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ON SPECTRAL APPROXIMATION
PART 2. ERROR ESTIMATES FOR THE GALERKIN METHOD (*)

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Abstract. — One considers an isolated eigenvalue A of finite multiplicity of an operator A which is approximated by a Galerkin method. Using Osborn's technics, one derives several error estimates for λ.

1. SITUATION AND RESULTS

In part 1 of this paper [3], we have been concerned with the problem of convergence in spectral approximation; since the theory we have developped has received concrete applications for non compact operators only in connection with the Galerkin method, we shall now restrict ourself to this case.

Let X be a complex Banach space of norm $\| \ \|$ and $\{ X_h \}$ be a sequence of finite dimensional subspaces of X. One gives two continuous sesquilinear forms a and b on X and one supposes a coercive. Then, by Lax-Milgram, one can define the continuous operators $A : X \to X$ and $A_h : X_h \to X_h$ by

$$a(Au, v) = b(u, v), \quad \forall u, v \in X, \quad a(A_h u, v) = b(u, v), \quad \forall u, v \in X_h.$$

All along this paper we shall suppose that the two following conditions are satisfied (*see* [3]):

$$\text{P1}: \lim_{h \to 0} \| (A - A_h) |_{X_h} \| = 0; \quad \text{P2}: \quad \forall x \in X, \quad \lim_{h \to 0} \inf_{x_h \in X_h} \| x - x_h \| = 0.$$

Let $\lambda \in \mathbf{C}$ be an isolated eigenvalue of A of finite algebraic multiplicity m; since a is coercive $\lambda \neq 0$ and there exists a closed disc Δ of center λ and boundary Γ such that $0 \notin \Delta$ and $\Delta \cap \sigma(A) = \{ \lambda \}$ where $\sigma(A)$ denotes the spectrum of A. Let $\mu_{1h}, \ldots, \mu_{m(h),h}$ be the eigenvalues of A_h, repeated following their algebraic multiplicities and contained in Δ. In [3], section 2,

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we have proved:

a) $m(h) = m$ for h small enough;

b) $\lim\limits_{h\to 0} \mu_{ih} = \lambda$, $i = 1, 2, \ldots, m$.

The purpose of this part 2 of our paper is to give estimates of λ by the μ_{ih}'s. In fact, we shall adapt to the situation described above Osborn's method [5]; note that, independently of the fact that A_h is a Galerkin approximation, we have simplified the presentation of Osborn's main argument and strengthened his results. See also Grigorieff [4].

At this point, we recall some standard notations. For an operator D, $R_z(D) = (z-D)^{-1}$ is the resolvent operator. Let Y and Z be closed subspaces of X; then for $x \in X$,

$$\delta(x, Z) = \inf_{z \in Z} \| x - z \|, \quad \delta(Y, Z) = \sup_{\substack{y \in Y \\ \| y \| = 1}} \delta(y, Z)$$

and

$$\hat{\delta}(Y, Z) = \max(\delta(Y, Z), \delta(Z, Y)).$$

Let us also open a short parenthesis on duality. Let X^* be the adjoint space of X, i. e. the set of antilinear continuous forms on X. By Lax-Milgram, the operator $C : X^* \to X$ defined by the relation $a(v, C\varphi) = \bar{\varphi}(v)$, $\forall v \in X$, $\varphi \in X^*$, is an isomorphism between X^* and X which allows to identify these two spaces. With this identification if $D : X \to X$ is a bounded linear operator, its adjoint $D^* : X \to X$ will be characterized by the relation $a(Du, v) = a(u, D^* v)$, $\forall u, v \in X$; one verifies also immediately the relation $\| D^* \| \leqq \| C \| \cdot \| C^{-1} \| \cdot \| D \|$.

We need, for the following, to introduce some further operators. $\Pi_h : X \to X$ is the projector with range X_h defined by the relation $a(\Pi_h u - u, v) = 0$, $\forall v \in X_h$. One has $A_h = \Pi_h A\,|_{X_h}$ and we set $B_h = \Pi_h A \Pi_h : X \to X$; exept for zero, B_h has the same spectrum as A_h and the same corresponding invariant subspaces. $E = (2\Pi i)^{-1} \int_\Gamma R_z(A)\,dz$ is the spectral projector of A relative to λ and, for h small enough, $F_h = (2\Pi i)^{-1} \int_\Gamma R_z(B_h)\,dz$ is the spectral projector of B_h relative to $\mu_{1h}, \ldots, \mu_{mh}$. Now consider the adjoints of these operators as defined above. A^* has the isolated eigenvalue $\bar{\lambda}$ of algebraic multiplicity m; Π_h^* will be the projector with range X_h satisfying the relation $a(v, \Pi_h^* u - u) = 0$, $\forall v \in X_h$; E^* and F_h^* will be the spectral projectors of A^* and $B_h^* = \Pi_h^* A^* \Pi_h^*$ associated respectively to $\bar{\lambda}$ and to the set $\bar{\mu}_{1h}, \ldots, \bar{\mu}_{mh}$;

