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NEAR-BEST APPROXIMATIONS TO THE SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND (*)

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Résume — *On considère des approximations par collocation de la solution d'equation intégrale de Fredholm de seconde espèce, et une approximation ponctuelle presque optimale est définie par collocation en des abscisses de formules de quadrature optimale On obtient une approximation globale presque optimale en ajoutant des termes correcteurs à l'approximation par collocation, a l'aide de propriétes de base du noyau resolvant Un procédé semblable aux itérations de Neumann améliore l'approximation par collocation même dans les cas où la série de Neumann diverge On discute en détail le cas de noyaux à singularité algébrique et on donne un exemple numérique*

Abstract — *Collocation approximations to the solution of Fredholm integral equation of the second kind are discussed, and a pointwise near-best approximation is defined by collocation at the abscissae of some best quadrature formula A global near-best approximation is obtained by adding some correction term to the collocation approximation, utilizing basic properties of the resolvent kernel A procedure similar to Neumann iterations is shown to improve the collocation approximation even in cases when the Neumann series diverges The case of kernels with algebraic singularity is discussed in detail and a numerical example is given*

1. POINTWISE NEAR-BEST APPROXIMATIONS

Consider Fredhom integral equation of the second kind

$$\left.\begin{aligned} & u(x) - \lambda \int_a^b K(x, t)\, u(t)\, dt = f(x), \quad f \in F \\ & \text{or, in operator form} \\ & u - \lambda K u = f \end{aligned}\right\}, \tag{1.1}$$

where F is a given continuity class of functions on $[a, b]$.

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R A I R O Analyse numérique/Numerical Analysis, 0399-0516/1982/129/$ 5 00

For any λ which is not an eigenvalue of K equation (1.1) has a unique solution, which can be represented in terms of the resolvent kernel Γ of (1.1) as

$$\left.\begin{aligned} &u(x) = \int_a^b \Gamma(x, t; \lambda)\, f(t)\, dt \\ &\text{or, in operator form} \\ &u = \Gamma f \end{aligned}\right\}. \tag{1.2}$$

For the ideal case when the resolvent kernel is known we can obtain approximations to $u(x)$ by using quadrature approximations to $\int_a^b \Gamma(x, t; \lambda)\, f(t)\, dt$. In this context, for a given $x \in [a, b]$, we introduce the class H_x of functions on $[a, b]$,

$$H_x = \{\, h \mid h(t) = \Gamma(x, t; \lambda)\, \phi(t),\ \phi \in F \,\}, \tag{1.3}$$

and define the following best n-point approximation to $u(x)$.

DEFINITION 1.1 : *Let* $\sum_{i=1}^{n} w_i^*\, h(t_i^*)$ *be the best quadrature formula, in some sense, for approximating* $\int_a^b h(t)\, dt$ *for* $h \in H_x$. *The best n-point approximation, in the same sense, to* $u(x)$ *is defined as*

$$u^*(x) = \sum_{i=1}^{n} w_i^*\, \Gamma(x, t_i^*; \lambda)\, f(t_i^*). \tag{1.4}$$

Let e_x^* be the error functional of the above best quadrature formula, i.e.

$$e_x^*(h) = \int_a^b h(t)\, dt - \sum_{i=1}^{n} w_i^*\, h(t_i^*). \tag{1.5}$$

Then the error in the best n-point approximation can be expressed in terms of e_x^* as

$$u(x) - u^*(x) = e_x^*(h) \tag{1.6}$$

where $h \in H_x$,

$$h(t) = \Gamma(x, t; \lambda)\, f(t) \quad t \in [a, b]. \tag{1.7}$$

Usually the resolvent kernel Γ is not known and the above best n-point approximations cannot be used. However, some near-best n-point approxima-

tions can be obtained by mere knowledge of the abscissae of a best n-point quadrature formula on H_x as follows :

