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APPROXIMATION OF A DEGENERATED ELLIPTIC BOUNDARY VALUE PROBLEM BY A FINITE ELEMENT METHOD (*) (**)

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Communique par P G CIARLET

Résume — *On considère un problème aux limites elliptique dégénéré, provenant d'un problème de Dirichlet ayant une symétrie de révolution Pour approcher la solution, nous construisons une méthode d'éléments finis basée sur une triangulation usuelle du domaine, mais avec des fonctions approchantes adaptées Cet aspect a deja ete mis en relief, en dimension un, par Crouzeix et Thomas Par une technique d'approximation de domaine et d'estimation à priori, nous donnons une démonstration directe, dans le cadre bidimensionnel, de la regularité de la solution Ceci permet de retrouver un comportement asymptotique de l'erreur, identique au cas standard*

Abstract — *We consider a degenerated elliptic boundary value problem arising in the (concrete) axisymmetric Dirichlet problem We construct a finite element approximation of the solution using a classical triangulation, but with special approximating functions This aspect has already been pointed out in the one-dimensional case by Crouzeix and Thomas Here we give a direct proof of the regularity of the solution in the two-dimensional case, using a technique of approximated domain and a priori estimates This leads to an asymptotic error behaviour which is the same as in the usual case*

1. STATEMENT OF THE PROBLEM

Let Ω be a convex bounded domain with a polygonal boundary contained in $\mathbb{R}^2_+ = \{ (r, z) \in \mathbb{R}^2 ; r > 0 \}$; $\Gamma = \partial\Omega \cap \mathbb{R}^2_+$ is the boundary of Ω in $\mathbb{R}^2_+$. We assume that it intersects the z-axis at right angles, i.e. there exists $R > 0$ such that

$$\Omega_R = \{ (r, z) \in \Omega ; 0 < r < R \} =]O, R[\times]O, H[.$$

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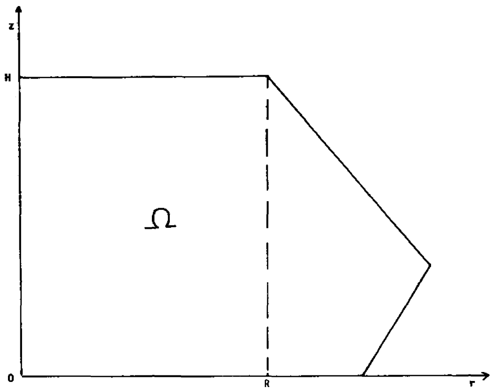

Figure 1.

Let f be a given function defined in Ω ; we consider the following boundary problem :

$$\left.\begin{aligned} -\Delta_r u &= -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\partial^2 u}{\partial z^2}=f \quad \text{in} \quad \Omega ; \\ u &= 0 \quad \text{on} \quad \Gamma . \end{aligned}\right\} \tag{1.1}$$

This problem has a (weak) variational formulation which needs the introduction of some functions spaces in order to be stated. The notations are those used in the partial differential equations theory (see e.g. Lions-Magenes [8]).

Let us denote by $L_1^2(\Omega)$ the weighted L^2 space of all Lebesgue measurable functions u such that

$$\int_\Omega |u|^2 \, r \, dr \, dz \quad \text{is finite}$$

and by

$$H_1^m(\Omega) = \{ u \in L_1^2(\Omega) ; D^\alpha u \in L_1^2(\Omega) ; \forall\alpha ; |\alpha| \leqslant m \},$$

the related Sobolev space of order m. All these spaces, equipped with their natural scalar products and norms, are Hilbert spaces. If A is a measurable subset of Ω, we shall use the semi-norm of $H_1^m(\Omega)$ defined for v in $H_1^m(\Omega)$ by

$$|v|_{j,A} = \left\{ \sum_{|\alpha|=j} \int_A |D^\alpha v|^2 \, r \, dr \, dz \right\}^{1/2} \quad \text{for} \quad j = 0, 1, -, m .$$

Then, if V is the closure of $\mathscr{D}(\Omega)$ in the $H_1^1(\Omega)$ norm, we can give a precise statement of (1.1).

