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PERTURBATION OF MIXED VARIATIONAL PROBLEMS. APPLICATION TO MIXED FINITE ELEMENT METHODS (*)

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Communiqué par P.-A. RAVIART

Abstract — *Degrees of freedom which are Lagrange multipliers arise in the finite element approximation of mixed variational problems. When these degrees of freedom are "local", the introduction of a small perturbation (corresponding by duality to a penalty function) enables the elimination of these unknowns at the element level. We first examine this method in the continuous case and show that the solution of the perturbed problem is close to that of the original one. We extend this result to the FEM. Several examples are given and the construction of a number of the element stiffness matrices is outlined.*

1. INTRODUCTION

Mixed finite element methods have been intensively used in the numerical solution of various physical problems; these include linear constraint problems such as the continuity equation in Stokes' equations.

The main feature of these methods is the introduction of a Lagrange multiplier in order to avoid the difficult problem of constructing basis functions which satisfy this constraint. This technique has two drawbacks: it introduces a large number of degrees of freedom corresponding to the multiplier, and the resulting stiffness matrix is not positive definite.

Moreover when the constraint is "local", i. e., imposed element by element, the corresponding Lagrange multiplier is also local, i. e. its basis functions have a support limited to one element. Thus it seems natural to try to eliminate the corresponding degrees of freedom at the element level (static condensation). However, this is not possible because of the non positivity of the element stiffness matrix. In this publication we show that the introduction of a small regular perturbation (or by duality an adequate penalty function) allows us to eliminate the unwanted degrees of freedom at the element level. Since this perturbation is

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regular we do not encounter the usual difficulties of the penalty function approach while avoiding the drawbacks of mixed finite element methods.

After recalling some notations we define our abstract variational problem and its perturbed version. We establish an approximation result relating the two sets of solutions and illustrate this by several examples. We then extend this theorem to mixed finite element methods. We finish by outlining the construction of some of the corresponding element stiffness matrices.

2. NOTATIONS AND PRELIMINARIES

In all our examples we consider a bounded open subset $\Omega \subset R^n$, with boundary Γ. We denote by $\mathbf{n}$ the unit outward normal along Γ. On Ω we introduce the Sobolev spaces:

$$H^m(\Omega) = \{ v \in L^2(\Omega) \mid \partial^\alpha v \in L^2(\Omega), \ |\alpha| \leqq m \},$$

where m is an integer and α an m-tuple integer with:

$$\alpha = (\alpha_1, \ldots, \alpha_n), \qquad \alpha_i \geqq 0, \qquad |\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$\partial^\alpha = \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^{\alpha_i}.$$

On $H^m(\Omega)$ we have the semi-norms:

$$|v|_{k,\Omega} = \left(\sum_{|\alpha|=k} \int |\partial^\alpha v|^2 \, dx \right)^{1/2}, \qquad k \leqq m$$

and the norm:

$$\| v \|_{m,\Omega} = \left(\sum_{k=0}^{m} |v|_{k,\Omega}^2 \right)^{1/2}.$$

We define:

$$H_0^1(\Omega) = \{ v \mid v \in H^1(\Omega), \ v|_\Gamma = 0 \}.$$

Given a vector valued function $\mathbf{q} \in \mathbf{X} = \prod_{i=1}^{s} X_i$, $\mathbf{q} = (q_1, \ldots, q_s)$ we set:

$$\| \mathbf{q} \|_{\mathbf{X}} = \left(\sum_{i=1}^{s} |q_i|_{X_i}^2 \right)^{1/2}.$$

We shall also need the following space:

$$\mathbf{H}(\text{div}; \Omega) = \{ \mathbf{q} \mid \mathbf{q} \in L^2(\Omega)^n; \ \text{div}\, \mathbf{q} \in L^2(\Omega) \},$$

with the norm:

$$\|\mathbf{q}\|_{H(\mathrm{div};\Omega)}=(\|\mathbf{q}\|_{0,\Omega}^2+\|\mathrm{div}\,\mathbf{q}\|_{0,\Omega})^{1/2}.$$

Let V be a real Hilbert space, V' its dual space, $(.,\ .)_V$ denotes the scalar product on V, $\|\ .\ \|_V$ the corresponding norm and $\langle\ .,\ .\ \rangle_{V',V}$ the duality product:

$$\langle y, x\rangle_{V',V}=y(x), \qquad y\in V', \quad x\in V.$$

When there is no risk of confusion, we will not indicate on which space the scalar product, the norm or the duality product are defined.

Given a second Hilbert space W, let $a(.,\ .)$ and $b(.,\ .)$ be two continuous bilinear forms on $V\times V$ and $V\times W$ respectively, we denote by $A\in\mathscr{L}(V;\ V')$ [resp. $B\in\mathscr{L}(V;\ W')$ and $B^*\in\mathscr{L}(W;\ V')$] the linear operator associated with the bilinear form $a(.,\ .)$ [resp. $b(.,\ .)$], i. e.:

$$\forall\,\psi,\quad \varphi\in V,\qquad \langle A\psi, \varphi\rangle=a(\psi, \varphi),$$
$$\forall(\psi, \mu)\in V\times W,\quad \langle B\psi, \mu\rangle=\langle\psi, B^*\mu\rangle=b(\psi, \mu).$$

Set:

$$\alpha=\sup_{\psi,\varphi\in V-\{0\}}\frac{|a(\psi, \varphi)|}{\|\psi\|\cdot\|\varphi\|},$$

we say that the bilinear form $a(.,\ .)$ is V =elliptic if $\alpha>0$.

We shall need the spaces:

$$Y=\mathrm{Ker}\,B=\{\psi\mid\psi\in V, b(\psi, \mu)=0, \forall\,\mu\in W\},$$
$$Z=\mathrm{Ker}\,B^*=\{\lambda\mid\lambda\in W, b(\psi, \lambda)=0, \forall\,\psi\in V\},$$
$$Z^0=\{g\mid g\in W', \langle g, \mu\rangle=0, \forall\,\mu\in Z\},$$
$$\widetilde{W}=W^{\perp}\text{ orthogonal set of }Z\text{ in }W.$$

REMARK 2.1: Since $b(.,\ .)$ is continuous, Z is a closed subset of W and we can make the following indentifications:

$$\widetilde{W}=\frac{W}{Z},$$
$$\widetilde{W}'=Z^0.$$

Our final aim is to construct an approximate solution of:

PROBLEM $P(f, g)$: *Given $f\in V'$ and $g\in Z^0$, find a pair $(\psi, \lambda)\in V\times W$ which satisfies:*

$$\left.\begin{aligned}\forall\,\varphi\in V,\quad a(\psi, \varphi)+b(\varphi, \lambda)&=\langle f, \varphi\rangle,\\ \forall\,\mu\in\widetilde{W},\quad b(\psi, \mu)&=\langle g, \mu\rangle.\end{aligned}\right\}\tag{2.1}$$

Note that this is equivalent to:

find $(\psi, \lambda) \in V \times \tilde{W}$ *such that:*

$$\left.\begin{aligned} A\psi + B^*\lambda &= f, \\ B\psi &= g. \end{aligned}\right\} \tag{2.2}$$

Introducing the closed affine manifold:

$$Y(g) = \{\psi \mid \psi \in V;\ B\psi = g\};$$

ψ can be characterized as a solution of:

find $\psi \in Y(g)$ *such that:*

$$\forall \varphi \in V, \quad a(\psi, \varphi) = \langle f, \varphi \rangle \tag{2.3}$$

Hence λ in (2.1) or (2.2) is the Lagrange multiplier corresponding to the linear constraint $\psi \in Y(g)$.

We can now state the basic abstract result:

THEOREM 2.1: *Under the following hypotheses:*

(*H* 1) *There exists a constant* $k \geqq 0$ *such that:*

$$\forall \mu \in \tilde{W}, \quad \sup_{\forall \varphi \in V - \{0\}} \frac{|b(\varphi, \mu)|}{\|\varphi\|_V} \geqq k \|\mu\|_W;$$

(*H* 2) *The bilinear form* $a(.,.)$ *is Y-elliptic;*

problem $P(f, g)$ *has a unique solution* $(\psi, \lambda) \in V \times \tilde{W}$.

This result is due to Brezzi [8] who actually proved that under (H 2), hypothesis (H 1) is a necessary and sufficient condition for problem $P(f, g)$ to be well posed. The importance of (H 1) was also pointed out by Babuška [2].

