with (3.5) this finishes the proof of Lemma 3.3. □

We can now state and prove a basic error estimate for the scheme (2.2).

THEOREM 3.1 : *Assume that the solution of* (2.1) *satisfies*

$$(u, \lambda) \in [W^{9/2,6/5}(\Omega)]^3 \times H^1(\Omega) .$$

Then if $(u_h, \lambda_h) \in V_h \times N_h^{\perp}$ *is a solution to* (2.2) *and* $\tilde{\lambda}$ *is the orthogonal projection of* λ *onto* $N_h^{\perp}$, *we have*

$$| u - u_h |_1 + | \lambda_h - \tilde{\lambda}_h |_h \leqslant Ch(\| u \|_{9/2,6/5} + \| \lambda \|_1) .$$

Proof : Let $\tilde{u} \in V_h$ be the interpolant of u. We first apply Lemma 3.1 and the general theory of Babuška [1] and Brezzi [2] (*cf.* also [7]) to conclude the existence of $(v, \mu) \in V_h \times N_h^{\perp}$ such that

$$| v |_1 + | \mu |_h \leqslant C,$$

and

$$\begin{aligned} | u_h - \tilde{u} |_1 + | \lambda_h - \tilde{\lambda} |_h \leqslant C \{ | (\nabla(u - \tilde{u}), \nabla v) | + \\ + | (\lambda - \tilde{\lambda}, \operatorname{div} v) | + | (\operatorname{div} (u - \tilde{u}), \mu) | \} . \end{aligned} \quad (3.9)$$

The first term on the right side of (3.9) obeys as usual (*cf.* [3]) the quasi-optimal bound

$$| (\nabla(u - \tilde{u}), \nabla v) | \leqslant | u - \tilde{u} |_1 | v |_1 \leqslant Ch | u |_2 . \quad (3.10)$$

The second term can be estimated by first noting that

$$(\tilde{\lambda}, \operatorname{div} v) = (\pi_h \lambda, \operatorname{div} v) \quad \forall v \in V_h ,$$

where $\pi_h \lambda$ is the orthogonal projection onto Q_h. Hence, by well-known approximation theory,

$$|(\lambda - \tilde{\lambda}, \operatorname{div} v)| \leqslant \| \lambda - \pi_h \lambda \|_0 \, | v |_1 \leqslant Ch \, | \lambda |_1 . \tag{3.11}$$

In estimating the third term on the right side of (3.9) we need the following « superapproximation » result, the proof of which is straightforward.

LEMMA 3.4 : *Defining for* $v \in [H^2(K)]^3$, $K = K_{ijk} \in C_h^0$,

$$L_l(v) = \int_K \operatorname{div}(v - \tilde{v}) \, \xi_{ijkl} \, dx , \quad l = 1, ..., 8 ,$$

where $\tilde{v}$ *denotes the piecewise trilinear interpolant of* v *on the eight subrectangles of* K, *we have*

$$L_l(v) = 0 , \quad l = 1, ..., 8 , \quad \text{if} \quad v \in [P_2]^3$$

and

$$L_8(v) = 0 , \quad \text{if} \quad v \in [P_5]^3 ,$$

so that, in particular,

$$| L_l(v) | \leqslant Ch^{7/2} \, | v |_{H^3(K)} , \quad l = 1, ..., 8 ,$$

and

$$| L_8(v) | \leqslant Ch^{k+2-3/p} \, | v |_{W^{k,p}(K)} , \quad 1 \leqslant p < \infty , \quad 4 \leqslant k \leqslant 6 .$$