Let f in $\mathscr{D}'(\Omega)$ be such that rf ranges in V' ; the problem is

$$\left.\begin{array}{l}\text{Find } u \text{ in } V\,;\forall v \in V\,;\\ a(u, v) = \langle\, rf, v\,\rangle\,,\end{array}\right\} \tag{1.2}$$

where

$$a(u, v) = \int_{\Omega} (\text{grad } u).(\text{grad } v)\, r\, dr\, dz\,, \tag{1.3}$$

$\langle\, .,\, .\,\rangle$ is the duality pairing between V' and V.

It is easily seen by a Poincaré's inequality (we shall later see that in a particular and fundamental aspect) that the continuous bilinear form given by (1.3) is V-elliptic. Therefore, the Lax-Milgram theorem insures the existence and uniqueness of the solution of problem (1.2).

Remark

1) Problem (1.2) solves (1.1) in the following sense :

$$\begin{cases} -\Delta_r u = f & \text{in } \mathscr{D}'(\Omega)\,,\\ u \in V\,. \end{cases}$$

2) If v belongs to $H_1^1(\Omega)$, the trace of u on Γ is defined as an element of $H_{\text{loc}}^{1/2}(\Gamma)$. u is in V iff $u = 0$ (in the sense of $H_{\text{loc}}^{1/2}(\Gamma)$) ; for all this, see Amirat [1], Atik [2], Lailly [7].

Now, with a variational formulation, we can make various conformal approximations of the space V in order to approximate the solution of (1.2). For the one considered here, we have an error behaviour which is the same as in the standard case.

We consider the following mesh of Ω denoted by $\mathscr{T}^h$; h is the maximum of diameters of the elements of $\mathscr{T}^h$ and is destined to tend to 0.

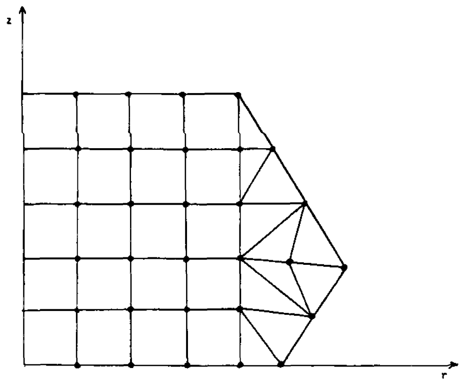

Figure 2.

K will be a generic element of $\mathcal{T}^h$, $\mathcal{N}$ is the set of all vertices of the elements K which do not lie on the z-axis.

We consider three kinds of approximating functions.

Case 1 : K is a rectangle with a side on the z-axis ; we have

$$P_K = \{ \alpha + \beta z ; \alpha, \beta \in \mathbb{R} \} ;$$

Σ_K represents the vertices of K which are not on the z-axis.

Case 2 : K is a rectangle and has not the above property ; then

$$P_K = \{ \alpha + \beta z + \gamma \log r + \delta r \log r ; \alpha, \beta, \gamma, \delta \in \mathbb{R} \} ;$$

Σ_K represents the set of the vertices of K.

Case 3 : K is a triangle (which happens far from the z-axis) ; we have

$$P_K = \mathbb{P}_1 = \{ \alpha + \beta z + \gamma r ; \alpha, \beta, \gamma \in \mathbb{R} \} ;$$

Σ_K represents the vertices of K.

