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A MIXED FINITE ELEMENT METHOD FOR THE BIHARMONIC PROBLEM (*)

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Communiqué par E. MAGENES

Résumé. — *L'objet de ce travail est l'étude d'une méthode d'éléments finis de type mixte pour le problème biharmonique. On étudie en particulier une méthode mixte du type Hellan-Hermann-Johnson avec des éléments finis rectangulaires. On reprend certains résultats de C. Johnson, que l'on démontre différemment et qu'on améliore. Ces résultats sont établis dans un cadre plus général, qui permet la construction d'un schéma d'approximation d'ordre plus élevé.*

Abstract. — *The purpose of this work is to study a mixed finite element method for the biharmonic problem. We study in particular a mixed method of Hellan-Hermann-Johnson type with rectangular finite elements. Some results of C. Johnson are given in a different proof, and they are improved. These results are established in a more general framework, which applies to the construction of an approximation scheme of higher order.*

INTRODUCTION

The subject of this paper is the study of a finite element method of mixed type for the approximation of the biharmonic problem.

This problem is a classical one and it has been studied from a theoretical point of view by many authors (*see* e. g. [10, 14, 19]). The use of finite element methods has contributed new developments to the numerical approach to the problem. Presently many and different types of approximation by means of finite element methods are used. In particular we recall displacement methods, various types of hybrid and mixed methods, and equilibrium methods.

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In this paper we have expecially studied the mixed methods of Hellan-Hermann-Johnson type with rectangular elements. As regards the error bounds these methods do not present the favourable circumstances that have permitted an optimal error bound, in many general conditions, with triangular elements. However it is possible, at least in particular cases, to obtain again the optimal convergence order.

More exactly here we reconsider some results obtained by Johnson (*see* [13]). We represent these results in a more simple form using the recent techniques to approximate saddle-points, we improve these results (for the approximation of the displacement we get an optimal error bound), and we arrange these results in a more general context that permits the extension to other schemes of higher order.

The outline of the paper is the following:

In paragraph 1 we study from an abstract point of view the properties of a class of saddle-point problems. In particular we give an abstract theorem of convergence (theorem 1) that should be useful, besides our case, for the study of error bounds in other schemes.

In paragraph 2 we introduce the "model problem".

In paragraph 3 we transform the "model problem" in a saddle-point problem. The formulation we obtain is not yet optimal for a discretisation by means of finite elements.

In paragraph 4 we briefly recall some of Green's formulas that we shall use later.

In paragraph 5, using Green's formulas, we give a new formulation of the initial problem, again of saddle-point type, that allows us a convenient discretisation.

In paragraph 6 we introduce two discretisation schemes.

In paragraph 7 we give convergence results using the abstract scheme introduced in paragraph 1.

1. ABSTRACT PROBLEM

Let V be a real Hilbert space with norm $\|.\|_V$ and W be a Banach space with norm $\|.\|_W$; let V' and W' be respectively the dual spaces of V and W. We denote by $(.,.)$ the duality between the spaces V' and V or between the spaces W' and W.

Let $a(.,.)$ and $b(.,.)$ be respectively two continuous bilinear forms on $V \times V$ and $W \times W$. We set

$$\|a\| = \sup_{u,\, v \in V} \frac{a(u, v)}{\|u\|_V \|v\|_V}, \tag{1.1}$$

$$\|b\| = \sup_{v \in V,\, \varphi \in W} \frac{b(v, \varphi)}{\|v\|_V \|\varphi\|_W}. \tag{1.2}$$

Given $f \in V'$ and $\mu \in W'$, we consider the following problem (continuous problem):

Find a pair $(u, \psi) \in V \times W$ such that:

$$\text{(P)} \qquad \begin{cases} \forall v \in V, \quad a(u, v) = b(v, \psi) + (f, v), \\ \quad \forall \varphi \in W, \quad b(u, \varphi) = (\mu, \varphi). \end{cases}$$

Let H be a real Hilbert space with norm $\|.\|_H$ and M be a Banach space with norm $\|.\|_M$, such that:

$$V \subset H, \ W \subset M, \text{ with continous and dense imbedding.} \tag{1.3}$$

We denote by σ_1, σ_2, the imbedding constants, that is

$$\forall v \in V, \quad \|v\|_H \leqq \sigma_1 \|v\|_V, \tag{1.4}$$

$$\forall \varphi \in W, \quad \|\varphi\|_M \leqq \sigma_2 \|\varphi\|_W. \tag{1.5}$$

We suppose that we can extend the bilinear form $a(.,.)$, defined on $V \times V$, to the space $H \times H$.

Let the bilinear form $a(.,.)$ be elliptic in H, that is

$$\exists \alpha > 0, \quad \forall v \in H, \qquad a(v, v) \geqq \alpha \|v\|_H^2. \tag{1.6}$$

Moreover we suppose that the bilinear form $b(.,.)$ satisfies the following stability condition (external ellipticity):

$$\exists \beta > 0, \quad \forall \varphi \in W, \qquad \sup_{v \in V} \frac{b(v, \varphi)}{\|v\|_V} \geqq \beta \|\varphi\|_M. \tag{1.7}$$

With simple considerations it's easy to verify that problem (P) has at most one solution. However we cannot deduce the existence of the solution from the assumptions we have made, and to prove the existence it will be necessary, in concrete situations, to use other properties.

We assume in addition that two finite-dimensional vectorial spaces V_h and W_h are given, such that

$$V_h \subset V, \quad W_h \subset W,$$

and we suppose that the following hypotheses hold:

(i) a linear continous operator π_h: $V \to V_h$ and a positive constant γ exist such that:

$$\left.\begin{array}{ll}\forall v \in V, & b(v-\pi_h v, \varphi_h)=0, \quad \forall \varphi_h \in W_h, \\ \forall v \in V, & \|\pi_h v\|_V \leqq \gamma \|v\|_V.\end{array}\right\} \tag{1.8}$$

(ii) $$\exists \beta_h > 0, \quad \forall \varphi_h \in W_h \quad \sup_{v_h \in V_h} \frac{b(v_h, \varphi_h)}{\|v_h\|_V} \geqq \beta_h \|\varphi_h\|_M. \tag{1.9}$$

Now we consider the following problem (discrete problem):

$$(\mathrm{P}_h) \quad \left\{\begin{array}{l}\text{Find a pair } (u_h, \psi_h) \in V_h \times W_h \text{ such that:} \\ \forall v_h \in V_h, \quad a(u_h, v_h) = b(v_h, \psi_h) + (f, v_h), \\ \forall \varphi_h \in W_h, \quad b(u_h, \varphi_h) = (\mu, \varphi_h).\end{array}\right.$$

It's easy to verify that problem (P_h) has a unique solution.

We define

$$B_h(\mu) = \{v_h \in V_h, \forall \varphi_h \in W_h, b(v_h, \varphi_h) = (\mu, \varphi_h)\}. \tag{1.10}$$

Then we have the following results:

THEOREM 1: *Assume that the hypotheses* (1.6) *and* (1.7) *hold, and that* (u, ψ) *and* (u_h, ψ_h) *are respectively the solutions of problems* (P) *and* (P_h). *Then we have:*

$$\|u-u_h\|_H \leqq C\left(\inf_{v_h \in B_h(\mu)} \|u-v_h\|_H + \sup_{v_h \in V_h} \inf_{\varphi_h \in W_h} \frac{b(v_h, \psi-\varphi_h}{\|v_h\|_H}\right), \tag{1.11}$$

where C *is a constant depending on* α, $\|a\|$, *but not depending on* (u, ψ), V_h *and* W_h.