REMARK 2.2: Consider the Lagrangian functional $\mathscr{L}: V \times \tilde{W} \to R$:

$$\mathscr{L}(\varphi, \mu) = \frac{1}{2} a(\varphi, \varphi) - \langle f, \varphi \rangle + b(\varphi, \mu) - \langle g, \mu \rangle.$$

If the bilinear form $a(.,.)$ is symmetric positive and Y-elliptic and if $b(.,.)$ satisfies (H 1) then the Lagrangian functional $\mathscr{L}(.,.)$ has a unique saddle point $(\psi, \lambda) \in V \times W$:

$$\forall (\psi, \mu) \in V \times W, \quad \mathscr{L}(\varphi, \mu) \leqq \mathscr{L}(\psi, \lambda) \leqq \mathscr{L}(\varphi, \lambda);$$

moreover (ψ, λ) is the solution of problem $P(f, g)$. By a standard technique of convex analysis [14], it is natural to introduce the Lagrangian functional:

$$\mathscr{L}_\varepsilon(\varphi, \mu) = \mathscr{L}(\varphi, \mu) - \frac{\varepsilon}{2} \|\mu\|_W^2;$$

so that we have also a quadratic form in μ.

A classic duality argument shows that this is equivalent to the introduction of a penalty function in the primal problem:

$$\min_{\varphi \in Y(g)} \frac{1}{2} a(\varphi, \varphi) - \langle f, \varphi \rangle,$$

which becomes:

$$\min_{\varphi \in V} \frac{1}{2} a(\varphi, \varphi) - \langle f, \varphi \rangle + \frac{1}{2\varepsilon} \| B_\varphi - g \|_W^2 .$$

It is these perturbation techniques which we want to develop for problem $P(f, g)$ and its numerical approximation.

3. REGULARIZATION OF A MIXED VARIATIONAL PROBLEM

We now replace problem $P(f, g)$ by a slightly more regular one:

PROBLEM $P_\varepsilon(f, g)$: *Given $f \in V'$ and $g \in Z^0$, find a pair $(\psi_\varepsilon, \lambda_\varepsilon) \in V \times W$ such that*:

$$\left.\begin{aligned} &\forall \varphi \in V, \quad a(\psi_\varepsilon, \varphi) + b(\varphi, \lambda_\varepsilon) = \langle f, \varphi \rangle, \\ &\forall \mu \in W, \quad -\varepsilon(\lambda_\varepsilon, \mu) + b(\psi_\varepsilon, \mu) = \langle g, \mu \rangle. \end{aligned}\right\} \tag{3.1}$$

REMARK 3.1 : Note that here we are looking for a solution on $V \times W$ and not on $V \times \tilde{W}$. We shall see that in some practical cases this brings an important simplification to the actual computation of approximate solutions.

REMARK 3.2: Since this is the case in all the applications we are going to study, we make the following convenient assumption: $W = W'$.

We then can write:

$$\lambda_\varepsilon = \frac{1}{\varepsilon}(B\psi_\varepsilon - g),$$

so that (3.1) is equivalent to:

$$\forall \varphi \in V, \quad a(\psi_\varepsilon, \varphi) + \frac{1}{\varepsilon}(B\varphi, B\psi_\varepsilon - g)_W = \langle f, \varphi \rangle.$$

Formally we can state problem $P_\varepsilon(f, g)$ in two equivalent forms:

1° *find* $(\psi_\varepsilon, \lambda_\varepsilon)$ *such that:*

$$\left.\begin{aligned} A\psi_\varepsilon + B^* \lambda_\varepsilon &= f, \\ -\varepsilon\lambda_\varepsilon + B\psi &= g; \end{aligned}\right\} \tag{3.2}$$

2° *find* ψ_ε *such that:*

$$A\psi_\varepsilon + \frac{1}{\varepsilon} B^* (B\psi_\varepsilon - g) = f, \tag{3.3}$$

(3.2) and (3.3) illustrate the relation between penalization and regularization approaches, as mentioned in remark 2.2. □

THEOREM 3.1: *Assume that hypotheses* (H 1) *and* (H 2) *hold and that there exists a constant* $\alpha > 0$ *such that:*

(H 3) $$\forall \varphi \in V, \quad (|a(\varphi, \varphi)| + \|B\varphi\|_W^2)^{1/2} \geqq \alpha \|\varphi\|_V.$$

Then problem $P_\varepsilon(f, g)$ *has a unique solution* $(\psi_\varepsilon, \lambda_\varepsilon) \in V \times W$; *moreover* $\lambda_\varepsilon \in \tilde{W}$ *and if* (ψ, λ) *is the unique solution of problem* $P(f, g)$, *there exists an* $\varepsilon_0 > 0$, *such that for* $\varepsilon < \varepsilon_0$:

$$\|\psi - \psi_\varepsilon\|_V + \|\lambda - \lambda_\varepsilon\|_W \leqq C\varepsilon, \tag{3.4}$$

where C is a constant depending on α, k, f *and* g *only.*

(In all that follows C may designate different constants.) The proof of this theorem rests on:

LEMMA 3.1: *Assume that hypotheses* (H 1), (H 2) *and* (H 3) *hold then there exists an* $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$, *the unique solution of problem* $P_\varepsilon(0, g)$, *say* $(\Phi_\varepsilon, \delta_\varepsilon)$, *satisfies:*

$$\|\Phi_\varepsilon\|_V + \|\delta_\varepsilon\|_W \leqq C\|g\|_W, \tag{3.5}$$

where C is a constant independent of ε.

Proof of theorem 3.1: Using remark 3.2, we transform problem $P_\varepsilon(f, g)$ into: *find* $\psi_\varepsilon \in V$ *such that:*

$$\forall \varphi \in V, \quad a(\psi_\varepsilon, \varphi) + \frac{1}{\varepsilon}(B\varphi, B\psi_\varepsilon)_W = \langle f, \varphi \rangle + \frac{1}{\varepsilon}(B\varphi, g)_W. \tag{3.6}$$

By hypothesis (H 3) and the continuity assumptions on the bilinear forms, there exists a unique ψ_ε satisfying (3.6). The second equation in (3.1) implies that λ_ε must be unique and

$$\lambda_\varepsilon = \frac{1}{\varepsilon}(B\psi_\varepsilon - g).$$

Since $g \in Z^0$, for all $\mu \in Z = \operatorname{Ker} B^*$ we have:

$$-\varepsilon(\lambda_\varepsilon, \mu) = -b(\psi_\varepsilon, \mu) + \langle g, \mu \rangle$$

and

$$b(\psi_\varepsilon, \mu) = \langle g, \mu \rangle = 0.$$

Hence:

$$\forall \mu \in Z, \quad (\lambda_\varepsilon, \mu) = 0,$$

so that according to remark 2.1 $\lambda_\varepsilon \in \tilde{W}$. It remains to establish (3.4). Subtraction of (3.1) from (2.1) leads to:

$$\left.\begin{aligned} \forall \varphi \in V, \quad & a(\psi-\psi_\varepsilon, \varphi)+b(\varphi, \lambda-\lambda_\varepsilon)=0, \\ \forall \mu \in W, \quad & -\varepsilon(\lambda-\lambda_\varepsilon, \mu)+b(\psi-\psi_\varepsilon, \mu)=-\varepsilon(\lambda, \mu). \end{aligned}\right\} \tag{3.7}$$

Let (α, β) be the unique solution of problem $P_\varepsilon(0, -\varepsilon\lambda)$:

$$\left.\begin{aligned} \forall \varphi \in V, \quad & a(\alpha, \varphi)+b(\varphi, \beta)=0, \\ \forall \mu \in W, \quad & -\varepsilon(\beta, \mu)+b(\alpha, \mu)=-\varepsilon(\lambda, \mu). \end{aligned}\right\} \tag{3.8}$$

Define:

$$\left.\begin{aligned} \chi &= \psi-\psi_\varepsilon+\alpha, \\ \eta &= \lambda-\lambda_\varepsilon+\beta, \end{aligned}\right\} \tag{3.9}$$

adding (3.7) and (3.8) it is clear that:

$$\chi \equiv 0 \quad \text{and} \quad \eta \equiv 0. \tag{3.10}$$

By lemma 3.1, for $\varepsilon < \varepsilon_0$:

$$\|\alpha\|_V+\|\beta\|_W \leqq \varepsilon C \|\lambda\|_W; \tag{3.11}$$

where $\|\lambda\|_W$ depends on f and g only. Inequality (3.4) now results from (3.9), (3.10) and (3.11). □

Proof of lemma 3.1: Lemma 3.1 is not used in the existence part of theorem 3.1 so that by a similar argument it is clear that there is a unique solution $(\Phi_\varepsilon, \delta_\varepsilon)$ to problem $P_\varepsilon(0, g)$.

