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ERROR ESTIMATES AND FREE-BOUNDARY CONVERGENCE FOR A FINITE DIFFERENCE DISCRETIZATION OF A PARABOLIC VARIATIONAL INEQUALITY (*)

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Abstract. — *We study a free-boundary problem of biological interest, for the heat equation; the problem is reduced to a variational inequality, which is solved by means of a finite difference method. We also give an error estimate, in the* L^∞*-norm, for the solution of the inequality, and a convergence theorem for the discrete free boundaries. The latter follows from a general result concerning the behaviour of the free boundaries of "perturbed" problems.* (see *Th.* 1.1).

1. INTRODUCTION

A diffusion-absorption problem (namely, of oxygen in a tissue; *see* e. g. the book [6], in particular page 319) leads to consider the following free-boundary problem:

PROBLEM 1.1: Find two "smooth" functions τ, u, such that:

$$\tau: \quad [0, 1] \to \mathbf{R} \text{ is strictly decreasing, with } \tau(1) = 0, \tag{1.1}$$

and, setting

$$\Omega = \{(x, t) \mid 0 < x < 1, 0 < t < \tau(x)\}, \tag{1.2}$$

it is [$\bar{\Omega}$ denoting the closure of Ω]:

$$u: \quad \bar{\Omega} \to \mathbf{R}; \qquad u \text{ is strictly positive in } \Omega; \tag{1.3}$$

$$u_{xx} - u_t = 1 \quad \text{in } \Omega; \tag{1.4}$$

$$u(x, \tau(x)) = u_x(x, \tau(x)) = 0 (0 \leqq x \leqq 1); \qquad u_x(0, t) = 0 (0 < t < \tau(0)); \tag{1.5}$$

$$u(x, 0) = \frac{1}{2}(1-x)^2, \qquad (0 \leqq x \leqq 1). \tag{1.6}$$

Because of their relevance, problems like the previous one have been widely studied both from the theoretical and from the numerical point of view;

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see e. g. [1, 2, 3, 7, 8, 9, 13]; *see* also [11, 17, 18, 19] for general results about parabolic free-boundary problems and for further references.

As pointed out e. g. in [1, 2, 3, 9], Problem 1.1 can be solved by means of a suitable variational inequality: let $\mathbf{R}^2_+$ be the open half-space:

$$\mathbf{R}^2_+ = \{ (x, t) \mid x \in \mathbf{R}, t > 0 \};$$

if we extend u to $\overline{\mathbf{R}}^2_+$ on setting:

$$U(x, t) = \begin{cases} u(|x|, t) & \text{for } (|x|, t) \in \overline{\Omega}, \\ 0 & \text{elsewhere,} \end{cases} \tag{1.7}$$

then it is:

$$U \geqq 0; \qquad U_{xx} - U_t \leqq 1; \qquad U(U_{xx} - U_t - 1) = 0 \quad \text{in } \mathbf{R}^2_+; \tag{1.8}$$

$$U(x, 0) = U_0(x) \quad \text{in } \mathbf{R} \tag{1.9}$$

[in the particular case of Probl. 1.1, it is

$$U_0(x) = \frac{1}{2}[(1 - |x|)^+]^2, \tag{1.10}$$

where $\theta^+ = 1/2(\theta + |\theta|)$; but we will treat more general initial data]. As we will see, if the initial value U_0 is such that

$$U_0 \in H^1_c(\mathbf{R}); \qquad U_0(x) \geqq 0, \qquad \forall x \in \mathbf{R} \tag{1.11}$$

[where $H^1(\mathbf{R})$ is the usual Sobolev space $\{ v \in L^2(\mathbf{R}) \mid v' \in L^2(\mathbf{R}) \}$, and the index c means "with compact support"], then problem (1.8), (1.9) is well posed if we look for a solution U with

$$U \in H^{2,1}_c(\mathbf{R}^2_+) \tag{1.12}$$

[say, $U \in H^{2,1}(\mathbf{R}^2_+)$ (3) and supp (U) is compact in $\overline{\mathbf{R}}^2_+$]. Moreover, if [as in (1.10)] $U_0(x) = U_0(-x)$, and $U''_0(x) \leqq 1$, $U'_0(x) \leqq 0$ for $x > 0$, the solution U satisfies $U(-x, t) = U(x, t)$ and $U_t(x, t) \leqq 0$, $U_x(x, t) \leqq 0$ in the first quadrant $Q = \{ (x, t) \mid x > 0, t > 0 \}$, so that, setting:

$$\left.\begin{aligned} \Omega = \{ (x, t) \in Q \mid U(x, t) > 0 \}; \quad & u = U_{|\overline{\Omega}}; \\ \tau(x) = \max\{ t \mid (x, t) \in \overline{\Omega} \}, & \end{aligned}\right\} \tag{1.13}$$

we get the solution of Problem 1.1.

This theoretical approach was developed in [1] by means of a semi-discretization of a problem like (1.8), (1.9) (actually, in the first quadrant Q); in

(3) We follow the notations of [15, 14], to which books we refer also for the properties of Sobolev spaces; in particular, $U \in H^{2,1}(\mathbf{R}^2_+)$ means that $U, U_x, U_{xx}, U_t \in L^2(\mathbf{R}^2_+)$.

the present paper we will study a complete (i. e., both in x and t) discretization of (1.8), (1.9), and give convergence theorems both for the family $\{ U_{h,k} \}$ approximating U and for the family $\{ \Omega_{h,k} \}$ approximating Ω.

The approximate solution $U_{h,k}$ will be constructed by combining with spline functions the values of discrete solutions (discretization in finite differences, h and k being the mesh-sizes in x and t respectively); the main tool for the estimates will be the maximum principle. The order of convergence will depend, in general, on the smoothness of the initial datum $U_0(x)$; e. g., for a "general" U_0 [i. e. satisfying just (1.11)] we will get

$$\| U_{1/n,\,1/n^2} - U \|_{L^\infty(\mathbf{R}^2_+)} = o(n^{-1/2+\varepsilon}), \qquad \forall \varepsilon > 0 \tag{1.14}$$

[the condition $k \sim h^2$ being however unnecessary for convergence].

Let us point out the interest of estimates like (1.14) in solving free boundary problems. In fact, consider, in a domain D, a sequence of functions $\{ f_n \}_{n \in \mathbf{N}}$ converging, in some topology, to a function f. In general, no matter of the topology in which $f_n \to f$ (4), we cannot be sure that the set of positivity of f_n will converge to the set of positivity of f. However, if we know an upper bound for $\| f_n - f \|_{L^\infty(D)}$, then we can construct an approximation of the set of positivity of f. In fact, let us prove the following theorem:

THEOREM 1.1: *Let f, f_n ($n = 1, 2, \ldots$) be continuous functions in a domain D, such that:*

$$\| f_n - f \|_{L^\infty(D)} = \varepsilon_n \to 0 \qquad (n \to +\infty). \tag{1.15}$$

Let δ_n be such that:

$$\delta_n > 0 \, \forall n; \qquad \delta_n \to 0 \qquad \text{and} \qquad \frac{\varepsilon_n}{\delta_n} \to 0 \qquad (n \to +\infty), \tag{1.16}$$

and let us set:

$$\Omega = \{ P \in D \mid f(P) > 0 \}; \qquad \Omega_n = \{ P \in D \mid f_n(P) > \delta_n \}. \tag{1.17}$$

Then $\{ \Omega_n \}_{n=1,2,\ldots}$ converges to Ω, say:

$$\Omega = \lim_{n \to \infty} \Omega_n \qquad [\text{in set theoretical sense}], \tag{1.18}$$

and the convergence is "from the interior", in the sense that there exists $\bar{n}$ (which can be actually computed) such that

$$\Omega_n \subset \Omega, \qquad \forall n \geqq \bar{n}. \tag{1.19}$$

Proof: Let $\bar{n}$ be such that $\delta_n \geqq \varepsilon_n$ for $n \geqq \bar{n}$; when $P \in \Omega_n$, $n \geqq \bar{n}$, we have

$$f(P) \geqq f_n(P) - \| f_n - f \|_{L^\infty(D)} = f_n(P) - \varepsilon_n > \delta_n - \varepsilon_n \geqq 0;$$

(4) e. g., for $f_n \equiv 1/n$, $f \equiv 0$, the set of positivity of f_n is the whole of D for each n, while the set of positivity of f is empty.

i. e. $P \in \Omega$, and (1.19) is proved. From (1.19) we get also $\limsup_{n \to \infty} \Omega_n \subset \Omega$, so that, in order to prove (1.18), we just need to show that $P \in \Omega \Rightarrow P \in \Omega_n$ if n is large enough; and this is obvious because, if $P \in \Omega$,

$$\lim_{n \to \infty} [f_n(P) - \delta_n] = f(P) > 0,$$

so that, for any n large enough, it is $f_n(P) > \delta_n$, i. e. $P \in \Omega_n$.

2. DISCRETIZATION

Let us now state in a precise form problem (1.8), (1.9), (1.12); to this aim, let us firstly recall that $H^1(\mathbf{R})$ consists of continuous functions on $\mathbf{R}$, and $H^{2,1}(\mathbf{R}^2_+)$ consists of continuous functions on $\mathbf{R}^2_+$, so that the meaning of relations like (2.2), (2.4), (2.5), (2.8), (2.12) in the sequel will be the classical one.