Proof: We have

$$\left.\begin{array}{r}\forall v_h \in B_h(\mu), \\ a(u-u_h, u-u_h) = a(u-u_h, u-v_h) + a(u-u_h, v_h-u_h) \\ = a(u-u_h, u-v_h) + b(v_h-u_h, \psi-\psi_h),\end{array}\right\} \tag{1.12}$$

and therefore

$$\|u-u_h\|_H^2 \leqq a(u-u_h, u-u_h) = a(u-u_h, u-v_h) + b(v_h-u_h, \psi-\psi_h). \tag{1.13}$$

As $v_h \in B_h(\mu)$ we observe that

$$\forall \varphi_h \in W_h, \quad b(v_h - u_h, \psi_h) = b(v_h - u_h, \varphi_h) = 0. \tag{1.14}$$

Then we have

$$\begin{aligned} b(v_h - u_h, \psi - \psi_h) &= b(v_h - u_h, \psi - \varphi_h) \leqq \inf_{\varphi_h \in W_h} b(v_h - u_h, \psi - \varphi_h) \\ &= \inf_{\varphi_h \in W_h} \|v_h - u_h\|_H \frac{b(v_h - u_h, \psi - \varphi_h)}{\|v_h - u_h\|_H} = \|v_h - u_h\|_H \inf_{\varphi_h \in W_h} \frac{b(v_h - u_h, \psi - \varphi_h)}{\|v_h - u_h\|_H} \\ &\leqq \|v_h - u_h\|_H \sup_{v_h \in V_h} \inf_{\varphi_h \in W_h} \frac{b(v_h - u_h, \psi - \varphi_h)}{\|v_h - u_h\|_H}. \end{aligned} \tag{1.15}$$

Setting

$$C_1(h) = \inf_{v_h \in B_h(\mu)} \|u - v_h\|_H, \tag{1.16}$$

$$C_2(h) = \sup_{v_h \in V_h} \inf_{\varphi_h \in W_h} \frac{b(v_h - u_h, \psi - \varphi_h)}{\|v_h - u_h\|_H}, \tag{1.17}$$

we have

$$\begin{aligned} \|u - u_h\|_H^2 &\leqq a(u - u_h, u - v_h) + b(v_h - u_h, \psi - \varphi_h) \\ &\leqq C_1(h)\|a\|.\|u - u_h\|_H + C_2(h)\|v_h - u_h\|_H \\ &\leqq C_1(h)\|a\|.\|u - u_h\|_H + C_2(h)\|u - v_h\|_H + C_2(h)\|u - u_h\|_H \\ &\leqq C_1(h)\|a\|.\|u - u_h\|_H + C_2(h).C_1(h) + C_2(h)\|u - u_h\|_H, \end{aligned} \tag{1.18}$$

from which we obtain

$$\|u - u_h\|_H = C(C_1(h) + C_2(h)), \tag{1.19}$$

that is, with (1.16) and (1.17), the desired inequality. □

THEOREM 2: *Assume that the hypotheses* (1.6), (1.7), (1.8), *hold, and that* (u, ψ) *and* (u_h, ψ_h) *are respectively the solutions of problems* (P) *and* (P_h). *Then we have:*

$$\|\psi - \psi_h\|_M \leqq C\left(\|u - u_h\|_H + \inf_{\varphi_h \in W_h} \sup_{v_h \in V_h} \frac{b(v_h, \psi - \varphi_h)}{\|v_h\|_V}\right), \tag{1.20}$$

where C *is a constant depending on* β_h, $\|a\|$, σ_1, *but not depending on* (u, ψ), V_h *and* W_h.

Proof: The following inequality is obvious:

$$\forall \varphi_h \in W_h, \quad \|\psi - \psi_h\|_M \leqq \|\psi - \varphi_h\|_M + \|\psi_h - \varphi_h\|_M. \tag{1.21}$$

Using (1.9) we can write:

$$\forall \varphi_h \in W_h, \quad \| \psi - \varphi_h \|_M \leqq \frac{1}{\beta_h} \sup_{v_h \in V_h} \frac{b(v_h, \psi - \varphi_h)}{\| v_h \|_V}, \tag{1.22}$$

$$\forall \varphi_h \in W_h, \quad \| \psi_h - \varphi_h \|_M \leqq \frac{1}{\beta_h} \sup_{v_h \in V_h} \frac{b(v_h, \psi_h - \varphi_h)}{\| v_h \|_V}. \tag{1.23}$$

Using (1.23) and a simple property of the supremum, we have:

$$\left.\begin{array}{c} \forall \varphi_h \in W_h, \\ \| \psi_h - \varphi_h \|_M \leqq \dfrac{1}{\beta_h} \left(\sup\limits_{v_h \in V_h} \dfrac{b(v_h, \psi - \varphi_h)}{\| v_h \|_V} + \sup\limits_{v_h \in V_h} \dfrac{b(v_h, \psi - \psi_h)}{\| v_h \|_V} \right). \end{array}\right\} \tag{1.24}$$

We recall that

$$\forall v_h \in V_h, \quad a(u, v_h) = b(v_h, \psi) + (f, v_h), \tag{1.25}$$

$$\forall v_h \in V_h, \quad a(u_h, v_h) = b(v_h, \psi_h) + (f, v_h), \tag{1.26}$$

and therefore

$$\forall v_h \in V_h, \quad b(v_h, \psi - \psi_h) = a(u - u_h, v_h), \tag{1.27}$$

and so we obtain

$$\left.\begin{array}{c} \forall \varphi_h \in W_h, \\ \| \psi_h - \varphi_h \|_M \leqq \dfrac{1}{\beta_h} \left(\sup\limits_{v_h \in V_h} \dfrac{b(v_h, \psi - \varphi_h)}{\| v_h \|_V} + \sup\limits_{v_h \in V_h} \dfrac{a(u - u_h, v_h)}{\| v_h \|_V} \right). \end{array}\right\} \tag{1.28}$$

Using (1.1) we have

$$\left.\begin{array}{c} \forall \varphi_h \in W_h, \\ \| \psi_h - \varphi_h \|_M \leqq \dfrac{1}{\beta_h} \left(\sup\limits_{v_h V_h} \dfrac{b(v_h, \psi - \varphi_h)}{\| v_h \|_V} + \| a \| . \| u - u_h \|_H \dfrac{\| v_h \|_H}{\| v_h \|_V} \right). \end{array}\right\} \tag{1.29}$$

and then, using (1.4),

$$\| \psi_h - \varphi_h \|_M \leqq \frac{1}{\beta_h} \left(\inf_{\varphi_h \in W_h} \sup_{v_h \in V_h} \frac{b(v_h, \psi - \varphi_h)}{\| v_h \|_V} + \sigma_1 \| a \| . \| u - u_h \|_H \right). \tag{1.30}$$

From (1.21), using (1.22) and (1.30), we obtain

$$\| \psi - \psi_h \|_M \leqq \frac{2}{\beta_h} \inf_{\varphi_h \in W_h} \sup_{v_h \in V_h} \frac{b(v_h, \psi - \varphi_h)}{\| v_h \|_V} + \frac{\sigma_1}{\beta_h} \| a \| . \| u - u_h \|_H . \tag{1.31}$$

We set

$$C=\max\left(\frac{2}{\beta_h},\frac{\sigma_1}{\beta_h}\|a\|\right), \tag{1.32}$$

and from (1.31) we obtain the desired inequality. □

In the sequel it will be very useful to consider the following:

LEMMA 1: *If the hypotheses* (1.3), (1.7), (1.8), *hold, then the condition* (1.9) *is satisfied.*

Proof: See Fortin [11]. □

The usefulness of lemma 1 consists in the possibility of obtaining (1.9) (discrete external ellipticity) from (1.7), (1.8), that are often in the practice more simple to verify than (1.9) itself.

2. THE MODEL PROBLEM

Let Ω be a bounded open subset of $\mathbb{R}^2$ with a sufficiently smooth boundary $\partial\Omega$, let f be a given function in $L^2(\Omega)$ and let us denote by $\partial/\partial n$ the outward normal derivative along $\partial\Omega$.

Now we consider the following problem (Dirichlet problem for the biharmonic operator):

$$\left.\begin{aligned}\Delta^2\psi=f\quad&\text{in }\Omega,\\ \psi=\partial\psi/\partial n=0\quad&\text{on }\partial\Omega.\end{aligned}\right\} \tag{2.1}$$

If the function f belongs, for simplicity, to $L^2(\Omega)$, and we seek for the solution in the space $H_0^2(\Omega)$, then the problem (2.1) is well-posed, that is the solution exists and is unique (*see* [14]).

The problem (2.1) is very interesting since it is found in numerous problems of physical mathematics, for instance in hydrodinamics problems and in plate bending problems.

The problem (2.1) is a classical one and it has been studied either from a theoretical point of view or from a numerical one by many authors (*see* e.g. [5, 10, 18, 19]). The use of finite element methods has improved the techniques to approximate numerically the solution of (2.1). At the present time there are many and different finite element methods for this approximation: conforming, non conforming, hybrid and mixed methods; the literature on this subject is quite large (*see* e.g. [3, 4, 6, 12]).