In each case, if v is a continuous function around the elements of Σ_K, we can define in a unique way the interpolate πv of v by

$$\begin{cases} \pi v \in P_K , \\ \pi v(a) = v(a) ; \forall a \in \Sigma_K . \end{cases}$$

Then, we define

$$S^h(\Omega) = \{ v^h \in C^0(\overline{\Omega}) ; v^h |_K \in P_K ; \forall K \in \mathcal{T}^h \} .$$

It is easily seen that we have the following result :

PROPOSITION 1.1 : *$S^h(\Omega)$ is a subspace of $H_1^1(\Omega)$. Every v^h in $S^h(\Omega)$ is biunivoquely determined by the collection of nodal parameters $v^h(a)$, $a \in \mathcal{N}$.*

If V^h denotes the subspace formed by v^h in $S^h(\Omega)$ such that $v^h(a) = 0$ when a lies in $\Gamma \cap \mathcal{N}$, then V^h is a subspace of V. ■

Now, we state the discrete problem :

$$\left.\begin{array}{l} \text{Find } u^h \in V^h ; \forall v^h \in V^h ; \text{such that} \\ a(v^h, v^h) = \langle rf, v^h \rangle . \end{array}\right\} \tag{1.4}$$

Cea's lemma leads to

$$\| u - u^h \|_{H_1^1(\Omega)} \leqslant c \| u - v^h \|_{H_1^1(\Omega)} ; \forall v^h \in V^h , \tag{1.5}$$

where c is a constant independent of u and h.

In the sequel, c will denote various pure constants and we shall indicate which elements it does not depend on, in each case.

To give the error estimate, we need some regularity results.

2. REGULARITY OF THE SOLUTION

We suppose that f is in $L_1^2(\Omega)$ (i.e. the data in the axisymmetric three-dimensional problem is L^2). We have the following main result : we set

$$D_1^2(\Omega) = \left\{ u \in H_1^2(\Omega) ; \frac{1}{r} \frac{\partial u}{\partial r} \in L_1^2(\Omega) \right\},$$

$$E(\Omega) = D_1^2(\Omega) \cap V ;$$

then, we have

THEOREM 2.1 : *The operator*

$$-\Delta_r : E(\Omega) \to L_1^2(\Omega)$$

is an isomorphism of Banach spaces. ∎

Remark : Since f is in $L_1^2(\Omega)$, we have $\langle rf, v \rangle = (f, v)$ where (f, v) denotes the scalar product of f and v in $L_1^2(\Omega)$, for v in V.

The proof of theorem 2.1 will be reduced to the non degenerated case by the following à priori estimates.

For $\varepsilon > 0$ small enough, Ω_ε is the approximated (convex) domain given by cutting in Ω a band of width ε around the z-axis :

$$\Omega_\varepsilon = \{ (r, z) \in \Omega ; r > \varepsilon \}$$
$$\Gamma_\varepsilon^D = \Gamma \cap \partial\Omega_\varepsilon$$
$$\Gamma_\varepsilon^N = \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon^D$$
$$E_\varepsilon = \left\{ v \in H^2(\Omega_\varepsilon) ; \frac{\partial v}{\partial r} = 0 \text{ on } \Gamma_\varepsilon^N ; v = 0 \text{ on } \Gamma_\varepsilon^D \right\}.$$

For suitable v, we shall use the following notation :

$$| v |_{j,\Omega_\varepsilon} = \left\{ \sum_{|\alpha|=j} \int_{\Omega_\varepsilon} | D^\alpha v |^2 \, r \, dr \, dz \right\}^{1/2}$$

LEMMA 2.1 : *There exists a constant c independent of $\varepsilon > 0$ and of v in E_ε, such that*

$$| v |_{0,\Omega_\varepsilon}^2 + | v |_{1,\Omega_\varepsilon}^2 \leqslant c^2 \, | \Delta_r v |_{0,\Omega_\varepsilon}^2 . \quad ∎ \tag{2.1}$$

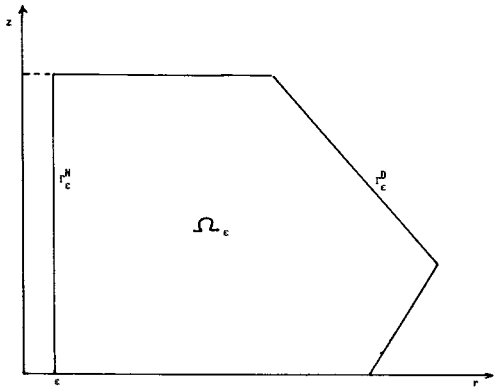

Figure 3.

