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CONVERGENCE OF AN ACCURATE SCHEME FOR FIRST ORDER QUASI LINEAR EQUATIONS (*)

by Alain Yves LE ROUX (1)

Communique par P-A RAVIART

Résumé — *La convergence d'un schéma quasi d'ordre deux, s'appliquant aux équations quasi linéaires du premier ordre est démontrée. Il s'agit du schéma de Godunov, corrigé par le schéma de Lax-Wendroff, chaque fois que la stabilité n'est pas compromise, c'est-à-dire partout sauf au niveau des changements de monotonie, en espace, de la solution. La technique reprend les travaux de D L Book, J P Boris et K Ham, en ce qui concerne la phase de correction. Il est également établi que la solution obtenue à la limite satisfait à la condition d'entropie, ce qui assure l'unicité.*

Abstract — *We prove the convergence of a quasi order two accuracy scheme applied to quasi linear first order equations. We use the scheme of Godunov, corrected by that of Lax-Wendroff when stability holds, i.e., everywhere except near the points where the solution loses locally its spatial monotony. This correction technique is based on the work of D L Book, J P Boris, K Ham. We prove that the solution obtained in the limit satisfies the entropy condition, which implies uniqueness.*

1. STATEMENT OF THE PROBLEMS

Let $T > 0$, we shall consider the two following problems, for a given function f , in $C^1(\mathbb{R})$. The first one is the Cauchy problem on $\mathbb{R} \times]0, T[$ with a prescribed initial condition u_0 in $L^\infty(\mathbb{R})$ assumed to be of locally bounded variation on \mathbb{R} : find a bounded function u satisfying the quasilinear equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad (1)$$

for (x, t) in $\mathbb{R} \times]0, T[$, and

$$u(x, 0) = u_0(x), \quad (2)$$

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for x in \mathbb{R} . The second one is posed on $]0, 1[\times]0, T[$ with initial and boundary conditions in space BV of bounded variation functions, and consists of finding a bounded function u satisfying equation (1) on $]0, 1[\times]0, T[$. Initial condition (2) on $]0, 1[$, with u_0 in $BV(]0, 1[)$, and when such a condition is required on the boundaries

$$u(0, t) = a(t), \quad u(1, t) = b(t), \tag{3}$$

for t in $]0, T[$, a and b in $BV(]0, T[)$.

It is well known that a weak solution of problem (1), (2), that is a function in $L^\infty(\mathbb{R} \times]0, T[)$ satisfying

$$\int \int_{\mathbb{R} \times]0, T[} \left(u \frac{\partial \phi}{\partial t} + f(u) \frac{\partial \phi}{\partial x} \right) dx dt + \int_{\mathbb{R}} u_0(x) \phi(x, 0) dx = 0 \tag{4}$$

for all test function ϕ in $C^2(\mathbb{R} \times [0, T[)$ with compact support in $\mathbb{R} \times [0, T[$, may be discontinuous and then Uniqueness may fail. Similar disadvantages arise for problem (1), (2), (3) on $]0, 1[\times]0, T[$. The correct solution is given by the vanishing viscosity method, as the limit of the solution u_ε of the parabolic equation

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u_\varepsilon) = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2}, \quad \varepsilon > 0, \tag{5}$$

when ε tends to zero, and is characterized by the following definition from S. N. Kruskov [4].

DÉFINITION 1 : $u \in L^\infty(\mathbb{R} \times]0, T[)$ is a solution of problem (1), (2) when it satisfies

$$\int \int_{\mathbb{R} \times]0, T[} \left\{ |u - k| \frac{\partial \phi}{\partial t} + \text{sg}(u - k) (f(u) - f(k)) \frac{\partial \phi}{\partial x} \right\} dx dt \geq 0 \tag{6}$$

for all $k \in \mathbb{R}$, and all non negative test functions $\phi \in C^2(\mathbb{R} \times]0, T[)$ with compact support, and when a negligible set $\mathcal{E} \subset]0, T[$ exists such that, for all $R > 0$,

$$\lim_{t \in \mathcal{E}, t \rightarrow 0} \int_{|x| < R} |u(x, t) - u_0(x)| dx = 0. \tag{7}$$

The sign function has been denoted by sg , defined by $\text{sg}(0) = 0$, $\text{sg}(x) = x/|x|$ for $x \neq 0$. In [1], or [8], this definition is extended to problem (1), (2), (3) as follows; γu is the trace of u on the boundary of $]0, 1[\times]0, T[$, which exists since u is of bounded variation on $]0, 1[\times]0, T[$ in the sense of Tonelli Cesari,

that is of bounded variation on $]0, 1[$ (resp. $]0, T[$) as function of x (resp. t) for almost all t in $]0, T[$ (resp. x in $]0, 1[$).

DÉFINITION 2 : $u \in BV(]0, 1[\times]0, T[)$ is a solution of problem (1), (2), (3) when it satisfies the initial condition (2) almost everywhere on $]0, 1[$, and

$$\begin{aligned} \iint_{]0, 1[\times]0, T[} \left\{ |u - k| \frac{\partial \phi}{\partial t} + \operatorname{sg}(u - k) (f(u) - f(k)) \frac{\partial \phi}{\partial x} \right\} dx dt &\geq \\ &\geq \int_0^T [\operatorname{sg}(a - k) (f(\gamma u(0, t)) - f(k)) \phi(0, t) - \\ &\quad - \operatorname{sg}(b - k) (f(\gamma u(1, t)) - f(k)) \phi(1, t)] dt, \quad (8) \end{aligned}$$

for all $k \in \mathbb{R}$ and all non negative test functions ϕ in $]0, 1[\times]0, T[$, with compact support.

Another definition, which is equivalent to the one above, may be stated by writing (8) for ϕ equal to zero at $x = 0$ and $x = 1$, and assuming that

$$\begin{aligned} \operatorname{Sup}_{k \in I(a, \gamma u(0, t))} \{ \operatorname{sg}(a - k) (f(\gamma u(0, t)) - f(k)) \} &= 0 \quad (\text{a.e. }]0, T[), \\ \operatorname{Inf}_{k \in I(b, \gamma u(1, t))} \{ \operatorname{sg}(b - k) (f(\gamma u(1, t)) - f(k)) \} &= 0 \quad (\text{a.e. }]0, T[). \end{aligned} \quad (9)$$

Here $I(\alpha, \beta)$ denotes the closed interval the bounds of which are α and β . All of these definitions may be generalized to similar problems for the multi-dimensional equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(f(u, x, t)) + g(u, x, t) = 0, \quad (10)$$

on an open set for the boundary value problem. Existence and Uniqueness for such problems are proved, the solutions of which always satisfy the entropy condition (E) of O. A. Oleinik (see [11]) on discontinuities

$$\frac{f(u(x + 0, t)) - f(u(x - 0, t))}{u(x + 0, t) - u(x - 0, t)} \geq \frac{f(u(x + 0, t)) - f(k)}{u(x + 0, t) - k} \quad (11)$$

for all k in $I(u(x + 0, t), u(x - 0, t))$. For problem (1), (2), we can also state that this solution is of locally bounded variation on $\mathbb{R} \times]0, T[$.