DEFINITION 1.2 : *Let $u_1, u_2, ..., u_n$ be n independent functions on $[a, b]$ such that*

$$u_j - \lambda K u_j \equiv f_j \in F , \tag{1.8}$$

and let $t_1^*, t_2^*, ..., t_n^*$ be the abscissae of the best n-point quadrature formula, in some sense, on H_x. Then the near-best n-point approximation to the solution $u(x)$ of the problem (1.1) at a given $x \in [a, b]$ is defined as

$$\bar{u}(x) = \sum_{j=1}^{n} \alpha_j u_j(x) \tag{1.9}$$

where the α_j's are chosen such that $\bar{u}$ collocates the integral equation at

$$t_1^*, t_2^*, ..., t_n^* .$$

The collocation property of $\bar{u}$ means that

$$\bar{u}(t_i^*) - \lambda \int_a^b K(t_i^*, t)\, \bar{u}(t)\, dt = f(t_i^*) , \quad i = 1, 2, ..., n , \tag{1.10}$$

or, using (1.8) and (1.9)

$$\sum_{j=1}^{n} \alpha_j f_j(t_i^*) = f(t_i^*) , \quad i = 1, 2, ..., n . \tag{1.11}$$

Solubility of the system (1.11) can, in general, be achieved by a proper choice of the basis functions $u_j, j = 1, 2, ..., n$.

The following theorem exhibits the " near-best property " of the near-best n-point approximation, i.e. that the error in it is given by the error functional e_x^* of the best n-point quadrature formula on H_x, operating on a function from H_x. The term "near-best" is being used in connection with the above best n-point approximation $u^*(x)$, which also satisfies the same property (eq. (1.6)).

THEOREM 1.1 : *Let $\bar{u}(x)$ be a near-best n-point approximation given by definition 2.1. Then*

$$u(x) - \bar{u}(x) = e_x^* \left(h - \sum_{j=1}^{n} \alpha_j h_j \right) \tag{1.12}$$

where h and the h_j's are in H_x, h given by (1.7) and

$$h_j(t) = \Gamma(x, t; \lambda)\, f_j(t) \quad t \in [a, b] , \quad j = 1, 2, ..., n . \tag{1.13}$$

Proof : The u_j's can be represented by means of the resolvent kernel as :

$$u_j(x) = \int_a^b \Gamma(x, t; \lambda)\, f_j(t)\, dt\,, \quad j = 1, 2, ..., n\,. \tag{1.14}$$

Using (1.2), (1.9) and (1.14) we obtain

$$u(x) - \bar{u}(x) = \int_a^b \Gamma(x, t; \lambda)\left[f(t) - \sum_{j=1}^{n} \alpha_j\, f_j(t)\right] dt\,. \tag{1.15}$$

Now applying the best n-point quadrature formula on H_x (whose abscissae are used in the definition of $\bar{u}$) to the integral in (1.15) we get

$$u(x) - \bar{u}(x) = \sum_{i=1}^{n} w_i^*\, \Gamma(x, t_i^*; \lambda)\left[f(t_i^*) - \sum_{j=1}^{n} \alpha_j\, f_j(t_i^*)\right] +$$

$$+ e_x^*\left(\Gamma(x, .; \lambda)\left(f - \sum_{j=1}^{n} \alpha_j\, f_j\right)\right). \tag{1.16}$$

The summation term in (1.16) vanishes by the collocation property (1.11) of $\bar{u}$, and the proof is completed by using (1.7) and (1.13).

In many practical applications it appears that the kernel $K(x, t)$ in (1.1) possesses some singularity, usually on some line in $[a, b] \times [a, b]$. In such cases, as it is shown in section 3, the resolvent kernel $\Gamma(x, t; \lambda)$ is also singular on the same line.

Moreover, even for a smooth kernel, the resolvent $\Gamma(x, t; \lambda)$ always possesses a $\delta(x - t)$ singularity, where δ is the Dirac-δ function. This strong x-dependence of the singular structure of $\Gamma(x, t; \lambda)$, considered as a function of t, is claimed to be the reason for the relative failure of the collocation method to produce efficient global approximations to the solution of Fredholm integral equations of the second kind. In our case the consequence is that the near-best n-point collocation approximation $\bar{u}$ can be highly efficient only at that point x to which it is assigned.