In the sequel we shall study a mixed method for the approximation of (2.1).

For these methods we can say that the theory is nearly complete when a subdivision of Ω with triangular elements is used (*see* e.g. [4, 9, 15]). As regards the use of rectangular elements we can say that while the extension of some mixed methods (*see* [9, 15]) is complete and it doesn't present supplementary difficulties, the extension of Hermann-Johnson scheme presents some further difficulties and at the present it's not complete.

The method we study here has been suggested to us by a previous paper of Johnson (*see* [13]). The results we obtain allow us to arrange the results of [13] in a more general context which is more adherent to the abstract scheme of [4].

In particular the present arrangement allows us:

(*a*) to semplify and improve the results of [13]: we prove an optimal error bound in $H^1(\Omega)$ for the displacements while in [13] Johnson proved only a non optimal error bound in $L^\infty(\Omega)$;

(*b*) to extend the theory to discretisation schemes of higher order, similar to those already obtained in [4] in the case of triangular elements.

3. FIRST TRANSFORMATION IN SADDLE-POINT PROBLEM

The first step we must do for the approximation of the problem (2.1) by means of a finite element method of mixed type is (*see* [4]) the transformation of the problem into the so-called Hellinger-Reissner form. For this purpose we introduce, as a new variable, the symmetric tensor-valued function $\mathbf{u} = (u_{ij})$, i, $j=1, 2$, $u_{12}=u_{21}$, with components given by

$$u_{ij} = \psi_{/ij} (^1), \qquad i, j = 1, 2, \tag{3.1}$$

where ψ is the solution of the problem (2.1).

For example, in plate bending problems $\mathbf{u}$, multiplied by a factor of proportionality, gives the tensor of the moments. One of the most important aspects of mixed methods is that of giving directly an approximation of $\mathbf{u}$ that is the most interesting unknown quantity in many practical problems and expecially in structural analysis. The tensor-valued function $\mathbf{u}$ belongs to the space

$$H = (L^2(\Omega))^4_s = \{\mathbf{v} : \mathbf{v} = (v_{ij}),\ v_{ij} \in L^2(\Omega),\ i, j = 1, 2,\ v_{12} = v_{21}\}. \tag{3.2}$$

([1]) The classical notation $_{/ij}$ indicates the derivative with respect to x_i and to x_j.

For convenience in the sequel the norm of a tensor $\mathbf{v} \in H$, defined by

$$\|\mathbf{v}\|_H = \left(\int_\Omega v_{ij} v_{ij} \, dx \right)^{1/2} \tag{3.3}$$

will simply be denoted by $\|\mathbf{v}\|_0$.

Now we consider the following saddle-point problem:

Find a pair $(\mathbf{u}, \psi) \in H \times H_0^2(\Omega)$ such that:

$$\left.\begin{aligned} &\forall \mathbf{v} \in H, \quad \int_\Omega u_{ij} v_{ij} \, dx = \int_\Omega v_{ij} \psi_{/ij} \, dx, \\ &\forall \varphi \, H_0^2(\Omega), \quad \int_\Omega u_{ij} \varphi_{/ij} \, dx = \int_\Omega f \varphi \, dx. \end{aligned}\right\} \tag{3.4}$$

It's easy to verify that if the pair $(\mathbf{u}, \psi)$ solves (3.4), with $\mathbf{u}$ defined by (3.1), then ψ is the solution of (2.1).

In fact from the first equation of (3.4) we easily obtain $u_{ij} = \psi_{/ij}$, $i, j = 1, 2$, while from the second equation, integrating by parts two times, we have $u_{ij/ij} = f$ and therefore we finally obtain $\Delta^2 \psi = f$. We observe that the problem (3.4) is a type (P) problem choosing

$$a(u, v) = \int_\Omega u_{ij} v_{ij} \, dx, \tag{3.5}$$

$$b(v, \varphi) = \int_\Omega v_{ij} \varphi_{/ij} \, dx \tag{3.6}$$

and therefore it will have at most one solution.

Since: (*a*) problem (3.4) has at most one solution; (*b*) if the pair $(\mathbf{u}, \psi)$ solves (3.4) then ψ is the solution of (2.1); (*c*) problem (2.1) is well-posed; we get that the solution of (3.4) will coincide with the solution of (2.1).

4. GREEN'S FORMULAS

The formulation (3.4) is not yet the optimal formulation that allows us to introduce an approximation scheme of mixed type. To pass from (3.4) to a new formulation, still of saddle-point type but in different spaces, we briefly recall some Green's formulas that we shall use in the sequel.

Let Ω be, for simplicity, a polygon with boundary $\partial\Omega$. We define the following space of tensor-valued functions:

$$V = \{ \mathbf{v} : \mathbf{v} = (v_{ij}),\ v_{ij} \in H^1(\Omega),\ i, j = 1, 2 \}. \tag{4.1}$$

([2]) We use (here and in the following pages) the convention of the summation of repeated indices.

If $\mathbf{v} \in V$ and $\varphi \in H^2(\Omega)$, then we have the following Green's formula:

$$\int_\Omega v_{ij} \varphi_{/ij} \, dx = - \int_\Omega v_{ij/i} \varphi_{/j} \, dx + \int_{\partial\Omega} (M_n(\mathbf{v}) \varphi_{/n} + M_{nt}(\mathbf{v}) \varphi_{/t}) \, ds, \quad (4.2)$$

and moreover if each component v_{ij} of $\mathbf{v}$ belongs to $H^2(\Omega)$ then we have

$$\int_\Omega v_{ij/i} \varphi_{/j} \, dx = - \int_\Omega v_{ij/ij} \varphi \, dx + \int_{\partial\Omega} Q_n(\mathbf{v}) \varphi \, ds, \quad (4.3)$$

where

$$M_n(\mathbf{v}) = v_{ij} n_i n_j, \quad (4.4)$$

$$M_{nt}(\mathbf{v}) = v_{ij} n_i t_j, \quad (4.5)$$

$$Q_n(\mathbf{v}) = v_{ij/i} n_j, \quad (4.6)$$

$\mathbf{n} = (n_1, n_2)$ is the unit outward normal and $\mathbf{t} = (t_1, t_2) = (n_2, -n_1)$ is the unit tangent along $\partial\Omega$, $\varphi_{/n} = \partial\varphi/\partial n$ and $\varphi_{/t} = \partial\varphi/\partial t$.

Combining (4.2) and (4.3) we obtain

$$\int_\Omega v_{ij} \varphi_{/ij} \, dx = \int_\Omega v_{ij/ij} \varphi \, dx + \int_{\partial\Omega} (M_n(\mathbf{v}) \varphi_{/n} + M_{nt}(\mathbf{v}) \varphi_{/t} - Q_n(\mathbf{v}) \varphi) \, ds. \quad (4.7)$$

Finally we observe that the term $\int_{\partial\Omega} M_{nt}(\mathbf{v}) \varphi_{/t} \, ds$ is valid also for functions $\varphi \in W^{1,p}(\Omega)$, $p > 2$.

5. SECOND TRANSFORMATION IN SADDLE-POINT PROBLEM

We cannot use the formulation (3.4) for an approximation of mixed type. In particular the spaces appearing in (3.4) are: H, on which no continuity requirements are made, and $H_0^2(\Omega)$, whose elements must have the second derivative in $L^2(\Omega)$. It's clear that, in an eventual discretisation of the problem (2.1) in the form (3.4), we could use discontinous "test functions" to approximate the space H, while we would almost be obliged to use continous "test functions" with their first derivative to approximate the space $H_0^2(\Omega)$. However it's well-known that this last circumstance, even if quite performable (*see* [8]), leads to remarkable formal and computational difficulties.

Another characteristic that makes mixed methods interesting is the fact that they permit the use of continous approximating functions but not necessairly of class C^1. For this purpose we must give a new formulation of the problem (3.4) in which, roughly speaking, we ask a greater regularity of the space which contains $\mathbf{u}$ in order to accept a smaller regularity of the space which contains ψ. In the sequel we shall refer to the formulation (3.4) of the problem (2.1), formulation that, as we have previously verified, is equivalent.

