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# GRADIENT METHODS FOR THE CONSTRUCTION OF LJUSTERNIK-SCHNIRELMANN CRITICAL VALUES (*)

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Abstract. — *In this paper gradient methods are proposed for the search for the Ljusternik-Schnirelmann critical values and the corresponding critical vectors of a functional g even with respect to the unit sphere. The paper describes a discretization of a continuous method proposed earlier by one of the authors.*

Résumé. — *Cet article propose les méthodes du gradient pour trouver les valeurs critiques et les vecteurs critiques correspondant de Ljusternik-Schnirelmann de la fonctionnelle g paire par rapport à la sphère unitée. Le papier représente la discrétisation d'une méthode continue proposée par un des auteurs.*

## 1. INTRODUCTION

Existence theorems for nonlinear eigenvalue problems in the form

$$\mu f'(x) - g'(x) = 0,$$

where $f$ and $g$ are functionals on a Hilbert space $H$, and $f'(x)$ and $g'(x)$ are the corresponding gradients, are considered in many papers (for an extensive list of references *see* S. Fučik, J. Nečas, J. Souček and V. Souček [2]). These existence theorems are based on the existence of a critical vector of $g(x)$ with respect to the manifold $M_r(f) = \{x \in H; f(x) = r\}$. Under suitable conditions it is proved that there exist at least one eigenvector, or an infinite number of eigenvectors, on the manifold $M_r(f)$.

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Iteration methods for the construction of an eigenvector and the corresponding eigenvalue are considered by M. A. Altman [1], J. Schröder [6], and W. Petry [5] where the Newton method, or the gradient method, is applied.

For all the Ljusternik-Schnirelmann critical values and critical vectors, a numerical approach was proposed in the paper by J. Nečas [4].

For the construction of the first Ljusternik-Schnirelmann critical value and the corresponding eigenvector and eigenvalue, the secant modulus method is used in the paper by the authors [3].

In this paper we shall consider, for the sake of simplicity, the eigenvalue problem

$$\mu x - g'(x) = 0,$$

in a Hilbert space $H$, where $g'(x)$ is the gradient of an even functional $g(x)$. For finding all the Ljusternik-Schnirelmann values of the functional $g(x)$ with respect to the sphere $S$, we shall construct some modifications of the method of steepest descent.

## 2. ITERATIVE CONSTRUCTION OF THE FIRST LJUSTERNIK-SCHNIRELMANN CRITICAL VALUE

Let $H$ be a real Hilbert space with the inner product $(.,.)$ and norm $\|.\|$. Further we set $S = \{ x \in H; \|x\| = 1 \}$. Let $g$ be an even functional (nonquadratic, generally) on $H$ possessing the Fréchet differential $g'(x)$ at each $x \in H$. Let $g'(x)$ be strongly continuous on $H$, i.e., for each sequence $\{x_n\}_{n=1}^{\infty} \subset H$ weakly converging to $x_0 \in H$, the sequence $\{g'(x_n)\}_{n=1}^{\infty}$ converges to $g'(x_0)$.

Let $M$ be a positive number. Suppose that for each $x, y \in H$, the following conditions are fulfilled:

$$(g'(x+h) - g'(x), h) \leqq M \|h\|^2, \tag{2.1}$$

$$(g'(x+h) - g'(x), h) > 0 \qquad \text{for} \quad h \neq 0, \tag{2.2}$$

$$g(0) = 0, \tag{2.3}$$

$$g'(0) = 0. \tag{2.4}$$

THEOREM 2.1: *Let the above assumptions be fulfilled. Let $x_1$ be an arbitrary initial approximation from S. If the sequence $\{x_n\}_{n=1}^{\infty} \subset S$ is defined by*

$$x_{n+1}=\frac{x_n+(1/2M)g'(x_n)}{\|x_n+(1/2M)g'(x_n)\|}, \tag{2.5}$$

*then each subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ contains a subsequence $\{x_{n_{k_j}}\}_{j=1}^{\infty}$ such that*

$$\lim_{j\to\infty}\|x_{n_{k_j}}-x_0\|=0,$$

$$\lim_{j\to\infty}\left\{\left[\left\|x_{n_{k_j}}+\frac{1}{2M}g'(x_{n_{k_j}})\right\|-1\right]2M-\mu\right\}=0,$$

$$\lim_{n\to\infty}(g(x_n)-g(x_0))=0,$$

*and*

$$\mu x_0-g'(x_0)=0. \tag{2.6}$$

*Proof*: From (2.2) we get

$$\left\|x_n+\frac{1}{2M}g'(x_n)\right\|>1 \tag{2.7}$$

for an arbitrary integer $n$.