they will satisfy the relations

$$E^* = (2\Pi i)^{-1} \int_{\bar{\Gamma}} R_z(A^*)\,dz \qquad \text{and} \qquad F_h^* = (2\Pi i)^{-1} \int_{\bar{\Gamma}} R_z(B_h^*)\,dz$$

where $\bar{\Gamma}$ is the conjugate circle of Γ (positively oriented).

In applications $E(X)$ and $E^*(X)$, the m-dimensional invariant subspaces of A and A^* corresponding to λ and $\bar{\lambda}$, will be often composed of smooth functions so that it is reasonable to introduce the quantities

$$\gamma_h = \delta(E(X), X_h), \qquad \gamma_h^* = \delta(E^*(X), X_h).$$

We can now state the results.

THEOREM 1 : *There exists a constant c, independent of h such that*

$$\hat{\delta}(F_h(X), E(X)) \leqq c\gamma_h; \quad \hat{\delta}(F_h^*(X), E^*(X)) \leqq c\gamma_h^*.$$

In section 2, we shall show that $F_h|_{E(X)}$ defines for h small enough, a bijection between $E(X)$ and $F_h(X)$; let Λ_h be this bijection; $\hat{A} = A|_{E(X)}$ and $\hat{B}_h = \Lambda_h^{-1} B_h \Lambda_h$ will be considered as operators in $E(X)$; $\hat{A}$ has the eigenvalue λ of algebraic multiplicity m and $\hat{B}_h$ has the eigenvalues $\mu_{1h}, \ldots, \mu_{mh}$.

THEOREM 2 : *There exists a constant c, independent of h such that*

$$\|\hat{A} - \hat{B}_h\|_{E(X)} \leqq c\gamma_h\gamma_h^*.$$

By the choice of a basis in $E(X)$, theorem 2 reduces our original task to a pure matricial problem. Let f be a holomorphic function defined in a neighborhood of λ; writting $f(\hat{A})$ and $f(\hat{B}_h)$ by the mean of Dunford integrals, one verifies immediately that

$$\|f(\hat{A}) - f(\hat{B}_h)\|_{E(X)} \leqq c\|\hat{A} - \hat{B}_h\|_{E(X)}$$

where c depends on f but not on h; using the classical properties of traces and determinants, one obtains theorem 3 a, b; theorem 3 c, d is a direct application of results quoted in [7], pp. 80-81; here α is the ascent of the eigenvalue λ of $\hat{A}$, β is the number of Jordan blocs of the canonical form of $\hat{A}$.

THEOREM 3 : *There exists a constant c, independent of h such that for h small enough:*

a) $$\left| f(\lambda) - \frac{1}{m}\sum_{i=1}^{m} f(\mu_{ih}) \right| \leqq c\,\gamma_h \gamma_h^*,$$

b) $$\left| f^m(\lambda) - \prod_{i=1}^{m} f(\mu_{ih}) \right| \leqq c\,\gamma_h \gamma_h^*,$$

c) $$\max_{i=1\ldots m} |\lambda - \mu_{ih}| \leqq c(\gamma_h \gamma_h^*)^{1/\alpha},$$

d) $$\min_{i=1\ldots m} |\lambda - \mu_{ih}| \leqq c(\gamma_h \gamma_h^*)^{\beta/m}.$$

REMARKS: 1) In his original work, in a difference context, Osborn [5] has obtained theorem 3 *a* for $f(z) = z$ and theorem 3 *c*; in another context also Chatelin [1] proves, theorem 3 *a* for $f(z) = z$.

2) For $f(z) = 1/z$, theorem 3 *a* gives an estimate of $1/\lambda$ by the arithmetic mean of the $1/\mu_{ih}$'s; the result has been already obtained by [2]; we are indebted to Chatelin who showed us that it can also be deduced by Osborn's method.