PROBLEM 2.1: Given $U_0(x)$ with:

$$U_0 \in H^1_c(\mathbf{R}); \tag{2.1}$$

$$U_0(x) \geqq 0, \qquad \forall x \in \mathbf{R}, \tag{2.2}$$

we look for a function $U(x, t)$ such that:

$$U \in H^{2,1}_c(\mathbf{R}^2_+); \tag{2.3}$$

$$U(x, 0) = U_0(x), \qquad \forall x \in \mathbf{R}; \tag{2.4}$$

$$U(x, t) \geqq 0, \qquad \forall (x, t) \in \overline{\mathbf{R}}^2_+; \tag{2.5}$$

$$U_{xx}(x, t) - U_t(x, t) \leqq 1, \qquad \text{a.e. in } \mathbf{R}^2_+; \tag{2.6}$$

$$[U_{xx}(x, t) - U_t(x, t) - 1].U(x, t) = 0, \qquad \text{a.e. in } \mathbf{R}^2_+. \tag{2.7}$$

Remark that the condition (2.1) is the "natural" one when we look for U satisfying (2.3), (2.4); however, we can expect that, for U_0 satisfying further properties, e. g. any of the following ones:

$$U_0(x) = U_0(-x), \qquad \forall x \in \mathbf{R}; \tag{2.8}$$

$$\left.\begin{array}{c} U_0 \text{ satisfies (2.8) and } \exists\, \alpha \geqq 0 \text{ (resp., } \exists\, \beta \geqq 0) \\ \text{such that } U_0'(x) \geqq -\alpha \text{ (resp., } U_0'(x) \leqq \beta) \text{ a.e. for } x \in \mathbf{R}_+; \end{array}\right\} \tag{2.9}$$

$$\exists\, \lambda \geqq 0 \quad \text{such that} \quad U_0'' \leqq 1 + \lambda \quad \text{in } \mathscr{D}'(\mathbf{R}); \tag{2.10}$$

$$\exists\, \mu \geqq 0 \quad \text{such that} \quad U_0'' \geqq 1 - \mu \quad \text{in } \mathscr{D}'(\mathbf{R}), \tag{2.11}$$

the solution $U(x, t)$ must satisfy similar properties. As a matter of fact, the following theorems follow directly from our results:

THEOREM 2.1: *Problem* 2.1 *has a unique solution.*

THEOREM 2.2: *If* U_0 *satisfies* (2.8), *the solution* U *of Problem* 2.1 *satisfies:*

$$U(x, t) = U(-x, t), \qquad \forall (x, t) \in \overline{\mathbf{R}}^2_+; \tag{2.12}$$

if (2.9) holds, U satisfies:

$$U_x(x, t) \geqq -\alpha \text{ (resp., } U_x(x, t) \leqq \beta), \qquad \text{a.e. in } Q; \tag{2.13}$$

if (2.10) holds, U satisfies:

$$U_{xx}(x, t) \leqq 1+\lambda;\ U_t(x, t) \leqq \lambda; \qquad \text{a.e. in } \mathbf{R}^2_+; \tag{2.14}$$

if (2.11) holds, U satisfies:

$$U_{xx}(x, t) \geqq -\mu;\ U_t(x, t) \geqq -\mu; \qquad \text{a.e. in } \mathbf{R}^2_+. \tag{2.15}$$

Remark 2.1: Theorem 2.1 could easily be deduced from results of [5]; *see* also [1] for problems directly posed in the first quadrant Q; for general parabolic inequalities, *see* the books [10, 14]. Some regularity results like those in Theorem 2 can be found in [12]. Let us point out that our results are of a "constructive" nature, say they can be used in order to prove both the existence theorem for Problem 2.1, and the regularity properties of Theorem 2.2.

It can easily be proved that (2.5), (2.6), (2.7) can be replaced with:

$$U_{xx}(x, t) - U_t(x, t) \in H(U(x, t)), \qquad \text{a.e. in } \mathbf{R}^2_+ \tag{2.16}$$

where $\theta \to H(\theta)$ is the multivalued maximal monotone operator associated to the Heaviside function, i. e.:

$$\left.\begin{array}{l} H:\ \mathbf{R} \to 2^{\mathbf{R}}; \qquad H(\theta) = \{0\} \quad \text{for} \quad \theta < 0; \\ H(\theta) = \{1\} \quad \text{for} \quad \theta > 1; \qquad H(0) = \{\tau \in \mathbf{R} \mid 0 \leqq \tau \leqq 1\}; \end{array}\right\} \tag{2.17}$$

and actually Problem 2.1 will be discretized when written in the form (2.16). More precisely, a semi-discretization in t, of mesh-size k, leads to study a family of problems of the type:

$$\left.\begin{array}{l} \text{given } u \geqq 0, \text{ find } U \text{ such that :} \\ U''(x) - \dfrac{1}{k} U(x) \in H(U(x)) - \dfrac{1}{k} u(x), \qquad \text{a.e. in } \mathbf{R} \end{array}\right\} \tag{2.18}$$

(for details *see* [1], where, by means of such a discretization, we studied a problem like Problem 2.1 in the first quadrant Q). A complete (i. e. both in x and t) discretization requires solving a family of problems of the following type (h denoting the mesh-size with respect to x):

$$\left.\begin{array}{l} \text{given } u \equiv (u_i)_{i\in \mathbf{Z}}, \quad \text{with} \quad u_i \geqq 0 \,\forall\, i, \text{ find :} \\ U \equiv (U_i)_{i\in \mathbf{Z}} \quad \text{such that} \quad -\frac{1}{h^2}(AU)_i - \frac{1}{k}U_i \in H(U_i) - \frac{1}{k}u_i \,\forall\, i \in \mathbf{Z} \end{array}\right\} \tag{2.19}$$

where the operator A is defined as follows:

$$\text{for} \quad v \equiv (v_i)_{i\in\mathbf{Z}}, \qquad \text{it is } (Av)_i = -v_{i+1} + 2v_i - v_{i-1}, \qquad \forall\, i \in \mathbf{Z}. \tag{2.20}$$

The following section is devoted to the study of problem (2.19).

3. THE DISCRETE PROBLEM

In the whole of this section, h and k are two fixed positive numbers. l^2 denotes the space of sequences $v \equiv (v_i)_{i\in\mathbf{Z}}$ from $\mathbf{Z}$ to $\mathbf{R}$, such that

$$\| v \|_{l^2}^2 = \sum_{i\in\mathbf{Z}} v_i^2 < +\infty;$$

we write $(u, v) = \sum_{i\in\mathbf{Z}} u_i v_i$ for the scalar product of $u \equiv (u_i)_{i\in\mathbf{Z}}$ and $v \equiv (v_i)_{i\in\mathbf{Z}}$. We also set:

$$a(u, v) = \sum_{i\in\mathbf{Z}} (u_{i-1} - u_i)(v_{i-1} - v_i), \qquad \forall\, u, v \in l^2; \tag{3.1}$$

remark that, with the notation of (2.20), it is:

$$a(u, v) = (Au, v), \qquad \forall\, u, v \in l^2. \tag{3.2}$$

We write $u \prec v$ if $u_i \leqq v_i \;\forall\, i \in \mathbf{Z}$; the following obvious "discrete maximum principle" will be often used:

$$\left.\begin{array}{l} \text{for any sequence } w \equiv (w_i)_{i\in\mathbf{Z}} \quad \text{with} \quad \limsup_{|i|\to+\infty} w_i \leqq 0 \\ \text{(in particular } \forall\, w \in l^2\text{), the following implication holds :} \\ \qquad w \not\prec 0(^5) \;\Rightarrow\; \exists\, i \in \mathbf{Z} \\ \text{such that} \quad w_i \geqq w_j \,\forall\, j \in \mathbf{Z}, \quad \text{and} \quad (Aw)_i > 0. \end{array}\right\} \tag{3.3}$$

Let us consider the following problem:

PROBLEM 3.1: Given $u \in l^2$, find $U \in l^2$ such that:

$$\frac{1}{h^2} a(U, U - v) + \frac{1}{k}(U, U - v) + \sum_{i\in\mathbf{Z}} (U_i)^+ \leqq \frac{1}{k}(u, U - v) + \sum_{i\in\mathbf{Z}} (v_i)^+ \;\forall\, v \in l^2 \tag{3.4}$$

[where $\theta^+ = (|\theta| + \theta)/2 \;\forall\, \theta \in \mathbf{R}$].

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(5) For any $r \in \mathbf{R}$, we denote by r also the sequence $(r_i)_{i\in\mathbf{Z}}$ defined by $r_i \equiv r \;\forall\, i \in \mathbf{Z}$.

LEMMA 3.1: *Problem* 3.1 *has a unique solution*; *for* $u, U \in l^2$, (3.4) *holds if and only if*:

$$-\frac{1}{h^2}(AU)_i - \frac{1}{k}U_i \in H(U_i) - \frac{1}{k}u_i, \qquad \forall i \in \mathbf{Z} \tag{3.5}$$

[*so that Probl.* 3.1 *is the variational form of Probl.* 2.19].

Proof: The bilinear form $\{u, v\} \to 1/h^2\, a(u, v) + 1/k\,(u, v)$ is a scalar product on l^2 equivalent to the original one; moreover, for any fixed $u \in l^2$, the functional $v \to -1/k\,(u, v) + \sum_{i \in \mathbf{Z}} (v_i)^+$ is convex, proper and l. s. c. on l^2; existence and uniqueness for Problem 3.1 then follow from [16].

The equivalence between (3.4) and (2.5) is an immediate consequence of (3.2) and of the fact that $\theta \to H(\theta)$ is the subdifferential map of $\theta \to \theta^+$.

Let us denote by $\mathcal{T}_{h,k}\, u$ the solution U of Problem 3.1; this defines a map $\mathcal{T}_{h,k}: l^2 \to l^2$. A first property of $\mathcal{T}_{h,k}$ is a monotonicity property:

LEMMA 3.2: $u \prec v \Rightarrow \mathcal{T}_{h,k}\, u \prec \mathcal{T}_{h,k}\, v$.

Proof: Setting $U = \mathcal{T}_{h,k}\, u$, $V = \mathcal{T}_{h,k}\, v$, $w = U - V$, let us assume, by contradiction, that $u \prec v$ and $w \not\prec 0$. By using (3.3) we get, for a suitable $i \in \mathbf{Z}$, $w_i > 0$ and $(A\, w)_i > 0$; from $w_i > 0$, i. e. $U_i > V_i$, we get, with an obvious meaning of symbols, $H(U_i) - H(V_i) \geqq 0$; from (3.5) and the similar relation involving v, V, we get:

$$-\frac{1}{h^2}(AU)_i - \frac{1}{k}U_i + \frac{1}{k}u_i \geqq -\frac{1}{h^2}(AV)_i - \frac{1}{k}V_i + \frac{1}{k}v_i,$$

say

$$\frac{1}{k^2}(A\, w)_i + \frac{1}{k}w_i \leqq \frac{1}{k}(u_i - v_i) \leqq 0,$$

and this relation gives a contradiction with $w_i > 0$, $(A\, w)_i > 0$.