Since:

$$\forall \varphi \in V, \quad a(\Phi_\varepsilon, \varphi)=-b(\varphi, \delta_\varepsilon),$$

then:

$$C\|\Phi_\varepsilon\|_V \cdot \|\varphi\|_V \geqq |b(\varphi, \delta_\varepsilon)|.$$

Applying (H 1) to this inequality gives:

$$\|\delta_\varepsilon\|_W \leqq C\|\Phi_\varepsilon\|_V. \tag{3.12}$$

By summing the two equations of problem $P_\varepsilon(0, g)$:

$$a(\Phi_\varepsilon, \Phi_\varepsilon)+b(\Phi_\varepsilon, \delta_\varepsilon)-b(\Phi_\varepsilon, \delta_\varepsilon)+\varepsilon\|\delta_\varepsilon\|_V^2=-\langle g, \delta_\varepsilon\rangle$$

and by (3.12):

$$|a(\Phi_\varepsilon, \Phi_\varepsilon)| \leqq (C_1\|g\|+\varepsilon C_2\|\Phi_\varepsilon\|_V)\|\Phi_\varepsilon\|_V.$$

Moreover:

$$\| B\Phi_\varepsilon \|_W \leqq C_3 \| g \| + \varepsilon C_1 \| \Phi_\varepsilon \|_V .$$

So that:

$$| a(\Phi_\varepsilon, \Phi_\varepsilon) | + \| B\Phi_\varepsilon \|_W^2 \leqq (\varepsilon^2 C_1^2 + \varepsilon C_2) \| \Phi_\varepsilon \|^2 + \| g \| (C_3 + 2\varepsilon C_4) \| \Phi_\varepsilon \| + C_4 \| g \|^2 .$$

Taking ε_0 such that:

$$\varepsilon_0^2 C_1^2 + \varepsilon_0 C_2 < \frac{1}{2}$$

and applying hypotheses (H 3) we get for $\varepsilon < \varepsilon_0$:

$$\| \Phi_\varepsilon \|_V \leqq C \| g \|_W ,$$

where C is independent of ε and Φ_ε for $\varepsilon < \varepsilon_0$ (3.5) is a consequence of this inequality and of (3.12). □

REMARK 3.3: Hypothesis (H 3) plays a central role in obtaining (3.4), if we limit ourselves to (H 1) and (H 2) problem P_ε (f, g) is still well posed but we have only:

$$\| \psi - \psi_\varepsilon \|_V + \| \lambda - \lambda_\varepsilon \|_W \leqq C \varepsilon^{1/2}. \tag{3.13}$$

We do not give here a demonstration of (3.13) since in all the examples we study (H 3) is satisfied. □

4. EXAMPLES

We start by the example which actually motivated this study:

Example 1: *Stokes stationary equations*. Let $\Omega \subset \mathrm{R}^N$ be a bounded open set: Let:

$$V = [H_0^1(\Omega)]^N,$$

$$W = L^2(\Omega),$$

$$\tilde{W} = \left\{ p \,|\, p \in L^2(\Omega), \int_\Omega p \, d\Omega = 0 \right\}$$

and the bilinear forms:

$$\forall \psi, \varphi \in V, \qquad a(\psi, \varphi) = \sum_{i,j} \int_\Omega \frac{\partial \psi_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j} d\Omega, \tag{4.1}$$

$$\forall \psi \in v, \quad \mu \in W, \qquad b(\psi, \mu) = \int_\Omega \mu \operatorname{div} \psi \, d\Omega. \tag{4.2}$$

Let $\mathbf{f} \in [L^2(\Omega)]^N$, if Ω has a smooth enough boundary Γ, then problem $P(\mathbf{f}, 0)$ is equivalent to:

find $\mathbf{u}=(u_1, \ldots, u_N)$ *and* λ *such that:*

$$\left.\begin{aligned} -\nu\Delta\mathbf{u}+\mathbf{grad}\, p=\mathbf{f} \quad & \text{in } \Omega, \\ \operatorname{div} \mathbf{u}=\mathbf{0} \quad & \text{in } \Omega, \\ \mathbf{u}=\mathbf{0} \quad & \text{on } \Gamma. \end{aligned}\right\} \tag{4.3}$$

The corresponding problem $P_\varepsilon(\mathbf{f}, 0)$ is equivalent to:

find ψ_ε *and* λ_ε *such that:*

$$\left.\begin{aligned} -\nu\Delta\mathbf{u}_\varepsilon+\mathbf{grad}\, p_\varepsilon=\mathbf{f} \quad & \text{in } \Omega, \\ -\varepsilon p_\varepsilon+\operatorname{div} \mathbf{u}_\varepsilon=0 \quad & \text{in } \Omega, \\ \mathbf{u}_\varepsilon=0 \quad & \text{on } \Gamma. \end{aligned}\right\} \tag{4.4}$$

Eliminating p_ε, we can restate (4.4) as:

$$\left.\begin{aligned} -\nu\Delta\mathbf{u}_\varepsilon+\frac{1}{\varepsilon}\mathbf{grad}\operatorname{div} \mathbf{u}_\varepsilon=\mathbf{f} \quad & \text{in } \Omega, \\ \mathbf{u}_\varepsilon=0 \quad & \text{on } \Gamma. \end{aligned}\right\} \tag{4.5}$$

Formulation (4.4) is the so called method of artificial compressibility introduced for Navier Stokes equations by Chorin [11] and Teman [18].

We have the:

PROPOSITION 4.1: *Problem* $P(\mathbf{f}, 0)$ [*resp.* $P_\varepsilon(\mathbf{f}, 0)$]. *has a unique solution* $(\mathbf{u}, p) \in V \times W$ [*resp.* $(\mathbf{u}_\varepsilon, p_\varepsilon) \in V \times W$] *and*

$$\left.\begin{aligned} \int_\Omega p_\varepsilon \, d\Omega=0, \\ \|\mathbf{u}-\mathbf{u}_\varepsilon\|_1+\|p-p_\varepsilon\|_W \leqq C\varepsilon. \end{aligned}\right\} \tag{4.6}$$

Proof: In order to use theorem 3.1 the only point we must check here is hypotheses (H 1).

Let $\lambda \in \tilde{W}$, then there exists $\psi \in V$, such that:

$$\operatorname{div} \psi=\lambda$$

and

$$\|\psi\|_V \leqq C \|\operatorname{div} \psi\|_{L^2(\Omega)},$$

where C is independent of ψ. Thus:

$$\sup_{\varphi \in V-\{0\}} \frac{|b(\varphi, \lambda)|}{\|\varphi\|_V} \geqq \frac{1}{c}\|\lambda\|_W.$$

The proposition is now a simple consequence of theorem 3.1. We have in fact (4.6) for all $\varepsilon > 0$, because $a(\varphi, \varphi)$ is V-elliptic, which is stronger than hypothesis (H 2)-Proving (4.6) in this case amounts to some simplifications in the proof of theorem 2.2 and we shall omit this proof. □

Proposition 4.1 is of course a classic result [19]. But we can apply the same technique to the Stokes' problem with mixed boundary conditions, a situation that arises in the analysis of incompressible linear isotropic materials.

Example 2 Herrmann's variational principle For a given elastic material of modulus of Young E and of modulus of Poisson σ the Lame' constants are:

$$\mu = \frac{E}{2(1+\sigma)}; \qquad \lambda = 2\mu \frac{\sigma}{(1-\sigma)}.$$

A body fills at rest an open set $\Omega \subset R^3$. In order to avoid rigid body motions, we suppose that this body is fixed to a rigid support on part of its boundary Γ_0 [meas $(\Gamma_0) \neq 0$]. On $\Gamma_1 = \Gamma - \Gamma_0$, we apply a given traction $\mathbf{T}$.

For the sake of simplicity we omit body forces and initial strains, which can be introduced without effecting the following theory.