In order to illustrate this theorem, we consider the example developped in section 4 of part 1 of this paper [3]; one can prove by Rappaz' method of elimination used in [6] the existence of an infinite number of isolated eigenvalues of finite multiplicities; by supposing the coefficients α, β, ... sufficiently smooth, one verifies that the corresponding eigensubspaces are subsets of $H^2 \times (H^1)^2$; consequently $\gamma_h = O(h)$, $\gamma_h^* = O(h)$ and the estimates of theorem 3 *a*, *b* are of order h^2.

We conclude this section by stating a very elementary result for the self-adjoint case. We suppose that the forms *a* and *b* are hermitian. Because of its coercivity, a is a scalar product for which X is a Hilbert space with norm $\|x\|_a^2 = a(x, x)$; then A, B_h and Π_h become hermitian. Let ν be an eigenvalue of A, which is not supposed isolated or of finite multiplicity, and G be the corresponding eigensubspace. For the distance $\delta(\nu, \sigma(B_h))$ from ν to the spectrum $\sigma(B_h)$ of B_h, one gets the estimate

$$\begin{aligned}\delta(\nu, \sigma(B_h)) = \inf_{\substack{y \in X \\ \|y\|=1}} \|(B_h - \nu)y\|_a &\leqq \inf_{\substack{y \in X_h \\ \|y\|=1}} \|(B_h - \nu)y\|_a \\ \leqq \inf_{\substack{y \in X_h \\ \|y\|=1}} \|(A - \nu)y\|_a &= \inf_{\substack{y \in X_h \\ \|y\|=1}} \|(A-\nu)(y-x)\|_a, \quad \forall x \in G,\end{aligned}$$

i. e., since the norms $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent:

$$\delta(\nu, \sigma(B_h)) \leqq c \inf_{\substack{x \in G \\ \|x\|=1}} \delta(x, X_h), \quad c \text{ independent of } h \text{ and } \nu. \tag{1}$$

REMARKS: 1) We have obtained the estimate (1) without supposing P1 or P2.

2) Examples show that it is not possible to replace the right member of (1), by $c\{\inf_{\substack{x \in G \\ \|x\|=1}} \delta(x, X_h)\}^{1+\varepsilon}$, $\varepsilon > 0$.

2. PROOFS

In this section we prove theorem 1 and 2. We use the definitions and notations of section 1 and we suppose hypotheses P1 and P2. c will denote a "generic" constant.

We first recall a well-known result. Since a is continuous and coercive, the projectors Π_h are bounded uniformly with respect to h and there exists a constant c such that $\|x-\Pi_h x\| \leqq c\,\delta(x, X_h)$, $\forall x \in X$; the Π_h^*'s possess the same properties.

Lemma 1 of section 2 of [3] shows that P1 implies the inequality $\sup_{\substack{x \in X_h \\ \|x\|=1}} \|R_z(A_h)x\| \leqq c$, $\forall z \in \Gamma$ for h small enough, c independent of h. We extend this result to B_h and B_h^*.

LEMMA 1: *There exists $h_0 > 0$ and c such that*

$$\|R_z(B_h)\| \leqq c, \quad h < h_0, \quad z \in \Gamma$$

and

$$\|R_z(B_h^*)\| \leqq c, \quad h < h_0, \quad z \in \bar{\Gamma}.$$

Proof: Since $R_{\bar{z}}(B_h^*) = (R_z(B_h))^*$ we need to prove only the first statement; since B_h is compact it suffices to verify that $\|(z-B_h)x\| \geqq c\|x\|$, $\forall x \in X$, $z \in \Gamma$. Taking in account the fact that $0 \notin \Gamma$ one has

$$\begin{aligned}\|x\| &\leqq \|\Pi_h x\| + \|x-\Pi_h x\| \\ &\leqq c\|(z-B_h)\Pi_h x\| + |z|^{-1}\|(z-B_h)(x-\Pi_h x)\| \leqq c\|(z-B_h)x\|. \quad \blacksquare\end{aligned}$$

We note that we shall not use any more P1 explicitely. Consequently, in the proofs of lemma 3 and theorem 1, the statements for the adjoints operators are obtained in the same way as for the direct operators.