Till now we have asked the solution ψ to belong to $H_0^2(\Omega)$ and the tensor $\mathbf{u}$ to have components belonging to $L^2(\Omega)$. Now we give a new formulation in which we ask ψ to belong to $W_0^{1,p}(\Omega)$, $p > 2$, hence with a loss of regularity, and we ask $\mathbf{u}$ to belong to a space V, space that we shall soon define, that is included in H and is more regular than H.

We suppose, for simplicity, that Ω is a convex polygon in the plain. Let $\mathcal{T}_h$ be a decomposition of Ω in convex subpolygons K and let h be the maximum diameter of the subpolygons. We say that $M_n(\mathbf{v})$ [*see* (4.4)] is "continous at the interelement boundaries" of the decomposition $\mathcal{T}_h$ if and only if, for any pair (K_1, K_2) of adjacent elements of $\mathcal{T}_h$, we have:

$$M_{n_1}(\mathbf{v}/_{K_1}) = M_{n_2}(\mathbf{v}/_{K_2}) \quad \text{on } K_1 \cap K_2, \tag{5.1}$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, are respectively the unit outward normals along ∂K_1, ∂K_2.

We define:

$$V = \{\mathbf{v} : \mathbf{v} = (v_{ij}),\ \forall K \in \mathcal{T}_h,\ v_{ij} \in H^1(K),\ i, j = 1, 2,\ v_{12} = v_{21}$$

$$\text{and } M_n(\mathbf{v}) \text{ is "continous at the interelement boundaries"}\}, \tag{5.2}$$

$$\|\mathbf{v}\|_V = \sum_{K \in \mathcal{T}_h} (\|\mathbf{v}\|_{1,K}^2)^{1/2}, \tag{5.3}$$

$$W = W_0^{1,p}(\Omega), \qquad p > 2, \tag{5.4}$$

$$b(\mathbf{v}, \varphi) = \sum_{K \in \mathcal{T}_h} \left(- \int_K v_{ij/i}\, \varphi_{/j}\, dx + \int_{\partial K} M_{nt}(\mathbf{v})\, \varphi_{/t}\, ds \right) \quad \text{with} \quad \mathbf{v} \in V, \quad \varphi \in W. \tag{5.5}$$

Now we consider the following problem:

Find a pair $(\mathbf{u}, \psi) \in V \times W$ such that:

$$\left.\begin{aligned} &\forall \mathbf{v} \in V, \\ &\int_\Omega u_{ij} v_{ij}\, dx = \sum_{K \in \mathcal{T}_h} \left(- \int_K v_{ij/i} \psi_{/j}\, dx + \int_{\partial K} M_{nt}(\mathbf{v}) \psi_{/t}\, ds \right), \\ &\forall \varphi \in W, \\ &\sum_{K \in \mathcal{T}_h} \left(- \int_K u_{ij/i} \varphi_{/j}\, dx + \int_{\partial K} M_{nt}(\mathbf{u}) \varphi_{/t}\, ds \right) = \int_\Omega f \varphi\, dx. \end{aligned}\right\} \tag{5.6}$$

We observe that (5.6) is a type (P) problem with $a(u, v)$ defined in (3.5) and $b(v, \varphi)$ defined in (5.5). As before the problem (5.6) will have at most one solution. As regards the existence of the solution we observe that if the first argument of the solution $(\mathbf{u}, \psi)$ of the problem (3.4) belongs to the space V, then the pair $(\mathbf{u}, \psi)$ is also the solution of the problem (5.6). We verify this assumption.

If $(\tilde{\mathbf{u}}, \tilde{\psi})$ is the solution of the problem (3.4) then

$$\forall \mathbf{v} \in H, \quad \int_\Omega \tilde{u}_{ij} v_{ij}\, dx = \int_\Omega v_{ij} \tilde{\psi}_{/ij}\, dx. \tag{5.7}$$

Since $V \subset H$, (5.7) is valid $\forall \mathbf{v} \in V$. Now integrating by parts the second member of (5.7), using the fact that $\mathbf{v} \in V$, $\tilde{\psi} \in H_0^2(\Omega)$, and using Green's formula (4.2), we obtain

$$\begin{aligned} \forall \mathbf{v} \in V, \quad \int_\Omega \tilde{u}_{ij} v_{ij}\, dx &= \sum_{K \in \mathcal{T}_h} \int_K v_{ij} \tilde{\psi}_{/ij}\, dx \\ &= \sum_{K \in \mathcal{T}_h} \left(- \int_K v_{ij/i} \tilde{\psi}_{/j}\, dx + \int_{\partial K} M_{nt}(\mathbf{v}) \tilde{\psi}_{/t}\, ds \right), \end{aligned} \tag{5.8}$$

that is the pair $(\tilde{\mathbf{u}}, \tilde{\psi})$ verifies the first equation of the problem (5.6). From the second equation of (3.4), if $\tilde{\mathbf{u}} \in V$, again integrating by parts, we have:

$$\left.\begin{aligned} &\forall \varphi \in H_0^2(\Omega), \\ &\sum_{K \in \mathcal{T}_h} \left(- \int_K \tilde{u}_{ij/i} \varphi_{/j}\, dx + \int_{\partial K} M_{nt}(\tilde{\mathbf{u}}) \varphi_{/t}\, ds \right) = \int_\Omega f \varphi\, dx, \end{aligned}\right\} \tag{5.9}$$

and hence the second equation of (5.6) is valid for all the functions $\varphi \in H_0^2(\Omega)$. As the space $H_0^2(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, $p > 2$, and since the application $\varphi \to b(\tilde{\mathbf{u}}, \varphi)$ is linear and continuous in the norm of $W^{1,p}(\Omega)$, $p > 2$, then (5.9) holds for all the functions $\varphi \in W_0^{1,p}(\Omega)$, $p > 2$. Therefore the pair $(\tilde{\mathbf{u}}, \tilde{\psi})$ is also the solution of the problem (5.6).

6. DISCRETISATION

In [13] Johnson, with a suitable discretisation, gives a bound of the first order for the error between the exact solution ($\mathbf{u}$, ψ) and the approximate solution ($\mathbf{u}_h$, ψ_h). The bound is obtained in the norm $\|.\|_0$ for the error $\mathbf{u}$-$\mathbf{u}_h$ and in the space $L^\infty(\Omega)$ for the error $\psi-\psi_h$. In this paragraph and in the following one we reconsider the discretisation of Johnson and we prove, in a different and more simple way, a bound of the first order in the norm $\|.\|_0$ for the error $\mathbf{u}$-$\mathbf{u}_h$ but in the space $H^1(\Omega)$ for the error $\psi-\psi_h$. Later we introduce a second discretisation which allows us to prove a convergence of the second order. Hence this paragraph and the following one have been divided in two parts: in the first we study the case of the linear convergence, in the second we study the case of the quadratic convergence. Let Ω be a bounded domain in the plane with a boundary consisting of a finite number of straight segments parallel to either the coordinate direction. Assume that Ω has been covered by a number of closed rectangles R_k such that any two rectangles are either disjoint or have a common vertex or side. We denote by h the maximum side lenght of the rectangles R_k covering Ω. Further, let λ_k denote the ratio of the lenghts of the non-parallel sides of R_k. We assume that there is a fixed positive number λ such that $1/\lambda \leqq \lambda_k \leqq \lambda$ for any k. This means of course that the rectangles R_k are not allowed to get very thin as $h \to 0$.

We define two finite-dimensional spaces V_h and W_h in the following way:

Discretisation 1

(*a*) V_h is the set of the tensor-valued functions $\mathbf{v}=(v_{ij})$, $i, j=1, 2$, such that:
(i) $v_{11} \in Q_0(x, y) \oplus$ (3) $\{x\}$; $v_{12} \in Q_0(x, y)$; $v_{22} \in Q_0(x, y) \oplus \{y\}$; on each R_k;
(ii) $M_n(\mathbf{v})$ is continous at the interelement boundaries.

(*b*) W_h is the set of the functions φ defined on Ω such that: (i) $\varphi \in Q_1(x, y)$ on each R_k, (ii) $\varphi \in C^0(\Omega)$, (iii) $\varphi=0$ on $\partial\Omega$.

The spaces V_h and W_h so defined are subspaces of V and W, respectively. It's easy to verify that we can choose the degrees of freedom (d. o. f.) in V_h and W_h as follows:

(*a*) d. o. f. in V_h. A function $\mathbf{v} \in V_h$ is uniquely determined by the value of $M_n(\mathbf{v})$ along each side and by the value of v_{12} in each rectangle (see *fig.* 1).