From now on we will confine ourselves to the case $u \succ 0$ [*see* (2.19)]; in particular [from $\mathcal{T}_{h,k}\, 0 = 0$ and Lemma 3.2] it will be $U \succ 0$, and we could replace (3.5) with:

$$\left.\begin{aligned} &U \succ 0; \qquad -\frac{1}{h^2}AU - \frac{1}{k}U + \frac{1}{k}u \prec 1; \\ &\forall i \in \mathbf{Z}, \qquad U_i > 0 \;\Rightarrow\; -\frac{1}{h^2}(AU)_i - \frac{1}{k}U_i + \frac{1}{k}u_i = 1. \end{aligned}\right\} \tag{3.6}$$

We will also write $U^0 \equiv (U_{i,0})_{i \in \mathbf{Z}}$ instead of $u \equiv (u_i)$; and we define:

$$\text{for } \; j > 0, \qquad U^j \equiv (U_{i,j})_{i \in \mathbf{Z}} \qquad \text{is given by} \;\; U^j = \mathcal{T}^j_{h,k}\, U_0; \tag{3.7}$$

in particular [*see* (3.6)] we have, for $j = 0, 1, \ldots$:

$$\left.\begin{array}{l} U^{j+1} \succ 0; \qquad -\frac{1}{h^2} A U^{j+1} - \frac{1}{k}(U^{j+1} - U^j) \prec 1; \\ U_{i,j+1} > 0 \\ \quad \Rightarrow \quad \frac{U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1}}{h^2} - \frac{U_{i,j+1} - U_{i,j}}{k} = 1 \end{array}\right\} \tag{3.8}$$

Remark 3.1: The formulation (3.8) gives the "natural" discretization of (2.5), (2.6), (2.7).

The following properties are related to the compactness of the support. From now on we will assume that U^0 verifies, for suitable $M, S \in R^+$:

$$0 \leqq U_{i,0} \leqq M \,\forall\, i \in \mathbf{Z}; \qquad U_{i,0} = 0 \qquad \text{for} \quad |i| \geqq \frac{S}{h}. \tag{3.9}$$

LEMMA 3.3: *Setting*

$$\tilde{S} = S + \sqrt{2M}, \tag{3.10}$$

we have:

$$U_{i,j} = 0 \qquad \text{for} \quad |i| \geqq \frac{\tilde{S}}{h}, \qquad j = 0, 1, \ldots \tag{3.11}$$

Proof: Let us define $V \equiv (V_i)_{i \in \mathbf{Z}}$ by setting:

$$V_i = \left[\left(\frac{\tilde{S}}{h} - |i|\right)^+\right]^2 \cdot \frac{h^2}{2};$$

we will prove, by induction on j, that $U^j \prec V$ for $j = 0, 1, \ldots$

For $|i| \geqq S/h$, it is obviously $U_{i,0} \leqq V_i$; for $|i| < S/h$ it is:

$$U_{i,0} \leqq M = \frac{1}{2}(\tilde{S} - S)^2 = \frac{h^2}{2}\left(\frac{\tilde{S}}{h} - \frac{S}{h}\right)^2 \leqq \frac{h^2}{2}\left(\frac{\tilde{S}}{h} - |i|\right)^2 = V_i;$$

so that $U^0 \prec V$. Now, in order to apply the induction argument we firstly remark that, as it follows from easy computations:

$$-\frac{1}{h^2}(AV)_i \begin{cases} \leqq 1 & \text{for} \quad |i| \leqq \frac{\tilde{S}}{h}, \\ < \frac{1}{2} & \text{for} \quad \frac{\tilde{S}}{h} < |i| < \frac{S}{h} + 1, \\ = 0 & \text{for} \quad |i| \geqq \frac{\tilde{S}}{h} + 1, \end{cases}$$

so that, globally, it is $-1/h^2 AV \prec 1$. From this relation from the induction assumption $U^j \prec V$, and from (3.3), a simple argument by absurd gives $U^{j+1} \prec V$.

Remark 3.2: Lemma 3.3 is the discrete analogue of a result of [4] concerning the compactness of the support for solution of (continuous) elliptic variational inequalities. For parabolic inequalities the compactness (both in x and t) of the support has been proved, in the continuous case, in [5]; the similar discrete result follows from Lemma 3.3 and Lemma 3.4 in the sequel.

Let us assume that $0 \leqq U_{i,0} \leqq M \; \forall i \in \mathbf{Z}$ [*see* (3.9); here the compactness of the support of U^0 is unnecessary].

LEMMA 3.4: *For* $j \geqq M/k$ *it is* $U^j \equiv 0$.

Proof: First of all we prove that, with the general notations of Problem 3.1, it is

$$0 \prec u \prec M \Rightarrow 0 \prec U \prec (M-k)^+. \tag{3.10}$$

In fact, if $M \leq k$, we can directly check that $U \equiv 0$ satisfies (3.5); and if $M > k$ we can apply (3.3) in order to derive a contradiction from $U \not\prec (M-k)^+$; moreover $U \succ 0$ follows from Lemma 3.2.

From (3.10) we get the lemma in the case $M \leq k$; in the general case, let m be an integer with $m \geqq M/k$; and let us assume, by contradiction, that $U^m \neq 0$. From (3.10), and since $U^m = \mathscr{T}_{h,k} U^{m-1}$, we get $U^{m-1} \not\prec k$; and, on iterating, $U^1 \not\prec (m-1)k$, $U^0 \not\prec mk$; and this relation contradicts $U^0 \prec M \leq mk$.

We will now prove some properties corresponding to the "continuous" statements of Theorem 2.2.

LEMMA 3.5: *If* $U_{i,0} = U_{-i,0} \; \forall i \in \mathbf{Z}$, *it is* $U_{i,j} = U_{-i,j} \; \forall i \in \mathbf{Z}$, $j > 0$; *if moreover* $\exists \alpha \geqq 0$ (resp., $\exists \beta \geqq 0$) *such that* $U_{i+1,0} - U_{i,0} \geqq -\alpha h$ (resp., $U_{i+1,0} - U_{i,0} \leqq \beta h$) $\forall i \geqq 0$, *it is* $U_{i+1,j} - U_{i,j} \geqq -\alpha h$ (resp., $U_{i+1,j} - U_{i,j} \leqq \beta h$) $\forall i, j \geqq 0$.

Proof: we obviously need the property just for $\mathscr{T}_{h,k}$ instead of $\mathscr{T}^j_{h,k} \; \forall j \geqq 0$. Given $u \in l^2$, set $U = \mathscr{T}_{h,k} u$; defining $v \equiv (v_i)_{i \in \mathbf{Z}}$ by means of $v_i \equiv u_{-i} \; \forall i \in \mathbf{Z}$ it is easily seen that, in general, $V = \mathscr{T}_{h,k} v$ is such that $V_i = U_{-i}$; in particular, if $u_i = u_{-i}$, it is $u = v$, hence $U = V$, that is, $U_i = U_{-i} \; \forall i \in \mathbf{Z}$.

Now let us assume that u satisfies (besides $u_i = u_{-i} \; \forall i \in \mathbf{Z}$) also $u_{i+1} - u_i \geqq -\alpha h$ ($i = 0, 1, \ldots$) for a suitable $\alpha \geqq 0$; defining $w \equiv (w_i)_{i \in \mathbf{Z}}$ by means of $w_i = -U_{i+1} + U_i + U_i - \alpha h$, we claim that $w_i \leqq 0$ when $i \geqq 0$. In fact, assume, by contradiction, that $\beta = \sup_{i \geqq 0} w_i > 0$; then it would be $\beta = w_r$ for a suitable $r \geqq 0$; and $w_r > 0$ implies that $U_r > U_{r+1} + \alpha h \geqq 0$, hence that $H(U_n) = \{1\}$. Then from (3.5) [written for $i = r$ and for $i = r+1$] it follows that

$$\frac{1}{h^2}(Aw)_r + \frac{1}{k} w_r \in 1 - H(U_r) + \frac{1}{k}(-u_{r+1} + u_r - \alpha h) \leqq 0,$$

so that $(Aw)_r < 0$, hence that $2w_r < w_{r-1} + w_{r+1}$. This is clearly absurd if $r \geqq 1$ (in which case, by definition, $w_{r-1} + w_{r+1} \leqq 2w_r$); if $r = 0$, it follows that $2w_0 < w_{-1} + w_1 \leqq w_{-1} + w_0$, that is, $w_0 < w_{-1}$, which implies that $U_0 < U_1$; but this is absurd since in this case it should be

$$w_0 = -U_1 + U_0 - \alpha h < 0,$$

contrary to the assumption.

The remaining part of this lemma can be proved with a similar argument.

Remark 3.3: A similar result holds if u is "symmetric with respect to 1/2 instead of "symmetric with respect to zero"; e. g. [for the sake of simplicity, we confine ourselves to a statement directly related to the biological problem mentioned in the Introduction] we can prove that

$$\left.\begin{array}{l} \text{if} \quad U_{i,0} = U_{1-i,0}, \quad \forall i \in \mathbf{Z}, \quad \text{then} \quad U_{i,j} = U_{1-i,j}, \\ \qquad \forall i \in \mathbf{Z}, \quad j \geqq 0; \\ \text{if, moreover,} \quad U_{i,0} \geqq U_{i+1,0}, \quad \forall i \geqq 0, \quad \text{then} \quad U_{i,j} \geqq U_{i+1,j}, \\ \qquad \forall i, \quad j \geqq 0. \end{array}\right\} \tag{3.11}$$

From the "practical" point of view, (3.11) can be more useful than Lemma 3.5 [*see* also §.6 in the sequel].

