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THREE REMARKS ON THE USE OF ČEBYŠEV POLYNOMIALS FOR SOLVING EQUATIONS OF THE SECOND KIND (*)

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Abstract — *Three methods are considered the Čebyšev-Euler method, the Čebyšev semi-iterative method and a refinement of the projection method for the approximation of the quasi inverse for self-adjoint operators A such that unity does not belong to the spectrum of A*

Résumé — *On considère trois méthodes la méthode de Čebyšev-Euler, la méthode semi-itérative de Čebyšev, et une amélioration de la méthode de projection, appliquées à l'approximation du quasi-inverse pour des opérateurs auto-adjoints A pour lesquels l'unité n'appartient pas au spectre de A*

In this communication we consider three methods for the approximate solution of equations of the second kind

$$x - Ax = y$$

in a (complex) Hilbert space with a selfadjoint bounded linear operator A. We only assume that unity does not belong to the spectrum of A.

We give a new proof and an error estimate for the Čebyšev-Euler method, we discuss the Čebyšev-semi-iterative method (*cf.* Varga [7]) and we consider a refinement of the projection method of type Q_v, introduced in [6].

1. INTRODUCTION

Let A be a bounded linear selfadjoint operator in a complex Hilbert space H. Let (E_λ) be its spectral decomposition, σ an interval containing the spectrum $\sigma(A)$ of A. Then

$$A = \int_\sigma \lambda \, dE$$

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and for each continuous function $p : \sigma \to \mathbb{R}$

$$p(A) = \int_\sigma p(\lambda)\, dE_\lambda$$

and

$$\| p(A) \| \leqq \sup_{\lambda \in \sigma} | p(\lambda) | .$$

Especially, if $1 \notin \sigma$,

$$\| (1 - A)^{-1} - p(A) \| \leqq \sup_{\lambda \in \sigma} \left| \frac{1}{1 - \lambda} - p(\lambda) \right| .$$

If p is an arbitrary polynomial of degree n, then $\| (1 - A)^{-1} - p(A) \|$ is minimal, if p is the proximum of r (with $r(\lambda) = (1 - \lambda)^{-1}$) in the space of all polynomials of degree n on the spectrum of A with respect of the sup-norm.

We call a method for the approximate solution of $x - Ax = y$ polynomial, if the approximate solution $\hat{x}$ is of the form $\hat{x} = p(A)\, y$, where $p(A)$ is an operator polynomial.

2. THE ČEBYŠEV-EULER METHOD

If $\sigma = [a, b]$, $b < 1$ then the Čebyšev-Euler method consists of determining the proximum p_n to r in $[a, b]$ by polynomials degree n. Then

$$x_n = p_n(A)\, y \tag{1}$$

is the Čebyšev-Euler approximation of the solution x of $x - Ax = y$.

This approximation is easy to calculate : it is known Čebyšev [3], Bernstein [2], Meinardus [4], that in the interval $[-1, 1]$ the proximum of

$$s_\alpha(\lambda) = \frac{1}{\lambda - \alpha} \qquad \alpha > 1$$

is given by the polynomials q_n of degree n which fulfill

$$\begin{gathered} \frac{1}{\lambda - \alpha} - q_n(\lambda) = \gamma_n \cos (n\varphi + \delta) \\ \gamma_n = \frac{(\alpha - \sqrt{\alpha^2 - 1})^n}{\alpha^2 - 1}, \quad \lambda = \cos \varphi , \quad \frac{\alpha\lambda - 1}{\lambda - 1} = \cos \delta . \end{gathered} \tag{2}$$

Using the Čebyšev polynomials t_n and v_n of first resp. second kind, (2) is equivalent to

$$(\alpha - \lambda)\, q_n(\lambda) = 1 - \gamma_n(\alpha\lambda - 1)\, t_n(\lambda) + \gamma_n \sqrt{\alpha^2 - 1}\, (1 - \lambda^2)\, v_{n-1}(\lambda) .$$