---

(3) Here and in the following pages we denote by $Q_k(x, y)$ the space of polynomials in x and y with a degree less or equal than k in each of the two variables x and y; we denote by $P_k(x, y)$ the space of polynomials in x and y with a degree less or equal than k in both the two variables x and y.

(*b*) d. o. f. in W_h. Each function $\varphi \in W_h$ is determined by its values at the four vertices of each rectangle.

Discretisation 2

(*a*) V_h is the set of the tensor-valued functions $\mathbf{v} = (v_{ij})$, $i, j = 1, 2$, such that:

(i) $v_{11} \in Q_1(x, y) \oplus \{x^2\}$; $v_{12} \in Q_1(x, y)$; $v_{22} \in Q_1(x, y) \oplus \{y^2\}$; on each R_k;

(ii) $M_n(\mathbf{v})$ is continous at the interelement boundaries.

(*b*) W_h is the set of the functions φ defined on Ω such that: (i) $\varphi \in Q_2(x, y)$ on each R_k, (ii) $\varphi \in C^0(\Omega)$, (iii) $\varphi = 0$ on $\partial\Omega$.

We can choose the d. o. f. as follows:

(*a*) d. o. f. in V_h. We take as d. o. f.: the value of v_{11} in two points of each vertical side; the value of v_{22} in two points of each horizontal side; the values $\int_{R_k} v_{11}\,dx$ and $\int_{R_k} v_{22}\,dx$ in each rectangle; the values of v_{12} at the vertices ([4]) of each rectangle (see *fig*. 2).

(*b*) d. o. f. in W_h. We choose the classical usual d. o. f. (values of at the vertices, at the midpoint of each side and at the centre).

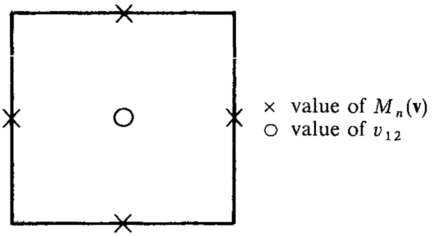

Figure 1

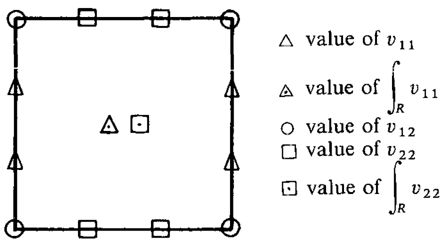

Figure 2

7. ERROR ESTIMATES

With the discretisation that we have introduced in the previous paragraph, the approximate formulation of the problem (5.6) assumes the following form:

([4]) *Remark:* This condition does not imply the continuity for v_{12}; for instance, if a node belongs to four different rectangles, four different d. o. f. will be present at this node.

Find a pair $(\mathbf{u}_h, \psi_h) \in V_h \times W_h$ such that:

$$\left.\begin{aligned} &\forall \mathbf{v} \in V_h, \\ &\int_\Omega u_{ij} v_{ij}\, dx = \sum_{R_k \in \mathcal{T}_h} \left(- \int_{R_k} v_{ij/i} \psi_{/j}\, dx + \int_{\partial R_k} M_{nt}(\mathbf{v}) \psi_{/t}\, ds \right), \\ &\forall \varphi \in W_h, \\ &\sum_{R_k \in \mathcal{T}_h} \left(- \int_{R_k} u_{ij/i} \varphi_{/j}\, dx + \int_{\partial R_k} M_{nt}(\mathbf{u}) \varphi_{/t}\, ds \right) = \int_\Omega f \varphi\, dx. \end{aligned}\right\} \tag{7.1}$$

We observe that the problem (7.1) is a type (P_h) problem with

$$a(u_h, v_h) = \int_\Omega u_{ij} v_{ij}\, dx, \tag{7.2}$$

$$b(v_h, \varphi_h) = \sum_{R_k \in \mathcal{T}_h} \left(- \int_{R_k} v_{ij/i} \varphi_{/j}\, dx + \int_{\partial R_k} M_{nt}(v) \varphi_{/t}\, ds \right). \tag{7.3}$$

In this paragraph we give bounds for the errors $\mathbf{u} - \mathbf{u}_h$ and $\psi - \psi_h$. To prove these bounds we shall refer to the abstract formulation that we have introduced in paragraph 1.

In particular we shall need to verify that the hypotheses (1.6) and (1.7) hold. Moreover, by lemma 1, it will be sufficient to prove (1.8).

Case 1

The condition (1.6), that is the ellipticity in H of the bilinear form $a(.,.)$, as we have already previously observed, is satisfied. For the other two conditions we prove the following lemmas:

LEMMA 2: *There exists a continous linear operator π_h: $V \to V_h$ which satisfies* (1.8) *and such that*

$$\|\mathbf{v} - \pi_h \mathbf{v}\|_0 \leqq C h, \tag{7.4}$$

where C is a constant independent of h.

Proof: We want to construct an application π_h that maps each $\mathbf{v} \in V$ into an element $\mathbf{w} = \pi_h \mathbf{v} \in V_h$ such that:

$$\forall \varphi \in W_h, \quad b(\mathbf{v} - \mathbf{w}, \varphi) = 0. \tag{7.5}$$

From (5.5), using Green's formula (4.2), we obtain the following equivalent expression for $b(\mathbf{v}, \varphi)$:

$$b(\mathbf{v}, \varphi) = \sum_{R_k \in \mathcal{T}_h} \left(\int_{R_k} v_{ij}\, \varphi_{/ij}\, dx - \int_{\partial R_k} M_n(\mathbf{v}) \varphi_{/n}\, ds \right). \tag{7.6}$$

As $\varphi \in W_h$, using (7.6), (7.5) becomes

$$\sum_{R_k \in \mathcal{T}_h} \left(\int_{R_k} (v_{12} - w_{12}) \varphi_{/xy} \, dx \, dy - \int_{\partial R_k} M_n(\mathbf{v} - \mathbf{w}) \varphi_{/n} \, ds \right) = 0. \tag{7.7}$$

To define $\mathbf{w}$ we must choose the value of $M_n(\mathbf{w})$ along each side and the value of w_{12} in each R_k. As $\varphi \in W_h$, setting $\varphi = a + bx + cy + dxy$ and denoting by l_i, $i = 1$, 4, the sides of R_k (with l_2 and l_4 vertical sides), on each R_k we have

$$b(\mathbf{v}, \varphi)\big|_{R_k} = d \int_{R_k} (v_{12} - w_{12}) \, dx \, dy - \int_{l_1, l_3} M_n(\mathbf{v} - \mathbf{w}) \, (c + dx) \, ds - \int_{l_2, l_4} M_n(\mathbf{v} - \mathbf{w}) \, (b + dy) \, ds.$$

For each side l_i we define $M_n(\mathbf{w})$ in the following way:

$$M_n(\mathbf{w})_{l_i} = \frac{1}{\mu(l_i)} \int_{l_i} M_n(\mathbf{v}) \, ds, \tag{7.8}$$

and so we easily have

$$c \int_{l_1, l_3} M_n(\mathbf{v} - \mathbf{w}) \, ds + b \int_{l_2, l_4} M_n(\mathbf{v} - \mathbf{w}) \, ds = 0.$$

Still we must choose w_{12} and we choose it so that

$$\int_{R_k} (v_{12} - w_{12}) \, dx \, dy = \int_{l_1, l_3} M_n(\mathbf{v} - \mathbf{w}) \, x \, ds + \int_{l_2, l_4} M_n(\mathbf{v} - \mathbf{w}) \, y \, ds, \tag{7.9}$$

that is, as $w_{12} = \text{Const.}$,

$$w_{12}\big|_{R_k} = \frac{1}{\mu(R_k)} \left(\int_{R_k} v_{12} \, dx \, dy - \int_{l_1, l_3} M_n(\mathbf{v} - \mathbf{w}) \, x \, ds - \int_{l_2, l_4} M_n(\mathbf{v} - \mathbf{w}) \, y \, ds \right). \tag{7.10}$$

Now we define π_h the operator that maps each $\mathbf{v} \in V$ into an element $\mathbf{w} \in V_h$ defined from (7.8), (7.10). By this choice (7.5) is satisfied. In a similar way we define on $\hat{R} = (0, 1) \times (0, 1)$ the operator $\hat{\pi}_h$ that maps each $\hat{\mathbf{v}} \in V$ into an element