Approximation will be first studied for increasing functions f , which is a simpler case, and for which two different corrections are possible, one converging towards a weak solution (section 2) and another towards the solution

characterized by definition 1 (section 3), for problem (1), (2). Approximations in the general case, and for problem (1), (2), (3), are treated in sections 4 and 5. Some comments are given in section 6.

2. APPROXIMATION OF WEAK SOLUTIONS OF PROBLEM (1), (2) FOR INCREASING f

Let $h > 0$ be the space meshsize ; the time meshsize is taken equal to qh , for a fixed positive real number q which will control the stability. A solution u of problem (1), (2) is approached by a function u_h defined on $\mathbb{R} \times]0, T[$, which has a constant value u_i^n on each set

$I_i \times J_n =](i - 1/2) h, (i + 1/2) h[\times](n - 1/2) qh, (n + 1/2) qh[$ for $i \in \mathbb{Z}$, $n \in \mathbb{N}$, $n \leq N = [T/qh] + 1$. Initial condition (2) is introduced through a L^2 projection on the space of functions with constant values on each I_i

$$u_i^0 = \frac{1}{h} \int_{I_i} u_0(x) dx . \tag{12}$$

The other values u_i^n are computed by using an explicit three points scheme. We shall combine two such schemes to obtain one which has an accuracy of order two as often as possible, and which we call a quasi order two scheme. An analysis of three points schemes is to be found in [3] or [6]. The procedure we follow to build such a quasi order two scheme mimics the technique developed in [2]. Other quasi order two schemes are proposed and analysed in [12].

In order to simplify some notations, the function f is assumed to be increasing in this section and in the next one ; this hypothesis will be suppressed later. We consider the decentered scheme (also called Donnor Cell, or Upwind Scheme)

$$u_i^{n+1} = u_i^n - q[f(u_i^n) - f(u_{i-1}^n)] . \tag{13}$$

This is a scheme whose accuracy is of order one, which is convergent (see [6]) and gives relatively good profiles for shocks under the Courant-Friedrichs-Lewy stability condition

$$qf'(k) \leq 1 \quad \text{for all } |k| \leq |u_0|_{L^\infty(\mathbb{R})} . \tag{14}$$

It is well known that the one step Lax-Wendroff scheme

$$u_i^{n+1} = u_i^n - q(f(u_{i+1}^n) - f(u_{i-1}^n))/2 + \frac{q^2}{2} [f'(\xi_{i+1/2}^n) (f(u_{i+1}^n) - f(u_i^n)) - f'(\xi_{i-1/2}^n) (f(u_i^n) - f(u_{i-1}^n))] \tag{15}$$

with $\xi_{i+1/2}^n$ defined in $I(u_i^n, u_{i+1}^n)$ by

$$f(u_{i+1}^n) - f(u_i^n) = f'(\xi_{i+1/2}^n) (u_{i+1}^n - u_i^n),$$

has an accuracy of order two, but is unstable for the L^∞ -norm, even when (14) is satisfied.

Examples of wrong convergence, towards weak solutions that do not satisfy the entropy condition (11) may be found in [3] or in [6]. The scheme (15) may be written in the form

$$u_i^{n+1} = u_i^n - q(f(u_i^n) - f(u_{i-1}^n)) - \tilde{a}_{i+1/2}^n + \tilde{a}_{i-1/2}^n, \tag{16}$$

for comparison with (13). The coefficients $\tilde{a}_{i+1/2}^n$ are defined by

$$\tilde{a}_{i+1/2}^n = \frac{q}{2} (1 - qf'(\xi_{i+1/2}^n)) (f(u_{i+1}^n) - f(u_i^n)). \tag{17}$$

Provided the C.F.L. condition (14) is satisfied, the difference $\tilde{a}_{i+1/2}^n - \tilde{a}_{i-1/2}^n$ is similar to a viscosity or a diffusion term. This shows that passing from the upwind scheme (13) to the Lax-Wendroff (15) is realized through a diminution of the amount of viscosity implicitly contained in the first scheme. From this remark, this step will be called the antidiffusion phase, as in [2] or [12].

We have now to correct the scheme (16), (17), which is exactly the Lax-Wendroff scheme, in such a way that stability will be preserved. We take

$$a_{i+1/2}^n = \text{sg}(u_{i+1}^n - u_i^n) \text{Max} \left\{ 0, \text{Min} \left[|\tilde{a}_{i+1/2}^n|, \frac{1}{2} \text{sg}(u_{i+1}^n - u_i^n) (u_i^n - u_{i-1}^n) \right] \right\}, \tag{18}$$

with $\tilde{a}_{i+1/2}^n$ defined in (17), and then write (16) with these quantities

$$u_i^{n+1} = u_i^n - q(f(u_i^n) - f(u_{i-1}^n)) - a_{i+1/2}^n + a_{i-1/2}^n. \tag{19}$$

The diffusion term ($a_{i+1/2}^n - a_{i-1/2}^n$) we take off is smaller than previously, since we always have $|a_{i+1/2}^n|$ less than $|\tilde{a}_{i+1/2}^n|$. It can easily be seen that $a_{i+1/2}^n$ is equal to zero when monotony is changing, since we have in such a case

$$\text{sg}(u_{i+1}^n - u_i^n) (u_i^n - u_{i-1}^n) \leq 0.$$

The coefficient 1/2 before this last term is introduced to preserve the monotony. Note that, when u is smooth enough, $|u_i^n - u_{i-1}^n|$ is close to $|u_{i+1}^n - u_i^n|$, and in (18) the coefficient of the first is 1/2, while it is always less than 1/8 for the second, provided that (14) is true. This shows that $|\tilde{a}_{i+1/2}^n|$ will often

realize the minimum, and then (19) is a scheme near the Lax-Wendroff one. This is the reason why (17), (18), (19) is called a quasi order two scheme. The scheme proposed in [2] differs little from this one ; the correction is performed on the values computed by (13) instead of the u^n , and the coefficient 1/2 in (18) is missing.

We shall prove the following result with more restricted stability condition than (14).

THEOREM 1 : *Suppose the stability condition*

$$q \sup_{|k| \leq |u_0|_{L^\infty(\mathbb{R})}} |f'(k)| \leq \frac{1}{2}, \tag{20}$$

and assume that f is an increasing function and u_0 lies in $L^\infty(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$, then the family $\{u_h\}$ built by (12) and the scheme (17), (18), (19) contains a sequence $\{u_{h_m}\}_m$ which converges in $L^1_{loc}(\mathbb{R} \times]0, T])$, as h_m tends to zero, towards a weak solution of the problem (1), (2).