For a given U^0, let us define:

$$\lambda(U^0) = \max_i \left[-1 - \frac{1}{h^2}(AU^0)_i \right]^+ \tag{3.12}$$

$$\mu(U^0) = \max_i \left[1 + \frac{1}{h^2}(AU^0)_i \right]^+ \tag{3.13}$$

LEMMA 3.6: *With the notation* (3.12), *it is:*

$$\left.\begin{array}{l} -\dfrac{1}{h^2}(AU^j)_i \leqq 1 + \lambda(U^0); \qquad \dfrac{1}{k}(U^{j+1} - U^j)_i \leqq \lambda(U^0), \\ \qquad \forall i \in \mathbf{Z}, \quad j \geqq 0. \end{array}\right\} \tag{3.14}$$

Proof: We can work just for $j = 1$, say with u, U instead of U^0, U^1. Set $w = 1/k\,(U-u) - \lambda(u)$, and let us assume, by contradiction, that $w_i > 0$

for some i; from (3.3) we would have, for a suitable $r \in Z$, $w_r > 0$ and $(A\,w)_r > 0$; but this is absurde since from $w_r > 0$ we get $H(U_r) = \{1\}$ and then, from (3.5) and the definition of $\lambda(u)$:

$$0 < (A\,w)_r = \frac{1}{k}[(AU)_r - (A\,u)_r]$$

$$= -\frac{h^2}{k}\left[1 + \frac{1}{k}(U_r - u_r)\right] + \frac{h^2}{k}\left[-\frac{1}{h^2}(A\,u)_r\right]$$

$$\leqq -\frac{h^2}{k}\left[1 + \frac{1}{k}(U_r - u_r)\right] + \frac{h^2}{k}[1 + \lambda(u)] \leqq -\frac{h^2}{k} w_r < 0.$$

Thus we got $1/k\,(U - u) \prec \lambda(u)$; from (3.5) we get now

$$-\frac{1}{h^2} A\,U \prec 1 + \lambda(u).$$

In a similar way [but adding also an induction argument on j] we could also prove that:

$$\left.\begin{aligned} -\frac{1}{h^2}(AU^j)_i \geqq -\mu(U^0); \qquad \frac{1}{k}(U^{j+1} - U^j)_i \geqq -\mu(U^0) \\ \forall\, i \in \mathbf{Z}, \quad j \geqq 0; \end{aligned}\right\} \tag{3.15}$$

for the sake of simplicity, we omit the details of the proof.

4. ESTIMATES

Let $U^0 \equiv (U_{i,0})_{i \in \mathbf{Z}}$ be any element of l^2 which, for some (positive) values of M, S, h, satisfies:

$$0 \leqq U_{i,0} \leqq M, \qquad \forall\, i \in \mathbf{Z}; \qquad U_{i,0} = 0 \qquad \text{for} \quad |i| \geqq \frac{S}{h} \tag{4.1}$$

and let $U^j \equiv (U_{i,j})_{i \in \mathbf{Z}}$ be defined through (3.7). we want to derive for $\{U^j\}_{j \geqq 0}$ some estimates which may depend on h, k, M, S, but not on U^0; with an eye to this aim, we use the convention of denoting with the same letter C any function such that:

$$\left.\begin{aligned} C = C(h, k, M, S) \text{ remains bounded} \\ \text{when } h, k, M, S \text{ remain bounded.} \end{aligned}\right\} \tag{4.2}$$

LEMMA 4.1: *The following estimates hold:*

$$\left.\begin{aligned} & hk\left\|\frac{1}{h^2}AU^1\right\|_{l^2}^2+hk\left\|\frac{1}{k}(U-U^0)\right\|_{l^2}^2+\frac{1}{h}a(U^1, U^1) \\ & \leqq C\left\{1+\frac{1}{h}a(U^0, U^0)\right\}; \end{aligned}\right\} \tag{4.3}$$

$$hk\sum_{j\geqq 0}\left\{\left\|\frac{1}{h^2}AU^{j+1}\right\|_{l^2}^2+\left\|\frac{1}{k}(U^{j+1}-U^j)\right\|_{l^2}^2\right\}\leqq C\left\{1+\frac{1}{h}a(U^0, U^0)\right\}. \tag{4.4}$$

Proof: We write (3.5) with $u = U^j$, $U = U^{j+1}$:

$$-\frac{1}{h^2}(AU^{j+1})_i-\frac{1}{k}(U^{j+1}-U^j)_i\in H(U_{i,j+1}), \qquad j\geqq 0, \quad i\in\mathbf{Z}. \tag{4.5}$$

Taking the square and summing in i, the left-hand member gives [*see* (3.2)]:

$$\left\|\frac{1}{h^2}AU^{j+1}\right\|_{l^2}^2+\left\|\frac{1}{k}(U^{j+1}-U^j)\right\|_{l^2}^2+\frac{2}{h^2k}a(U^{j+1}, U^{j+1}-U^j),$$

while the right-hand member is not greater than 1 for $|i| < (S+\sqrt{2M})/h+1$, and can be replaced by 0 for the remaining i's [recall Lemma 3.3: $(AU^{j+1})_i = (U^{j+1}-U^j)_i = 0$ for $|i| \geqq (S+\sqrt{2M})/h+1$]; so that, after summing in i, we can estimate with $2(S+\sqrt{2M})/h+1$.

Now, remarking that:

$$\begin{aligned} & 2a(U^{j+1}, U^{j+1}-U^j) \\ & \quad = 2a(U^{j+1}, U^{j+1})-2a(U^{j+1}, U^j) \\ & \quad \geqq 2a(U^{j+1}, U^{j+1})-2[a(U^{j+1}, U^{j+1})]^{1/2}[a(U^j, U^j)]^{1/2} \\ & \quad \geqq 2a(U^{j+1}, U^{j+1})-[a(U^{j+1}, U^{j+1})+a(U^j, U^j)] \\ & \quad = a(U^{j+1}, U^{j+1})-a(U^j, U^j) \end{aligned}$$

we can rewrite our relation under the form:

$$\left.\begin{aligned} & \left\|\frac{1}{h^2}AU^{j+1}\right\|_{l^2}^2+\left\|\frac{1}{k}(U^{j+1}-U^j)\right\|_{l^2}^2+\frac{1}{h^2k}a(U^{j+1}, U^{j+1}) \\ & \leqq \frac{1}{h^2k}a(U^j, U^j)+2\frac{S+\sqrt{2M}}{h}+1. \end{aligned}\right\} \tag{4.6}$$

From (4.6) with $j = 0$ we get obviously (4.3); in order to get (4.4), it is sufficient to sum in j (from 0 to M/k) the relations (4.6) and recall that (in view of Lemma 3.4) it is $AU^{j+1} = U^{j+1}-U^j = 0$ for $j > M/k$.

We will often use the spline functions of order 0, 1, 2, defined respectively by: S_0 = characteristic function of the interval $[-1/2, 1/2[$; $S_1 = S_0 \bigstar S_0$; $S_2 = S_1 \bigstar S_0$; or, otherwise stated:

$$\left.\begin{aligned} S_0(x) &= \begin{cases} 1 & \text{for } -\frac{1}{2} \leqq x < \frac{1}{2}; \\ S_1(x) = (1-|x|)^+ & \text{for } x \in \mathbf{R}; \\ 0 & \text{elsewhere on } \mathbf{R}, \end{cases} \\ S_2(x) &= \begin{cases} \frac{3}{4} - x^2 & \text{for } |x| \leqq \frac{1}{2}, \\ \frac{1}{2}\left(\frac{3}{2} - |x|\right)^2 & \text{for } \frac{1}{2} < x \leqq \frac{3}{2}, \\ 0 & \text{elsewhere on } \mathbf{R}. \end{cases} \end{aligned}\right\} \tag{4.7}$$

Let us remark that, if we define:

$$\left.\begin{aligned} f_{h,k}^{(1)}(x, t) &= \sum_{i \in \mathbf{Z}} \sum_{j \geqq 0} \left(-\frac{1}{h^2} A U^{j+1}\right)_i S_0\left(\frac{x}{h} - i\right) S_0\left(\frac{t}{k} - j - \frac{1}{2}\right); \\ f_{h,k}^{(2)}(x, t) &= \sum_{i \in \mathbf{Z}} \sum_{j \geqq 0} \left[-\frac{1}{k}(U^{j+1} - U^j)_i\right] S_0\left(\frac{x}{h} - i\right) S_0\left(\frac{t}{k} - j - \frac{1}{2}\right); \\ g_{h,k}^{(1)}(x, t) &= \sum_{i \in \mathbf{Z}} \sum_{j \geqq 0} \left(-\frac{1}{h^2} A U^{j+1}\right)_i S_0\left(\frac{x}{h} - i\right) S_1\left(\frac{t}{k} - j\right); \\ g_{h,k}^{(2)}(x, t) &= \sum_{i \in \mathbf{Z}} \sum_{j \geqq 0} \left[-\frac{1}{k}(U^{j+1} - U^j)_i\right] S_2\left(\frac{x}{h} - i\right) S_0\left(\frac{t}{k} - j - \frac{1}{2}\right), \end{aligned}\right\} \tag{4.8}$$

the relation (4.4) can be rewritten in the form:

$$\|f_{h,k}^{(1)}\|_{L^2(\mathbf{R}_+)}^2 + \|f_{h,k}^{(2)}\|_{L^2(\mathbf{R}_+)}^2 \leqq C\left\{1 + \frac{1}{h} a(U^0, U^0)\right\} \tag{4.9}$$

and, from well known properties of spline functions:

$$\|f_{h,k}^{(1)} - g_{h,k}^{(1)}\|_{L_x^2(H_t^{-1})} \leqq C\left\{1 + \frac{1}{h} a(U^0, U^0)\right\}^{1/2} . k, \quad (^6) \tag{4.10}$$

$$\|f_{h,k}^{(2)} - g_{h,k}^{(2)}\|_{L_t^2(H_x^{-1})} \leqq C\left\{1 + \frac{1}{h} a(U^0, U^0)\right\}^{1/2} . h \tag{4.11}$$

(6) $u \in L_x^2(H_t^{-1})$ means that $x \to u(x, .)$ is square-summable on $\mathbf{R}$ as a function valued in $H^{-1}(\mathbf{R}_+)$ [with the obvious norm]; the definition of $L^2(H_x^{-1})$ is similar.