$\hat{\mathbf{w}} \in V_h$ again defined from (7.8), (7.10), with $R_k = \hat{R}$. It's easy to verify that:

$$\left.\begin{array}{ll}\text{(i)} & \hat{\pi}_h \hat{\mathbf{v}} = \hat{\hat{\pi}}_h \hat{\hat{\mathbf{v}}}, \\ \text{(ii)} & \forall \mathbf{v} \in P_0(x, y), \quad \hat{\pi}_h \hat{\mathbf{v}} = \hat{\mathbf{v}}.\end{array}\right\} \tag{7.11}$$

In these conditions we can use the Bramble-Hilbert lemma (*see* [1]) obtaining

$$\| \hat{\mathbf{v}} - \pi_h \hat{\mathbf{v}} \|_{0,\hat{R}} \leqq C | \hat{\mathbf{v}} |_{1,\hat{R}}, \tag{7.12}$$

and therefore

$$\| \mathbf{v} - \mathbf{w} \|_{0,R_k}^2 = h^2 \| \hat{\mathbf{v}} - \hat{\mathbf{w}} \|_{0,\hat{R}}^2 \leqq C h^2 | \hat{\mathbf{v}} |_{1,\hat{R}}^2 = C h^2 | \mathbf{v} |_{1,R_k}^2, \tag{7.13}$$

from which

$$\| \mathbf{v} - \mathbf{w} \|_{0,R_k} \leqq C h | \mathbf{v} |_{1,R_k}. \tag{7.14}$$

Moreover in a similar way we obtain

$$\| \pi_h \mathbf{v} \|_{1,R_k} \leqq C \| \mathbf{v} \|_{1,R_k}. \quad \square \tag{7.15}$$

LEMMA 3: *Let $W = W_0^{1,p}(\Omega)$, $p > 2$, and $M = H_0^1(\Omega)$; then condition (1.7) holds.*

Proof: Let $\varphi \in W$; we choose $\tilde{\mathbf{v}} \in V$ so defined:

$$\tilde{v}_{11} = \tilde{v}_{22} = -\varphi, \qquad \tilde{v}_{12} = \tilde{v}_{21} = 0. \tag{7.16}$$

Then we have

$$b(\tilde{\mathbf{v}}, \varphi) = \sum_{R_k \in \mathcal{T}_h} \left(- \int_{R_k} (-\varphi_{/x}^2 - \varphi_{/y}^2)\, dx\, dy \right) = | \varphi |_{1,\Omega}^2. \tag{7.17}$$

But

$$\| \tilde{\mathbf{v}} \|_V^2 = \sum_{R_k \in \mathcal{T}_h} (\| \tilde{\mathbf{v}} \|_{1,R_k}^2) = 2 \| \varphi \|_{1,\Omega}^2, \tag{7.18}$$

from which, by Poincaré's inequality, we have

$$\| \tilde{\mathbf{v}} \|_V = \sqrt{2} \| \varphi \|_{1,\Omega} \leqq \sqrt{2}\, C_p | \varphi |_{1,\Omega}, \tag{7.19}$$

where C_p is the Poincaré constant. Hence

$$b(\tilde{\mathbf{v}}, \varphi) = | \varphi |_{1,\Omega}^2 \geqq \frac{1}{\sqrt{2}\, C_p} | \varphi |_{1,\Omega} \| \tilde{\mathbf{v}} \|_V, \tag{7.20}$$

from which

$$\frac{b(\tilde{\mathbf{v}}, \varphi)}{\| \tilde{\mathbf{v}} \|_V} \geqq \beta \| \varphi \|_{1,\Omega}, \tag{7.21}$$

with $\beta = 1/(\sqrt{2}\, C_p^2)$, and finally

$$\forall \varphi \in W, \quad \sup_{\mathbf{v} \in V} \frac{b(\mathbf{v}, \varphi)}{\|\mathbf{v}\|_V} \geqq \frac{b(\tilde{\mathbf{v}}, \varphi)}{\|\tilde{\mathbf{v}}\|_V} \geqq \beta \|\varphi\|_{1,\Omega}, \tag{7.22}$$

that is (1.7). □

Using lemma 2 and lemma 3, and recalling the abstract scheme of paragraph 1, we have the following theorems:

THEOREM 3: *If* $(\mathbf{u}, \psi)$ *and* $(\mathbf{u}_h, \psi_h)$ *are respectively the solutions of the problems* (5.6) *and* (7.1), *then we have:*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leqq C\left(\inf_{\mathbf{v}_h \in B_h(f)} \|\mathbf{u} - \mathbf{v}_h\|_0 + \sup_{\mathbf{v}_h \in V_h} \inf_{\varphi_h \in W_h} \frac{b(\mathbf{v}_h, \psi - \varphi_h)}{\|\mathbf{v}_h\|_0} \right), \tag{7.23}$$

where C is a constant depending on α, $\|a\|$, *but not depending on* $(\mathbf{u}, \psi)$, V_h *and* W_h.

Proof: Since the bilinear form $a(.,.)$ defined in (3.5) is elliptic in the space H defined from (3.2), and since the bilinear form $b(.,.)$ defined in (5.5) satisfies, using lemma 3, condition (1.7), theorem 3 is an obvious application of theorem 1. □

THEOREM 4: *If* $(\mathbf{u}, \psi)$ *and* $(\mathbf{u}_h, \psi_h)$ *are respectively the solutions of the problems* (5.6) *and* (7.1), *then we have:*

$$\|\psi - \psi_h\|_1 \leqq C\left(\|\mathbf{u} - \mathbf{u}_h\|_0 + \inf_{\varphi_h \in W_h} \sup_{\mathbf{v}_h \in V_h} \frac{b(\mathbf{v}_h, \psi - \varphi_h)}{\|\mathbf{v}_h\|_1} \right), \tag{7.24}$$

where C is a constant depending on β_h, $\|a\|$, σ_1, *but not depending on* $(\mathbf{u}, \psi)$, V_h *and* W_h.

Proof: Using lemma 2 and the same arguments used for theorem 3, theorem 4 is an obvious application of theorem 2. □

Now the problem consists in founding a bound for the terms that appear in the inequalities (7.23) and (7.24). For this we prove the following lemmas:

LEMMA 4: *There exists a positive constant C such that:*

$$\inf_{\mathbf{v}_h \in B_h(f)} \|\mathbf{u} - \mathbf{v}_h\|_0 \leqq C h. \tag{7.25}$$

Proof: The proof is contained in lemma 2 [*see* (7.12)], as $\mathbf{u}_h \in B_h(f)$ and $\mathbf{u}_h = \pi_h \mathbf{u}$. □

LEMMA 5: *There exists a positive constant C such that*

$$\sup_{\mathbf{v}\in V_h} \inf_{\varphi\in W_h} \frac{b(\mathbf{v}, \psi-\varphi)}{\|\mathbf{v}\|_0} \leqq Ch. \tag{7.26}$$

Proof At first we prove that if $\mathbf{v}\in V_h$, ψ is "smooth" and ψ^I is the interpolate of ψ, then

$$b(\mathbf{v}, \psi-\psi^I) \leqq Ch\|\mathbf{v}\|_0 |\psi|_3. \tag{7.27}$$

Let $\hat{R}=(0, 1)\times(0, 1)$. We set

$$b(\mathbf{v}, \psi-\psi^I)=\sum_{R_k\in\mathscr{T}_h} b_{R_k}(\mathbf{v}, \psi-\psi^I). \tag{7.28}$$

It's easy to verify that

$$b_{R_k}(\mathbf{v}, \psi-\psi^I)=b_{\hat{R}}(\hat{\mathbf{v}}, \hat{\psi}-\hat{\psi}^I). \tag{7.29}$$

For each $\hat{\mathbf{v}}\in V_h$ let us consider the following functional:

$$\pi: \quad \hat{\psi}\to b_{\hat{R}}(\hat{\mathbf{v}}, \hat{\psi}-\hat{\psi}^I)=\pi\hat{\psi}. \tag{7.30}$$