Proof : We first state that the L^∞ -norm is preserved. For each $i \in \mathbb{Z}$, $n \in \{0, \dots, N\}$ we introduce the real numbers $\lambda_{i+1/2}^n, \mu_{i+1/2}^n$ defined by

$$2 a_{i+1/2}^n = \lambda_{i+1/2}^n (u_i^n - u_{i-1}^n) = \mu_{i+1/2}^n (1 - qf'(\xi_{i+1/2}^n)) q(f(u_{i+1}^n) - f(u_i^n)). \tag{21}$$

We have from (17), (18) and (20) that they both belong to $[0, 1]$. Scheme (19) becomes

$$u_i^{n+1} = u_i^n \left[1 - \frac{1}{2} \lambda_{i+1/2}^n - \frac{q}{2} f'(\xi_{i-1/2}^n) [2 - \mu_{i-1/2}^n (1 - qf'(\xi_{i-1/2}^n))] \right] + u_{i-1}^n \left[\frac{1}{2} \lambda_{i+1/2}^n + \frac{q}{2} f'(\xi_{i-1/2}^n) [2 - \mu_{i-1/2}^n (1 - qf'(\xi_{i-1/2}^n))] \right],$$

where both coefficients of u_i^n and u_{i-1}^n are non negative from (20). We deduce now

$$\text{Min}(u_i^n, u_{i-1}^n) \leq u_i^{n+1} \leq \text{Max}(u_i^n, u_{i-1}^n),$$

which gives, by induction,

$$|u_h|_{L^\infty(\mathbb{R} \times \{0, T\})} \leq |u_0|_{L^\infty(\mathbb{R})}. \tag{22}$$

To get the strong convergence (needed by the nonlinear term), we state that the locally bounded variation property is preserved by the scheme. We have

$$u_{i+1}^{n+1} - u_i^{n+1} = (u_{i+1}^n - u_i^n) \left[1 - \frac{1}{2} \lambda_{i+3/2}^n - \frac{q}{2} f'(\xi_{i+1/2}^n) [2 - \mu_{i+1/2}^n (1 - qf'(\xi_{i+1/2}^n))] \right] + (u_i^n - u_{i-1}^n) \left[\frac{1}{2} \lambda_{i+1/2}^n + \frac{q}{2} f'(\xi_{i-1/2}^n) [2 - \mu_{i-1/2}^n (1 - qf'(\xi_{i-1/2}^n))] \right],$$

where the two coefficients of $(u_{i+1}^n - u_i^n)$ and $(u_i^n - u_{i-1}^n)$ are non negative from (20). For any integer I , we take the absolute values and sum for $|i| \leq I$, to get for all $n \leq N$.

$$\sum_{|i| \leq I} |u_{i+1}^{n+1} - u_i^{n+1}| \leq \sum_{|i| \leq I+1} |u_{i+1}^n - u_i^n| \leq \dots \leq \sum_{|i| \leq I+n} |u_{i+1}^0 - u_i^0|. \tag{23}$$

For all $R > 0$, we set $I = [R/h]$, and then this last term is bounded by the variation of u_0 on $] - R, R[$, which is finite.

Furthermore, we have from (19)

$$|u_i^{n+1} - u_i^n| \leq |u_i^n - u_{i-1}^n|, \tag{24}$$

and then an estimation on time variation may be derived from (23). We apply now Helly's theorem, which states that a sequence $\{u_{h_m}\}_m$ can be extracted out of the family $\{u_h\}$, and converges towards a function u in $L^1_{loc}(\mathbb{R} \times]0, T[)$. Obviously u belongs to $L^\infty(\mathbb{R} \times]0, T[)$.

We introduce now a test function $\phi \in C^2(\mathbb{R} \times [0, T])$, with compact support, and approach it by its L^2 projection on the space of constant functions on each set $I_i \times J_n$

$$\phi_h(x, y) = \phi_i^n = \frac{1}{qh^2} \iint_{I_i \times J_n} \phi(x, t) dx dt \quad \text{if } (x, t) \in I_i \times J_n.$$

After multiplying (19) by ϕ_i^n and summing, we get

$$\left| \sum_i \sum_n [u_i^n (\phi_i^n - \phi_i^{n-1}) + qf(u_i^n) (\phi_{i+1}^n - \phi_i^n)] h + \sum_i u_i^0 \phi_i^0 h \right| \leq \sum_i \sum_n |a_{i+1/2}^n| |\phi_{i+1}^n - \phi_i^n| h. \tag{25}$$

From (18) and (23), we have

$$\sum_i \sum_n |a_{i+1/2}^n| |\phi_{i+1}^n - \phi_i^n| h \leq \frac{T}{8q} \left| \frac{\partial \phi}{\partial x} \right|_{L^\infty} \sum_{|i| \leq I+N} |u_{i+1}^0 - u_i^0| h,$$

where I is such that the support of ϕ lies in $] - Ih, Ih[\times]0, T[$ and Ih is bounded.

It follows that the first member of (25) tends to zero, which gives (4) at the limit for $h = h_m$ tending towards zero. Theorem 1 is proved.

Note that we can deduce from (22), (23), (24) that u verifies the initial condition (2) in a strong sense, and that u has a locally bounded variation on $\mathbb{R} \times]0, T[$.

Although no counter-example is known by the author, of a convergence of the scheme (17), (18), (19) towards a wrong weak solution, additional assumptions will appear to get the convergence towards the solution characterized by definition 1.

3. APPROXIMATION OF THE RIGHT SOLUTION OF PROBLEM (1), (2)

The function f is still supposed to be increasing. We note that a second order of accuracy is a nonsense when the solution is not smooth enough. More precisely, the coefficients $a_{i+1/2}^n$ defined in (18) have the same order than h , if u lies in C^1 . We shall impose only that they tend to zero by writing for some $C > 0$ and α in $]0, 1[$,

$$|a_{i+1/2}^n| \leq Ch^\alpha. \quad (26)$$

For instance, $C = 2$ and $\alpha = 1/6$ make this constraint active only for h less than 10^{-7} in classical examples. A program of this numerical method has never to test (26) in practice.

We also change the definition of $a_{i+1/2}^n$ by introducing qf into the minimum ; we shall take

$$a_{i+1/2}^n = \text{sg}(u_{i+1}^n - u_i^n) \times \\ \times \text{Max} \{ 0, \text{Min} [| \tilde{a}_{i+1/2}^n |, Ch^\alpha, \text{sg}(u_{i+1}^n - u_i^n) q(f(u_i^n) - f(u_{i-1}^n))] \}. \quad (27)$$

The scheme (17), (27), (19) remains of quasi order two, since the coefficient of $q | f(u_{i+1}^n) - f(u_i^n) |$, arising in $\tilde{a}_{i+1/2}^n$ in (27), is equal to $(1 - qf'(\xi_{i+1/2}^n))/2$, which is less than $1/2$, while the coefficient of $q | f(u_i^n) - f(u_{i-1}^n) |$ is one if the monotony does not change at the point ih .