Proof : Since v is in E_ε and using Green's formula, we obtain

$$-\int_{\Omega_\varepsilon} (\Delta_r v)\, vr\, dr\, dz = -\int_{\Omega_\varepsilon} v \operatorname{div}(r \operatorname{grad} v)\, dr\, dz = | v |_{1,\Omega_\varepsilon}^2 .$$

Let $\Phi :]O, H[\to \mathbb{R}$ be the continuous piecewise affine function describing Ω by

$$\Omega = \{ (r, z) \in \mathbb{R}^2 ; 0 < z < H ; 0 < r < \Phi(z) \} ; \tag{2.2}$$

we have

$$| v(r, z) | \leqslant \int_r^{\Phi(z)} \left| \frac{\partial v}{\partial r}(s, z) \right| ds ,$$

$$\sqrt{r}\, | v(r, z) | \leqslant \int_r^{\Phi(z)} \sqrt{s} \left| \frac{\partial v}{\partial r}(s, z) \right| ds \leqslant c \left\{ \int_r^{\Phi(z)} \left| \frac{\partial v}{\partial r} \right|^2 s\, ds \right\}^{1/2} .$$

Here c can only be the square root of the diameter of Ω and does not depend on ε and v (see the above remark on the coerciveness of the bilinear form of problem (1.2)). Integrating in z gives (2.1) (with another constant). ∎

LEMMA 2.2 : *For all $v \in E_\varepsilon$, we have*

$$\left| \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v}{\partial r} \right) \right|_{0,\Omega_\varepsilon}^2 + \left| \frac{\partial^2 u}{\partial r\, \partial z} \right|_{0,\Omega_\varepsilon}^2 + \left| \frac{\partial^2 u}{\partial z^2} \right|_{0,\Omega_\varepsilon} \leqslant | \Delta_r u |_{0,\Omega_\varepsilon}^2 . \quad \blacksquare \tag{2.3}$$

Proof : Note that it is sufficient to prove (2.2) for v in $H^3(\Omega_\varepsilon) \cap E_\varepsilon$.

By a density argument (see Grisvard [6]), the general case is true. Then, if v is in $H^3(\Omega_\varepsilon) \cap E_\varepsilon$, we have

$$| \Delta_r v |_{0,\Omega_\varepsilon}^2 = \left| \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v}{\partial r} \right) \right|_{0,\Omega_\varepsilon}^2 + \left| \frac{\partial^2 v}{\partial z^2} \right|_{0,\Omega_\varepsilon}^2 + 2 \int_{\Omega_\varepsilon} \frac{\partial}{\partial r}(r v_1) \frac{\partial v_2}{\partial z} \, dr \, dz ,$$

where

$$v_1 = \frac{\partial v}{\partial r} ; \quad v_2 = \frac{\partial v}{\partial z} . \tag{2.4}$$

Let us consider the one differential form $\overline{\omega} = r v_1 \, dv_2$; by differentiating, we obtain

$$d\overline{\omega} = \frac{\partial}{\partial r}(r v_1) \frac{\partial v_2}{\partial z} \, dr \wedge dz - \frac{\partial}{\partial z}(r v_1) \frac{\partial v_2}{\partial r} \, dr \wedge dz .$$

Using Stokes' formula, we can write

$$\int_{\Omega_\varepsilon} \frac{\partial}{\partial r}(r v_1) \frac{\partial v_2}{\partial z} \, dr \, dz = \int_{\partial\Omega_\varepsilon \circlearrowleft} \overline{\omega} + \int_{\Omega_\varepsilon} \frac{\partial v_1}{\partial z} \frac{\partial v_2}{\partial r} \, r \, dr \, dz .$$

By (2.4) and the boundary value of v in E_ε, the curvilinear integral is reduced to the integration of $\overline{\omega}$ along the oriented piecewise regular path given by

$$\gamma :]O, H[\to \mathbb{R}^2 ; \quad \gamma(z) = (\Phi(z), z) .$$

There is a subdivision of $]O, H[$, $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_{P+1} = H$, such that

$$\Phi_\iota = \Phi \,|_{[\alpha_\iota, \alpha_{\iota+1}]} \text{ is linear affine} .$$