To show that let us suppose that a near-best approximation, $\bar{u} = \sum_{j=1}^{n} \alpha_j\, u_j$ obtained for some $x \in [a, b]$, is being used to approximate $u(y)$ at some $y \neq x$. Then, as in theorem 1.1 it can be shown that

$$u(y) - \bar{u}(y) = e_x^*\left(h - \sum_{j=1}^{n} \alpha_j\, h_j\right). \tag{1.17}$$

But here h and the h_j's are in the class H_y. Thus, the error in $\bar{u}(y)$ is a result of the error functional of a best quadrature formula on H_x operating on a function in H_y. This error is not expected to be very small since best quadrature approximations on H_x are not at all suitable for functions in H_y, whose singular structure is completely different from that of the functions in H_x.

In the following section we show that a global near-best approximation to u, with a global near-best property of the kind (1.12), can be obtained by adding an appropriate correction term to $\bar{u}$.

2. GLOBAL NEAR-BEST APPROXIMATIONS

Let us assume that the resolvent kernel Γ can be decomposed as

$$\Gamma(x, t; \lambda) = S(x, t; \lambda) + R(x, t; \lambda) \tag{2.1}$$

where S contains the "main" singularities of Γ on $[a, b]$, as specified in definition 2.1 below. Let us also define some sets of functions on $[a, b]$ associated with this decomposition,

$$H_x^R = \{ h \mid h(t) = R(x, t; \lambda)\, \phi(t),\ \phi \in F,\ t \in [a, b] \}, \quad x \in [a, b]. \tag{2.2}$$

DEFINITION 2.1 : *Let* (2.1) *be a decomposition of* Γ *such that the best n-point quadrature formulae on the associated sets* H_x^R *are independent of* x *for* $x \in [a, b]$ *and let* $\sum_{i=1}^{n} w_i^*\, h(t_i^*)$ *be such a best formula with abscissae*

$$t_i^* \in [a, b], \quad i = 1, 2, ..., n.$$

A global nth order approximation $\bar{u}_c$ *to* u *on* $[a, b]$ *is defined as*

$$\bar{u}_c(x) = \bar{u}(x) + \int_a^b S(x, t; \lambda) \left[f(t) - \sum_{j=1}^{n} \alpha_j\, f_j(t) \right] dt \tag{2.3}$$

where $\bar{u}$ *is the collocation approximation to* u *collocating* (1.1) *at*

$$t_i^*,\ i = 1, 2, ..., n,$$

i.e.,

$$\bar{u}(x) = \sum_{j=1}^{n} \alpha_j\, u_j(x) \tag{2.4}$$

where the α_j's are determined by

$$\sum_{j=1}^{n} \alpha_j f_j(t_i^*) = f(t_i^*), \quad i = 1, 2, ..., n, \tag{2.5}$$

where $f_j = u_j - \lambda K u_j$.

Remark : The c in $\bar{u}_c$ stands for the correction of $\bar{u}$ with the correction term

$$\int_a^b S(x, t; \lambda) \left[f(t) - \sum_{j=1}^{n} \alpha_j f_j(t) \right] dt. \tag{2.6}$$

The evaluation of this correction term involves merely direct integration since the α_j's appearing in it are the same as those defining $\bar{u}$ in (2.4).

THEOREM 2.1 : *The error in the global near-best approximation is given by*

$$u(x) - \bar{u}_c(x) = e^* \left(h - \sum_{j=1}^{n} h_j \right) \quad \forall x \in [a, b] \tag{2.7}$$

where $h \in H_x^R$, $h_j \in H_x^R$, $j = 1, 2, ..., n$,

$$h(t) = R(x, t; \lambda) f(t), \tag{2.8}$$

$$h_j(t) = R(x, t; \lambda) f_j(t), \tag{2.9}$$

and e^* is the error functional of the associated best n-point quadrature formula on the sets H_x^R, $x \in [a, b]$.

Proof : Using (2.1) in (1.15) and rearranging the terms using (2.3) we obtain

$$u(x) - \bar{u}_c(x) = \int_a^b R(x, t; \lambda) \left[f(t) - \sum_{j=1}^{n} \alpha_j f_j(t) \right] dt. \tag{2.10}$$

Applying the best n-point quadrature formula from definition 2.1 to the integral in (2.10) we get

$$u(x) - \bar{u}_c(x) = \sum_{i=1}^{n} w_i^* R(x, t_i^*; \lambda) \left[f(t_i^*) - \sum_{j=1}^{n} \alpha_j f_j(t_i^*) \right] + \\ + e^* \left(R(x, .; \lambda) \left(f - \sum_{j=1}^{n} \alpha_j f_j \right) \right). \tag{2.11}$$

The summation term vanishes using (2.5) and the proof is completed by using (2.8) and (2.9).