Another hypothesis will be done on the initial condition u_0 . We suppose that for all real $R > 0$, a finite sequence $\{x_i\}$ of $K(R)$ terms exists such that

$$x_1 = -R < x_2 < x_3 < \dots < x_{K(R)} = R,$$

and u_0 is monotonous on each interval $]x_{i-1}, x_i[$. We shall say that u_0 has a locally finite number of local extrema. This property is a little more restrictive than the local bounded variation, but will not induce any change in practice ; it will be preserved by the scheme (17), (27), (19) when (20) is satisfied.

THEOREM 2 : *Suppose that the stability condition (20) is verified, and assume that f is an increasing function, and that u_0 has a locally finite number of local extrema, then the whole family $\{ u_h \}$ built by the scheme (17), (27), (19) converges towards the solution of problem (1), (2) characterized by definition 1, as h tends to zero.*

Proof : The estimates (22), (23), (24) are proved in the same way than for theorem 1, since $a_{i+1/2}^n$ constructed by (27) is, in absolute value, less than this one given in (18). Thus we get the convergence in $L^1_{loc}(\mathbb{R} \times]0, T[)$ of a sequence $\{ u_{h_m} \}_m$, towards a function u which is a weak solution of problem (1), (2). It remains to prove that u satisfies definition 1, since Uniqueness ensures the convergence of the whole family $\{ u_h \}$. The initial condition (2) is verified in a strong sense by the trace of u at $t = 0$, as in a previous remark, and thus is (7) stated.

Let $k \in \mathbb{R}$, and ϕ a non negative test function as required in (6). We have

$$u_i^{n+1} - k = u_i^n - k - q\gamma_i^n \{ f(u_i^n) - f(k) - (f(u_{i-1}^n) - f(k)) \},$$

for all (i, n) , (28)

with

$$\gamma_i^n = 1 + v_{i+1/2}^n - v_{i-1/2}^n (1 - qf'(\xi_{i-1/2}^n))/2,$$

where $v_{i+1/2}^n$ is defined by

$$a_{i+1/2}^n = v_{i+1/2}^n q(f(u_i^n) - f(u_{i-1}^n))$$

and $\mu_{i-1/2}^n$ as in (21). Note that we have, from (20),

$$0 \leq \lambda_{i+1/2}^n = 2 qf'(\xi_{i-1/2}^n) v_{i+1/2}^n \leq v_{i+1/2}^n \leq 1,$$

which gives

$$\frac{1}{2} \leq \gamma_i^n \leq 2.$$

We now define k_i^n in $I(k, u_i^n)$ by

$$f(u_i^n) - f(k) = f'(k_i^n) (u_i^n - k),$$

and then (28) becomes

$$u_i^{n+1} - k = (u_i^n - k) (1 - qf'(k_i^n) \gamma_i^n) + (u_{i-1}^n - k) qf'(k_{i-1}^n) \gamma_i^n.$$

The two coefficients of $(u_i^n - k)$ and $(u_{i-1}^n - k)$ are non negative, for

$$|k| \leq |u_0|_{L^\infty}.$$

Taking the absolute values we obtain :

$$|u_i^{n+1} - k| \leq |u_i^n - k| - q \{ |f(u_i^n) - f(k)| - |f(u_{i-1}^n) - f(k)| \} \gamma_i^n. \quad (29)$$

For $|k| > |u_0|_{L^\infty}$, this inequality is directly obtained by multiplying by $\text{sg}(u_i^{n+1} - k)$. We now introduce the non negative quantities $S_{i+1/2}^n$ defined by

$$S_{i+1/2}^n = \begin{cases} 1 & \text{if } k \geq u_{i+1}^n \geq u_i^n \text{ or } k \leq u_{i+1}^n \leq u_i^n, \\ q \frac{f(k) - f(u_i^n)}{f(u_{i+1}^n) - f(u_i^n)} & \text{if } k \in I(u_i^n, u_{i+1}^n), \\ 0 & \text{if } k \geq u_i^n \geq u_{i+1}^n \text{ or } k \leq u_i^n \leq u_{i+1}^n. \end{cases} \quad (30)$$

We take it equal to $1/2$ for $k = u_i^n = u_{i+1}^n$. We can write now

$$|f(u_{i+1}^n) - f(k)| - |f(u_i^n) - f(k)| = (1 - 2S_{i+1/2}^n) |f(u_{i+1}^n) - f(u_i^n)|,$$

and show the inequality

$$\mu_{i+1/2}^n (S_{i-1/2}^n - S_{i+1/2}^n) \geq 0, \quad (31)$$

which is obvious when one of the two factors is zero. Otherwise, k belongs to one of the intervals $I(u_i^n, u_{i-1}^n)$ or $I(u_i^n, u_{i+1}^n)$. In the first case $S_{i+1/2}^n$ is zero, and in the second $S_{i-1/2}^n$ is one, greater than $S_{i+1/2}^n$ by (20), thus (31) follows.

We take out the diffusion from (29), by writing

$$\begin{aligned} |f(u_i^n) - f(k)| - |f(u_{i-1}^n) - f(k)| &= \\ &= \frac{1}{2} [|f(u_{i+1}^n) - f(k)| - |f(u_{i-1}^n) - f(k)|] \\ &\quad - \frac{1}{2} [|f(u_{i+1}^n) - f(k)| - 2|f(u_i^n) - f(k)| + |f(u_{i-1}^n) - f(k)|]. \end{aligned}$$

We get :

$$\begin{aligned} |u_i^{n+1} - k| &\leq |u_i^n - k| - \frac{q}{2} \{ |f(u_{i+1}^n) - f(k)| - |f(u_{i-1}^n) - f(k)| \} + \\ &+ \frac{q}{2} |f(u_{i+1}^n) - f(u_i^n)| (1 - 2S_{i+1/2}^n) - q\nu_{i+1/2}^n |f(u_i^n) - f(u_{i-1}^n)| (1 - 2S_{i-1/2}^n) \\ &- \frac{q}{2} |f(u_i^n) - f(u_{i-1}^n)| (1 - 2S_{i-1/2}^n) [1 - \mu_{i-1/2}^n (1 - qf'(\xi_{i-1/2}^n))]. \end{aligned} \quad (32)$$

Define $\beta_{i+1/2}^n$ by

$$\beta_{i+1/2}^n = \frac{q}{2} f'(\xi_{i+1/2}^n) [1 - \mu_{i+1/2}^n (1 - qf'(\xi_{i+1/2}^n))] (1 - 2S_{i+1/2}^n),$$

then we get

$$|\beta_{i+1/2}^n| \leq 1. \tag{33}$$

Note that

$$qv_{i+1/2}^n |f(u_i^n) - f(u_{i-1}^n)| = \frac{q}{2} |f(u_{i+1}^n) - f(u_i^n)| (1 - qf'(\xi_{i+1/2}^n)) \mu_{i+1/2}^n,$$

then the second line of (32) may be written

$$\beta_{i+1/2}^n |u_{i+1}^n - u_i^n| + \mu_{i+1/2}^n (S_{i-1/2}^n - S_{i+1/2}^n) qf'(\xi_{i+1/2}^n) (1 - qf'(\xi_{i+1/2}^n)).$$