If γ_ι is the oriented path associated with Φ_ι, we have

$$\int_{\partial\Omega_\varepsilon \circlearrowleft} \overline{\omega} = \sum_{\iota=0}^{P} \int_{\gamma_\iota} \overline{\omega} .$$

But we have $v = 0$ on γ_ι ; hence

$$v_1 \, \Phi'_\iota + v_2 = 0 .$$

Since Φ_ι is linear affine, its derivative Φ'_ι is constant and we obtain

$$\int_{\gamma_\iota} \overline{\omega} = - \Phi'_\iota \int_{\gamma_\iota} r v_1 \, dv_1 = \frac{1}{2} \Phi'_\iota \int_{\gamma_\iota} v_1^2 \, dr .$$

using the fact that v_1 vanishes at the endpoints of γ_i, since the gradient of v vanishes in two noncolinear directions. Finally, noticing that $dr = \Phi_i' \, dz$ on γ_i, we obtain in all cases :

$$\int_{\gamma_i} \overline{\omega} = \frac{1}{2} \int_{\gamma_i} v_2^2 \, dz ,$$

and then

$$\int_{\partial\Omega_\varepsilon} \overline{\omega} = \frac{1}{2} \int_0^H v_2^2(\Phi(z), z) \, dz \geqslant 0 .$$

This inequality gives (2.3). ■

Remark : The proof is valid in all the cases where Ω is represented by (2.2). Geometrically this means that the axisymmetric domain of $\mathbb{R}^3$, obtained by the rotation of Ω around the z-axis, is convex. The three-dimensional regularity is then completely recovered.

LEMMA 2.4 : *For all v in E_ε, we have*

$$\left| \frac{1}{r} \frac{\partial v}{\partial r} \right|_{0,\Omega_\varepsilon}^2 + \left| \frac{\partial^2 v}{\partial r^2} \right|_{0,\Omega_\varepsilon}^2 \leqslant c^2 \left| \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) \right|_{0,\Omega_\varepsilon}^2 \tag{2.5}$$

where c is a constant independent of v and ε. ■

Proof : We set

$$w = r \frac{\partial v}{\partial r} ; \quad w = 0 \quad \text{on} \quad \Gamma_\varepsilon^N ;$$

extending w by 0 in Ω, we can assume that w is in $H^1(\Omega)$, and write

$$w(r, z) = \int_0^r \frac{\partial w}{\partial r}(s, z) \, ds .$$

An easy majoration gives

$$r^{-3/2} \mid w(r, z) \mid \leqslant \frac{1}{r} \int_0^r \frac{1}{\sqrt{s}} \left| \frac{\partial w}{\partial r} \right| ds .$$

Denote by $\tilde{w}$ et $\partial\tilde{w}/\partial r$ the extension by 0 of w and $\partial w/\partial r$ respectively ; Hardy's inequality leads to

$$\int_0^{+\infty} r^{-3} \mid \tilde{w}(r, z) \mid^2 dr \leqslant \int_0^{+\infty} \frac{1}{r} \left| \frac{\partial \tilde{w}}{\partial r} \right|^2 dr .$$

After integrating in z and taking into account that $\tilde{w}$ and $\partial\tilde{w}/\partial r$ vanish outside of Ω_ε, we obtain (2.5). ∎

Proof of theorem 2.1 : $-\Delta_r$ is a continuous injective operator from $E(\Omega)$ into $L^2_1(\Omega)$. Theorem 2.1 will be proved if we show $-\Delta_r$ is surjective as well. At the same time, we will give an estimate of the continuity constant of the inverse operator.

Let f be in $L^2_1(\Omega)$. For $\varepsilon > 0$ small enough, we denote by $f_\varepsilon = f|_{\Omega_\varepsilon}$ the restriction of f to Ω_ε.