Since it is assumed in definition 2.1 that the best n-point quadrature formula there is independent of x for $x \in [a, b]$, so are also the α_j's in $\bar{u}$ and $\bar{u}_c$. This allows the definition of $\bar{u}_c$ as a global approximation. However, more important is the global nature of the near-best property (2.7) of $\bar{u}_c$, i.e., for any $x \in [a, b]$ the error in $\bar{u}_c$ is expressed by the error functional of the best quadrature formula on H_x^R operating on a function for which this best quadrature approximation is suited.

In the following section we present some basic properties of the resolvent kernel which are relevant to the application of the pointwise and global near-best approximations presented above. We also outline a procedure for calculating appropriate singular parts S of Γ for the case of kernels with algebraic singularity.

3. THE SINGULARITIES OF THE RESOLVENT KERNEL

In operator form we have the relations $u - \lambda K u = f$ and $u = \Gamma f$. Hence

$$\begin{aligned}\Gamma f &= u \\ &= f + \lambda K u \\ &= f + \lambda K(f + \lambda K u) \\ &\vdots \\ &= f + \lambda K f + \lambda^2 K^2 f + \cdots + \lambda^m K^m f + \lambda^{m+1} K^{m+1} u .\end{aligned}$$

Replacing u by Γf in the last term of the above expression we obtain an operator identity for Γ

$$\Gamma = I + \lambda K + \lambda^2 K^2 + \cdots + \lambda^m K^m + \lambda^{m+1} K^{m+1} \Gamma . \tag{3.1}$$

In (3.1) the product of two integral operators A and B, with kernels $A(x, t)$ and $B(x, t)$ respectively, is the integral operator AB with the kernel

$$C(x, t) = \int_a^b A(x, y)\, B(y, t)\, dy , \tag{3.2}$$

and I is the identity operator.

Integral operators are smoothing operators, and thus, in general, $K^{j+1}(x, t)$ is smoother than $K^j(x, t)$. Therefore, the most severe singularities of $\Gamma(x, t; \lambda)$ are imbedded in the first terms of the expression (3.1). The identity operator contributes a $\delta(x - t)$ singularity to $\Gamma(x, t; \lambda)$ where δ is the Dirac-δ function.

As an example we consider the important class of kernels $K(x, t)$ with jump discontinuities on $x = t$,

$$K(x, t) = K_1(x, t) + \sum_{j=1}^{\infty} k_j^{(1)} \frac{(x - t)_+^j}{j!}, \tag{3 3}$$

where $K_1 \in C^\infty([a, b] \times [a, b])$, and $k_j^{(1)}$ is the jump in the jth derivative of K, i e,

$$k_j^{(1)} = \frac{\partial^j K}{\partial x^j}(t^+, t) - \frac{\partial^j K}{\partial x^j}(t^-, t) \tag{3 4}$$

The use of the additional index 1 is to be clarified below

Using the equality

$$\int_a^b \frac{(x - y)_+^k}{k!} \frac{(y - t)_+^l}{l!} dy = \frac{(x - t)_+^{k+l+1}}{(k + l + 1)!}, \quad x, t \in [a, b], \tag{3 5}$$

it can be proved, by induction on n, that the kernel of the operator K^n can be represented as

$$K^n(x, t) = K_n(x, t) + \sum_{j=n-1}^{\infty} k_j^{(n)} \frac{(x - t)_+^j}{j!}, \quad n \geqslant 1, \tag{3 6}$$

where $K_n \in C^\infty([a, b] \times [a, b])$ The $k_j^{(n)}$'s can be obtained recursively from the $k_j^{(1)}$'s by the relations

$$\left.\begin{aligned} k_j^{(m)} &= 0 && j < m - 1 \\ k_j^{(m)} &= \sum_{i=0}^{j-1} k_i^{(1)} k_{j-i-1}^{(m-1)} && j \geqslant m - 1 \end{aligned}\right\} \tag{3 7}$$