Thus (32) becomes now

$$\begin{aligned} |u_i^{n+1} - k| &\leq |u_i^n - k| - \frac{q}{2} \{ |f(u_{i+1}^n) - f(k)| - |f(u_i^n) - f(k)| \} + \\ &\quad + \beta_{i+1/2}^n |u_{i+1}^n - u_i^n| - \beta_{i-1/2}^n |u_i^n - u_{i-1}^n| \\ &\quad + q |f(u_{i+1}^n) - f(u_i^n)| (1 - qf'(\xi_{i+1/2}^n)) \mu_{i+1/2}^n (S_{i-1/2}^n - S_{i+1/2}^n), \end{aligned}$$

which is a discrete analogue of (6). We multiply by ϕ_i^n , defined as for theorem 1, and sum for $i \in \mathbb{Z}$, $n \in \{0, \dots, N\}$ to get at the limit for h in $\{h_m\}_m$

$$\begin{aligned} \int \int_{\mathbb{R} \times]0, T[} \left[|u - k| \frac{\partial \phi}{\partial t} + \text{sg}(u - k) (f(u) - f(k)) \frac{\partial \phi}{\partial x} \right] dx dt &\geq \\ &\geq \lim_{h \rightarrow 0} \sum_i \sum_n \beta_{i+1/2}^n |u_{i+1}^n - u_i^n| (\phi_{i+1}^n - \phi_i^n) h - \\ - \lim_{h \rightarrow 0} \sum_i \sum_n |f(u_{i+1}^n) - f(u_i^n)| (1 - qf'(\xi_{i+1/2}^n)) \mu_{i+1/2}^n (S_{i-1/2}^n - S_{i+1/2}^n) \phi_i^n qh. \end{aligned} \tag{34}$$

Since the local bounded variation is preserved, and by (33), the first limit in the second member of (34) is zero. From (31), this second member is now non positive, and we have to prove that it is non negative ; thus the only possibility is to be equal to zero, and to prove it, we shall use the additional hypothesis.

Using that u_i^{n+1} lies in $I(u_i^n, u_{i-1}^n)$, we obtain that the locally finite number of extrema of u_0 is conserved for all n .

Thus $\mu_{i+1/2}^n(S_{i-1/2}^n - S_{i+1/2}^n)$ is zero but when both k and u_i^n belong to $I(u_{i-1}^n, u_{i+1}^n)$, which arises at most for two values of i between two extrema. This means that the sum for i is reduced to $2K(R + T/q)$ terms, with R such that the set $] - R, R[\times]0, T[$ contains the support of ϕ . Moreover, non zero terms satisfy

$$0 \leq |f(u_{i+1}^n) - f(u_i^n)| (1 - qf'(\xi_{i+1/2}^n)) \mu_{i+1/2}^n(S_{i-1/2}^n - S_{i+1/2}^n) \leq 2 |a_{i+1/2}^n| / q.$$

We have proved that the second member of (34) is always greater than

$$- 4CTK(R + T/q) |\phi|_{L^\infty} h^2/q,$$

which tends to zero with h . Theorem 2 is proved.

4. APPROXIMATION FOR NON MONOTONOUS FUNCTION f

The function f is now any element in $C^1(\mathbb{R})$, and then we have to consider the Godunov scheme instead of (13), before the antidiffusion phase. This scheme is the following, at time nqh and at point ih .

$$\left. \begin{aligned} u_{i+1/2}^n \in I(u_i^n, u_{i+1}^n) \text{ realizes } \text{Min}_{k \in I(u_i^n, u_{i+1}^n)} \{ \text{sg}(u_{i+1}^n - u_i^n) f(k) \}, \\ u_i^{n+1} = u_i^n - q[f(u_{i+1/2}^n) - f(u_{i-1/2}^n)]. \end{aligned} \right\} \quad (35)$$

The first step selects the stationary value at point $(i + 1/2)h$, for t in $]nqh, (n + 1)qh[$, of the correct solution of (1), equal to $u_t(x, nqh)$ as Cauchy condition. The second step is the L^2 projection of this solution on the space of functions constant on each I_v , at time $(n + 1)qh$. The initial data (2) are introduced by (12), as previously. The convergence for the scheme (35) was proved in [7].

A quasi order two version of the scheme of Godunov may be the following

$$u_{i+1/2}^n \text{ defined as in (35),} \tag{36.i}$$

$$\begin{aligned} \tilde{a}_{i+1/2}^n = \frac{q}{2} [f(u_{i+1}^n) - 2f(u_{i+1/2}^n) + f(u_i^n) - \\ - qf'(\xi_{i+1/2}^n)(f(u_{i+1}^n) - f(u_i^n))], \end{aligned} \tag{36.ii}$$

$$\begin{aligned} a_{i+1/2}^n = \text{sg}(u_{i+1}^n - u_i^n) \text{Max} \{ 0, \text{Min} [|\tilde{a}_{i+1/2}^n|, Ch^n, \\ \text{sg}(u_{i+2}^n - u_{i+1}^n) \text{sg}(u_{i+1}^n - u_i^n) q | f(u_{i+3/2}^n) - f(u_{i+1/2}^n) |, \\ \text{sg}(u_{i+1}^n - u_i^n) \text{sg}(u_i^n - u_{i-1}^n) q | f(u_{i+1/2}^n) - f(u_{i-1/2}^n) |] \}, \end{aligned} \tag{36.iii}$$

$$u_i^{n+1} = u_i^n - q[f(u_{i+1/2}^n) - f(u_{i-1/2}^n)] - a_{i+1/2}^n + a_{i-1/2}^n. \tag{36.iv}$$

This is a generalization of (17), (27), (19) for non monotonous f ; if f is an increasing function and for $\text{sg}(u_{i+2}^n - u_{i+1}^n) = \text{sg}(u_{i+1}^n - u_i^n)$, since $u_{i+1/2}^n$ is equal to u_i^n for all i , we get

$$q | f(u_{i+3/2}^n) - f(u_{i+1/2}^n) | = qf'(\xi_{i+1/2}^n) | u_{i+1}^n - u_i^n | \geq | \tilde{a}_{i+1/2}^n |,$$

and $\tilde{a}_{i+1/2}^n$ is exactly the same as in (17). The third argument cannot realize the minimum, thus (36.iii) is identically (27) in that case.

A generalization of the scheme (17), (18), (19) will take $1/2 u_{i+1/2}^n$ instead of $qf(u_{i+1/2}^n)$ in (36.iii), but the definition of $u_{i+1/2}^n$ cannot be used in that case, and L^∞ stability may fail.

We show now that $\tilde{a}_{i+1/2}^n$ in (36.iii) has the same sign than $(u_{i+1}^n - u_i^n)$, as required in a diffusion term. We write it as follows

$$\begin{aligned} \tilde{a}_{i+1/2}^n &= \frac{q}{2} (f(u_{i+1}^n) - f(u_{i+1/2}^n)) (1 - qf'(\xi_{i+1/2}^n)) + \\ &\quad + \frac{q}{2} (f(u_i^n) - f(u_{i+1/2}^n)) (1 + qf'(\xi_{i+1/2}^n)), \end{aligned}$$

and then deduce from (35) that $\text{sg}(u_{i+1}^n - u_i^n) a_{i+1/2}^n$ is non negative.