By a simple calculation we obtain

$$\begin{aligned}g(x_{n+1})-g(x_n)&=(g'(x_n+\tau(x_{n+1}-x_n)),\ x_{n+1}-x_n)\\ &\geqq (g'(x_n),\ x_{n+1}-x_n)-M\|x_{n+1}-x_n\|^2\\ &=2M\left\{\left\|x_n+\frac{1}{2M}g'(x_n)\right\|.\|x_n\|\right.\\ &\qquad\left.-\left(x_n+\frac{1}{2M}g'(x_n),\ x_n\right)\right\}\geqq 0,\end{aligned}$$

in virtue of (2.1), (2.2), and (2.5) and thus

$$\left.\begin{aligned}g(x_n)&\leqq g(x_{n+1}),\\ (g'(x_n),\ x_{n+1}-x_n)&\geqq M\|x_{n+1}-x_n\|^2.\end{aligned}\right\} \tag{2.8}$$

From the last inequality,

$$M\|x_{n+1}-x_n\|^2\leqq (g'(x_n),\ x_{n+1}-x_n)<g(x_{n+1})-g(x_n) \tag{2.9}$$

follows with respect to (2.2). The functional $g$ is bounded and we thus obtain

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0, \tag{2.10}$$

from (2.8) and (2.9).

The sequence $\{x_n\}_{n=1}^{\infty}$ is bounded; thus there exists a subsequence (in the sequel we shall denote each subsequence as the original) converging weakly to some $x_0 \in H$. Therefore $g'(x_n)$ and $g(x_n)$ converge to $g'(x_0)$ and $g(x_0)$, respectively.

From (2.2), (2.3), (2.4), and (2.8) we get

$$0 < g(x_n) \leqq g(x_0).$$

In virtue of (2.3) then

$$x_0 \neq 0. \tag{2.11}$$

There exists a subsequence of

$$\left\{ \left\| x_n + \frac{1}{2M} g'(x_n) \right\| \right\}_{n=1}^{\infty},$$

such that

$$\lim_{n\to\infty} \left\| x_n + \frac{1}{2M} g'(x_n) \right\| = c_0 \geqq 1, \tag{2.12}$$

with respect to (2.7).

Suppose that $c_0 = 1$. Then

$$\lim_{n\to\infty} \left\| x_n + \frac{1}{2M} g'(x_n) \right\|^2 \quad \lim_{n\to\infty} \left[ 1 + \frac{1}{M}(g'(x_n), x_n) + \frac{1}{4M^2} \| g'(x_n) \|^2 \right] = 1,$$

i.e.

$$\frac{1}{M}(g'(x_0), x_0) + \frac{1}{4M^2} \| g'(x_0) \|^2 = 0,$$

which contradicts (2.2) in virtue of (2.11). Thus

$$c_0 > 1. \tag{2.13}$$

From (2.5) we have

$$x_n = \frac{1}{\|x_n+(1/2\,M)g'(x_n)\|-1} \times \left[\frac{1}{2\,M}g'(x_n) - \left\|x_n + \frac{1}{2\,M}g'(x_n)\right\|(x_{n+1}-x_n)\right],$$

and thus in virtue of (2.10), (2.12), and (2.13) the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x_0$.

THEOREM 2.2: *Let the assumptions of theorem* 2.1 *be satisfied. Moreover let*

$$\sup_{x\in S}\|g'(x)\|^2 \leqq 2\,M^2. \tag{2.14}$$

*Let* $x_1$ *be an arbitrary initial approximation from* $S$. *If the sequence* $\{x_n\}_{n=1}^{\infty} \subset S$ *is defined by*

$$x_{n+1} = \lambda_n x_n + \frac{1}{2\,M}g'(x_n), \tag{2.15}$$

*with*

$$\lambda_n = -\frac{1}{2\,M}[(g'(x_n),\, x_n) - \sqrt{(g'(x_n),\, x_n)^2 - \|g'(x_n)\|^2 + 4\,M^2}\,], \tag{2.16}$$

*then each subsequence* $\{x_{n_k}\}_{k=1}^{\infty}$ *contains a subsequence* $\{x_{n_{k_j}}\}_{j=1}^{\infty}$ *such that*

$$\lim_{j\to\infty}\|x_{n_{k_j}} - x_0\| = 0,$$

$$\lim_{j\to\infty}(\|g'(x_{n_{k_j}})\| - \mu) = 0,$$

$$\lim_{n\to\infty}(g(x_n) - g(x_0)) = 0,$$

*and*

$$\mu x_0 - g'(x_0) = 0. \tag{2.6}$$

*Proof:* It is easy to see that

$$\left\|\lambda_n x_n + \frac{1}{2\,M}g'(x_n)\right\|^2 = 1,$$

for $x_n \in S$ with respect to (2.14), (2.15), and (2.16), i.e. $x_{n+1} \in S$.