Let us define:

$$U_{h,k}(x, t) = \sum_{i \in \mathbf{Z}} \sum_{j=0} U_{i,j+1} \, S_2\left(\frac{x}{h} - i\right) S_1\left(\frac{t}{k} - j\right), \qquad \forall (x, t) \in \overline{\mathbf{R}^2_+}; \tag{4.12}$$

as an obvious consequence of Lemma 3.3, Lemma 3.4 and properties of S_0, S_1, S_2, we get:

$$\left.\begin{array}{c} U_{h,k}(x, t) \geqq 0, \qquad \forall (x, t) \in \mathbf{R}^2_+; \\ U_{h,k}(x, t) = 0 \text{ whenever} \\ |x| \geqq S + \sqrt{2M} + \frac{3}{2} h, \qquad \text{or} \qquad t \geqq M, \end{array}\right\} \tag{4.13}$$

$$\frac{\partial^2 U_{h,k}}{\partial x^2} = g^{(1)}_{h,k}; \qquad \frac{\partial U_{h,k}}{\partial t} = g^{(2)}_{h,k}, \qquad \text{a.e. in } \mathbf{R}^2_+ \tag{4.14}$$

LEMMA 4.2: *The following estimate holds:*

$$\| U_{h,k} \|^2_{H^{2,1}(\mathbf{R}^2_+)} \leqq C \left\{ 1 + \frac{1}{h} a(U^0, U^0) \right\} \tag{4.15}$$

Proof: We can choose

$$\| u \|^2_{H^{2,1}(\mathbf{R}^2_+)} = \| u_{xx} \|^2_{L^2(\mathbf{R}^2_+)} + \| u_t \|^2_{L^2(\mathbf{R}^2_+)} + \| u(x, 0) \|^2_{H^1(\mathbf{R})},$$

[this will just change the value of C in (4.15)]; so that, in view of (4.9), (4.10), (4.11), (4.14), we only need and estimate of $\| U_{h,k}(x, 0) \|^2_{H^1(\mathbf{R})}$. Because of (4.13) [which implies that $\| U_{h,k}(x, 0) \|^2_{L^2(\mathbf{R})} \leqq C \| D_x U_{h,k}(x, 0) \|^2_{L^2(\mathbf{R})}$] we just have to show that $\| D_x U_{h,k}(x, 0) \|^2_{L^2(\mathbf{R})} \leqq C \{ 1 + (1/h) a(U^0, U^0) \}$.

In order to prove the last relation, we remark that

$$D_x U_{h,k}(x, 0) = \frac{1}{h} \sum_{i \in \mathbf{Z}} (U_{i,1} - U_{i+1,1}) \, S_1\left(\frac{x}{h} - i - \frac{1}{2}\right),$$

so that (7):

$$\| D_x U_{h,k}(x, 0) \|^2_{L^2(\mathbf{R})} \leqq \frac{C}{h^2} \sum_{i \in \mathbf{Z}} (U_{i,1} - U_{i+1,1})^2 \int_{\mathbf{R}} S_1^2\left(\frac{x}{h} - i - \frac{1}{2}\right) dx$$

$$\leqq \frac{C}{h} \sum_{i \in \mathbf{Z}} (U_{i,1} - U_{i+1,1})^2 = \frac{C}{h} a(U^1, U^1)$$

and we can conclude by using (4.3).

---

(7) Because of the properties of the supports of $S_m(\xi - i)$, $S_m(\xi - i')$, we have

$$\left(\sum_{i \in \mathbf{Z}} \alpha_i S_m(\xi - i)^2\right) \leq C_m \sum_{i \in \mathbf{Z}} \alpha_i^2 S_m^2(\xi - i)$$

where, e. g., $C_m = m - 1$.

LEMMA 4.3: *Setting:*

$$U_h(x) = \sum_{i \in \mathbf{Z}} U_{i,0} S_2\left(\frac{x}{h} - i\right), \qquad \forall x \in \mathbf{R}, \tag{4.16}$$

it is:

$$\| U_{h,k}(x, 0) - U_h(x) \|^2_{L^2(\mathbf{R})} \leqq C\left\{1 + \frac{1}{h} a(U^0, U^0)\right\}.k. \tag{4.17}$$

Proof: By using the remark in footnote (7), we get:

$$\| U_{h,k}(x, 0) - U_h(x) \|^2_{L^2(\mathbf{R})}$$
$$= \int_{\mathbf{R}} \left| \sum_{i \in \mathbf{Z}} (U_{i,1} - U_{i,0}) S_2\left(\frac{x}{h} - i\right)\right|^2 dx$$
$$\leqq C \sum_{i \in \mathbf{Z}} (U_{i,1} - U_{i,0})^2 \int_{\mathbf{R}} S_2^2\left(\frac{x}{h} - i\right) dx \leqq C h k^2 \left\| \frac{1}{k}(U^1 - U^0)\right\|^2_{l^2};$$

(4.17) then follows from (4.3).

Now let us remark that, from Lemma 3.6, we have:

$$U_{i,j+1} \leqq U_{i,j} + k\lambda(U^0), \qquad \forall i \in \mathbf{Z}, \quad j \geqq 0; \tag{4.18}$$

$$U_{i+1,j} \leqq 2 U_{i,j} + [1 + \lambda(U^0)] h^2, \qquad \forall i \in \mathbf{Z}, \quad j \geqq 0; \tag{4.19}$$

and from $U_{i+1,j+1} \geqq 0$, we get, in view of (4.5):

$$U_{i,j} \leqq k + \left(1 + \frac{2k}{h^2}\right) U_{i,j+1}, \qquad \forall i \in \mathbf{Z}, \quad j \geqq 0. \tag{4.20}$$

LEMMA 4.4: *Let* (r, s) *be such that* $U_{r,s} = 0$; *and let* (i, j) *be such that* $|i - r| \leqq 2$, $|j - s| \leqq 1$; *then it is:*

$$U_{i,j} \leqq 2\{3 + 2\lambda(U^0)\}\{k + h^2\}.$$

Proof: From $U_{r,s} = 0$, and by applying twice (4.19), we get:

$$U_{r+1,s} \leqq [1 + \lambda(U^0)] h^2; \qquad U_{r+2,s} \leqq 3[1 + \lambda(U^0)] h^2;$$

then (4.18) gives

$$\max[U_{r+2,s+1}, U_{r+1,s+1}, U_{r,s+1}] \leqq k\lambda(U^0) + 3[1 + \lambda(U^0)] h^2.$$

We need now a similar estimate for $U_{i,s-1}$, if $s > 0$; from $U_{r,s} = 0$, $s > 0$, and (4.20), it follows that $U_{r,s-1} \leqq k$; then from $U_{r+1,s} \leqq [1 + \lambda(U^0)] h^2$ (relation that we already checked), and from (4.18) we get:

$$U_{r+1,s-1} \leqq k + [1 + \lambda(U^0)][h^2 + k];$$

finally, from this last relation and (4.19) we get

$$U_{r+2,s-1} \leqq 2k + [1 + \lambda(U^0)][3h^2 + 4k].$$

Remark 4.1: We could get a similar formula by using $\mu(U^0)$ instead of $\lambda(U^0)$; and another one involving both $\lambda(U^0)$ and $\mu(U^0)$.

LEMMA 4.5: *Let* $(\bar{x}, \bar{t})$ *be any point in* $\bar{\mathbf{R}}^2_+$; *from*

$$U_{h,k}(\bar{x}, \bar{t}) > 2\{3+2\lambda(U^0)\}\{h^2+k\}$$

it follows that

$$U_{i,j} > 0 \quad \text{for} \quad \left|\frac{\bar{x}}{h} - i\right| \leqq 2, \quad \left|\frac{\bar{t}}{k} - j\right| \leqq 1.$$

Proof: In the opposite case, we would have $U_{r,s} = 0$ for a suitable pair (r, s) with

$$\left|\frac{\bar{x}}{h} - r\right| \leqq 2, \quad \left|\frac{\bar{t}}{k} - s\right| \leqq 1;$$

then $U_{i,j} \leqq 2\{3+2\lambda(U^0)\}\{h^2+k\}$ for (i, j) "near" (r, s); but this is absurde because $U_{h,k}(\bar{x}, \bar{t})$ is a convex combination of such $U_{i,j}$ and by assumption $U_{h,k}(x, t) > 2\{3+2\lambda(U^0)\}\{h^2+k\}$.

5. CONVERGENCE RESULTS

We will denote by $\Theta_{h,k}$, Θ_h, respectively, the maps defined by:

$$\left.\begin{aligned} \Theta_{h,k} U^0 &= U_{h,k}(x, t), \quad U_{h,k} \text{ given by (4.12);} \\ \Theta_h U^0 &= U_h(x), \quad U_h \text{ given by (4.16).} \end{aligned}\right\} \tag{5.1}$$

Starting from $U_0(x)$ such that:

$$U_0(x) \in H^1_c(\mathbf{R}); \quad U_0(x) \geqq 0, \quad \forall x \in \mathbf{R}, \tag{5.2}$$

let us define the family $\{U^0_h\}_{h>0}$ by means of:

$$U^0_h \equiv (U_{i,0,h})_{i\in\mathbf{Z}}; \quad U_{i,0,h} = \frac{1}{h}\int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - i\right) dx, \quad \forall i \in \mathbf{Z}. \tag{5.3}$$

LEMMA 5.1: *Let* S^*, M^* *be such that:*

$$U_0(x) \leqq M^*; \quad U_0(x) = 0 \quad \text{for} \quad |x| \geqq S^*. \tag{5.4}$$

Then, for each $h > 0$, the vector U^0_h defined through (5.3) satisfies (4.1) with $M = M^*$, $S = S^* + (3h)/2$; moreover it is:

$$\frac{1}{h} a(U^0_h, U^0_h) \leqq C \| U_0(x) \|^2_{H^1(\mathbf{R})}, \tag{5.5}$$

$$\| \Theta_h U^0_h - U_0(x) \|_{L^2(\mathbf{R})} \leqq C h \| U_0(x) \|_{H^1(\mathbf{R})}. \tag{5.6}$$

Proof: All properties are classical and very easy to check; let us simply remark that (5.5) can be proved by means of:

$$\frac{1}{h} a(U_h^0, U_h^0) = \left\| D_x \sum_{i \in \mathbf{Z}} U_{i,0,h} S_1\left(\frac{x}{h} - i\right)\right\|^2_{L^2(\mathbf{R})}$$

and $\sum_{i \in \mathbf{Z}} U_{i,0,h} S_1(x/h - i)$ converges in $H^1(\mathbf{R})$ to $U_0(x)$.