For $a_{i+1/2}^n = \tilde{a}_{i+1/2}^n$, and $a_{i-1/2}^n = \tilde{a}_{i-1/2}^n$, (36.iv) is exactly the Lax-Wendroff scheme. The quasi order two accuracy is justified by the same arguments than above, since the most general case occurs when both f and u_h are locally monotonous. The results of theorem 2 are extended in the next theorem.

THEOREM 3 : *Suppose that the stability condition (20) is verified, and u_0 has a locally finite number of local extrema, then the family $\{u_h\}$ built by the scheme (36) converges towards the solution of problem (1), (2), satisfying definition 1, in $L^1_{\text{loc}}(\mathbb{R} \times]0, T[)$ as h tends to zero.*

Proof : As previously, we introduce three quantities for all i and n by

$$\begin{aligned} a_{i+1/2}^n &= \lambda_{i+1/2}^n q(f(u_{i+3/2}^n) - f(u_{i+1/2}^n)) \\ &= \mu_{i+1/2}^n \tilde{a}_{i+1/2}^n = \nu_{i+1/2}^n q(f(u_{i+1/2}^n) - f(u_{i-1/2}^n)). \end{aligned} \tag{37}$$

We have

$$0 \leq | \lambda_{i+1/2}^n | \leq 1, \quad 0 \leq \mu_{i+1/2}^n \leq 1, \quad 0 \leq | \nu_{i+1/2}^n | \leq 1,$$

and $\lambda_{i-1/2}^n$ has the same sign than $\nu_{i+1/2}^n$, and they both verify

$$0 \leq 1 + \nu_{i+1/2}^n - \lambda_{i-1/2}^n \leq 2.$$

The scheme may be written now

$$u_i^{n+1} = u_i^n - q(f(u_{i+1/2}^n) - f(u_{i-1/2}^n))(1 + v_{i+1/2}^n - \lambda_{i-1/2}^n).$$

To prove the stability, we have to consider all combination of $u_{i-1/2}^n$, u_i^n and $u_{i+1/2}^n$, which may be resumed by two cases, $u_{i-1/2}^n \leq u_i^n \leq u_{i+1/2}^n$ and $u_{i-1/2}^n \leq u_{i+1/2}^n \leq u_i^n$ since others are similar to one of them.

We also define the quantities $\xi_{i\pm 1/4}^n$ in $I(u_{i\pm 1/2}^n, u_i^n)$ by

$$f(u_{i\pm 1/2}^n) - f(u_i^n) = f'(\xi_{i\pm 1/4}^n)(u_{i\pm 1/2}^n - u_i^n),$$

and then, for $u_{i-1/2}^n \leq u_i^n \leq u_{i+1/2}^n$, we get

$$u_i^{n+1} = u_i^n - qf'(\xi_{i+1/4}^n)(1 + v_{i+1/2}^n - \lambda_{i-1/2}^n)(u_{i+1/2}^n - u_i^n) - qf'(\xi_{i-1/4}^n)(1 + v_{i+1/2}^n - \lambda_{i-1/2}^n)(u_i^n - u_{i-1/2}^n), \quad (38)$$

where the coefficients of $u_{i+1/2}^n$ and $u_{i-1/2}^n$ are non negative. We first write that $u_{i-1/2}^n$ is less than u_i^n , and find a non negative coefficient for u_i^n , which is less than $u_{i+1/2}^n$. A similar procedure is used to minorate, thus we have

$$u_{i-1/2}^n \leq u_i^{n+1} \leq u_{i+1/2}^n.$$

For $u_{i-1/2}^n \leq u_{i+1/2}^n \leq u_i^n$ ($a_{i+1/2}^n$ is zero, but not $a_{i-1/2}^n$ in this case), we have from (38) $u_i^{n+1} \leq u_i^n$ obviously. We write now that $u_{i+1/2}^n$ is greater than u_i^n , to get

$$u_i^{n+1} \geq u_{i+1/2}^n - q(f(u_{i+1/2}^n) - f(u_{i-1/2}^n))(1 + v_{i+1/2}^n - \lambda_{i-1/2}^n).$$

Since the function $u - qf(u)(1 + v_{i+1/2}^n - \lambda_{i-1/2}^n)$ is increasing by (20), we get u_i^{n+1} greater than $u_{i-1/2}^n$, which gives the stability in the second main case.

To prove that the bounded variation is preserved, we write

$$\begin{aligned} u_{i+1}^{n+1} - u_i^{n+1} &= (u_{i+3/2}^n - u_{i+1}^n) [-qf'(\xi_{i+5/4}^n)(1 + v_{i+3/2}^n - \lambda_{i+1/2}^n)] + \\ &\quad + (u_{i+1}^n - u_{i+1/2}^n) [1 - qf'(\xi_{i+3/4}^n)(1 + v_{i+3/2}^n - \lambda_{i+1/2}^n)] \\ &\quad + (u_{i+1/2}^n - u_i^n) [1 + qf'(\xi_{i+1/4}^n)(1 + v_{i+1/2}^n - \lambda_{i-1/2}^n)] \\ &\quad + (u_i^n - u_{i-1/2}^n) [+qf'(\xi_{i-1/4}^n)(1 + v_{i+1/2}^n - \lambda_{i-1/2}^n)]. \end{aligned}$$

where the coefficients are non negative. We take the absolute values, and sum for $|i| \leq I$ to get

$$\sum_{|i| \leq I} |u_{i+1}^{n+1} - u_i^{n+1}| \leq \sum_{|i| \leq I+1} \{ |u_{i+1}^n - u_{i+1/2}^n| + |u_{i+1/2}^n - u_i^n| \},$$

which gives (23). The estimate of the time variation is obvious ; by (20) we have

$$|u_i^{n+1} - u_i^n| \leq 2 q [|f(u_{i+1/2}^n) - f(u_i^n)| + |f(u_i^n) - f(u_{i-1/2}^n)|],$$

to obtain (24). As in theorem 1, these estimates yield the L^V convergence of a sequence $\{u_{h_m}\}$, towards $u \in L^\infty(\mathbb{R} \times]0, T[) \cap BV_{loc}(\mathbb{R} \times]0, T[)$, satisfying the initial condition (2). It remains to prove that u verifies Definition 1 ; we use for it the same notations than above.