Now writing $\mu = \sum_{i,j,k,l} \alpha_{ijkl} \, \xi_{ijkl} = \sum_l \mu_l$ we have

$$\begin{aligned} \left| \left(\operatorname{div}(u - \tilde{u}), \sum_{l=1}^{4} \mu_l\right) \right| &\leqslant C \, | u - \tilde{u} |_1 \, |\mu|_h \\ &\leqslant C_1 \, h \, | u |_2 , \end{aligned} \tag{3.12}$$

and, applying Lemma 3.4 and Lemma 3.3,

$$\begin{aligned} \left| \left(\operatorname{div}(u - \tilde{u}), \sum_{l=5}^{7} \mu_l\right) \right| - = & \left| \sum_{i,j,k} \sum_{l=5}^{7} \alpha_{ijkl} \int_{K_{ijk}} \operatorname{div}(u - \tilde{u}) \, \xi_{ijkl} \, dx \right| \\ &\leqslant Ch^2 \, | u |_3 \sum_{l=5}^{7} \| \mu_l \|_0 \leqslant C_1 \, h \, | u |_3 . \end{aligned} \tag{3.13}$$

Similarly, applying the Holder inequality and Lemma 3 4 we find that

$$| (\operatorname{div} (u - \tilde{u}), \mu_8) | \leqslant Ch^{k-1} \, | u |_{k\ p} \, \| \mu_8 \|_{0\ q} ,$$
$$1 \leqslant p < \infty , \quad p^{-1} + q^{-1} = 1 , \quad 4 \leqslant k \leqslant 6 \qquad (3\ 14)$$

Choosing here $p = 6/5$, we have $q = 6$ and so, by Lemma 3 3,

$$\| \mu_8 \|_{0\ q} \leqslant Ch^{\ 5/2} \, | \mu |_h \leqslant C_1 \, h^{-5/2}$$

By interpolating in (3 14) we then obtain

$$| (\operatorname{div} (u - \tilde{u}), \mu_8) | \leqslant Ch^{7/2} \, \| u \|_{9/2\ 6/5} \, \| \mu \|_{0\ 6}$$
$$\leqslant C_1 \, h \, \| u \|_{9/2\ 6/5} \qquad (3\ 15)$$

From (3 12), (3 13) and (3 15) we see, applying the Sobolev embedding, that

$$| (\operatorname{div} (u - \tilde{u}), \mu) | \leqslant Ch \, \| u \|_{9/2\ 6/5}$$

Combining this with (3 9) through (3 11) and finally applying the triangle inequality together with the usual bound for $| u - \tilde{u} |_1$, we obtain the desired estimates for $| u - u_h |_1$ and $| \lambda_h - \tilde{\lambda} |_h$, and the proof of Theorem 3 1 is complete □

Remark The regularity assumption in Theorem 3 1 is not quite realistic even in the simple geometry considered, since there are in general singularities in the solution near the adges and vertices of Ω Taking the leading edge singulalarity into account, we conjecture from [6, 9] that u can satisfy

$$u \in [W^{s\ 6\ 5}(\Omega)]^3 \quad \text{for} \quad s \lessapprox 4{,}4$$

if f in (2 1) is sufficiently smooth With this regularity assumption, we would obtain $\| u - u_h \|_1 \approx 0(h^{0\ 9})$ □

Remark One cannot obtain any convergence rate for the pressure in L_2 from Theorem 3 1, since Lemma 3 3 only implies that

$$| \lambda_h - \tilde{\lambda} |_h \geqslant Ch^{5/2} \, \| \lambda_h - \tilde{\lambda} \|_0$$

However, as in [7], it follows easily from the definition of $| \,.\, |_h$ that if λ_h is first averaged over each $K_{ijk} \in C_h^0$ then the resulting smoothed pressure $\pi_h^0 \lambda_h$ converges

$$\| \lambda - \pi_h^0 \lambda_h \|_0 \leqslant Ch(\| u \|_{9/2\ 6/5} + \| \lambda \|_1) \qquad \square$$

Remark : Assuming that we have for Eq. (2.1) the a prıorı estımate

$$\| u \|_2 + \| \lambda \|_1 \leqslant C \| f \|_0 ,$$

whıch ıs generally conȷectured for a convex polyhedral domaın, one can prove usıng the technıque of [7] that

$$\| u - u_h \|_0 \leqslant Ch^2(\| u \|_{9/2,6/5} + \| \lambda \|_1) . \quad \square$$

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