Proof of theorems 2.1, 2.2: From (4.15), (5.5) we get:

$$\left.\begin{array}{c}\text{the family } \{\Theta_{h,k} U_h^0\}_{h,k=0} \text{ has at least a cluster point} \\ U(x, t) \in H^{2,1}(\mathbf{R}_+^2)[\text{when } h, k \to 0^+];\end{array}\right\} \tag{5.7}$$

for any such cluster point U we must have, from (4.13):

$$U \in H_c^{2,1}(\mathbf{R}_+^2); \qquad U(x, t) \geqq 0, \qquad \forall (x, t) \in \overline{\mathbf{R}_+^2}; \tag{5.8}$$

moreover, from (4.10), (4.11), (5.5), taking into account (4.5) and $0 \leqq H(U_{i,j+1}) \leqq 1$, we get:

$$0 \leqq U_{xx} - U_t \leqq 1, \qquad \text{a.e. in } \mathbf{R}_+^2. \tag{5.9}$$

Still from (4.5), by using classical arguments of monotonicity type, we get also [8]:

$$U_{xx} - U_t = 1 \text{ in the [open] set } \{(x, t) \in \mathbf{R}_+^2 \mid U(x, t) > 0\} \tag{5.10}$$

so that in order to prove that U solves Problem 2.1 we just need (2.4); and this relation follows from (5.6), (4.17). The uniqueness being obvious, Theorem 2.1 is proved; and Theorem 2.2 is an obvious consequence of Lemma 3.5, Lemma 3.6 (3.15), and of easy properties of the map $U_0(x) \to \{U_h^0\}_{h>0}$ defined in (5.3).

Let us remark that, as we have just seen, any cluster point $U(x, t)$ of $\{\Theta_{h,k} U_h^0\}_{h,k>0}$ must coincide with the (unique) solution of Problem 2.1; this obviously implies that the whole family $\{\Theta_{h,k} U_h^0\}_{h,k>0}$ converges [weakly in $H^{2,1}(\mathbf{R}_+^2)$ and strongly in $C^0(\overline{\mathbf{R}_+^2})$] to $U(x, t)$; we want now to derive some estimates for $\| \Theta_{h,k} U_h^0 - U \|_{L^\infty(\mathbf{R}_+^2)}$.

To this aim, let us recall some results on the Cauchy-Dirichlet problem for the heat equation in a family of spaces of Sobolev type (*see* e. g. [15], also for notations); let us consider a rectangle

$$R = \{(x, t) \mid -L < x < L; 0 < t < T\} \tag{5.11}$$

---

(8) By extracting a sub-family we can assume that $U_{h,k} \to U$ uniformly, because of the uniform compactness of supp $(U_{h,k})$ and of the compact embedding $H^{2,1}(E) \subset C^0(\bar{E})$ for E regular bounded.

with L, T given positive numbers; for $\theta \in [0, 1]$ we denote by $H^{2\theta, \theta}(R)$ the space $L^2(0, T; H^{2\theta}(-L, L)) \cap H^{\theta}(0, T; L^2(-L, L))$; and by $H^{-2\theta, -\theta}(R)$ the space $L^2(0, T; H^{-2\theta}(-L, L)) + H^{-\theta}(0, T; L^2(-L, L))$ [remark that $H^{-2\theta, -\theta}(R) = (H_0^{2\theta, \theta}(R))'$, where, as usual, the subscript o added to the symbol of a function space denotes the closure of C^∞-compactly supported functions].

With these notations, it is (*see* e. g. [15]).

THEOREM 5.1: *Let* θ *be given with* $3/4 < \theta < 1$. *For any* $\{F, G\}$ *such that:*

$$F \in H^{2\theta-2, \theta-1}(R), \qquad G \in H_0^{2\theta-1}(-L, L) \tag{5.12}$$

there exists a unique V *such that:*

$$\left.\begin{array}{ll} V \in H^{2\theta, \theta}(R); \quad V(\pm L, t) = 0 & \text{for } 0 \leqq t \leqq T; \\ V(x, 0) = G(x) \quad \text{for } -L \leqq x \leqq L; & V_{xx} - V_t = F \quad \text{in } \mathscr{D}'(R). \end{array}\right\} \tag{5.13}$$

Remark 5.1: Let us recall that:

$$H^{2\theta, \theta}(R) \subset C^0(\bar{R}) \qquad \text{for } \theta > \frac{3}{4} \tag{5.14}$$

so that the meaning of boundary conditions in (5.13) is the classical one.

Remark 5.2: In Theorem 5.1, as well as in (5.14), there is a continuous dependence; in particular, with a constant C depending only on θ, L, T, it is:

$$\| V \|_{L^\infty(R)} \leqq C \{ \| F \|_{H^{2\theta-2, \theta-1}(R)} + \| G \|_{H^{2\theta-1}(-L, L)} \}, \tag{5.15}$$

We will apply these results on choosing L, T such that, for h sufficiently small, $\Theta_{h,k} U_h^0$ is supported in R (⁹) [then also $U(x, t)$ will be supported in R]; with the notations in (4.8), we will choose in (5.12):

$$F = g_{h,k}^{(1)} - g_{h,k}^{(2)} + f_{h,k}^{(2)} - f_{h,k}^{(1)}; \qquad G = (\Theta_{h,k} U_h^0)(x, 0) = (\Theta_h U_h^0)(x) \tag{5.16}$$

and we will denote by $v_{h,k}$ the corresponding solution V of (5.13).

LEMMA 5.2: *The following estimate holds:*

$$\| v_{h,k} \|_{L^\infty(R)} \leqq C_\theta (h^{2(1-\theta)} + k^{1-\theta}), \qquad \frac{3}{4} < \theta \leqq 1 \tag{5.17}$$

where C_θ *may depend on* $U_0(x)$, h, k, *but remains bounded for* $h, k \to 0^+$.

Proof: We want to apply (5.14); by means of the interpolation inequality (*see* [15] again):

$$\|f\|_{L^2(0, T; H^{2\theta-2}(-L, L))} \leqq C \|f\|_{L^2(R)}^{2\theta-1} \|f\|_{L^2(0, T; H^{-1}(-L, L))}^{2(1-\theta)}, \qquad \frac{3}{4} < \theta \leqq 1$$

(⁹) e. g., for $h \leq 1$, we can choose $L = S + \sqrt{2}\, M + 3/2$, $T = M$ [*see* (4.13)]; remark that the corresponding R depends only on $U_0(x)$.

and by using (4.9), (4.11), (5.5), we get:

$$\| f_{h,k}^{(2)} - g_{h,k}^{(2)} \|_{L^2(0,T,H^{2\theta-2}(-L,L))} \leqq C h^{2(1-\theta)} \{ 1 + \| U_0(x) \|_{H^1(\mathbf{R})} \}$$

and, in a similar way, by using (4.10) instead of (4.11):

$$\| f_{h,k}^{(1)} - g_{h,k}^{(1)} \|_{H^{\theta-1}(0,T;L^2(-L,L))} \leqq C k^{1-\theta} \{ 1 + \| U_0(x) \|_{H^1(\mathbf{R})} \},$$

so that, with the choice (5.16) of F, the norm $\| F \|_{H^{2\theta-2,\theta-1}(R)}$ can be estimated by the right hand member of (5.17).

From (4.15), (4.17), (5.5), (5.6), still *via* interpolation inequalities, we get for G given in (5.16),

$$\| G \|_{H_0^{2\theta-1}(-L,L)} \leqq C \{ k^{1-\theta} + h^{2(1-\theta)} \} . \{ 1 + \| U_0(x) \|_{H^1(\mathbf{R})} \};$$

so that we can conclude by using (5.15).

LEMMA 5.3: *The following estimates hold,* $\forall (x,t) \in \overline{R}$:

$$(\Theta_{h,k} U_h^0)(x,t) \leqq U(x,t) + v_{h,k}(x,t) + \| v_{h,k} \|_{L^\infty(R)} + 2[3 + 2\lambda(U_h^0)](h^2 + k), \tag{5.18}$$

$$(\Theta_{h,k} U_h^0)(x,t) \geqq U(x,t) + v_{h,k}(x,t) - \| v_{h,k} \|_{L^\infty(R)}. \tag{5.19}$$

Proof: Let us define:

$$\delta = \Theta_{h,k} U_h^0 - U - v_{h,k} - \| v_{h,k} \|_{L^\infty(R)} - 2[3 + 2\lambda(U_h)](h^2 + k);$$

and

$$\omega = \{ (x,t) \in R \mid \delta(x,t) > 0 \}.$$

From

$$U(x,t) \geqq 0, \; v_{h,k}(x,t) + \| v_{h,k} \|_{L^\infty(R)} \geqq 0,$$

we must have in ω:

$$(\Theta_{h,k} U_h^0)(x,t) > 2[3 + 2\lambda(U_h^0)](h^2 + k),$$

so that we can apply Lemma 4.5 and, by using the definition of $v_{h,k}$, we get:

$$\delta_{xx} - \delta_t = 1 - (U_{xx} - U_t) \quad \text{in } \omega.$$

In particular, $\delta_{xx} - \delta_t \geqq 0$ in ω; and (by continuity) $\delta = 0$ on $\delta\omega \cap R$; moreover, on the parabolic boundary of R it is obviously $\delta \leqq 0$; the maximum principle then implies that $\omega = \varnothing$, i. e. (5.18).

In a similar way, setting now $\delta = U + v_{h,k} - \| v_{h,k} \|_{L^\infty(R)} - \Theta_{h,k} U_h^0$, and $\omega = \{ (x,t) \in R \mid \delta(x,t) > 0 \}$, in ω it is $U(x,t) > 0$, so that $U_{xx} - U_t = 1$, and then $\delta_{xx} - \delta_t \geqq 0$ in ω; still *via* the maximum principle we get $\omega = \varnothing$.