Let:

$$\mathscr{V} = \{ u \mid u \in H^1(\Omega),\ \gamma_0 u = 0 \text{ on } \Gamma_0 \},$$
$$V = [\mathscr{V}]^3.$$

On $V \times V$ we define:

$$a(\mathbf{u}, \mathbf{v}) = \sum_{i,j} \int_\Omega (\varepsilon_{ij}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v}))\, d\Omega,$$

where:

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

and the norm:

$$\|\mathbf{u}\|^2 = \sum_i \int_\Omega \mathbf{grad}\, u_i\ \mathbf{grad}\, u_i\ d\Omega.$$

Let:

$$W = L^2(\Omega).$$

On $V \times W$ we introduce:

$$b(\mathbf{u}, p) = \int_\Omega \operatorname{div} \mathbf{u} \,.\, p\, d\Omega.$$

We define:

PROBLEM $P_{\sigma,\sigma}$ (**T**, 0):
find $(\mathbf{u}, p) \in V \times W$ *such that:*

$$\left.\begin{aligned} \forall \mathbf{v} \in V, \quad & 2\mu a(\mathbf{u}, \mathbf{v}) + 2\sigma\mu b(\mathbf{v}, p) = \int_{\Gamma_1} \mathbf{T}.\mathbf{v}\, d\Gamma, \\ \forall q \in W, \quad & -(1-2\sigma)(p, q) + b(\mathbf{u}, q) = 0. \end{aligned}\right\} \tag{4.7}$$

This is Herrmann's variational formulation for elasticity problems [16]. Now for the incompressible case ($\sigma = 0.5$), we have:

PROBLEM P (**T**, 0):
find $(\mathbf{u}, p) \in V \times W$ *such that:*

$$\forall \mathbf{v} \in V, \quad 2\mu_0 a(\mathbf{u}, \mathbf{v}) + \mu_0 b(\mathbf{v}, p) = \int_{\Gamma_1} \mathbf{T}.\mathbf{v}\, d\Gamma,$$

$$\forall q \in W, \quad b(\mathbf{u}, q) = 0,$$

where $\mu_0 = E/3$.

REMARK 4.1: Setting $p = \operatorname{div} \mathbf{u}/(1-2\sigma)$ in (4.7) we get the penalty type formulation:

$$2\mu a(\mathbf{u}, \mathbf{v}) + \lambda \int_{\Omega} \operatorname{div} \mathbf{u} . \operatorname{div} \mathbf{v}\, d\Omega = \int_{\Gamma_1} \mathbf{T}.\mathbf{v}\, d\Gamma, \tag{4.9}$$

which is the standard variational formulation of elasticity, where λ is now the penalty parameter.

Since we have $0 \leqq \sigma \leqq 0.5$, we can define:

$$\varepsilon = 1 - 2\sigma.$$

Then:

$$\mu = \mu_0 + \varepsilon\mu_1, \qquad \mu_1 = \frac{E}{\sigma(1+\sigma)},$$

and (4.7) becomes:

$$\left.\begin{aligned} & 2\mu_0 a(\mathbf{u}, \mathbf{v}) + 2\varepsilon\mu_1 a(\mathbf{u}, \mathbf{v}) \\ & \quad + (1-\varepsilon)\mu_0 b(\mathbf{v}, p) + \varepsilon(1-\varepsilon)\mu_1 b(\mathbf{v}, p) = \int_{\Gamma_1} \mathbf{T}.\mathbf{v}\, d\Gamma \\ & \quad -\varepsilon(p, q) + b(\mathbf{u}, q) = 0. \end{aligned}\right\} \tag{4.10}$$

Now consider:

PROBLEM P_ε (**T**, 0):
find (**u**, p) *such that:*

$$\left.\begin{aligned} \forall \mathbf{v} \in V, \quad 2\mu_0 a(\mathbf{u}, \mathbf{v}) + \mu_0 b(\mathbf{v}, p) = \int_{\Gamma_1} \mathbf{T}.\mathbf{v}\, d\Gamma, \\ \forall q \in W, \quad -\varepsilon(p, q) + b(\mathbf{u}, q) = 0. \end{aligned}\right\} \tag{4.11}$$

We have the:

THEOREM 4.1: *If* $\mathbf{T} \in [H^{-1/2}(\Gamma_1)]^3$, *and if* meas $(\Gamma_1) \neq 0$ and meas $(\Gamma_0) \neq 0$ *problems* $P_{\sigma,\sigma}(T, 0)$, $P_\varepsilon(\mathbf{T}, 0)$ *and* $P(T, 0)$ *have each one a unique solution in* $V \times W$ *respectively* $(\mathbf{u}_\sigma, p_\sigma)$, $(\mathbf{u}_\varepsilon, p_\varepsilon)$ *and* $(\mathbf{u}, p)$. *Moreover:*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\sigma\| + \|p_\sigma - p_\varepsilon\| \leqq C_1 \varepsilon, \tag{4.12}$$

$$\|\mathbf{u}_\sigma - \mathbf{u}\| + \|p_\sigma - p\| \leqq C_2 \varepsilon. \tag{4.13}$$

Proof of theorem 4.1: Since most of the proof is rather technical, we shall only outline it. First we check hypotheses (H 1). We have here:

$$\text{Ker } B^* = \{0\}.$$

For any $p \in L^2(\Omega)$, consider the (unique) solution of the mixed problem:

$$\begin{aligned} \Delta\theta &= p, \\ \theta &= 0 \quad \text{on } \Gamma_1, \\ \frac{\partial\theta}{\partial \mathbf{n}} &= 0 \quad \text{on } \Gamma_0. \end{aligned}$$

Set $\mathbf{v} = (\partial\theta/\partial x_1, \partial\theta/\partial x_2, \partial\theta/\partial x_3)$, div $\mathbf{v} = p$, and

$$\|\mathbf{v}\|_V \leqq C \|p\|_{L^2(\Omega)}.$$

So that:

$$\frac{|b\,\mathbf{v}, p)|}{\|\mathbf{v}\|} \geqq (1/C)\|p\|,$$

where C is a constant depending on Ω and Γ_0 only. Next by Korn's inequality [13]: $\forall u \in V$, $a(\mathbf{u}, \mathbf{u}) \geqq \alpha \|\mathbf{u}\|_V^2$.

Both hypotheses (H 2) and (H 3) hold, implying existence and uniqueness of a solution for all three problems. Clearly

$$\|\mathbf{u}_\sigma\| + \|p_\sigma\| \leqq C,$$

where C is a constant independent of σ. Consider $(\chi, \xi) \in V \times W$, solution of:

$$\begin{aligned} &2\mu_0 a(\chi, \mathbf{v}) + \mu_0 b(\mathbf{v}, \xi) \\ &\quad = 2\varepsilon\mu_1 a(\mathbf{u}_\sigma, v) - \varepsilon\mu_0 b(\mathbf{v}, p_\sigma) - \varepsilon(1-\varepsilon)\mu_1 b(\mathbf{v}, p_\sigma) \\ &\qquad\qquad - \varepsilon(\xi, q) + b(\chi, q) = 0. \end{aligned} \tag{4.14}$$

We have:

$$\|\chi\| + \|\xi\| \leqq C\varepsilon(\|\mathbf{u}_\sigma\| + \|p_\sigma\|),$$

where C does not depend on ε.

Taking an appropriate linear combination of (4.10), (4.11) and (4.14) we get:

$$\mathbf{u}_\sigma - \mathbf{u}_\varepsilon + \chi = 0; \qquad p_\sigma - p_\varepsilon + \xi = 0;$$

hence (4.12). Now applying theorem 2.2 to problem $P_\varepsilon(\mathbf{T}, 0)$, we have:

$$\|\mathbf{u} - \mathbf{u}_\varepsilon\| + \|p - p_\varepsilon\| \leqq C\,\varepsilon;$$

and (4.13) results from this inequality and from (4.12). □

REMARK 4.2: The hypothesis, meas $(\Gamma_1) \neq 0$ is not necessary in theorem 2.1. We used it because it becomes essential when we consider the case:

$$\mathbf{u} = \mathbf{u}_d \quad \text{on } \Gamma_0.$$

which is then a simple corollary to the theorem. It can be seen that for the non homogeneous case, and meas $(\Gamma_1) = 0$ problem $P_\varepsilon(\mathbf{T}, 0)$ is still well posed while problem $P(T, 0)$ will have no solution, unless:

$$\int_\Gamma \mathbf{u} \,.\, \mathbf{n}\, d\Gamma = 0.$$

Example 3: *Mixed formulation of Dirichlet's problem*. Let $\Omega \subset R^N$ be an open bounded set of class C^2. Let:

$$V = \mathbf{H}(\text{div}; \Omega); \qquad W = L^2(\Omega).$$

For any $\psi \in V$ and $\varphi \in H^1(\Omega)$ the following Green's formula is true [17]:

$$\int_\Omega \psi\, \mathbf{grad}\, \varphi + \int_\Omega \text{div}\, \psi \,.\, \varphi = \int_\Gamma \varphi \,.\, (\psi \,.\, \mathbf{n})\, d\Gamma. \tag{4.15}$$

We consider Dirichlet's problem:

$$\left.\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned}\right\} \tag{4.16}$$

where $f \in L^2(\Omega)$. We introduce the bilinear forms:

$$\forall \psi, \quad \varphi \in V, \qquad a(\psi, \varphi) = \int_\Omega \psi . \varphi \, d\Omega,$$

$$\forall \psi \in V, \quad \mu \in W, \qquad b(\psi, \mu) = \int_\Omega \mu \operatorname{div} \psi \, d\Omega.$$

Note that Ker $B^* = \{0\}$, so that $\tilde{W} = W$. The norm on V is actually:

$$(a(\psi, \psi) + \| B\psi \|^2)^{1/2} \qquad \text{and} \qquad \operatorname{Ker} B = \{ \psi \mid \psi \in V, \operatorname{div} \psi = 0 \},$$

so that hypotheses (H 2) and (H 3) are satisfied. Let us prove that (H 1) holds for $b(.,\ .)$. Given $\mu \in L^2(\Omega)$, we define v as the solution of:

$$\begin{aligned} -\Delta v &= \mu \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \Gamma. \end{aligned}$$

Now set:

$$\varphi = \mathbf{grad}\, v, \qquad \| \varphi \|_V \leqq C \| \mu \|_W$$

and

$$b(\varphi, \mu) = \int_\Omega \mu \operatorname{div} \varphi \, d\Omega = \| \mu \|_W^2$$

so that (H 1) is satisfied.