We prove the following statement:

$$\pi\hat{\psi}=0, \qquad \forall\,\hat{\psi}\in P_2(x, y). \tag{7.31}$$

In fact, recalling the spaces V_h and W_h, if $\hat{\psi}\in P_2(x, y)$ and $\hat{\psi}^I\in Q_1(x, y)$, then $\hat{\psi}-\hat{\psi}^I$ is a polynomial of the type $r.x.(1-x)+s.y.(1-y)$, with r, s, constants. Hence we have:

$$\begin{aligned} b_{\hat{R}}(\hat{\mathbf{v}}, \hat{\psi}-\hat{\psi}^I) &= -\int_{\hat{R}} \hat{v}_{ij/i}(\hat{\psi}-\hat{\psi}^I)_{/j}\,dx\,dy+\int_{\partial\hat{R}} M_{nt}(\mathbf{v})\,(\hat{\psi}-\hat{\psi}^I)_{/t}\,ds \\ &= -\int_{\hat{R}} \hat{v}_{11/x}(\hat{\psi}-\hat{\psi}^I)_{/x}\,dx\,dy-\int_{\hat{R}} \hat{v}_{22/y}(\hat{\psi}-\hat{\psi}^I)_{/y}\,dx\,dy \\ &\quad +\int_{\partial\hat{R}} \hat{v}_{12}(n_2^2-n_1^2)(\hat{\psi}-\hat{\psi}^I)_{/t}\,ds \\ &= -ar\int_{\hat{R}}(1-2x)\,dx\,dy-bs\int_{\hat{R}}(1-2y)\,dx\,dy \\ &\quad +c\left(r\int_{l_1\,l_3}(1-2x)\,dx-s\int_{l_2\,l_4}(1-2y)\,dy\right), \end{aligned} \tag{7.32}$$

where a, b, c, are constants depending on $\hat{\mathbf{v}}$. Now it's easy to verify that the terms of $b_{\hat{R}}(\hat{\mathbf{v}}, \hat{\psi}-\hat{\psi}^I)$ all vanish, both the terms on $\hat{\mathbf{R}}$ and the terms on $\partial\hat{R}=\bigcup_{i=1}^{4} l_i$, and so (7.31) is proved

From (7.29), using the Bramble-Hilbert lemma, we obtain

$$b_{\hat{R}}(\hat{\mathbf{v}}, \hat{\psi}-\hat{\psi}^i) \leqq c(\hat{\mathbf{v}}) \, |\hat{\psi}|_{3,\hat{R}}, \tag{7.33}$$

setting

$$c(\hat{\mathbf{v}}) = \sup_{\hat{\psi}' \in P_2(x,y)} \frac{b_{\hat{R}}(\hat{v}, \hat{\psi}-\hat{\psi}^i)}{|\hat{\psi}|_{3,\hat{R}}}. \tag{7.34}$$

It's easy to prove that

$$c(\hat{\mathbf{v}}) \leqq C \|\hat{\mathbf{v}}\|_{0,\hat{R}}, \tag{7.35}$$

and so

$$b_{\hat{R}}(\hat{\mathbf{v}}, \hat{\psi}-\hat{\psi}^i) \leqq C \|\hat{\mathbf{v}}\|_{0,\hat{R}} \, |\hat{\psi}|_{3,\hat{R}} = C h \|\mathbf{v}\|_{0,R_k} \, |\psi|_{3,R_k} \tag{7.36}$$

that is, with (7.28), (7.27). Now we have:

$$\forall \mathbf{v} \in V_h, \quad \frac{b(\mathbf{v}, \psi-\psi^i)}{\|\mathbf{v}\|_0} \leqq C h |\psi|_3, \tag{7.37}$$

$$\forall \mathbf{v} \in V_h, \quad \inf_{\varphi \in W_h} \frac{b(\mathbf{v}, \psi-\varphi)}{\|\mathbf{v}\|_0} \leqq \frac{b(\mathbf{v}, \psi-\psi^i)}{\|\mathbf{v}\|_0}, \tag{7.38}$$

and hence we finally obtain (7.26). □

LEMMA 6: *There exists a positive constant C such that:*

$$\inf_{\varphi \in W_h} \sup_{\mathbf{v} \in V_h} \frac{b(\mathbf{v}, \psi-\varphi)}{\|\mathbf{v}\|_1} \leqq C h. \tag{7.39}$$

Proof: Let us recall the expression of $b(\mathbf{v}, \psi-\varphi)$:

$$b(\mathbf{v}, \psi-\varphi) = \sum_{R_k \in \mathscr{T}_h} \left(-\int_{R_k} v_{ij/i} (\psi-\varphi)_{/j} \, dx + \int_{\partial R_k} M_{nt}(\mathbf{v}) \, (\psi-\varphi)_{/t} \, ds \right). \tag{7.40}$$

We can choose $\varphi = \tilde{\psi}^i \in Q_1(x, y)$ a convenient interpolate of ψ so that the terms of $b(\mathbf{v}, \psi-\varphi)$ along ∂R_k vanish in each R_k. More exactly in each R_k we have

$$\int_{\partial R_k} M_{nt}(\mathbf{v}) \, (\psi-\tilde{\psi}^i)_{/t} \, ds = \sum_{i=1}^{4} \int_{l_i} M_{nt}(\mathbf{v}) \, (\psi-\tilde{\psi}^i)_{/t} \, ds, \tag{7.41}$$

from which, integrating by parts, we obtain

$$\sum_{i=1}^{4}\int_{l_i} M_{nt}(\mathbf{v})\,(\psi-\tilde{\psi}^I)_{/t}\,ds$$
$$=\sum_{i=1}^{4}\left(-\int_{l_i}\frac{\partial M_{nt}(\mathbf{v})}{\partial t}(\psi-\tilde{\psi}^I)\,ds+[M_{nt}(\mathbf{v})\,(\psi-\tilde{\psi}^I)]_{P_i}^{P_{i+1}}\right), \quad (7.42)$$

where P_i, $i=1,4$, are the vertices of R_k $(P_5=P_1)$. But $\partial M_{nt}(\mathbf{v})/\partial t=0$, and so it is sufficient to choose $\tilde{\psi}^I=\psi$ at the vertices P_i so that (7.42) vanishes. So we obtain

$$b(\mathbf{v},\ \psi-\tilde{\psi}^I)=\sum_{R_k\in\mathcal{T}_l}\left(-\int_{R_k} v_{ij/i}(\psi-\tilde{\psi}^I)_{/j}\,dx\right), \quad (7.43)$$

from which follows

$$b(\mathbf{v},\ \psi-\tilde{\psi}^I)\leqq C\|\mathbf{v}\|_1\,\|\psi-\tilde{\psi}^I\|_1. \quad (7.44)$$

We set

$$G(\varphi)=\sup_{\mathbf{v}\in V_h}\frac{b(\mathbf{v},\ \psi-\varphi)}{\|\mathbf{v}\|_1}=\sup_{\mathbf{v}\in V_h}H(\varphi,\ \mathbf{v}). \quad (7.45)$$

We have

$$\inf_{\varphi\in W_h}G(\varphi)\leqq G(\tilde{\psi}^I)=\sup_{\mathbf{v}\in V_h}H(\tilde{\psi}^I,\mathbf{v}), \quad (7.46)$$

$$\forall\,\mathbf{v}\in V_h,\quad H(\tilde{\psi}^I,\ \mathbf{v})\leqq\|\psi-\tilde{\psi}^I\|_1, \quad (7.47)$$

from which

$$\sup_{\mathbf{v}\in V_h}H(\tilde{\psi}^I,\ \mathbf{v})=G(\tilde{\psi}^I)\leqq\|\psi-\tilde{\psi}^I\|_1, \quad (7.48)$$

and finally

$$\inf_{\varphi\in W_h}\ \sup_{\mathbf{v}\in V_h}\frac{b(\mathbf{v},\ \psi-\varphi)}{\|\mathbf{v}\|_1}\leqq\|\psi-\tilde{\psi}^I\|_1, \quad (7.49)$$

but

$$\|\psi-\tilde{\psi}^I\|_1\leqq Ch \quad (7.50)$$

from which (7.39) easily follows. □

Now we can give the main result:

THEOREM 5: *If theorem 3, theorem 4, and lemma 4, lemma 5, lemma 6, hold, then we have:*

$$\|\mathbf{u}-\mathbf{u}_h\|_0+\|\psi-\psi_h\|_1\leqq Ch, \quad (7.51)$$

where C is a constant not depending on the decomposition. □

Case 2

Now we extend the so far obtained results to the case in which the spaces V_h and W_h are the spaces defined in the discretisation 2. The condition (1.6) obviously still holds. Lemma 3 is valid as it does not depend on the spaces V_h and W_h. We must prove the validity of the condition (1.8) and hence we give the following lemma:

LEMMA 7: *There exists a continous linear operator π_h: $V \to V_h$ which satisfies* (1.8) *and such that:*

$$\|\mathbf{v}-\pi_h \mathbf{v}\|_0 \leqq C h^2, \tag{7.52}$$

where C is a constant not depending on h.