We can collect both Lemma 5.2 and Lemma 5.3 in the form:

$$\left.\begin{aligned} \|\Theta_{h,k} U_h^0 - U\|_{L^\infty(R)} \leq C\{h^2+k\}^{1-\theta} + 2\{3+2\lambda(U_h^0)\}\{h^2+k\}, \\ \frac{3}{4} < \theta \leq 1; \end{aligned}\right\} \tag{5.20}$$

and we need now an estimate on $\lambda(U_h^0)$.

LEMMA 5.4: $\lambda(U_h^0) \leq C h^{-3/2}$, *C depending only on* $U_0(x)$.

Proof:

$$\begin{aligned} \lambda(U_h^0) &= \left\| \left[-1 - \frac{1}{h^2}(A U_h^0)_i \right]^+ \right\|_{l^\infty} \\ &\leq 1 + \frac{1}{h^2} \left\| A\left(\frac{1}{h}\int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - i\right) dx \right) \right\|_{l^\infty} \\ &= 1 + \frac{1}{h^3} \left\| \int_{\mathbf{R}} \{U_0(x)\, S_2\}\left(\frac{x}{h} - i - 1\right) dx \right. \\ &\qquad \left. -2\int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - i\right) dx + \int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - i + 1\right) dx \right\|_{l^\infty} \\ &= 1 + \frac{1}{h^3} \left\| \int_{\mathbf{R}} \{U_0(x+h) - 2U_0(x) + U_0(x-h)\}\, S_2\left(\frac{x}{h} - i\right) dx \right\|_{l^\infty} \\ &\leq 1 + \frac{1}{h^3} \sup_x |U_0(x+h) - 2U_0(x) + U_0(x-h)| \cdot \left\| \int_{\mathbf{R}} S_2\left(\frac{x}{h} - i\right) dx \right\|_{l^\infty} \\ &= 1 + \sup_x \left| \frac{U_0(x+h) - 2U_0(x) + U_0(x-h)}{h^2} \right| ; \end{aligned}$$

taking into account that the elements of $H_c^1(R)$ are Hölder continuous of exponent 1/2, we get the Lemma.

We can collect all previous result under the form:

THEOREM 5.2: *For* $h, k \to 0^+$ *the family* $\Theta_{h,k} U_h^0$ *converges (weakly in* $H^{2,1}(\mathbf{R}_+^2)$ *and uniformly in* $\overline{\mathbf{R}}_+^2$*) to* $U(x, t)$*; moreover we have the error bound* (10):

$$\left.\begin{aligned} &\exists C, \quad \forall \varepsilon > 0\, \exists C_\varepsilon [C, C_\varepsilon \text{ depending on } U_0(x)] \text{ such that} \\ &\|\Theta_{h,k} U_h^0 - U\|_{L^\infty(\mathbf{R}_+^2)} \leq C_\varepsilon (h^2+k)^{1/4-\varepsilon} + C h^{-3/2}(h^2+k); \end{aligned}\right\} \tag{5.21}$$

(10) It is an "interesting" bound just for $k = o(h^{3/2})$; *see* however the following section 6.

in particular, on choosing $h = 1/n$, $k = 1/n^2$, *and setting*

$$U_n(x, t) = (\Theta_{1/n,\,1/n^2}\; U^0_{1/n})(x, t),$$

it is:

$$\| U_n - U \|_{L^\infty(\mathbf{R}^2_+)} = o(n^{-1/2+\varepsilon}), \qquad \forall \varepsilon > 0. \tag{5.22}$$

Remark 5.3: Coming back to notations (4.12), let us remark that $U_{i,j+1}$ is an average of values of $U_{h,k}(x, t)$ in points near $(i/h, j/k)$ [e. g. we have

$$U_{i,j+1} = \int_{\mathbf{R}} U_{h,k}\left(x, \frac{j}{k}\right) h.\beta\left(\frac{x}{h} - i\right) dx,$$

where $\beta(\xi)$ is the constant by parts which equals

$$\frac{12}{7} \text{ on } \left[-\frac{1}{2}, \frac{1}{2}\right[, \ 0 \text{ outside } \left[-\frac{3}{2}, \frac{3}{2}\right[, \quad \text{and} \quad -\frac{3}{7} \text{ elsewhere}\Big].$$

In particular, if we define

$$\tilde{U}_{h,k}(x, t) = \sum_{i \in \mathbf{Z}} \sum_{j \geqq 0} U_{i,j+1}\, S_0\left(\frac{x}{h} - i - \frac{1}{2}\right) S_0\left(\frac{t}{k} - j - \frac{1}{2}\right),$$

we will have

$$\begin{aligned} &| U_{h,k}(x, t) - \tilde{U}_{h,k}(x, t) | \\ &\quad \leqq C \sup \{ | U_{h,k}(x, t) - U_{h,k}(\xi, \tau) |;\ |x - \xi| \leqq 3h,\ |t - \tau| \leqq 2k \}, \end{aligned}$$

so that we can estimate $\| \tilde{U}_{h,k} - U_{h,k} \|_{L^\infty(\mathbf{R}^2_+)}$ by means of Hölder norms of $U_{h,k}$; in particular, with obvious notations, we can replace (5.22) by:

$$\| \tilde{U}_n - U \|_{L^\infty(\mathbf{R}^2_+)} = o(n^{-1/2+\varepsilon}), \qquad \forall \varepsilon > 0, \tag{5.23}$$

because of Hölder continuity of elements of $H^{1,2}(\mathbf{R}^2_+)$.

6. FURTHER RESULTS AND FINAL REMARKS

Let us firstly give a complement to Lemma 5.4.

LEMMA 6.1: *If* $U_0(x)$ *satisfies* (2.10), *then* [*with the same* λ] U^0_h *defined through* (5.3) *satisfies* $\lambda(U^0_h) \leqq \lambda$ *for any* h.

Proof: Let us firstly remark that, starting from $U_0(x) = 1/2\, x^2$, formula (5.3) gives a $\{ U^0_h \}$ such that $(A U^0_h)_i \equiv -1\ \forall i \in \mathbf{Z}$, $\forall h > 0$; so that, by substraction of $1/2(1+\lambda)x^2$, we can start from $U_0(x)$ which is a concave function, and we need to show that $(AU_h^0)_i \geqq 0 \; \forall\, i \in \mathbf{Z}, \; \forall\, h > 0$. Now we have, $\forall\, h > 0, \; \forall\, i \in \mathbf{Z}$:

$$\begin{aligned}(AU_h^0)_i &= -U_{i+1,0,h} + 2\,U_{i,0,h} - U_{i-1,0,h} \\ &= -\frac{1}{h}\int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - i - 1\right) dx \\ &\quad + \frac{2}{h}\int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - i\right) dx - \frac{1}{h}\int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - i + 1\right) dx \\ &= \frac{2}{h}\int_{\mathbf{R}} S_2\left(\frac{x}{h} - i\right) . \frac{2\,U_0(x) - U_0(x-h) - U_0(x+h)}{2}\, dx \geqq 0\end{aligned}$$

because of the concavity of $U_0(x)$.

Remark 6.1: In a similar way, we could prove that, if U_0 satisfies (2.11), it is $\mu(U_h^0) \leqq \mu$ [the same μ]; the proof will be omitted.

It is now clear that for a "smooth" $U_0(x)$ we can improve the convergence of $\Theta_{h,k}\, U_h^0$ to $U(x, y)$; let us prove, e. g., the following theorem:

THEOREM 6.1: *If* $U_0(x) \in W^{2,\infty}(\mathbf{R})$ (11) *the family* $\Theta_{h,k}\, U_{h,k}$ *converges to* $U(x, y)$ *in the weak star topology of* $W^{2,1,\infty}(\mathbf{R}_+^2)$ (12); *moreover it is*

$$\| \Theta_{h,k}\, U_h^0 - U \|_{L^\infty(\mathbf{R}_+^2)} = o((h^2 + k)^{1-\varepsilon}), \qquad \forall\, \varepsilon > 0. \tag{6.1}$$

Proof: The first part of the thorem follows from (4.14), Lemma 3.6, (3.15), Lemma 6.1 and Rem. 6.1; in order to prove (6.1) it is sufficient to study problem (5.13) in spaces like $W^{2\theta,\theta\, p}(\mathbf{R}_+^2)$ (13) instead of $H^{2\theta,\theta}(\mathbf{R}_+^2)$; we can go up to $\theta > 3/(2\,p)$. (14), but p can be choosen as large as we want, so that we can choose $\theta = \varepsilon$.

Remark 6.2: The existence theorem in $W^{2,1,\infty}$ for inequalities with $U_0 \in W^{2,\infty}$ is contained in [12]. Starting with U_0 in intermediate spaces between $W^{2,\infty}(\mathbf{R})$ and $H^1(\mathbf{R})$, we could get estimates which are intermediate between (6.1) and (5.21); however the meaning of such estimates is not obvious because of the terme $h^{-3/2}\, k$ which appears in (5.21); *see* Remark 6.3 in the sequel.

We can also use Lemma 6.1 in order to avoid the assumption $k = o(h^{3/2})$ [*see* footnote (10)]; in fact, from Lemma 5.3 and Lemma 6.1 it follows that:

$$\left.\begin{aligned} &\| \Theta_{h,k}\, U_h^0 - U \|_{L^\infty(\mathbf{R}_+^2)} = O(\| v_{h,k} \|_{L^\infty(R)} + h^2 + k) \\ &\qquad \text{if } \; U_0'' \text{ is bounded from above;}\end{aligned}\right\} \tag{6.2}$$

(11) Say U_0, U_0', $U_0'' \in L^\infty(\mathbf{R})$.

(12) For $1 \leqq p \leqq +\infty$, $W^{2,1,p}(\mathbf{R}_+^2) = \{ v \mid v, v_x, v_{xx}, v_t \in L^p(\mathbf{R}_+^2) \}$.

(13) For $\theta \in]0, 1[$, $W^{2\theta,\theta,p}(\mathbf{R}_+^2)$ is an interpolate space between $W^{2,1,p}(\mathbf{R}_+^2)$ and $L^p(\mathbf{R}_+^2)$.

(14) Remark that (5.14) must be replaced by $W^{2\theta,\theta,p}(\mathbf{R}_+^2) \subset C^0(\overline{\mathbf{R}}_+^2)$ for $\theta > 3/(2\,p)$.