From (2.2), (2.4), and (2.14), $\lambda_n > 0$ follows. Analogously as in theorem 2.1 we get

$$g(x_{n+1}) - g(x_n) \geqq (g'(x_n), x_{n+1} - x_n) - M \| x_{n+1} - x_n \|^2$$
$$= -M(\lambda_n - 1)^2 + \frac{1}{4M} \| g'(x_n) \|^2,$$

from (2.1), (2.2), and (2.15). We wish to show that

$$g(x_{n+1}) \geqq g(x_n).$$

It follows from (2.2) and (2.4) that

$$2M(\lambda_n - 1) < 0;$$

thus according to the above inequality it is sufficient to show that

$$\| g'(x_n) \| \geqq 2M(1 - \lambda_n).$$

This inequality is equivalent to

$$[(g'(x_n), x_n) + 2M - \| g'(x_n) \|]^2 \leqq (g'(x_n), x_n)^2 - \| g'(x_n) \|^2 + 4M^2,$$

in virtue of (2.2), (2.4), (2.14), and (2.16). According to (2.14),

$$(g'(x_n), x_n)^2 + 4M^2 + \| g'(x_n) \|^2 + 4M(g'(x_n), x_n)$$
$$-2(g'(x_n), x_n) \| g'(x_n) \| - 4M \| g'(x_n) \| - (g'(x_n), x_n)^2$$
$$+ \| g'(x_n) \|^2 - 4M^2$$
$$= 2[\| g'(x_n) \| - (g'(x_n), x_n)] [\| g'(x_n) \| - 2M] \leqq 0,$$

and thus

$$g(x_{n+1}) - g(x_n) \geqq 0.$$

The rest of the proof now follows as in theorem 2.1.

COROLLARY 2.1: *If, in addition to the assumptions of theorems* 2.1 *or* 2.2, *we assume that* (2.6) *has only isolated solutions on S, then the whole sequence* $\{x_n\}_{n=1}^{\infty}$ *converges to an element* $x_0$ *satisfying* (2.6), *moreover the whole sequences*

$$\left\{ \left[ \left\| x_n + \frac{1}{2M} g'(x_n) \right\| - 1 \right] 2M \right\}_{n=1}^{\infty} \quad \text{or} \quad \{ \| g'(x_n) \| \}_{n=1}^{\infty},$$

*respectively, converge to a number* $\mu$ *satisfying* (2.6).

*Proof:* The assertion follows analogously as in the paper [3] by the authors.

COROLLARY 2.2: *In addition to the assumptions of theorems 2.1 or 2.2, we assume that $\gamma_1$ is the first critical value of the functional g with respect to the sphere S. Furthermore, let there exist a constant $\varepsilon > 0$ such that there is no critical value in the interval $(\gamma_1 - \varepsilon, \gamma_1)$. Let $x_1 \in H$, $g(x_1) > \gamma_1 - \varepsilon$.*

*Then for each limit point $x_0$ of the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (2.5) or (2.15), respectively, we have*

$$g(x_0) = \operatorname*{Max}_{x \in S} g(x) = \gamma_1.$$

*Proof:* The assertion follows from theorems 2.1, or 2.2, respectively.

## 3. ITERATIVE CONSTRUCTION OF THE LJUSTERNIK-SCHNIRELMANN CRITICAL VALUES

For the convenience of the reader we shall briefly recall principal definitions and results concerning the Ljusternik-Schnirelmann theory in a Hilbert space.

Let $K$ be a symmetric closed set in $H$. We say that ord $K = 0$ if $K$ is empty; that ord $K = 1$ if $K = K_1 \cup K_2$, where the $K_i$ are closed subsets of $K$ and neither $K_1$ nor $K_2$ contains antipodal points; that ord $K = n$ if $K = \bigcup_{i=1}^{n+1} K_i$, where the $K_i$ are closed subsets of $K$ not containing antipodal points and $n$ is the least possible number; and that ord $K = \infty$ if no such $n$ exists.