From the expression (3 1) for Γ and the relations in (3 6) it is clear that $\Gamma(x, t, \lambda)$ is infinitely smooth on $[a, b] \times [a, b]$ apart from a $\delta(x - t)$ singularity and jump discontinuities across $x = t$

The near-best approximations $\bar{u}$ (definition 1 2) are defined by means of best quadrature formulae on the set H_x in (1 3) Let $F = C^\infty[a, b]$, then H_x is the class of infinitely smooth functions on $[a, b]$ apart from a $\delta(x - t)$ singularity and jump discontinuities at $t = x$ A best n-point quadrature formula on this H_x can be composed of two Gaussian quadrature formulae, one on $[a, x]$ and the other on $[x, b]$, and an additional term, $\phi(x)$ (where $h(t) = \Gamma(x, t, \lambda)\, \phi(t)$ is the integrand), to take care of the $\delta(x - t)$ singularity of Γ Let $e_k^{[c\,d]}$ denote the error functional of the k-point Gaussian quadrature formula on $[c, d]$ Then, for a given $x \in [a, b]$ we choose k_x such that the k_x-

point Gaussian quadrature approximation on $[a, x]$ is of the same order as the $(n - k_x - 1)$-point Gaussian quadrature approximation on $[x, b]$, i.e.

$$O(e_{k_x}^{[a,x]}(g)) \approx O(e_{n-k_x-1}^{[x,b]}(g)), \quad g \in C^\infty[a, b].$$

The near-best n-point approximation is then defined by collocation at the k_x Gaussian points in $[a, x]$, at x, and at the $n - k_x - 1$ Gaussian points in $[x, b]$. From theorem 1.1 then follows :

COROLLARY 3.1 : *The error in the near-best n-point approximation*

$$u(x) = \sum_{j=1}^{n} \alpha_j u_j(x)$$

to the solution $u(x)$ *of the integral equation* (1.1) *with a kernel of the form* (3.3) *and* $f \in C^\infty[a, b]$ *is given by*

$$u(x) - \bar{u}(x) = e_{k_x}^{[a,x]}\left(h - \sum_{j=1}^{n} \alpha_j h_j\right) + e_{n-k_x-1}^{[x,b]}\left(h - \sum_{j=1}^{n} \alpha_j h_j\right) \quad (3.8)$$

where h *and the* h_j*'s are as in* (1.12), *hence when restricted to* $[a, x)$ *they are in* $C^\infty[a, x)$ *and restricted to* $(x, b]$ *they are in* $C^\infty(x, b]$.

For a global near-best approximation we need to find a suitable singular part S of Γ as described in definition 2.1. A possible S can be induced from (3.1) as the kernel of the operator

$$S_m = I + \lambda K + \lambda^2 K^2 + \cdots + \lambda^m K^m \quad (3.9)$$

where m can be chosen so that the " remainder "

$$R_m = \lambda^{m+1} K^{m+1} \Gamma \quad (3.10)$$

has a sufficiently smooth kernel. For kernels of the form (3.3) it can be shown, using (3.6), that the kernel of R_m has at least $m - 1$ continuous derivatives. Furthermore, for this type of kernels a more practical S can be obtained by replacing each K^n in (3.9) by its representation (3.6), retaining only the terms which contribute to the jump discontinuities in $\partial^j S/\partial x^j$, $j=0, 1, ..., m-1$. This provides us a singular part $\bar{S}_m$ of Γ,

$$\bar{S}_m(x, t; \lambda) = \delta(x - t) + \sum_{j=0}^{m-1} \left[\sum_{i=1}^{m} \lambda^i k_j^{(i)}\right] \frac{(x - t)_+^j}{j!}, \quad (3.11)$$

whose associated " remainder " $\bar{R}_m = \Gamma - \bar{S}_m$, just like R_m, has a kernel

in $C^{(m-1)}[a, b]$, for any $x \in [a, b]$. The sets $H_x^{R_m}$ and $H_x^{\bar{R}_m}$ associated with S_m and $\bar{S}_m$ are therefore in $C^{(m-1)}[a, b]$ for any $x \in [a, b]$ if $F = C^{\infty}[a, b]$. Choosing $S = \bar{S}_{2n+1}$ we get $H_x^R \subset C^{(2n)}[a, b]$ and obviously the n-point Gaussian quadrature formula can be taken as a best n-point formula on H_x^R for any $x \in [a, b]$. The global nth order near-best approximation for this case is thus defined as