In particular, from Lemma 5.2 and (6.2) or, which is the same, from (5.20) and Lemma 6.1 we get:

LEMMA 6.2: *If U''_0 is bounded from above, it is:*

$$\| \Theta_{h,k} U_h^0 - U \|_{L^\infty(\mathbf{R}_+^2)} = o((h^2+k)^{1/4-\varepsilon}), \qquad \forall \varepsilon > 0. \tag{6.3}$$

Remark 6.3: As already seen, problems like (5.13) can be studied in many types of spaces; and for the solution $v_{h,k}$ corresponding to the choice (5.16) of F, G, we could state the estimate:

$$\left\{ \begin{array}{l} \| v_{h,k} \|_{L^\infty(R)} \leqq C_{\theta,p}(h^2+k)^{1-\theta}, \qquad \theta > \dfrac{p+1}{2p} \\ \qquad \text{if} \quad U_0(x) \in W^{2\theta-1,p}(\mathbf{R}); \end{array} \right.$$

combining with (6.2), we get:

$$\left\{ \begin{array}{l} \text{if } U''_0 \text{ is bounded from above, and if } U_0 \text{ belongs to} \\ \qquad W^{2\theta-1,p}(\mathbf{R}), \qquad \text{with} \quad \theta > \dfrac{p+1}{2p}, \text{ it is} \\ \| \Theta_{h,k} U_h^0 - U \|_{L^\infty(\mathbf{R}_+^2)} = o((h^2+k)^{(1-1/p)/2-\varepsilon}), \qquad \forall \varepsilon > 0; \end{array} \right.$$

in particular, for the most interesting biological problem:

$$\left. \begin{array}{r} \text{on choosing } U_0(x) = \dfrac{[(1-|x|)^+]^2}{2} \ \text{(15), it is} \\ \| \Theta_{h,k} U_h^0 - U \|_{L^\infty(\mathbf{R}_+^2)} = o((h^2+k)^{1/2-\varepsilon}), \qquad \forall \varepsilon > 0. \end{array} \right\} \tag{6.4}$$

A final remark is concerned with the choice of $\{ U_h^0 \}_{h>0}$ given by (5.3). It is quite obvious that, with different choices, the convergence results still hold if we have similar estimates for $\Theta_h U_h^0 - U_0$; e. g. we could define U_h^0 by means of:

$$U_h^0 \equiv (U_{i,0,h})_{i\in\mathbf{Z}}; \qquad U_{i,0,h} = \frac{1}{h}\int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - i + \frac{1}{2}\right) dx. \tag{6.5}$$

Remark that, with the choice (6.5), we will have:

$$\left. \begin{array}{l} \text{if } U_0(x) \equiv U_0(-x), \qquad \text{then} \quad U_h^0 \text{ defined through (6.5) satisfies} \\ \qquad\qquad U_{i,0,h} \equiv U_{1-i,0,h}; \end{array} \right\} \tag{6.6}$$

(15) Remark that such a $U_0(x)$ satisfies

$$U''_0 \leqq 1 \qquad \text{and} \qquad U_0 \in W^{1+1/p-\varepsilon,p}(\mathbf{R})\, \forall p \in [1, +\infty], \qquad \forall \varepsilon > 0.$$

in fact

$$U_{1-i,0,h} = \frac{1}{h}\int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - 1 + i + \frac{1}{2}\right) dx$$

$$= [\text{because of the symmetry of } U_0,\ S_2]$$

$$= \frac{1}{h}\int_{\mathbf{R}} U_0(x)\, S_2\left(\frac{x}{h} - i + \frac{1}{2}\right) dx = U_{i,0,h}.$$

Also remark that:

$$\left.\begin{array}{l} \text{if } U_0(x) = U_0(-x) \quad \text{and} \quad U_0' \leqq 0 \quad \text{for} \quad x > 0, \\ \quad \text{then } U_h^0 \text{ defined through (6.5) satisfies} \\ \quad U_{i+1,0,h} \leqq U_{i,0,h} \quad \text{for} \quad i \geqq 0 \end{array}\right\} \tag{6.7}$$

[for $i = 0$ we have equality because of (6.6); for $i > 0$ the property follows from the monotonicity of $U_0(x)$].

Properties (6.6), (6.7) are interesting from a "practical" point of view, in fact, in solving Problem 2.1 with a symmetric $U_0(x)$, we can ask for a method which solves just half a problem, in the first quadrant Q, and then duplicate the solution by symmetry [*see* (2.12)]. Now, on choosing U_h^0 given by (5.3), we get in fact a symmetric discrete solution [*see* Lemma 3.5; it is obvious that, from $U_0(x) = U_0(-x)$, formula (5.3) gives U_h^0 such that $U_{i,0,h} = U_{-i,0,h}$]; however, it is difficult to solve just half a problem in the discrete case corresponding to the data as in Lemma 3.5. On the contrary, on choosing U_h^0 given by (6.5), the discrete solution $U_{i,j}$ will satisfy $U_{i,j} = U_{1-i,j}$ [*see* Rem. 3.3 and (6.6)]; and we will see that in this case we can solve just the problem for $i, j \geqq 0$ (instead of $i \in \mathbf{Z}$, $j \geqq 0$).

In fact, let us consider the following problem where

$$l_+^2 = \left\{ v \mid v \equiv (v_i)_{i \geqq 0};\ \sum_{i=0}^{\infty} v_i^2 < +\infty \right\}:$$

PROBLEM 6.1: Given $w \in l_+^2$, find $W \in l_+^2$ such that:

$$\left.\begin{array}{l} -\dfrac{1}{h^2}(A\,W)_i - \dfrac{1}{k}\,W_i \in H(W_i) - \dfrac{1}{k}\,w_i, \qquad i = 1, 2, 3, \ldots \\ W_0 = W_1 \end{array}\right\} \tag{6.8}$$

It is easy to check that Problem 6.1 has a unique solution, and that, setting:

$$\left.\begin{array}{llll} u_i = w_i & \text{for} \quad i > 0; & u_i = w_{1-i} & \text{for} \quad i \leqq 0, \\ U_i = W_i & \text{for} \quad i > 0; & U_i = W_{1-i} & \text{for} \quad i \leqq 0, \end{array}\right\} \tag{6.9}$$

the vector U is the solution of Problem (2.18) corresponding to u. We can also state Problem 6.1 in following the variational form [similar to the one given in Probl. 3.1]:

PROBLEM 6.2: Given $w \in l^2_+$, find $W \in l^2_+$ such that, $\forall\, v \in l^2_+$

$$\frac{1}{h^2} b(W, W-v) + \frac{1}{k}(W, W-v)_{l^2_+} + \sum_{i>0} W_i^+ \leqq \frac{1}{k}(w, W-v)_{l^2_+} + \sum_{i>0} v_i^+ \tag{6.10}$$

where

$$(v, z)_{l^2_+} = \sum_{i \geqq 0} v_i z_i \qquad \text{and} \qquad b(v, z) = \sum_{i \geqq 1} (v_{i-1} - v_i)(z_{i-1} - z_i).$$

Obviously, if we denote by $\mathscr{T}^+_{h,k}$ the map $w \longrightarrow W$, we will have for $\mathscr{T}^+_{h,k}$ properties quite similar to the ones stated for $\mathscr{T}_{h,k}$ in Section 3; by analogy to definition (3.7), we could also define:

$$W^0 = w; \qquad W^j = (\mathscr{T}^+_{h,k})^j\, W^0, \qquad (j = 1, 2, \ldots) \tag{6.11}$$

and with a slight modification with respect to (5.11) ([16]):

$$\left.\begin{aligned}(\Theta^+_{h,k}\, W^0)(x, t) = \sum_{i, j \geqq 0} W_{i, j+1}\, S_2\left(\frac{x}{h} - i + \frac{1}{2}\right) S_1\left(\frac{t}{k} - j\right)\\ \text{for} \quad x \geqq 0, \quad t \geqq 0.\end{aligned}\right\} \tag{6.12}$$

We will confine ourselves to state the result corresponding to Problem 1.1:

THEOREM 6.2: Let $\{ W^0_h \}_{h>0}$ be defined through:

$$\left.\begin{aligned} W^0_h \equiv (W_{i,0,h})_{i \geqq 0}; \qquad & W_{i,0,h}\\ &= \frac{1}{h}\int_{\mathbf{R}} \frac{1}{2}[(1-|x|)^+]^2\, S_2\left(\frac{x}{h} - i + \frac{1}{2}\right) dx,\end{aligned}\right\} \tag{6.13}$$

and let $u(x, t)$ be the solution of Problem 1.1. Then:

$$\Theta^+_{h,k}\, W^0_h \text{ converges to } u(x, t) \text{ uniformly in } \overline{Q} \text{ and weakly in } H^{2,1}(Q); \tag{6.14}$$

$$\| \Theta^+_{h,k}\, W^0_h - u \|_{L^\infty(Q)} = o((h^2 + k)^{1/2 - \varepsilon}), \qquad \forall\, \varepsilon > 0. \tag{6.15}$$

Remark 6.4: Setting, for a fixed C:

$$\left.\begin{aligned}\Omega_{h,k} = \{(x, t) \in Q \mid (\Theta^+_{h,k}\, W^0_h)(x, t) > C(h^2 + k)^{1/2}\},\\ \tau_{h,k}(x) = \max\{ t \mid (x, t) \in \Omega_{h,k}\},\end{aligned}\right\} \tag{6.16}$$

we will have

$$\Omega_{h,k} = \{(x, t) \mid 0 < x < l;\ 0 < t < \tau_{h,k}(x)\},$$

$$\tau_{h,k} \text{ is monotone nonincreasing in } x$$

([16]) We translated of 1/2 the origin in x, in order to have $(D_x \Theta^+{}_k\, W^0_h)(0, t) = 0$.

and $\Omega_{h,k}$ will converge from the interior to Ω [*see* Th. 1.1] for any choice of C in (6.16).

Remark 6.5: If we are interested just to an approximation of Ω, and we need not a "regular" approximation of $u(x, t)$,we can avoid the construction of $\Theta_{h,k}^{+} W_h^0$ and use just the values of $W_{i,j}$; *see* Remark 5.3.

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