The recursion formulas for t_n and v_n lead to the recursion formula for q_n

$$q_{n+1}(\lambda) = 2\,\gamma\lambda q_n(\lambda) - \gamma^2\, q_{n-1}(\lambda) - 2\,\gamma$$

$$\gamma = \alpha - \sqrt{\alpha^2 - 1}\,, \quad q_0(\lambda) = \frac{\alpha}{1 - \alpha^2}\,, \quad q_1(\lambda) = \frac{\lambda + \sqrt{\alpha^2 - 1}}{\alpha^2 - 1}\,.$$

A linear transformation of the interval $[-1, 1]$ onto $[a, b]$ gives the polynomials p_n of best approximation of r by

$$p_n(\lambda) = -\frac{b - a}{2}\, q_n\!\left(\frac{2\,\lambda - b - a}{b - a}\right)$$

which leads to the recursion formula

$$p_{n+1}(\lambda) = -\frac{2}{b - a}\left[\frac{2\,\gamma}{b - a}(2\,\lambda - b - a)\,p_n(\lambda) - \gamma^2\, p_{n-1}(\lambda) - 2\,\gamma\right]$$

$$p_0(\lambda) = \frac{b - a}{2}\,\frac{\alpha}{\alpha^2 - 1}$$

$$p_1(\lambda) = \frac{1}{\alpha^2 - 1}\left(-\lambda + \frac{b + a}{2} - \frac{b - a}{2}\sqrt{\alpha^2 - 1}\right)$$

$$\alpha = \frac{2 - b - a}{b - a}$$

$$\gamma = \alpha - \sqrt{\alpha^2 - 1}\,.$$

The error estimate is

$$\max_{\lambda \in [a,b]}\left|\frac{1}{1 - \lambda} - p_n(\lambda)\right| = \max_{\lambda \in [-1,1]}\frac{b - a}{2}\left|\frac{1}{\lambda - \alpha} - q_n(\lambda)\right|$$

$$= \frac{b - a}{2}\,\frac{\gamma^n}{\alpha^2 - 1}\max|\cos(n\varphi + \delta)| \leqq \frac{b - a}{2(\alpha^2 - 1)}\cdot\gamma^n\,.$$

If we replace λ by A, we obtain the following result :

If $\sigma(A) \subset [a, b]$, $b < 1$, then the best polynomial approximation method of $x - Ax = y$ is given by the following semi-iterative method.

$$x_{n+1} = -\frac{2}{b - a}\left[\frac{2\,\gamma}{b - a}(2\,Ax_n - (b + a)\,x_n) - \gamma^2\, x_{n-1} - 2\,\gamma y\right]$$

$$x_0 = \frac{b - a}{2}\cdot\frac{\alpha}{\alpha^2 - 1}\,y\,, \quad x_1 = \frac{1}{\alpha^2 - 1}\left[-Ay + \left(\frac{b + a}{2} - \frac{b - a}{2}\sqrt{\alpha^2 - 1}\right)y\right]$$

$$\alpha = \frac{2 - b - a}{b - a}\,, \quad \gamma = \alpha - \sqrt{\alpha^2 - 1}$$

with the error estimate

$$\| x - x_n \| \leqq \frac{b - a}{2(\alpha^2 - 1)} \gamma^n .$$

This method also can be obtained by using methods of summability theory (*cf.* Niethammer [5]).

3. THE ČEBYŠEV SEMI-ITERATIVE METHOD

Let A be a linear selfadjoint bounded operator with $1 \notin \sigma(A)$ and (x_n) the Picard iteration sequence

$$x_{n+1} = Ax_n + y , \qquad x_0 = y .$$

This method calculates a linear combination

$$\tilde{x}_n = \sum_{j=0}^{n} \gamma_j x_j$$

such that $x - \tilde{x}_n$ has a small norm.