We know (see Grisvard [6]; at this step, the convexity of Ω is needed) $-\Delta$ is an isomorphism of Banach spaces from E_ε onto $L^2(\Omega_\varepsilon)$.

$-\Delta_r = -\Delta - \dfrac{1}{r}\dfrac{\partial}{\partial r}$ appears as a perturbation of an isomorphism by a compact operator $-\dfrac{1}{r}\dfrac{\partial}{\partial r}$; hence, it is a Fredholm operator of index 0. Since the kernel of $-\Delta_r$ is reduced to 0, $-\Delta_r$ is also onto. Then, the problem

$$\begin{cases} -\Delta_r u_\varepsilon = f_\varepsilon & \text{in } \mathscr{D}'(\Omega_\varepsilon), \\ u_\varepsilon \in E_\varepsilon, \end{cases}$$

has one (and only one) solution. Using the estimates given by the above lemmas, we have

$$\sum_{j=0}^{2} |u_\varepsilon|^2_{j,\Omega_\varepsilon} + \left|\frac{1}{r}\frac{\partial u_\varepsilon}{\partial r}\right|^2_{0,\Omega_\varepsilon} \leqslant c^2 |f_\varepsilon|^2_{0,\Omega_\varepsilon} \leqslant c^2 \|f\|_{L^2_1(\Omega)}, \tag{2.6}$$

where c is a constant independent of f and $\varepsilon > 0$.

Extending by 0 all the partial derivatives of u_ε to Ω, and using the well-known properties of weak compactness of finite balls in Hilbert spaces, as well as the continuity of partial derivatives in $\mathscr{D}'(\Omega)$, we obtain that there exists a function u in $D^2_1(\Omega)$ which satisfies $-\Delta_r u = f$ in $\mathscr{D}'(\Omega)$.

The compactness of the injection of $H^2_{\text{loc}}(\Omega)$ into $H^1_{\text{loc}}(\Omega)$ implies also that u is in V (hence, it is in $E(\Omega)$). Finally, the inferior semi-continuity of the norm of $L^2_1(\Omega)$ insures that

$$\|u\|_{D^2_1(\Omega)} \leqslant c\,\|f\|_{L^2_1(\Omega)},$$

where c is a constant depending only on the diameter of Ω by Poincaré's inequality. The constant of Hardy's inequality is independent of Ω. ∎

3. ERROR ESTIMATE

The solution u of problem (1.2) belongs to $D^2_1(\Omega)$ if f is in $L^2_1(\Omega)$; hence, it is continuous on $\Omega \cup \Gamma$. This enables us to define the interpolate of u by.

$$\left.\begin{array}{l}\pi^h u \in V^h ,\\ \pi^h(a) = u(a); \quad \forall a \in \mathcal{N}\end{array}\right\}. \tag{3.1}$$

Thus, to estimate $\| u - u^h \|_{H_1^1(\Omega)}$, we need only approximating the interpolate error $\| u - \pi^h u \|_{H_1^1(\Omega)}$.

Let us denote by h_K and ρ_K the diameter of $K \in \mathcal{T}^h$ and the maximum of radius of the balls inscribed in K, respectively. Assume the regularity hypothesis on the mesh $\mathcal{T}^h$:

$$\max_{K \in \mathcal{T}^h} \frac{h_K}{\rho_K} \leqslant c , \tag{3.2}$$

where c is a constant independent of h and $h = \max\limits_{K \in \mathcal{T}^h} h_K$. Then we have :

THEOREM 3.1 : *Under the condition* (3.2), *if u is the solution of problem* (1.2) *with the data f in $L_1^2(\Omega)$ and if u^h denotes the solution of the discrete problem* (1.4), *we have*

$$\| u - u^h \|_{H_1^1(\Omega)} \leqslant ch \| f \|_{L_1^2(\Omega)}$$

where c is a constant independent of h and f in $L_1^2(\Omega)$. ■

Standard techniques give the proof, using the following lemmas.