$$\bar{u}_c(x) = \sum_{j=1}^{n} \alpha_j u_j(x) + \int_a^b \bar{S}_{2n+1}(x, t; \lambda) \left[f(t) - \sum_{j=1}^{n} \alpha_j f_j(t) \right] dt \quad (3.12)$$

where the α_j's are determined by (2.5) with the t_i^*'s taken as the n Gaussian points in $[a, b]$.

Remark : The choice of Gaussian points as collocation points is also suggested in other works; see e.g. Pruess [4]. However, the motivation there is based on the attempt of making $f - \sum_{j=1}^{n} \alpha_j f_j$ nearly orthogonal to all polynomials of degree $< n$. In the present work, the discussion leading to collocation at Gaussian points emphasizes the significance of the singular structure of K in this context, and reveals the necessity of adding a correction term to the collocation approximation. With this correction term we have, using theorem 2.1, the following promising result :

COROLLARY 3.2 : *The error in the global nth order near-best approximation* $\bar{u}_c$ *to the solution* u *of* (1.1) *with a kernel of the form* (3.3) *and* $F = C^{\infty}[a, b]$ *is given by*

$$u(x) - \bar{u}_c(x) = e_n^{[a,b]}\left(h - \sum_{j=1}^{n} \alpha_j h_j \right) \quad \forall x \in [a, b] \quad (3.13)$$

where $h, h_1, h_2, \ldots, h_n \in C^{(2n)}[a, b]$ *and* $e_n^{[a,b]}$ *is the error functional of the n-point Gaussian quadrature formula on* $[a, b]$.

The main contribution of the correction term in (3.12) is

$$\int_a^b \delta(x - t) \left[f(t) - \sum_{j=1}^{n} \alpha_j f_j(t) \right] dt = f(x) - \sum_{j=1}^{n} \alpha_j f_j(x) \quad (3.14)$$

due to the first term in $\bar{S}_{2n+1}$. This correction seems to be essential even for smooth K's. By this simple correction we get the approximation

$$\bar{u}_c^{(0)} = \sum_{j=1}^{n} \alpha_j u_j(x) + f(x) - \sum_{j=1}^{n} \alpha_j f_j(x) \quad (3.15)$$

which can be interpreted as a one stage Neumann iteration from the "point"

$$\bar{u} = \sum_{j=1}^{n} \alpha_j u_j .$$

It is mentioned in Baker [1] that this simple correction frequently improves expansion approximations.

If in (3.12) the $\bar{S}_{2n+1}$ is replaced by the kernel of S_m of (3.9) then $\bar{u}_c$ becomes simply the mth Neumann iterate starting from $\bar{u}$. It is important to notice the difference between the present motivation of obtaining this approximation and the simpler motivation based on Neumann iterations. Since the present motivation is based solely upon the fact that the kernel $K^{j+1}(x, t)$ is smoother than the kernel $K^j(x, t)$, it does not require the convergence of the Neumann series.

It can be shown that corollary 3.2 holds for the $(2n + 1)$th Neumann iterate from $\bar{u}$. Therefore, even for a divergent Neumann series we expect the first $2n + 1$ iterations to improve the collocation approximation $\bar{u}$, although further iterations might destroy this improvement.

Repeated Neumann iterations are not commonly used since their computation is expensive. However, for kernels of the form (3.3), the approximation $\bar{u}_c$, using the computationally simple kernel $\bar{S}_{2n+1}$, has been shown to play the same role as the $(2n + 1)$th Neumann iterate. The technique used for kernels of the form (3.3) can be easily extended to deal with more general kernels of the form

$$K(x, t) = C(x, t) + \sum_{j=0}^{\infty} c_j (x - t)_+^{r_j} \tag{3.16}$$

with real exponents $-1 < r_0 < r_1 < r_2 < \ldots$ where $C \in C^\infty([a, b] \times [a, b])$. Here, a generalization of equality (3.5) should be used,

$$\int_a^b (x - y)_+^r (y - t)_+^s \, dy = B(r + 1, s + 1)(x - t)_+^{r+s+1} \tag{3.17}$$

where B is the Beta function.