Let $V_n = \{K;\ K \subset S$ is a symmetric compact set and ord $K \geqq n\}$. Let

$$\gamma_k = \sup_{K \in V_k} \min_{x \in K} g(x).$$

The fundamental theorem of Ljusternik and Schnirelmann is the following:

THEOREM 3.1: *Under the assumptions of theorem 2.1 there exist $x_k \in S$, $k = 1, 2, \ldots$ such that*

$$g'(x_k) - \mu_k x_k = 0,$$

$$g(x_k) = \gamma_k, \qquad \gamma_k \searrow 0, \qquad x_k \to 0 \ (\textit{weakly}).$$

The proof, which is in a very easy version given in the paper [4] by J. Nečas, is based on the Ljusternik-Schnirelmann principle of critical values which, roughly speaking, means that for every $\gamma_k$, there exists a saddle point $x_k$ such that $\gamma_k = g(x_k)$ and

$$g'(x_k) - \mu_k x_k = 0.$$

The proof of the Ljusternik-Schnirelmann principle can be done as in paper [4] by deformations of sets of prescribed order along the trajectories of the solutions of differential equations on $S$,

$$\dot{x} = g'(x) - x(x, g'(x)), \qquad x(0) = x_0 \in S,$$

(for this equation, see also M. M. Vajnberg [7], theorem 14.1). For details, see e.g. S. Fučik, J. Nečas, J. Souček and V. Souček [2].

Let the assumptions of theorem 2.1 hold for a functional $g$. Let $\gamma_1$ and $\gamma_2$ be the first and second Ljusternik-Schnirelmann critical values of the functional $g$ with respect to the sphere $S$, $\gamma_1 > \gamma_2$. Furthermore, let there exist a constant $\varepsilon > 0$ such that there is no critical value in the interval $(\gamma_2 - \varepsilon, \gamma_2)$. Let $K_1$ be a compact symmetric subset of $S$, ord $K_1 \geqq 2$ (e.g. $K_1 = L \cap S$, $L$ is a subspace of $H$, and dim $L = 2$),

$$\gamma_2 - \varepsilon < \operatorname*{Min}_{x \in K_1} g(x) < \gamma_2. \tag{3.1}$$

For $x \in K_1$, put

$$x_{n+1}(x) = \frac{x_n(x) + (1/2\, M)\, g'(x_n(x))}{\| x_n(x) + (1/2\, M)\, g'(x_n(x)) \|}, \tag{3.2}$$

where $x_1(x) = x$.

Let $x_n^{(0)}$ be a vector from $K_1$ such that

$$\operatorname*{Min}_{x \in K_1} g(x_n(x)) = g(x_n(x_n^{(0)})) \tag{3.3}$$

for an arbitrary integer $n$.

THEOREM 3.1: *Let the above assumptions be fulfilled. Then the following assertions hold:*

(i) $$\lim_{n \to \infty} g(x_n(x_n^{(0)})) = \gamma_2;$$

(ii) *there exists* $x^{(0)} \in K_1$ *such that*

$$\lim_{n \to \infty} g(x_n(x^{(0)})) = \gamma_2;$$

(iii) *each subsequence* $\{x_{n_k}^{(0)}\}_{k=1}^{\infty}$ *contains a subsequence* $\{x_{n_{k_j}}^{(0)}\}_{j=1}^{\infty}$ *such that*

$$\lim_{j \to \infty} \| x_{n_{k_j}}^{(0)} - x^{(0)} \| = 0,$$

*and* $x^{(0)}$ *satisfies* (ii);

(iv) *for each* $x^{(0)}$ *satisfying* (ii), *each subsequence* $\{x_{n_k}(x^{(0)})\}_{k=1}^{\infty}$ *contains a subsequence* $\{x_{n_{k_j}}(x^{(0)})\}_{j=1}^{\infty}$ *such that*

$$\lim_{j \to \infty} \| x_{n_{k_j}}(x^{(0)}) - x_0 \| = 0,$$

$$\lim_{j \to \infty} \left[ \left( \left\| x_{n_{k_j}}(x^{(0)}) + \frac{1}{2M} g'(x_{n_{k_j}}(x^{(0)})) \right\| - 1 \right) 2M - \mu \right] = 0,$$

*and*

$$\mu x_0 - g'(x_0) = 0. \tag{2.6}$$

*Proof*: From theorem 2.1 we have

$$g(x_n(x)) \leqq g(x_{n+1}(x)), \tag{3.4}$$

for each $x \in K_1$.

Put

$$\varphi(x) = \frac{x + (1/2M) g'(x)}{\| x + (1/2M) g'(x) \|},$$

for $x \in S$.