Let $k \in \mathbb{R}$, $i \in \mathbb{Z}$; as a main case, we mind that we have the configuration $u_{i-1/2}^n \leq k \leq u_i^n \leq u_{i+1/2}^n$. Other cases are solved by similar arguments, and a more complete proof of such an inequality is to be found in [6] or [10]. We have

$$\begin{aligned} u_i^{n+1} - k &= (u_i^n - k) \{ 1 - qf'(k_i^n) (1 + v_{i+1/2}^n - \lambda_{i-1/2}^n) \} + \\ &\quad + (u_{i+1/2}^n - u_i^n) \{ -qf'(\xi_{i+1/4}^n) (1 + v_{i+1/2}^n - \lambda_{i-1/2}^n) \} \\ &\quad + (u_{i-1/2}^n - k) \{ qf'(k_{i-1/2}^n) (1 + v_{i+1/2}^n - \lambda_{i-1/2}^n) \}, \end{aligned}$$

where all coefficients are non negative. We take the absolute values, and obtain

$$|u_i^{n+1} - k| \leq |u_i^n - k| - q \{ \text{sg}(u_{i+1/2}^n - k) (f(u_{i+1/2}^n) - f(k)) - \text{sg}(u_{i-1/2}^n - k) (f(u_{i-1/2}^n) - f(k)) \} (1 + v_{i+1/2}^n - \lambda_{i-1/2}^n). \quad (39)$$

This inequality may be established for all other combinations of $u_{i-1/2}^n$, u_i^n , $u_{i+1/2}^n$ and k .

By introducing a non negative test function ϕ , that we approach as previously, we get

$$\begin{aligned} \sum_i \sum_n [|u_i^n - k| (\phi_i^n - \phi_i^{n-1}) h + \text{sg}(u_{i+1/2}^n - k) (f(u_{i+1/2}^n) - f(k)) (\phi_{i+1}^n - \phi_i^n) qh] &\geq \\ &\geq \sum_i \sum_n \phi_i^n (v_{i+1/2}^n - \lambda_{i-1/2}^n) [\text{sg}(u_{i+1/2}^n - k) (f(u_{i+1/2}^n) - f(k)) - \\ &\quad - \text{sg}(u_{i-1/2}^n - k) (f(u_{i-1/2}^n) - f(k))] qh. \quad (40) \end{aligned}$$

Passing to the limit on the first member of (40) is obvious ; that gives the first member of (6). For the second member, we define the quantities S_i^n in $[0, 1]$, for all (i, n) by

$$S_i^n = \begin{cases} 1 & \text{if } k > u_{i+1/2}^n > u_{i-1/2}^n \text{ or } k < u_{i+1/2}^n \leq u_{i-1/2}^n, \\ (f(k) - f(u_{i-1/2}^n)) / (f(u_{i+1/2}^n) - f(u_{i-1/2}^n)) & \text{if } k \in I(u_{i-1/2}^n, u_{i+1/2}^n), \\ 1 & \text{if } k < u_{i-1/2}^n < u_{i+1/2}^n \text{ or } k > u_{i-1/2}^n \geq u_{i+1/2}^n \end{cases}$$

which are similar to the $S_{i+1/2}^n$ for the monotonous case. We can write

$$\begin{aligned} \text{sg}(u_{i+1/2}^n - k) (f(u_{i+1/2}^n) - f(k)) - \text{sg}(u_{i-1/2}^n - k) (f(u_{i-1/2}^n) - f(k)) &= \\ = \text{sg}(u_{i+1/2}^n - u_{i-1/2}^n) (f(u_{i+1/2}^n) - f(u_{i-1/2}^n)) (1 - 2 S_i^n), \end{aligned}$$

with

$$| a_{i+1/2}^n | (S_{i+1}^n - S_i^n) \leq 0 .$$

The second member of (40) becomes

$$\begin{aligned} \sum_i \sum_n qh\phi_i^n \{ & \mu_{i+1/2}^n \tilde{a}_{i+1/2}^n [\text{sg}(u_{i+3/2}^n - u_{i+1/2}^n) (1 - 2 S_{i+1}^n) + \\ & + \text{sg}(u_{i+1/2}^n - u_{i-1/2}^n) (1 - 2 S_i^n) - \text{sg}(u_{i+3/2}^n - u_{i+1/2}^n) (1 - 2 S_{i+1}^n)] \\ & - \mu_{i-1/2}^n a_{i-1/2}^n \text{sg}(u_{i+1/2}^n - u_{i-1/2}^n) (1 - 2 S_i^n) \} . \end{aligned}$$

which is the sum of a diffusion term, which tends to zero after an integration by parts, and another term which is non positive. We have effectively

$$\begin{aligned} \mu_{i+1/2}^n \text{sg}(u_{i+1/2}^n - u_{i-1/2}^n) &= \mu_{i+1/2}^n \text{sg}(u_{i+3/2}^n - u_{i+1/2}^n) \\ &= \mu_{i+1/2}^n \text{sg}(u_{i+1}^n - u_i^n) , \end{aligned}$$

which gives as remaining term

$$\sum_i \sum_n 2 qh\phi_i^n | a_{i+1/2}^n | (S_{i+1}^n - S_i^n) ;$$

this sum tends to zero, by the same arguments than for theorem 2.

Theorem 3 is proved.

5. APPROXIMATION OF PROBLEM (1), (2), (3)

We consider the same scheme than in section 4, and f is not assumed to be monotonous. The boundary values are introduced through

$$u_0^n = \frac{1}{qh} \int_{J_n} a(t) dt , \quad u_1^n = \frac{1}{qh} \int_{J_n} b(t) dt , \quad n \leq N , \quad (41)$$

with $I = 1/h$, supposed to be an integer. The quasi order two character of this scheme is kept up, since the boundaries have a measure equal to zero. Theorem 3 may be extended to this problem. The set $]0, 1[\times]0, T[$ is denoted by Ω .

THEOREM 4 : *Assume the stability condition,*

$$q \text{ Sup}_{|k| \leq M} | f'(k) | \leq \frac{1}{2} \quad \text{with} \quad M = \text{Max} (| u_0 |_{L^\infty} , | a |_{L^\infty} , | b |_{L^\infty}) , \quad (42)$$

and that u_0, a and b have a finite number of local extrema on $]0, 1[$ or $]0, T[$, then the family $\{ u_h \}$ built by the scheme (36), and (12), (41) converges in $L^1(\Omega)$ towards the solution of problem (1), (2), (3), characterized by definition 2.

Proof. : The estimates for convergence are the same than for the previous theorems, but for the bounded variation, which is here

$$\sum_{i=0}^{I-1} |u_{i+1}^{n+1} - u_i^{n+1}| \leq \sum_{i=0}^{I-1} |u_{i+1}^n - u_i^n| + |u_1^{n+1} - u_I^n| + |u_0^{n+1} - u_0^n|.$$

They yield the convergence of a sequence $\{u_{h_m}\}_m$ in $L^1(\Omega)$ towards $u \in BV(\Omega)$, which satisfies to the initial condition (2). As in theorem 3, we have also (8) for all test function ϕ which is equal to zero at $x = 0$ and $x = 1$. The hypothesis of finite number of local extrema that we have done on a and b is used as above for u_0 only.