We can now state the following result.

THEOREM 4.2: *Given $f \in L^2(\Omega)$, problem $P(0, f)$ has a unique solution (ψ, λ) in $V \times W$. Moreover λ is the solution of (4.16) and we have:*

$$\psi = \mathbf{grad}\ \lambda,$$

problem $P_\varepsilon(0, f)$ has a unique solution $(\psi_\varepsilon, \lambda_\varepsilon)$ in $V \times W$ and

$$\| \psi - \psi_\varepsilon \|_V + \| \lambda - \lambda_\varepsilon \|_W \leqq C\varepsilon,$$

moreover λ_ε is the solution of:

$$\left.\begin{aligned} \varepsilon \lambda_\varepsilon - \Delta \lambda_\varepsilon &= f \quad \text{in} \quad \Omega, \\ \lambda_\varepsilon &= 0 \quad \text{on} \quad \Gamma, \end{aligned}\right\} \tag{4.17}$$

and

$$\psi_\varepsilon = \mathbf{grad}\ \lambda_\varepsilon .$$

Proof: The first part of the theorem can be found in Raviart-Thomas [17]. The second part is a consequence of theorem 2.2. The only point that remains to be proved is (4.17). Let λ_ε be the unique solution of (4.17). Since $f \in L^2(\Omega)$:

$$\psi_\varepsilon = \mathbf{grad}\ \lambda_\varepsilon;$$

in V and

$$-\Delta\lambda_\varepsilon = \operatorname{div}\psi_\varepsilon = f - \varepsilon\lambda_\varepsilon .$$

Applying Green's formula (4.15) to this last equality shows that $(\psi_\varepsilon, \lambda_\varepsilon)$ is a solution of problem $P_\varepsilon(0, f)$, hence $(\mathbf{grad}\,\lambda_\varepsilon, \lambda_\varepsilon)$ is the unique solution of the perturbed problem.

Note that problem $P_\varepsilon(0, f)$ is equivalent to:

$$\forall \varphi \in V, \quad a(\psi_\varepsilon, \varphi) + \frac{1}{\varepsilon}\int_\Omega (\operatorname{div}\psi_\varepsilon - f)\operatorname{div}\varphi \, d\Omega = 0. \tag{4.18}$$

We use (4.18) in the finite element method. From the duality relationship between penalization and regularization we can build a large number of examples. We give here only one such example for which the corresponding penalty formulation has been studied, in relation with the finite element method by Aubin [1] and Babuška [3].

Example 4: Approximation of a Dirichlet's problem by a penalization on the boundary. Here (4.16) is replaced by

$$\left.\begin{aligned} -\Delta u_\varepsilon &= f && \text{in } \Omega, \\ \frac{1}{\varepsilon}u_\varepsilon + \frac{\partial u_\varepsilon}{\partial \mathbf{n}} &= 0 && \text{on } \Gamma. \end{aligned}\right\} \tag{4.19}$$

Define:

$$V = H^1(\Omega),$$
$$W = L^2(\Gamma).$$

We still denote by v the trace $\gamma_0 v$ of $v \in V$ on the boundary Γ, of Ω, we have, for Γ smooth enough, $v \in H^{1/2}(\Gamma)$. We define the bilinear forms:

$$\forall \psi, \varphi \in V, \qquad a(\psi, \varphi) = \int_\Omega \mathbf{grad}\,\psi\, \mathbf{grad}\,\varphi \, d\Omega,$$

$$\forall \psi \in V, \quad \forall \lambda \in W, \qquad b(\psi, \lambda) = \int_\Gamma \psi\lambda \, d\Gamma.$$

Note that if we take $W' = W$, $B\psi$ is nothing but $\gamma_0\psi$. Now:

$$Y = H_0^1(\Omega),$$

so that $a(.,.)$ is V-elliptic.

By the Poincare's inequality:

$$\left(a(\psi, \psi) + \int_\Gamma |\psi|^2 \, d\Gamma\right)^{1/2},$$

is a norm equivalent to the usual one on $H^1(\Omega)$.

Next we show that (H 1) holds on $b(.,.)$. Given $\mu \in L^2(\Gamma)$, let w be the solution of:

$$\Delta w = 0,$$
$$w|_\Gamma = \mu$$

and

$$\|w\|_{H^1(\Omega)} \leqq C \|\mu\|_{L^2(\Gamma)}.$$

Taking $\varphi = w$, we obtain:

$$b(\varphi, \mu) = \int_\Gamma \mu^2 \, d\Gamma$$

and (H 1) is satisfied. Applying theorem 2.2, we get the following result.

PROPOSITION 4.3: *Problem $P_\varepsilon(f, 0)$ has a unique solution*

$$(u_\varepsilon, \lambda_\varepsilon) \in H^1(\Omega) \times L^2(\Gamma).$$

If u is the solution of (4.16) *then:*

$$\|u - u_\varepsilon\|_{H^1(\Omega)} + \left\| \frac{\partial u_\varepsilon}{\partial \mathbf{n}} + \lambda_\varepsilon \right\| \leqq C \varepsilon$$

and

$$-\Delta u_\varepsilon = f \quad \text{in } \Omega,$$

$$\varepsilon \lambda_\varepsilon = u_\varepsilon \qquad \text{and} \qquad -\frac{\partial u_\varepsilon}{\partial \mathbf{n}} = \lambda_\varepsilon \quad \text{on } \Gamma.$$

Next we consider the approximation of problems P and P_ε.

5. APPROXIMATION

The abstract setting will be as follows:

We consider V_h and W_h, finite dimensional subspaces of V and W. We introduce a new bilinear form on $V \times W_h$, $b_h(.,.)$ related to $b(.,.)$ by:

$$\forall \varphi \in V, \quad \forall \lambda_h \in W_h, \qquad b_h(\varphi, \lambda_h) = b(\varphi, \lambda_h) \tag{5.1}$$

The discrete analogues of hypotheses (H 1) and (H 2) are:

$$\forall \lambda_h \in W_h, \quad \sup_{\varphi_h \in V_h - \{0\}} \frac{|b(\varphi_h, \lambda_h)|}{\|\varphi_h\|} \geqq \tilde{k} \|\lambda_h\|. \tag{5.2}$$

with $\tilde{k} > 0$ independent of λ_h and $a(.,.)$ is Y_h-elliptic, where:

$$Y_h = \operatorname{Ker} B_h = \{\varphi_h \mid \varphi_h \in V_h, \forall \mu_h \in W_h, b_h(\varphi_h, \mu_h) = 0\}. \tag{5.3}$$

We introduce the space:

$$Z_h = \operatorname{Ker} B_h^* = \{ \mu_h \mid \mu_h \in W_h, \ \forall \varphi_h \in V_h, \ b_h(\varphi_h, \mu_h) = 0 \}.$$

We shall need the following hypothesis (satisfied in all the examples, we have in mind):

(H 4) $$\operatorname{Proj}_{W_h} Z = Z_h.$$

Then as a consequence:

$$\tilde{W}_h = Z_h^\perp \subset \operatorname{Proj}_{W_h} \tilde{W}.$$

This hypothesis allows us to define the approximate problem:

PROBLEM $P_h(f, g)$: *Given $(f, g) \in V' \times Z^0$, find a pair $(\psi_h, \lambda_h) \in V_h \times \tilde{W}_h$ such that:*

$$\left.\begin{aligned} \forall \varphi_h \in V_h, \quad & a(\psi_h, \varphi_h) + b_h(\varphi_h, \lambda_h) = \langle f, \varphi_h \rangle, \\ \forall \mu_h \in W_h, \quad & b_h(\psi_h, \mu_h) = \langle g, \mu_h \rangle \end{aligned}\right\} \tag{5.4}$$

and its corresponding perturbed version:

PROBLEM $P_{h,\varepsilon}(f, g)$: *Given $(f, g) \in V' \times Z^0$, find a pair $(\psi_{h,\varepsilon}, \lambda_{h,\varepsilon}) \in V_h \times W_h$ such that:*

$$\left.\begin{aligned} \forall \varphi_h \in V_h, \quad & a(\psi_{h,\varepsilon}, \varphi_h) + b_h(\varphi_h, \lambda_{h,\varepsilon}) = \langle f, \varphi_h \rangle, \\ \forall \mu_h \in W_h, \quad & -\varepsilon(\lambda_{h,\varepsilon}, \mu_h) + b_h(\psi_{h,\varepsilon}, \mu_h) = \langle g, \mu_h \rangle, \end{aligned}\right\} \tag{5.5}$$

or in its penalty form:

$$\left.\begin{aligned} \forall \varphi_h \in V_h, \quad & a(\psi_{h,\varepsilon}, \varphi_h) + \frac{1}{\varepsilon}(B_h \psi_{h,\varepsilon}, B_h \varphi_h) = \langle f, \varphi_h \rangle, \\ & \lambda_{h,\varepsilon} = \frac{1}{\varepsilon}(B_h \psi_{h,\varepsilon} - \operatorname{Proj}_{W_h} g). \end{aligned}\right\} \tag{5.6}$$

Existence and uniqueness of a solution to problem $P_h(f, g)$ results from theorem 2.1, giving us the following approximation theorem:

THEOREM 5.1: *If hypotheses* (H 1), (H 3) *are satisfied for problem $P(f, g)$, if* (5.2) *holds for problem $P_h(f, g)$ and if in addition there exist $\alpha > 0$ such that:*

$$\forall \varphi_h \in V_h, \quad |a(\varphi_h, \varphi_h)| + \| B_h \varphi_h \|^2 \geqq \alpha^2 \| \varphi_h \|^2.$$

Then problem $P_{h,\varepsilon}(f, g)$ [resp. $P_h(f, g)$] *has a unique solution $(\psi_{h,\varepsilon}, \lambda_{h,\varepsilon}) \in V_h \times W_h$,* [*resp.* $(\psi_h, \lambda_h) \in V_h \times \tilde{W}_h$].

In addition if (ψ, λ) *is the solution of problem* $P(f, g)$ *we have:*

$$\|\psi-\psi_h\|+\|\lambda-\lambda_h\| \leqq C \inf_{\varphi_h \in Y_h} \|\psi-\varphi_h\|+\inf_{\mu_h \in W_h} \|\lambda-\lambda_h\|, \tag{5.7}$$

$$\|\psi-\psi_{h,\varepsilon}\|+\|\lambda-\lambda_{h,\varepsilon}\| \leqq \|\psi-\psi_h\|+\|\lambda-\lambda_h\|+C\varepsilon. \tag{5.8}$$

Proof: (5.7) is a particular case of a general result of Brezzi ([8]. th. 3.1). Applying theorem 3.1 to problems $P_h(f, g)$ and $P_{h,\varepsilon}(f, g)$ we obtain:

$$\|\psi_h-\psi_{h,\varepsilon}\|+\|\lambda_h-\lambda_{h,\varepsilon}\| \leqq C\varepsilon.$$

Since the constant in hypotheses (H 1), (H 2) and (H 3) are independent of h, C is also independent of h. (5.8) is then a consequence of the trianguler inequality applied to $\psi-\psi_h+\psi_h-\psi_{h,\varepsilon}$ and $\lambda-\lambda_h+\lambda_h-\lambda_{h,\varepsilon}$. □

REMARK 5.1: By (5.8) we can replace problem $P_h(f, g)$ by its perturbed version $P_{h,\varepsilon}(f, g)$ without loosing any accuracy, provided ε is small enough. Using the penalty formula (5.6) instead of (5.5) we do not need to introduce the Lagrange multiplier at all. □

REMARK 5.2: The critical point in theorems 5.1 and 5.2 is hypothesis (H 1). On one hand it is a necessary condition for problem $P_h(f, g)$ to be well posed, while on the other hand without it we cannot apply theorem 3.2. This hypothesis is in general easily verified on $V \times W$. There are cases (such as examples 1 and 2 for instance) where it does not hold on $V_h \times W_h$, hence the necessity to introduce an approximate bilinear form $b_h(.,\ .)$. Even then is not easy in general to prove the existence of a k independent of h. We give now a sufficient condition for this hypothesis to hold, [*cf.* Fortin [15 *bis*]].

LEMMA 5.1: *Assume that hypothesis* (H 1) *holds on* $V \times W$. *Assume that there exist an operator* $\pi_h : D \rightarrow V_h$ *where* D *is a dense subset of* V, *and a constant* $C>0$, *such that:*

$$\forall \psi \in D, \quad \forall \mu_h \in W_h, \qquad b_h(\psi-\pi_h \psi, \mu_h)=0,$$
$$\forall \psi \in D, \|\pi_h \psi\| \leqq C\|\psi\|.$$

Then hypothesis (H 1) *holds.*

REMARK 5.3: If (H 1) does not hold for $b(.,\ .)$ or (5.2) for $b_h(.,\ .)$ on $V_h \times W_h$, then we have the typical behavior of a penalty method where for example:

$$\|\psi-\psi_{h,\varepsilon}\| \leqq C_1 \varepsilon+\frac{C_2}{\sqrt{\varepsilon}}[\inf_{\psi_h \in V_h} \|\psi-\psi_h\|]. \tag{5.9}$$

Whereas in theorem 5.2 ε can be chosen independently of h, there is an optimal ε for every h. For a demonstration of (5.2) and corresponding numerical methods we refer to [5].

In order to give an intuitive illustration of the "miracle" of (5.8) as compared to (5.9) we give a simple formal example.

6. A FORMAL ILLUSTRATION

In all that follows Ω is a bounded polygonal set in R^2. On Ω we define a regular admissible triangulation $\mathscr{K}$ of "size h". We assume that all the underlying concepts of the finite element are familiar to the reader and we refer to [10] for these. We denote by $P_s(K)$ the space of all polynomials of degree $\leqq s$ on the triangle $K \in \mathscr{K}$.

Consider example 4 (§ 4). Let Γ be the boundary of Ω. Γ is a polygonal line through the boundary node set $\Gamma_h \{ \Gamma_h = i \mid i$ is a node of $\mathscr{K}$, $i \in \Gamma \}$. Now the corresponding problem $P_{h,\varepsilon}(f, 0)$ is:

$$\min_{\varphi_h \in V_h} \frac{1}{2} \int_\Omega \operatorname{grad}^2 \varphi_h \, d\Omega + \frac{1}{2\varepsilon} \int_\Gamma \varphi_h^2 \, d\Gamma - \int_\Omega f \cdot \varphi_h \, d\Omega. \tag{6.1}$$

Take $\Omega = (]0, 1[)$ and a uniform triangulation of step size h (*fig.* 1 *a*), and the standard (linear) P_1 elements. Consider the node $C \in \Gamma_h$ (*fig.* 1 *b*), (6.1) gives the following finite difference "scheme":

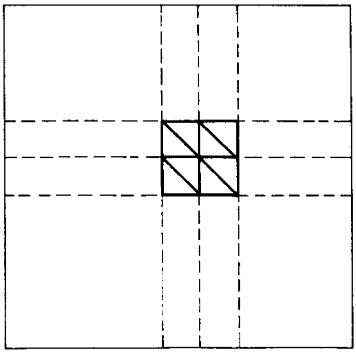
$$2\left(1 + \frac{h^2}{3\varepsilon}\right) u(C) - \left(\frac{1}{2} - \frac{h^2}{6\varepsilon}\right) u(S) - \left(\frac{1}{2} - \frac{h^2}{6\varepsilon}\right) u(N) - u(W) = h^2 f_h(C).$$

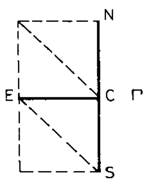
In most standard finite element softwares in engineering one finds:

$$2\left(1 + \frac{1}{\varepsilon}\right) i(C) - \frac{1}{2} u(S) - \frac{1}{2} u(N) - u(W) = h^2 f_h(C). \tag{6.2}$$

Formulation (6.1) leads to equations with coefficients depending on h^2/ε while in (6.2) they depend on ε but not on h. The latter formulation is thus stable as $\varepsilon \to 0$.

(1a)





(1b)

Figure 1.

This simple example illustrates in some sense the behavior of problem $P_{h,\varepsilon}(f, g)$ under hypothesis (H 1). It shows that optimal conditioning of the system is obtained by the proper choice of a penalty formulation on the approximate problem rather than on the original one. [This is equivalent to the construction of a bilinear form $b_h(.,\ .)$].