Since for

$$\tilde{w}_n = \tilde{x}_n - x = \sum_{j=0}^{n} \gamma_j (x_j - x) + \sum_{j=0}^{n} (\gamma_j - 1) x$$

it is

$$\tilde{w}_n = \sum_{j=0}^{n} \gamma_j A^j (x_0 - x) = p(A)(x_0 - x)$$

with the condition $p(1) = \sum_{j=0}^{n} \gamma_j = 1$, so

$$\tilde{w}_n = p(A)(y - (I - A)^{-1} y) = \int_\sigma p(\lambda) \frac{-\lambda}{1 - \lambda} dE_\lambda y$$

and $\| \tilde{w}_n \|$ is minimal, if p is a polynomial of degree n with $p(1) = 1$ and

$$\max_{\lambda \in \sigma(A)} \left| p(\lambda) \frac{\lambda}{1 - \lambda} \right| \leqq \max_{\lambda \in \sigma(A)} \left| q(\lambda) \frac{\lambda}{1 - \lambda} \right| ,$$

where q is an arbitrary polynomial of degree n with $q(1) = 1$. If both 1 and 0 do not belong to the spectrum of A, then p is up to a constant the same polynomial as the polynomial q with $q(1) = 1$ and $\max_{\lambda \in \sigma(A)} | q(\lambda) |$ is minimal.

In each case, this minimal polynomial does not lead to an easy semi-iterative method, so the usual minimization condition is to determine the polynomial p of degree n with $p(1) = 1$ and minimal norm.

It is well known that the transformed Čebyšev polynomials have this property that their norm on an interval is minimal. So one has to consider three cases

1° $\sigma(A) \subset [a, b], b < 1$
2° $\sigma(A) \subset [a, b], a > 1$
3° $\sigma(A) \subset [a_1, b_1] \cup [a_2, b_2]$.

In the first and second case

$$p_n(\lambda) = \frac{t_n\left(\dfrac{2\lambda - b - a}{b - a}\right)}{t_n\left(\dfrac{2 - b - a}{b - a}\right)}$$

is the minimal polynomial with

$$\| p_n \| = \max_{\lambda \in [a,b]} | p_n(\lambda) | = \left| \frac{1}{t_n\left(\dfrac{2 - b - a}{b - a}\right)} \right| .$$

Using the recursing formula for the Čebyšev polynomials we obtain for

$$\rho_n = t_n\left(\frac{2 - b - a}{b - a}\right)^{-1}$$

$$\rho_{n+1}^{-1} = 2\frac{2 - b - a}{b - a}\rho_n^{-1} - \rho_{n-1}^{-1}, \quad \rho_0 = 1, \quad \rho_1^{-1} = \frac{2 - b - a}{b - a}$$

and

$$p_{n+1}(\lambda) = \frac{2\rho_{n+1}}{\rho_n}\frac{2\lambda - b - a}{b - a}p_n(\lambda) - \frac{\rho_{n+1}}{\rho_{n-1}}p_{n-1}(\lambda)$$

$$p_0(\lambda) = 1, \quad p_1(\lambda) = \frac{2\lambda - b - a}{2 - b - a}.$$

This gives after a short calculation, using the recursion formulas and

$$\tilde{w}_{n+1} = \tilde{x}_{n+1} - x = p_n(A)\, w_0$$

the semi-iterative method

$$\tilde{x}_{n+1} = \frac{4\,\rho_{n+1}}{\rho_n(b-a)}\left(A\tilde{x}_n + y - \frac{b+a}{2}\tilde{x}_n\right) - \frac{\rho_{n+1}}{\rho_{n-1}}\tilde{x}_{n-1}$$

$$\tilde{x}_0 = y\,, \quad \tilde{x}_1 = \frac{2}{2-b-a}Ay + y$$

and the error estimate

$$\| x - \tilde{x}_n \| \leqq \| p_n(A)(x-y) \| \leqq | \rho_n | \; \| x - y \| \,.$$