LEMMA 3.2 : *Let $K =]0, h[\times]0, h[$. There exists a constant c independent of v in $D_1^2(K)$ and of h such that*

$$| v - \pi v |_{j,K} \leqslant ch^{2-j}\left(| v |_{2,K}^2 + \left| \frac{1}{r} \frac{\partial v}{\partial r} \right|_{0,K}^2\right)^{1/2} ; \quad j = 0,1 . \tag{3.3}$$

Proof : We set $\hat{K} =]0, 1[\times]0, 1[$. With a linear change of each variable, the proof is a consequence of

There exists a constant c independent of $\hat{v} \in D_1^2(\hat{K})$ such that

$$\| \hat{v} \|_{D_1^2(\hat{K})}^2 \leqslant c^2 \left(| \hat{v}(\hat{a}_1) |^2 + | \hat{v}(\hat{a}_2) |^2 + | \hat{v} |_{2,K}^2 + \left| \frac{1}{r} \frac{\partial \hat{v}}{\partial r} \right|_{0,\hat{K}}^2 \right), \tag{3.4}$$

where $\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (1, 1)$ are the vertices of $\hat{K}$ out of the z-axis. To prove (3.4), we assume the contrary. Then, there exists a sequence $\{ \hat{v}_n \}$ such that

$$\| \hat{v}_n \|_{D_1^2(\hat{K})} = 1 ; \tag{3.5}$$

$$\lim | \hat{v}_n(\hat{a}_i)| = 0 ; i = 1, 2 ; \tag{3.6}$$

$$\lim | \hat{v}_n |_{2,\hat{K}} = 0 ; \tag{3.7}$$

$$\lim \left| \frac{1}{r} \frac{\partial \hat{v}_n}{\partial r} \right|_{0,\hat{K}} = 0 . \tag{3.8}$$

Using a subsequence, we can assume by (3.5) that there exists a function $\hat{v}$ in $D_1^2(\hat{K})$ such that

$$\lim \hat{v}_n = \hat{v} \quad \text{weakly in} \quad D_1^2(K) .$$

Using (3.7), we obtain

$$\hat{v} = \alpha r + \beta z + \gamma, \quad \alpha, \beta, \gamma \in \mathbb{R} .$$

(3.8) gives $\alpha = 0$ and (3.6), $\beta = \gamma = 0$. Therefore, the sequence $\{ \hat{v}_n \}$ converges weakly to 0 in $D_1^2(\hat{K})$. The compactness of the unit ball of $H_1^2(\hat{K})$ in $H_1^1(\hat{K})$ (see e.g. El Kolli [5]) gives

$$\lim \hat{v}_n = 0 \quad \text{in} \quad H_1^1(\hat{K}) \text{ strongly} .$$

But this result, together with (3.7) and (3.8) contradicts (3.5). Thus, (3.4) and the lemma are proved. ■

LEMMA 3.3 : *Let* $K =]\bar{r}, \bar{r} + h[\times]0, h[$

$$h \leqslant c\bar{r}, \tag{3.9}$$

where c is independent of h and $\bar{r}$. *Then, there exists a constant c independent of h and* $\bar{r}$ *such that*

$$| v - \pi v |_{j,K} \leqslant ch^{2-j} \left(| v |_{2,K}^2 + \left| \frac{1}{r} \frac{\partial v}{\partial r} \right|_{0,K}^2 \right)^{1/2} ; \quad j = 0, 1 ,$$

for all v in $D_1^2(K)$. ■

Proof : We prove the assertion for $j = 1$, the proof for $j = 0$ being analogous. We set (Euler's variable change) :

$$r = e^s ; \qquad z = \mu t ,$$

where μ is a constant chosen in such a way that K is transformed in a square $\hat{K}$, i.e.,

$$\mu = h/\log\left(1 + \frac{h}{\bar{r}}\right) = \bar{r} + \theta h ; \quad 0 < \theta < 1 .$$