7. APPLICATION TO THE FINITE ELEMENT METHOD

Examples (1) and (2) are closely related since in both cases we are dealing with the continuity equation $\operatorname{div}\psi = 0$. As a matter of fact we can construct the same operator $\operatorname{div}_h(.)$ for both applications. The first successful construction of such an operator is due to Fortin [15]. Extensions of this method to a large family of conforming and non-conforming elements can be found in the paper of Crouzeix and Raviart [12].

These authors have solved the delicate problem of defining consistent approximations satisfying in some approximate sense hypothesis (H 1). But the actual construction of the operator $\operatorname{div}_h(.)$ can be by itself quite difficult. In the first part of this paragraph we show how the judicious use of numerical integration rules leads to simple procedures.

Finite elements for Stokes equations and incompressible materials

1) *Parabolic element with constant pressure*

We consider the six node triangle, (*fig.* 2) whose shape functions are:

$$\varphi_i=\lambda_i(2\lambda_i-1),\qquad 1\leqq i\leqq 3,\quad 1\leqq i<j\leqq 3.$$
$$\varphi_{ij}=4\lambda_i\lambda_j,$$

where λ_i is the linear shape function associated to the vertex i. The pressure is assumed to be constant on each element.

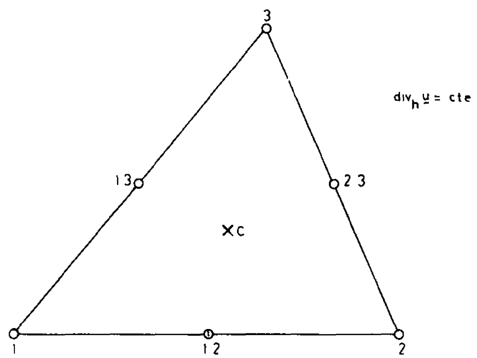


Figure 2.

We define the following spaces:

$$\mathscr{V}_h = \{ \psi_h \mid \psi_h \in C^0(\Omega),\ \psi_h|_K \in P_2(K) \},$$

$$V_h = (\mathscr{V}_h)^2 \subset V[H_0^1(\Omega)]^2,$$

$$W_h = \{ \mu_h \mid \mu_h \in P_0(K) \},$$

$$W_h = \left\{ \mu_h \,\middle|\, \mu_h \in W_h,\ \int_\Omega \mu_h \, d\Omega = 0 \right\}.$$

We introduce the bilinear form on $V \times W_h$:

$$\forall \mu_h \in W_h, \quad \forall \varphi \in V\, b_h(\varphi, \mu_h) = b(\varphi, \mu_h).$$

On each $K \in \mathscr{K}$, by definition:

$$b_h(\psi, \mu)|_K = \int_K \operatorname{div}_h \psi \,.\, \mu \, dK = \int_K \operatorname{div} \psi \,.\, \mu \, dK,$$

for all constants μ, hence:

$$\operatorname{div}_h \psi|_K = \frac{1}{\operatorname{meas}(K)} \int_K \operatorname{div} \psi \, dK. \tag{7.1}$$

In the finite element approximation $\psi_h|_K \in P_2(K)$ so that $\operatorname{div} \psi_h \in P_1(K)$ and (7.1) can be computed exactly by a 1 point integration formula whose node is at the center c, of the triangle:

$$\operatorname{div}_h \psi_h|_K = \operatorname{div} \psi_h(c). \tag{7.2}$$

We consider directly problem $P_{h,\varepsilon}(f, 0)$ (example 1, § 4), in its penalty formulation (5.6). We have to compute the stiffness matrix on K corresponding to:

$$\frac{1}{\varepsilon} \int_K (\operatorname{div}_h \psi_h)^2 \, dK,$$

by (7.2) we see that this amounts to computing:

$$\frac{1}{\varepsilon} \int_K (\operatorname{div} \psi_h)^2 \, dK, \tag{7.3}$$

by *means of the center point rule*.

Hence the new divergence operator is defined by a *reduced* integration since in order to compute (7.3) accurately we would have had to use a three point rule at least. If we have at hand a finite element software including the computation by numerical integration of

$$\int_K (\operatorname{div} \psi_h)^2 \, dK.$$

Then we need just a minor modification in order to use it for problem $P_{h\,\varepsilon}(f, 0)$ (examples 1 and 2)

By a result of Crouzeix-Raviart [12] we know that there exists an operator $\pi_h \in \mathscr{L}(V \cap H^2(\overline{\Omega}), V_h)$ such that

$$\forall \varphi \in V \cap H^2(\Omega), \quad b_h(\varphi - \pi_h \varphi, \mu_h) = 0 \quad \Rightarrow \quad \mu_h = 0,$$

and

$$\|\pi_h \varphi\|_V \leqq C \|\varphi\|_V$$

According to lemma 5 1 hypothesis (H 1) holds uniformly in h Since [*cf* [12]] for $\psi \in Y \cap H^2(\Omega)$ and $\mu \in H^1(\Omega) \cap \tilde{W}$

$$\inf_{\varphi_h \in Y_h} \|\psi - \varphi_h\| \leqq Ch, \qquad \inf_{\mu_h \in \tilde{W}_h} \|\mu - \mu_h\| \leqq Ch$$

we can state the following results

THEOREM 7 1 *Let* $\Psi_{h\,\varepsilon}$ *be the unique solution of the problem*

$$\min_{\varphi_h \in V_h} \frac{1}{2} \sum_{i=1}^{2} \int_\Omega (\mathbf{grad}\, \varphi_{ih\,\varepsilon})\, d\Omega + \frac{1}{2\varepsilon} \int_\Omega (\mathrm{div}_h\, \varphi_h)^2\, d\Omega - \int_\Omega \mathbf{f}\ \ \varphi_h\, d\Omega \tag{7 1}$$

Let $(\psi, \lambda) \in V \times W$ *be the unique solution of problem* $P(f, 0)$ (*example* 1, § 4) *suppose that* $\psi \in [H^3(\Omega)]^2$ *and* $\lambda \in H^1(\Omega)$ *then*

$$\|\psi - \psi_{h,\varepsilon}\|_V + \left\|\lambda - \frac{1}{\varepsilon} \mathrm{div}_h\, \psi_h\right\|_{\tilde{W}} \leqq C_1\, \varepsilon + C_2\, h \tag{7 3}$$

The same result can be stated for example 2, § 4

2) 7 *node element with linear pressure*

Given a triangle $K \in \mathscr{K}$, let c be its center Consider the polynomial space $P_s(K)$ spanned by the following shape functions (cf *fig* 3)

$$\varphi_i = \lambda_i (2\lambda_i - 1) + 3\lambda_1 \lambda_2 \lambda_3, \qquad 1 \leqq i \leqq 3,$$
$$\varphi_{ij} = 4\lambda_i \lambda_j - 12 \lambda_1 \lambda_2 \lambda_3, \qquad 1 \leqq i < j \leqq 3,$$
$$\varphi_c = 27 \lambda_1 \lambda_2 \lambda_3$$

Note that $P_2(K) \subset P_s(K) \subset P_3(K)$

We define:

$$\mathscr{V}_h=\{\varphi_h|\,\varphi_h\in C^0(\overline{\Omega}),\ \varphi_h|_K\in P_s(K)\},$$
$$W_h=\{\mu_h|\,\mu_h|_K\in P_1(K)\}$$

and the spaces V_h and $\widetilde{W}_h$ as before.

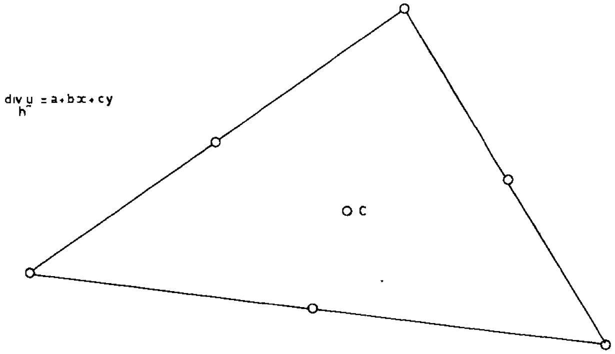


Figure 3.