We shall state (9) for u . Let $k \in \mathbb{R}$ and $i \in \{1, \dots, i - 1\}$; we have for all non negative test function $\psi \in C^1(]0, T[)$, with compact support and approached as in (41), the inequality

$$\begin{aligned} \sum_n [\text{sg}(u_{i+1/2}^n - k) (f(u_{i+1/2}^n) - f(k)) + \beta_{i+1/2}^n] \psi^n qh &\leq \\ &\leq \sum_n (|u_i^n - k| - |u_i^{n+1} - k|) \psi^n h + \\ &\quad + \sum_n [\text{sg}(u_{i-1/2}^n - k) (f(u_{i-1/2}^n) - f(k)) + \beta_{i-1/2}^n] \psi^n qh \\ &\quad + 2 \sum_n |a_{i+1/2}^n| (S_{i+1}^n - S_i^n) \psi^n qh. \end{aligned}$$

This comes from (39), where we have put

$$\beta_{i+1/2}^n = \mu_{i+1/2}^n \tilde{a}_{i+1/2}^n \text{sg}(u_{i+3/2}^n - u_{i+1/2}^n) (1 - 2 S_{i+1}^n).$$

We write the same inequality for all $j \leq i$, to get the following inequality :

$$\begin{aligned} \sum_n [\text{sg}(u_{i+1/2}^n - k) (f(u_{i+1/2}^n) - f(k)) + \beta_{i+1/2}^n] \psi^n qh &\leq \\ &\leq \sum_n (\text{sg}(u_{1/2}^n - k) (f(u_{1/2}^n) - f(k)) + \beta_{1/2}^n) \psi^n qh + \\ &\quad + \sum_n \sum_{j \leq i} \{ 2 q |a_{j+1/2}^n| (S_{j+1}^n - S_j^n) \psi^n + |u_j^n - k| (\psi^n - \psi^{n-1}) \} h, \end{aligned} \tag{43}$$

the last term of the right hand side of (43) is bounded by :

$$2 |\psi|_{L^1(0,T)} KCh^\alpha + (M + |k|) |\psi'|_{L^\infty(0,T)} T ih,$$

where K is the total number of local extrema for u_0 , a and b . We also have for all $j \geq 0$,

$$|\sum_n \beta_{j+1/2}^n \psi^n qh| \leq Ch^\alpha |\psi|_{L^1(0,T)},$$

and then all those terms tend to zero, with h .

From the sequence $\{ \tilde{u}_{h_m} \}$, associated to the same h_m than $\{ u_{h_m} \}$ and defined by

$$\tilde{u}_{h_m}(x, t) = u_{j+1/2}^n \text{ if } x \in I_j, \quad t \in J_n,$$

we can extract a subsequence which converges towards u almost everywhere on Ω . Thus, for almost all x in $]0, 1[$, we can write that a sequence $u_{h_m}(x, \cdot)$ converges towards $u(x, \cdot)$ almost everywhere on Ω . Since all these functions are uniformly bounded, the convergence is worth also in $L^1(0, T)$ by Lebesgue's theorem.

Note also that for all n , we have from (34. i)

$$\text{sg}(u_{1/2}^n - k)(f(u_{1/2}^n) - f(k)) \leq \text{sg}(u_0^n - k)(f(u_{1/2}^n) - f(k)),$$

what we put in (43). Thus we have now to know the limit of

$$\sum_n \text{sg}(u_0^n - k)(f(u_{1/2}^n) - f(k)) \psi^n qh.$$

Denoting by λ the weak-star limit in $L^\infty(0, T)$ of $f(\tilde{u}_{h_m}(0, t))$, and using a regularization argument to treat the sign function, we get the following limit

$$\int_0^T \text{sg}(a(t) - k)(\lambda(t) - f(k)) \psi(t) dt.$$

Up to now, we have at the limit on (43) the inequality,

$$\begin{aligned} \int_0^T \text{sg}(u(x, t) - k)(f(u(x, t)) - f(k)) dt &\leq \\ &\leq \int_0^T \text{sg}(a(t) - k)(\lambda(t) - f(k)) \psi(t) dt + T(M + |k|) \|\psi'\|_{L^\infty} x, \end{aligned}$$

for $x = \lim ih'_m$, such that $\tilde{u}_{h_m}(x, \cdot)$ converges, which is true almost everywhere on $]0, 1[$. Let now x tend to zero, and we find that λ is equal to $(\gamma u(0, \cdot))$ almost everywhere on $]0, T[$. Next we take k in $I(a(t), \gamma u(0, t))$, for almost all t , which gives

$$\text{sg}(\gamma u(0, \cdot) - a) [f(\gamma u(0, \cdot)) - f(k)] \leq 0. \tag{44}$$

A similar result may be stated at $x = 1$, by the same way. Now (9) follows obviously and theorem 4 is proved.

6. CONCLUSION

The results proved in the above four theorems may be theoretically generalized to any scheme of lower order of accuracy which is corrected up to an higher order scheme, since stability and monotony are respected, and if the convergence is already stated for the lower order scheme. A well known scheme of order one is the Lax-Friedrichs scheme

$$u_i^{n+1} = \frac{1}{2}(u_{i-1}^n + u_{i+1}^n) - \frac{q}{2}(f(u_{i+1}^n) - f(u_{i-1}^n)), \tag{45}$$

which may be expounded as the L^2 projection on a decentered mesh. Nevertheless, we have to be care with the quasioorder two schemes associated to (45). We get, after correction of this scheme by the Lax-Wendroff one, in the same way as above, the following scheme, where f is assumed to be increasing,

$$a_{i+1/2}^n = \text{sg}(u_{i+1}^n - u_i^n) \text{Max} \left\{ 0, \text{Min} \left[\frac{1}{2}(1 - q^2 f'^2(\xi_{i+1/2}^n)) |u_{i+1}^n - u_i^n|, ch^\alpha, \right. \right. \\ \left. \left. q \text{sg}(u_{i+1}^n - u_i^n)(f(u_i^n) - f(u_{i-1}^n)), q \text{sg}(u_{i+1}^n - u_i^n)(f(u_{i+2}^n) - f(u_i^n)) \right] \right\},$$

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{q}{2}(f(u_{i+1}^n) - f(u_{i-1}^n)) - a_{i+1/2}^n + a_{i-1/2}^n.$$

This is exactly the Lax-Wendroff scheme when the minimum is realized by the first argument, but this will occur only for u satisfying

$$\sqrt{2} - 1 \leqq qf'(u) \leqq \frac{1}{2},$$

which is too restrictive. The correction up to the second order will be seldom performed, and this can be explained by the two large difference of the diffusion lying implicetely in both of these schemes.

Theorem 2 may be extended to the multidimensionnal problem on an open set with initial and boundary conditions, and for equation (10), where the components f_j of f are assumed to be monotonous functions with respect to the variable u , but we have to suppose that the finite number of local extrema is preserved in each direction, since it is not guaranteed by the scheme. The total variation is still kept, from the monotony of the f_j .

Other methods to obtain second order schemes which are stable, are described by B. Van Leer; see [13]. Average and slope of the solution are used on each slab I_i , without assuming the continuity.

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