For other classes of kernels it might be more difficult to find a convenient expression for the singular part S of Γ. However, the results obtained with the near-best approximations for kernels of the form (3.3) indicate that the study of other types of singular kernels deserves a strong consideration.

4. NUMERICAL EXAMPLE

We consider the Fredholm equation

$$u(x) - \lambda \int_0^1 K(x, t)\, u(t)\, dt = f(x) \tag{4.1}$$

with $f \in C^\infty[0, 1]$ and with kernel

$$K(x, t) = \begin{cases} x(1 - t) & 0 \leqslant x \leqslant t \leqslant 1 \\ t(1 - x) & 0 \leqslant t \leqslant x \leqslant 1 \end{cases}. \tag{4.2}$$

This kernel can be rewritten as

$$K(x, t) = x(1 - t) - (x - t)_+ , \tag{4.3}$$

i.e., in the form (3.3) with $K_1(x, t) = x(1 - t)$, $k_0^{(1)} = 0$, $k_1^{(1)} = -1$ and $k_i^{(1)} = 0$ for $i \geqslant 2$. Hence we can use (3.7) and (3.11) to compute the singular parts $\bar{S}_m$ of Γ.

To demonstrate the power of the corrected approximation $\bar{u}_c$ we consider corrections to a low order collocation approximation, a 4th order in this case. The 4-point collocation approximation $\bar{u}$ is taken as the third degree polynomial collocating the integral equation at the four Gaussian points in [0, 1]. In $\bar{u}_c$, defined by (3.12), the computation of the f_j's is performed analytically by (1.8) and the correction term is approximated by using Simpson's rule with $h = 0.01$.

To investigate the influence of particular singularities of Γ upon the correction term we compute a sequence of approximations, $\bar{u}_c^{(m)}$, $m = 0, 1, 2, \ldots$, corresponding to correction terms with $\bar{S}_m$. Thus $\bar{u}_c^{(m)}$ takes care of the jump discontinuities in the first $m - 1$ derivatives of Γ. We note that in this case $\bar{S}_{2i+1} = \bar{S}_{2i}$ and therefore $\bar{u}_c^{(2i+1)} = \bar{u}_c^{(2i)}$.

We tested the problem (4.1) with $\lambda = 1$ and $f(x) = x$ having the solution

$$u(x) = \sin x / \sin 1 .$$

In table 4.1 we give results at $x = 0$, 0.2, 0.4, 0.6, 0.8, and 1.0 computed from the collocation approximation $\bar{u}$ and the corrected approximations $\bar{u}_c^{(0)}$, $\bar{u}_c^{(2)}$ and $\bar{u}_c^{(4)}$. These are compared with values computed from the analytic solution $u(x)$.

TABLE 4.1.

x	0.0	0.2	0.4	0.6	0.8	1.0
$\bar{u}(x)$	− 0.000271694	0.236219378	0.462707145	0.670937092	0.852654706	0.999605471
$\bar{u}_c^{(0)}(x)$	0.000000000	0.236098504	0.462781577	0.671017224	0.852503761	1.000000000
$\bar{u}_c^{(2)}(x)$	0.000000000	0.236097644	0.462782799	0.671018304	0.852502461	1.000000000
$\bar{u}_c^{(4)}(x)$	0.000000000	0.236097651	0.462782835	0.671018328	0.852502439	0.999999967
$u(x)$	0.000000000	0.237097660	0.462782852	0.671018352	0.852502467	1.000000000

A clear improvement is already achieved by $\bar{u}_c^{(0)}$ using only the first correction term. From three correct significant figures in $\bar{u}$ the accuracy is improving to five correct figures in $\bar{u}_c^{(0)}$, and to seven correct figures in $\bar{u}_c^{(4)}$.

We note that the development of the near-best approximations presented here is based upon the representation (1.2) of the solution of problem (1.1). In fact, similar near-best approximations can be obtained for many problems whose solution has a representation of the form (1.2) (see [2]). In [3] this is done for the harmonic Dirichlet problem.

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