Then $\varphi$ is an odd continuous operator from $S$ into $S$ and thus

$$\text{ord } K_{n+1} = \text{ord } \varphi(K_n) \geqq \ldots \geqq \text{ord } K_1 = 2,$$

where

$$K_{n+1} = \left\{ x_{n+1} \in H;\ x_{n+1} \frac{x_n + (1/2M) g'(x_n)}{\| x_n + (1/2M) g'(x_n) \|},\ x_n \in K_n \right\}.$$

From this we immediately get

$$\lim_{n\to\infty} \operatorname*{Min}_{x\in K_1} g(x_n(x)) \leqq \gamma_2, \tag{3.5}$$

in virtue of the definition of $\gamma_2$. This limit exists because of (3.4).

Put

$$\lim_{n\to\infty} \operatorname*{Min}_{x\in K_1} g(x_n(x)) = \varkappa. \tag{3.6}$$

Furthermore, there exist a subsequence of $\{x_n^{(0)}\}_{n=1}^{\infty}$ (we use the same notation for it as for the previous one) and $x^{(0)} \in K_1$ such that

$$\lim_{n\to\infty} \| x_n^{(0)} - x^{(0)} \| = 0 \tag{3.7}$$

and, with respect to (3.3) and (3.6),

$$\lim_{n\to\infty} g(x_n(x_n^{(0)})) = \varkappa. \tag{3.8}$$

According to theorem 2.1,

$$\lim_{n\to\infty} g(x_n(x^{(0)})),$$

is a critical value of the functional $g$ with respect to $S$, thus in virtue of (3.1), (3.4) and the assumption that there is no critical value in the interval $(\gamma_2 - \varepsilon, \gamma_2)$, we obtain

$$\lim_{n\to\infty} g(x_n(x^{(0)})) \geqq \gamma_2.$$

Hence with respect to (3.7) there exist integers $n_0$ and $n_1$ such that

$$g(x_{n_0}(x_n^{(0)})) \geqq \gamma_2 - \eta, \tag{3.9}$$

for each $\eta > 0$ and each $n \geqq n_1$.

According to (3.4) and (3.9) this implies that there exists an integer $n_1 \geqq n_0$ such that

$$g(x_n(x_n^{(0)})) \geqq g(x_{n_0}(x_n^{(0)})) \geqq \gamma_2 - \eta, \tag{3.10}$$

for each integer $n \geqq n_1$.

From (3.8) and (3.10) we obtain

$$\varkappa = \lim_{n\to\infty} g(x_n(x_n^{(0)})) \geqq \gamma_2 - \eta$$

for each $\eta > 0$ and thus

$$\lim_{n\to\infty} g(x_n(x_n^{(0)}))=\gamma_2$$

in virtue of (3.3) and (3.5).

The rest of the proof now follows as in theorem 2.1.

COROLLARY 3.1: *Let the assumptions of theorem* 2.1 *hold for a functional g. Let*

$$\gamma_1 \geqq \ldots \geqq \gamma_k > \gamma_{k+1} = \ldots = \gamma_{k+l} > \gamma_{k+l+1},$$

*be the Ljusternik-Schnirelmann critical values of the functional g with respect to the sphere S.*

*Let there exist a constant* $\varepsilon > 0$ *such that there is no critical value in the interval* $(\gamma_{k+l}-\varepsilon, \gamma_{k+l})$. *Let* $K_1$ *be a compact symmetric subset of S,*

$$\text{ord } K_1 \geqq k+1,$$

$$\gamma_{k+l}-\varepsilon < \min_{x\in K_1} g(x) < \gamma_{k+l}.$$

*For* $x \in K$, *let the sequences* $\{x_n(x)\}_{n=1}^{\infty}$ *and* $\{x_n^{(0)}\}_{n=1}^{\infty}$ *be defined by* (3.2) *and* (3.3), *respectively.*

*Then*

$$\lim_{n\to\infty} g(x_n(x_n^{(0)}))=\gamma_{k+l},$$

*and there exists* $x^{(0)} \in K_1$ *such that*

$$\lim_{n\to\infty} g(x_n(x^{(0)}))=\gamma_{k+l}.$$

*Moreover, the assertions* (iii) *and* (iv) *of theorem* 3.1 *hold.*

*Proof:* The proof is analogous to the proof of theorem 3.1.

COROLLARY 3.2: *If, in addition to the assumptions of theorem* 3.1 *or corollary* 3.1 *we assume that* (2.6) *has only isolated solutions on S, then the whole sequence* $\{x_n(x^{(0)})\}_{n=1}^{\infty}$ *converges to a vector* $x^{(0)}$ *satisfying* (ii) *and, moreover,*

$$\lim_{n\to\infty}\left[\left(\left\|x_n(x^{(0)})+\frac{1}{2M}g'(x_n(x^{(0)}))\right\|-1\right)2M-\mu\right]=0,$$

*where* $\mu$ *is a number satisfying* (2.6).

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