Proof: Denoting by $\mathbf{w}=\pi_h \mathbf{v}$, we must construct an application π_h such that, for each $\mathbf{v}$,

$$\forall \varphi \in W_h, \quad b(\mathbf{v}-\mathbf{w}, \varphi)=0. \tag{7.53}$$

In each $R_k \in \mathcal{T}_h$ we must choose 14 d. o. f. to determine $\mathbf{w} \in V_h$. At first we choose the following 8 d. o. f.:

$$\int_{l_i} M_n(\mathbf{v}-\mathbf{w})\, p\, ds, \qquad i=1,\ 4, \quad p \in P_1. \tag{7.54}$$

We observe that with this choice of $M_n(\mathbf{w})$ along ∂R_k the equation (7.53) is satisfied $\forall \varphi \in P_1$.

Moreover when $\varphi \in W_h$ assumes respectively the values x^2, y^2, xy, $x^2 y$, xy^2, $x^2 y^2$, we obtain the further following six conditions:

$$\varphi=x^2, \qquad \int_{R_k} (v_{11}-w_{11})\, dx\, dy = \int_{l_2, l_4} M_n(\mathbf{v}-\mathbf{w})\, ds, \tag{1}$$

$$\varphi=y^2, \qquad \int_{R_k} (v_{22}-w_{22})\, dx\, dy = \int_{l_1, l_3} M_n(\mathbf{v}-\mathbf{w})\, ds, \tag{2}$$

$$\left.\begin{array}{c} \varphi=xy, \\ \displaystyle\int_{R_k} (v_{12}-w_{12})\, dx\, dy = \int_{l_1, l_3} M_n(\mathbf{v}-\mathbf{w})\, x\, ds + \int_{l_2, l_4} M_n(\mathbf{v}-\mathbf{w})\, y\, ds, \end{array}\right\} \tag{3}$$

$$\left.\begin{array}{c} \varphi=x^2 y, \\ \displaystyle\int_{R_k} (v_{11}-w_{11})\, y\, dx\, dy + \int_{R_k} (v_{12}-w_{12})\, x\, dx\, dy = \int_{l_1, l_3} M_n(\mathbf{v}-\mathbf{w})\, x^2\, ds, \end{array}\right\} \tag{4}$$

$$\left.\begin{aligned}&\varphi = xy^2,\\&\int_{R_k}(v_{12}-w_{12})\,y\,dx\,dy+\int_{R_k}(v_{22}-w_{22})\,x\,dx\,dy = \int_{l_2,l_4} M_n(\mathbf{v}-\mathbf{w})\,y^2\,ds\end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned}&\varphi = x^2y^2,\\&\int_{R_k}(v_{11}-w_{11})\,y^2\,dx\,dy\\&\quad+\int_{R_k}(\mathbf{v}_{12}-\mathbf{w}_{12})\,xy\,dx\,dy+\int_{R_k}(\mathbf{v}_{22}-\mathbf{w}_{22})\,x^2\,dx\,dy\\&\quad=\int_{l_1\,l_3} M_n(\mathbf{v}-\mathbf{w})\,x^2\,ds+\int_{l_2,l_4} M_n(\mathbf{v}-\mathbf{w})\,y^2\,ds.\end{aligned}\right\} \tag{6}$$

We observe that (7.54) and (1) determine the value of w_{11} in each R_k; in a similar way (7.54) and (2) determine the value of w_{22}. The other four conditions can be considered conditions for w_{12} and determine its value.

By this choice we obtain the desired function $\mathbf{w}=\pi_h\mathbf{v}$ such that (7.53) is satisfied. In a similar way we define the application $\hat{\pi}_h$ in $\hat{R}$. It is easy to verify that:

$$\left.\begin{aligned}&\text{(i)} && \widehat{\pi_h\mathbf{v}}=\hat{\pi}_h\hat{\mathbf{v}},\\&\text{(ii)} && \forall\,\mathbf{v}\in P_1(x,y),\quad \hat{\pi}_h\hat{\mathbf{v}}=\hat{\mathbf{v}}.\end{aligned}\right\} \tag{7.55}$$

Now we can use the Bramble-Hilbert lemma and we obtain

$$\|\hat{\mathbf{v}}-\pi_h\hat{\mathbf{v}}\|_{1,\hat{R}} \leqq C\,|\hat{\mathbf{v}}|_{1,\hat{R}}, \tag{7.56}$$

and therefore

$$\|\mathbf{v}-\mathbf{w}\|_{0,R}^2=h^2\,\|\hat{\mathbf{v}}-\hat{\mathbf{w}}\|_{0,\hat{R}}^2 \leqq h^4\,\|\hat{\mathbf{v}}-\hat{\mathbf{w}}\|_{1,\hat{R}}^2 \leqq C\,h^4\,|\hat{\mathbf{v}}|_{1,\hat{R}}^2=C\,h^4\,|\hat{\mathbf{v}}|_{1,R}^2, \tag{7.57}$$

from which

$$\|\mathbf{v}-\mathbf{w}\|_{0,R} \leqq C\,h^2\,|\mathbf{v}|_{1,R}.\quad \square \tag{7.58}$$

Using this lemma and what we previously said, theorem 3 and theorem 4 are still valid.

We must now bound again the terms which appear in the inequalities (7.23) and (7.24). For this we give the following lemmas:

LEMMA 8: *There exists a positive constant C such that:*

$$\inf_{\mathbf{v}_h \in B_h(f)} \|\mathbf{u}-\mathbf{v}_h\|_0 \leqq C h^2. \tag{7.59}$$

Proof: The proof is contained in lemma 7 [*see* (7.52)]. □

LEMMA 9: *There exists a positive constant C such that:*

$$\sup_{\mathbf{v}_h \in V_h} \inf_{\varphi \in W_h} \frac{b(\mathbf{v}_h, \psi-\varphi)}{\|\mathbf{v}_h\|_0} \leqq C h^2. \tag{7.60}$$

Proof: The proof is the same as in lemma 5 with the only difference that now we have

$$\pi\hat{\psi}=0, \qquad \forall\, \hat{\psi}\, P_3(x, y). \quad \square \tag{7.61}$$

LEMMA 10: *There exists a positive constant C such that:*

$$\inf_{\varphi \in W_h} \sup_{\mathbf{v}_h \in V_h} \frac{b(\mathbf{v}_h, \psi-\varphi)}{\|\mathbf{v}_h\|_1} \leqq C h^2. \tag{7.62}$$

Proof: Again we choose $\varphi=\tilde{\psi}^i \in Q_2(x, y)$ a convenient interpolate of in order to vanish, in each R_k, the component along ∂R_k of $b(\mathbf{v}, \psi-\tilde{\psi}^i)$ [*see* (7.40)]. Now $v_{12} \in Q_1(x, y)$, $v_{12}|_{l_1, l_3} \in P_1(x)$, $v_{12}|_{l_2, l_4} \in P_1(y)$, so that $\partial M_{nt}(\mathbf{v})/\partial t$ [*see* (7.42)] is constant along each side l_i, $i=1, 4$. Choosing $\tilde{\psi}^i|_{l_i}$ such that $\int_{l_i} \tilde{\psi}^i\, ds=\int_{l_i} \psi\, ds$ and $\tilde{\psi}^i=\psi$ at the vertices of each rectangle we obtain the desired function $\tilde{\psi}^i \in Q_2(x, y)$. Proceeding as in lemma 6 we obtain (7.62). □

Now we can give the following result:

THEOREM 6: *If theorem* 3, *theorem* 4, *and lemma* 8, *lemma* 9, *lemma* 10, *hold, then we have:*

$$\|\mathbf{u}-\mathbf{u}_h\|_0+\|\psi-\psi_h\|_1 \leqq C h^2, \tag{7.63}$$

where C is a costant not depending on the decomposition. □

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