$w = v - \pi v$ is transformed in $\hat{w}$ by $w(r, z) = \hat{w}(s, t)$ and $\hat{w} = \hat{v} - \hat{\pi}\hat{v}$, $\hat{\pi}$ being the usual interpolation operator by polynomials of partial degree less than 1 from the values on the vertex. Then, we have

$$| w |_{1,\hat{K}}^2 = \mu \int_{\hat{K}} \left(\left| \frac{\partial \hat{w}}{\partial s} \right|^2 + \left(\frac{e^s}{\mu} \right)^2 \left| \frac{\partial \hat{w}}{\partial t} \right|^2 \right) ds\, dt\,.$$

Since

$$\frac{e^s}{\mu} \leqslant \frac{(\bar{r} + h)}{\bar{r}} \leqslant c\,;\ c \text{ independent of } h \text{ and } \bar{r}\,;$$

we obtain

$$| w |_{1,K}^2 \leqslant c\mu \int_{\hat{K}} \left(\left| \frac{\partial \hat{w}}{\partial s} \right|^2 + \left| \frac{\partial \hat{w}}{\partial t} \right|^2 \right) ds\, dt\,.$$

Using some standard results of error interpolation (see Ciarlet [4], Raviart [9]), we have

$$| w |_{1,K}^2 \leqslant c^2 \log^2 \left(1 + \frac{h}{\bar{r}} \right) \mu \int_{\hat{K}} \left(\left| \frac{\partial^2 \hat{v}}{\partial s^2} \right|^2 + \left| \frac{\partial^2 \hat{v}}{\partial s\, \partial t} \right|^2 + \left| \frac{\partial^2 \hat{v}}{\partial t^2} \right|^2 \right) ds\, dt\,.$$

Taking again the previous variables, we obtain

$$| w |_{1,K}^2 \leqslant c^2 \log^2 \left(1 + \frac{h}{\bar{r}} \right) \times$$
$$\times \int_{\hat{K}} \left\{ r^2 \left| \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) \right|^2 + \mu^2 \left| \frac{\partial^2 v}{\partial r\, \partial z} \right| + \left(\frac{\mu}{r} \right)^2 \mu^2 \left| \frac{\partial^2 v}{\partial z^2} \right|^2 \right\} r\, dr\, dz\,.$$

Using

$$\frac{\mu}{r} \leqslant \frac{\bar{r} + \theta h}{\bar{r}} \leqslant c\,;\ c \text{ constant independent of } h \text{ and } \bar{r}\,,$$

and with some easy majorations, we come to

$$| w |_{1,K}^2 \leqslant c^2 (\bar{r} + h)^2 \log^2 \left(1 + \frac{h}{\bar{r}} \right) \left(| v |_{2,K}^2 + \left| \frac{1}{r} \frac{\partial v}{\partial r} \right|_{0,K}^2 \right),$$

where c is a constant independent of h, $\bar{r}$ and v. Finally, (3.10) follows from

$$\bar{r} \left(1 + \frac{h}{\bar{r}} \right) \log \left(1 + \frac{h}{\bar{r}} \right) \leqslant \left(1 + \frac{h}{\bar{r}} \right) h\,. \quad \blacksquare$$

LEMMA 3.4 : *If K is a triangle satisfying the following conditions : h_K : diameter of K less than h ; least interior angle of K greater than $\theta_0 > 0$; θ_0 independent of h; K : contained in the half space $\{(r, z) \in \mathbb{R}^2 ; r \geqslant R\}$, R independent of h ; then, there exists a constant c independent of h such that*

$$| v - \pi v |_{j,K} \leqslant c h^{2-j} | v |_{2,K} ; \quad j = 0, 1 , \tag{3.11}$$

for all v in $D_1^2(K)$. ■

Proof : It is sufficient to remark that both semi-norms with and without weight are uniformly equivalent far from the z-axis and then to apply standard error interpolation estimates. ■

The proof of theorem 3.1 results from the above lemmas by standard techniques of majoration of the error over each element which can be reduced by a linear affine transformation keeping the estimates under assumption (3.2) to the cases (3.3), (3.10) and (3.11). ■

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