We omit the proof of the following trivial:

LEMMA 2: *Let Y and Z be two subspaces of X with the same finite dimension;* 1 *et $P: Y \to Z$ be a linear operator such that $\|Py-y\| \leqq 0.5\|y\|$, $\forall y \in Y$.*

Then P is bijective,

$$\|P^{-1}z\| \leqq 2\|z\|, \quad \forall z \in Z$$

and

$$\sup_{\substack{z\in Z\\ \|z\|=1}} \|P^{-1}z-z\| \leqq 2 \sup_{\substack{y\in Y\\ \|y\|=1}} \|Py-y\|.$$

LEMMA 3:

$$\|(E-F_h)\ |_{E(X)}\| \leqq c\|(A-B_h)\ |_{E(X)}\| \leqq c\gamma_h,$$
$$\|(E^*-F_h^*)\ |_{E^*(X)}\| \leqq c\|(A^*-B_h^*)\ |_{E^*(X)}\| \leqq c\gamma_h^*.$$

Proof: For h small enough, by lemma 1, one has

$$\begin{aligned}\|(E-F_h)\ |_{E(X)}\| &\leqq (2\Pi)^{-1}\int_\Gamma \|R_z(B_h)\|.\|(A-B_h)R_z(A)\ |_{E(X)}\|.|dz|\\ &\leqq c\|(A-B_h)\ |_{E(X)}\|;\end{aligned}$$

$$\begin{aligned}\|(A-B_h)\ |_{E(X)}\| &\leqq \|(I-\Pi_h)A\ |_{E(X)}\|+\|\Pi_h A(I-\Pi_h)\ |_{E(X)}\|\\ &\leqq c\|(I-\Pi_h)\ |_{E(X)}\| \leqq c\gamma_h. \quad \blacksquare\end{aligned}$$

Proof of theorem 1: Lemma 3 implies that $\delta(E(X), F_h(X)) \leqq c\,\gamma_h$. Set, as in Section 1, $\Lambda_h = F_h\,|_{E(X)} : E(X) \to F_h(X)$; for h small enough $E(X)$ and $F_h(X)$ have the same dimension m; on the other side P2 implies $\lim_{h\to 0}\gamma_h = 0$; by lemma 2, Λ_h^{-1} exists for h small enough and is uniformly bounded with respect to h; furthermore $\sup_{\substack{x\in F_h(x)\\ \|x\|=1}} \|\Lambda_h^{-1}x-x\| \leqq c\,\gamma_h$, i. e. $\delta(F_h(X), E(X)) \leqq c\,\gamma_h$. $\blacksquare$

Proof of theorem 2: Let $S_h = \Lambda_h^{-1}F_h - I : X \to X$; S_h is uniformly bounded with respect to h (*see* proof of theorem 1); from the identity

$$(\hat{A}-\hat{B}_h)x = (A-B_h)x + S_h(A-B_h)x, \ x\in E(X),$$

one obtains for $x \in E(X)$, $y \in E^*(X)$, since $F_h S_h = 0$,

$$a((\hat{A}-\hat{B}_h)x, y) = a((A-B_h)x, y) + a(S_h(A-B_h)x, (I-F_h^*)y); \tag{2}$$

$$\begin{aligned}\|\hat{A}-\hat{B}_h\|_{E(X)} &\leqq c \sup_{\substack{x\in E(X),\, y\in X\\ \|x\|=\|y\|=1}} a((\hat{A}-\hat{B}_h)x, y)\\ &\leqq c \sup_{\substack{x\in E(X),\, y\in E^*(X)\\ \|x\|=\|y\|=1}} a((\hat{A}-\hat{B}_h)x, y);\end{aligned} \tag{3}$$

for $x \in E(X)$, $y \in E^*(X)$, $\|x\| = \|y\| = 1$, one has (using lemma 3):

$$|a(S_h(A-B_h)x, (I-F_h^*)y)| \leqq c\|(A-B_h)x\| . \|(I-F_h^*)y\| \leqq c\gamma_h\gamma_h^*; \quad (4)$$

$$\begin{aligned} a((A-B_h)x, y) = {} & a((I-\Pi_h)Ax, (I-\Pi_h^*)y) \\ & + a((I-\Pi_h)x, (I-\Pi_h^*)A^*y) \\ & + a((I-\Pi_h)x, A^*(\Pi_h^*-I)y); \end{aligned}$$

$$|a((A-B_h)x, y)| \leqq c\gamma_h\gamma_h^*; \quad (5)$$

theorem 2 follows from (2), (3), (4) and (5). ∎

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