In the third case we assume that there is known a number η such that

$$\sigma(A) \subset [-\rho, 1-\eta] \cup [1+\eta, \rho]\,.$$

Since the polynomials

$$q_{2n}(\lambda) = t_n\left(\frac{2\lambda^2 - 1 - \alpha^2}{1-\alpha^2}\right)$$

are the polynomials of minimal norm on the intervals $[-1, -\alpha] \cup [\alpha, 1]$ of degree $2n$ with $q_{2n}(1) = 1$ (*cf.* Achieser [1], p. 287) a linear transformation of $[-1, -\alpha] \cup [\alpha, 1]$ onto $[-\rho, 1-\eta] \cup [1+\eta, \rho+2]$ (resp. $[2-\rho, 1-\eta] \cup [1+\eta, \eta]$ if more convenient) and the substitution of λ by A leads to the semi-iterative method

$$x_{n+1} = \frac{4\,\tau_{n+1}}{\tau_n((\rho+1)^2 - \eta^2)}\left[(A^2 x_n - 2Ax_n + x_n - Ay - y) - \frac{1}{2}((\rho+1)^2 + \eta^2)\,x_n\right] - \frac{\tau_{n+1}}{\tau_{n-1}}x_{n-1}$$

$$x_0 = y$$

$$x_1 = -\frac{2}{(\rho+1)^2 + \eta^2}(A^2 y - Ay) + y$$

$$\tau_{n+1}^{-1} = t_n\left(-\frac{(\rho+1)^2 + \eta^2}{(\rho+1)^2 - \eta^2}\right)$$

$$\tau_{n+1}^{-1} = -2\frac{(\rho+1)^2 + \eta^2}{(\rho+1)^2 - \eta^2}\tau_n^{-1} - \tau_{n-1}^{-1}$$

$$\tau_0 = 1$$

$$\tau_1^{-1} = -\frac{(\rho+1)^2 + \eta^2}{(\rho+1)^2 - \eta^2}\,.$$

The order of convergence of this method is

$$\| x_n - x \| \leqq \| p(A)(y - x) \| = O(\tau_n^{-1}) .$$

4. A ČEBYŠEV PROJECTION METHOD

Let A be again a bounded linear selfadjoint operator in a Hilbert space H with spectrum in $[a, b]$, $b < 1$.

Let $p_n : [a, b] \to \mathbb{R}$ be the polynomials from section 2, which are the proxima of $(1 - \lambda)^{-1}$ of degree n in $[a, b]$. Then for linear independent elements $z_1, \ldots, z_k$ of H we determine

$$z = \sum_{j=1}^{n} \gamma_j z_j$$

from the system of linear equations

$$\langle z - Az - y + (1 - A) p_n(A) y, z_j \rangle = 0$$

for $j = 1, 2, \ldots, k$. Then

$$\hat{x}_n = p_n(A) y + z$$

is an approximation for the solution x of $x - Ax = y$.

If $p_n = 0$, then this method is the usual Ritz-Galerkin method, if

$$p_n(\lambda) = \sum_{j=0}^{n} \lambda^j ,$$

then this method is the projection method of type Q_{n+1} introduced in [6].

As usual in the theory of the Ritz-Galerkin method, the optimal rate of convergence for compact A is obtained, if $z_1, \ldots, z_k$ are eigenvectors of A. In this case we get with

$$x = p_n(A) y + ((1 - A)^{-1} - p_n(A)) y$$
$$\hat{x}_n = p_n(A) y + z$$

and a simple Hilbert space calculation

$$z = \sum_{j=1}^{k} \left(\frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right) \langle y, z_j \rangle z_j$$

$$\| x - x_n \|^2 = \sum_{j=k+1}^{\infty} \left| \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right|^2 | \langle y, z_j \rangle |^2$$

so

$$\| x - \hat{x}_n \| \leqq \sup_{j \geqslant k+1} \left| \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right| \leqq \frac{b - a}{2(\alpha^2 - 1)} \gamma^n$$

where

$$\alpha = \frac{2 - b - a}{b - a}, \quad \gamma = \alpha - \sqrt{\alpha^2 - 1}$$

as in a section 2. Also as in section 2 is shown $p_n(A)$ y can be calculated by a semi-iterative method.

5. CONCLUDING REMARKS

Niethammer [5] has shown that the order of convergence of the Čebyšev semi-iterative method tends to the order of convergence of the Čebyšev-Euler method. In [8] M. Wolf has demonstrated that the Čebyšev projection method in general gives quite better approximations than the usual Ritz-Galerkin method.

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