On $V_h\times W_h$ we introduce the bilinear form:

$$\forall\,\mu_h\in W_h,\quad \forall\,\varphi\in V,\qquad b_h(\varphi,\,\mu_h)=b(\varphi,\,\mu_h).$$

Then by [12], there exists an operator $\pi_h\in\mathscr{L}(V\cap H^3(\overline{\Omega});\ V_h)$ such that:

$$\forall\,\varphi\in V\cap H^3(\overline{\Omega}),\quad b_h(\varphi-\pi_h\varphi,\mu_h)=0\ \Rightarrow\ \mu_h=0$$

and

$$\|\pi_h\varphi\|\leqq C\|\varphi\|.$$

Moreover:

$$\forall\,\psi\in Y\cap H^3(\Omega),\quad \inf_{\varphi_h\in Y_h}\|\psi-\varphi_h\|\leqq C\,h^2$$

and

$$\forall\,\lambda\in\widetilde{W}\cap H^2(\Omega),\quad \inf_{\mu_h\in\widetilde{W}_h}\|\lambda-\mu_h\|\leqq C\,h^2.$$

So that we can state the following result:

THEOREM 7.2: *For the 7 node triangular element, under the assumptions of theorem 7.1 and provided that $\psi\in[H^2(\Omega)]^2$ and $\lambda\in H^2(\Omega)$ we have the error estimate:*

$$\|\psi-\psi_{h,\,\varepsilon}\|_V+\|\lambda-\frac{1}{\varepsilon}\operatorname{div}_h\psi_{h,\,\varepsilon}\|\leqq C_1\,\varepsilon+C_2\,h^2. \tag{7.4}$$

We are now going to show how to construct the operator $\mathrm{div}_h(\,.\,)$. It is not possible to use directly a reduced integration technique here, but we can use numerical integration in order to obtain a simple procedure. We recall that there exists a 3-point rule based on the mid side nodes exact for $P_2(K)$ and a 7 point rule based on the seven nodes of the element exact for $P_3(K)$. Now by definition:

$$\int_K \mathrm{div}_h \psi \,.\, \mu \, dK = \int_K \mathrm{div}\, \psi \,.\, \mu \, dK, \tag{7.5}$$

for all $\mu \in P_1(K)$. Let η_{ij} $(1 \leqq i < j)$ be the three *linear* shape functions associated with the 3 mid side nodes. It is clear that they form a basis of $P_1(K)$. Since $\mathrm{div}_h \psi \in P_1(K)$ we can write that:

$$\mathrm{div}_h \psi = \sum_{1 \leqq i < j \leqq 3} \eta_{ij} \, \mathrm{div}_h \psi(ij) \tag{7.6}$$

and by (7.5):

$$\frac{\mathrm{meas}(K)}{3} \mathrm{div}_h \psi(ij) = \int_K \eta_{ij} \, \mathrm{div}\, \psi \, dx, \quad 1 \leqq i < j \leqq 3, \tag{7.7}$$

the right hand side of (7.6) can be computed exactly by the 7 point rule, set $\mathbf{q} = (\psi(1), \ldots, \psi(c))^T$, we have:

$$\int_K \eta_{ij} \, \mathrm{div}\, \psi \, dx = \mathbf{S}_{ij}^T \mathbf{q}, \tag{7.8}$$

where $\mathbf{S}_{ij}$ is a (14×1) matrix. Now from (7.6) and the three point rule:

$$\int_K \mathrm{div}_h^2 \, \psi \, dx = \Big(\sum_{1 \leqq i < j \leqq 3} \mathrm{div}_h^2 \, \psi(ij) \Big) \frac{\mathrm{meas}(K)}{3};$$

so that by (7.8):

$$\int_K \mathrm{div}_h^2 \, \psi \, dx = \frac{3}{\mathrm{meas}(K)} \mathbf{q} \Big[\sum_{1 \leqq i < j \leqq 3} \mathbf{S}_{ij} \mathbf{S}_{ij}^T \Big] \mathbf{q}$$

and the stiffness matrix we wanted to obtain is:

$$\frac{1}{\varepsilon} \frac{3}{\mathrm{meas}(K)} \sum_{1 \leqq i < j \leqq 3} \mathbf{S}_{ij} \mathbf{S}_{ij}^T,$$

so that the only computing effort is in establishing (7.8).

REMARK 7.1: Note that the perturbed approximate problem is solved on W_h and not on $\tilde{W}_h$. Hence in Stokes' equations the approximate pressure is given up to a constant, and without any additional constraint the actual linear systems deduced from example 1, paragraph 4 will be singular. This difficulty does not appear in the penalty approach. □

REMARK 7.2: For quadrilateral elements Gauss Legendre product rules enable us to use the reduced integration technique in order to define $\text{div}_h(.)$ directly from div (.) as was the case for the first element (*).

This technique has been used with great success for the solution of Navier-Stokes equations [6]. □

Of course we can extend this penalty approach to all elements described in Crouzier-Raviart [12]. We refer to [4] for more examples.

We can also extend this method to mixed FEM for Dirichlet's problem (example 3, § 3) we refer to [5 *bis*] for the corresponding results.

8. CONCLUSION

We have shown how the introduction of a perturbation can simplify the handling of the linear constraint in mixed variational formulations. This method is efficient in all cases where this constraint reduces to a local one in the corresponding FEM formulation; that is, the approximate operator $B_h(.)$ is defined element by element.

Our results are still valid when this operator is global such as in the hybrid methods studied by Thomas [20] for second order partial differential equations or such as the mixed methods for 4th order equations given in [9]. But in this case little is gained in using the penalty approach above since the introduction of $B_h(.)$ is not simple. Still in these cases the perturbation theorems do show that methods of solution of "augmented Lagrangian" type should be very powerful indeed.

Our method has been successfully used for numerical analysis of incompressible or nearly incompressible materials [7], as well as for the computation of viscous incompressible flows [6].

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(*) Added in proof: for Herrmann's problem this has also been done by D. S. Malkus in *A Finite Element Displacement model valid for any value of the compressibility*, I. J. Solids and Structures, vol. 12, 1976.

REFERENCES

1 J P AUBIN, *Approximation of Elliptic Boundary Value Problems*, Wiley, N Y , 1972

2 I BABUŠKA, *The Finite Element Method with Lagranian Multipliers*, Num Math , Vol 20, 1973, pp 179-192

3 I BABUŠKA, *The Finite Element Method with Penalty*, Math Comp , Vol 27, 1973, pp 221-228

4 M BERCOVIER, *A Family of Finite Elements with Penalization for the Numerical Solution of Stokes and Navier-Stokes Equations*, *in* Proc I F I P Conf 1977, North-Holland, Amsterdam, 1977

5 M BERCOVIER, *On the Penalty and Extrapolation Method* (to appear)

5 *bis.* M BERCOVIER, *Thèse de Doctorat d'État*, Rouen, 1976

6 M BERCOVIER, and M ENGELMAN *A Finite Element for the Numerical Solution of Viscous Incompressible flows* (to appear in J Comp Phys)

7 M BERCOVIER and F LIVNE *A 4 CST Quadrilateral Element for Incompressible and Nearly Incompressible Materials* (to appear in CALCOLO)

8 F BREZZI *On the Existence, Uniqueness and Approximation of saddle Point Problems Arising from Lagrangian Multipliers*, R A I R O Vol 8 R-2 1974 pp 129-151

9 F BREZZI, and P A RAVIART, *Mixed Finite Element Methods for 4th Order Elliptic Equations*, Topics in Numerical Analysis, Vol III, J J H MILLER Ed , Academic Press

10 P G CIARLET, *Numerical Analysis of the Finite Element Method for Elliptic Boundary Value Problems*, North Holland, Amsterdam, 1977

11 A J CHORIN, *A Numerical Method for Solving Incompressible Problems*, J Comp Phys , Vol 2, 1967

12 M CROUZEIX and P A RAVIART, *Conforming and non Conforming Finite Elements Methods for Solving the Stationary Stokes Equation* R A I R O , Vol 7, R-3, 1973, pp 33-76

13 G DUVAUT and J L LIONS, *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972

14 I EKELAND and R TEMAM, *Analyse convexe et problèmes variationnels*, Dunod, Paris, 1974

15 M FORTIN, *Thèse de Doctorat d'État*, Paris, 1972

15 *bis.* M FORTIN, *An Analysis of the Convergence of Mixed Finite Element Methods*, R A I R O , Vol 11, R-3, 1977, pp 341-354

16 L R HERRMANN *Elasticity Equations for Incompressible or Nearly Incompressible Materials by a Variational Theorem* A I A A Journal, Vol 3, 1965, pp 1896-1900

17 P A RAVIART and J M THOMAS, *A Mixed Finite Element Method for 2nd Order Elliptic Problems*, *in Proc Symp on the Mathematical aspects of the FEM* Rome, December 1975, Lecture notes in Mathematics 606, Springer Verlag, pp 292-315

18 R TEMAM, *Une méthode d'approximation de la solution des équations de Navier-Stokes*, Bull Soc Math Fr , Vol 96, 1968

19 R TEMAN, *Navier-Stokes equations*, North Holland, Amsterdam, 1977

20 J M THOMAS, *Thèse de Doctorat d'État*, Paris, 1977